LECTURE ON QUARKYONIC EFFECTIVE FIELD THEORY*

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I review the essential features of Quarkyonic Matter. I argue how such features can be included in a field theoretical description. This field theory has nucleons and quarks because close to the Fermi surface, the degrees of freedom of Quarkyonic Matter are nucleons and inside the Fermi sea, they are quarks. Ghost nucleon fields are needed to avoid over-counting degrees of freedom, and to allow the physical nucleon degrees of freedom not to extend within the quark Fermi sea.

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1. Introduction: Review of basic features of Quarkyonic Matter

Quarkyonic Matter [1] was proposed based on an interesting observation about Quantum Chromodynamics in the large-$N_{\rm c}$ limit. [2, 3]. The effect of fermion loops is reduced by $1/N_{\rm c}$ in this limit. This observation has a direct consequence concerning de-confinement in finite baryon number density systems.

Suppose we have a Fermi gas at finite density corresponding to a quark chemical potential μ_Q and temperature T. For a two-flavor system, $\mu_Q = \mu_u + \mu_d$, where $\mu_{u,d}$ are the up and down quark chemical potentials (or Fermi energies). The baryon number chemical potential is $\mu_B = \mu_p + \mu_n = N_{\rm c}\mu_Q$, where $\mu_{p,n}$ are the proton and neutron chemical potentials. This equation follows because a baryon is made of $N_{\rm c}$ quarks. The large-$N_{\rm c}$ limit is taken by fixing

$$g^2_{\rm 't\,Hooft} = g^2 N_{\rm c} \tag{1}$$

as $N_{\rm c} \to \infty$.

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Let us consider the long-range force between two heavy test quarks for a finite temperature and finite density system. The quark loop contribution to this potential is shown in Fig. 1. The gluon loops contribute with a factor of $g^2 N_c$ and remain in the large-N_c limit. On the other hand, the quark loops are proportional to g^2, one power of N_c less because there are only N_c quarks compared to N_c^2 gluon degrees of freedom. Therefore, as far as the potential is concerned, it is only controlled by gluon loops, and these gluon loops do not at all feel the effect of finite density. Therefore, the system remains confined independent of density, until the temperature is raised to the deconfinement temperature. The deconfinement temperature is density-independent!

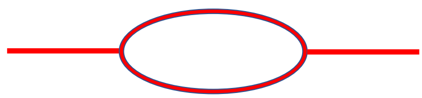

Fig. 1. The quark loop contribution to the static potential between two heavy test quarks.

This of course breaks down when the chemical potential in the quark loop is sufficiently large. The quark loop is contributing to the Debye mass, and so, when this mass is as large as $\Lambda_{\rm QCD}$, then screening effects occur and there is deconfinement

$$m^2_{\rm Debye} \sim g^2_{\rm 't\,Hooft} \frac{\mu_Q^2}{N_c} \sim \Lambda^2_{\rm QCD}\,. \tag{2}$$

Only when the quark chemical potential becomes $\mu_Q \gg \sqrt{N_c}\Lambda_{\rm QCD}$ does deconfinement occur. Due to the factor of $\sqrt{N_c}$, this takes place at densities that are large compared to a typical QCD scale $\Lambda^3_{\rm QCD}$. Within the Fermi sea, final states are Pauli blocked so that interactions are only allowed when quarks change places in the Fermi sea. The interactions typically occur when the momentum scale is large. As it is the case in jet physics, when the momentum scale of interactions is much larger than $\Lambda_{\rm QCD}$, such interactions may be described perturbatively, in spite of the fact that the media remains confined. This means that the degrees of freedom deep inside the Fermi sea are those of an almost free gas of quarks. The quarks form a Fermi sphere. On the other hand, interactions are permitted for small angles and small momentum transfer for particles near the Fermi surface. Such Fermi surface states are not blocked from scattering by filled energy levels. Here, infrared confining effects are important, and quarks are confined into nucleons and mesons.

It is important to observe that in the Quarkyonic picture, confinement effects take place at the top of the Fermi sea, not the bottom. This is because long-range interactions, which are sensitive to the confinement scale, take place near the Fermi surface. Another way to see this is to observe that the nucleons themselves do not saturate all possible fermionic states, but a filled quark Fermi sea does. The lowest energy states should be completely filled, and this corresponds to the zero temperature Fermi sea of quarks. Near the Fermi surface, one can make a continuous transition to low occupation density, and this is what the confined degrees of freedom at the Fermi surface do.

In the large-N_c limit, the probability to see a nucleon is of the order of $e^{-\beta(M-\mu_B)}$ which is of the order of e^{-N_c} in the large-N_c limit since both the baryon mass and the baryon chemical potential are of the order of $N_c \Lambda_{QCD}$, so long as the temperature is not too large, $T \leq \Lambda_{QCD}$, and the baryon chemical potentials are less than the nucleon mass M. There are three separate possible phases of matter. When the temperature and density are low, matter is confined and there are no baryons. When the baryon chemical potential exceeds the nucleon mass, but the temperature remains low, matter is confined and there are nucleons present. This matter is Quarkyonic. Of course at high temperatures, or at super high density and low temperature, matter may become deconfined. A cartoon of such possible phases is shown in Fig. 2.

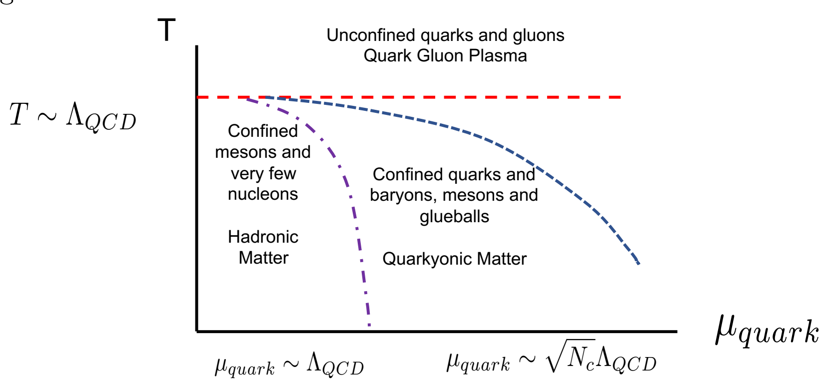

Fig. 2. (Color online) A cartoon for the various phases of high density matter. The gray/red dashed line is the deconfinement temperature at zero density. The three phases shown are deconfined matter, hadronic matter and Quarkyonic Matter.

Quarkyonic Matter may be envisioned as a Fermi sea of quarks surrounded by a Fermi shell of nucleons. This is illustrated in Fig. 3. This picture is reinforced by recent computations that demonstrate how this shell structure may dynamically emerge [4, 5]. Below, we describe the origin of this picture.

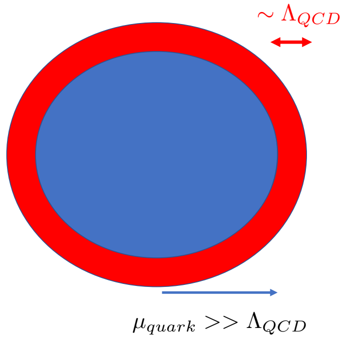

Fig. 3. The quark Fermi sea surrounded by a shell of nucleonic matter.

We will first argue that at low densities, it is preferable to have a free gas of nucleons compared to a free gas of quarks. We will work in the additive quark parton model where quark masses $m_Q = m_N/N_\mathrm{c}$. This means that quark Fermi momenta and nucleon Fermi momenta are related since

$$k_Q^\mathrm{F} = \sqrt{\mu_Q^2 - M_Q^2} = \frac{\sqrt{N_\mathrm{c}^2\mu_Q^2 - N_\mathrm{c}^2 M_Q^2}}{N_\mathrm{c}} = \frac{\sqrt{\mu_B^2 - M_N^2}}{N_\mathrm{c}} = \frac{k_N^\mathrm{F}}{N_\mathrm{c}}\,. \tag{3}$$

For a free Fermi gas, the baryon density of a gas of an isospin degenerate nucleons and a gas of quarks is

$$\begin{aligned} \rho_N &= \frac{2}{3\pi^2}\left(k_N^\mathrm{F}\right)^3\,, \\ \rho_Q &= \frac{2}{3\pi^2}\left(k_Q^\mathrm{F}\right)^3\,. \end{aligned} \tag{4}$$

We see that if there was equilibrium between quarks, and that quarks and nucleons were free gases, then the quark contribution would be suppressed by $1/N_\mathrm{c}^3$, so the pressure for a gas at fixed baryon number chemical potential would be enormously higher than that for the corresponding gas of quarks.

We can ask the question a different way. Suppose we compare the energy per nucleon of a free gas of quarks to that for a free gas of nucleons at equal baryon number density. In this case, the chemical potentials of the quark and nucleon gases that we compare would be different. Equality $\rho_N = \rho_Q$ would require that the Fermi momenta of nucleons in the gas of nucleons would be equal to the Fermi momenta in the gas of quarks. On the other hand, the energy of a baryon per unit baryon number of nuclear matter at low densities is very close to M_N, but for quarks, for equal baryon number density it is of the order of $N_\mathrm{c}\sqrt{(k_Q^\mathrm{F})^2+M_Q^2} \sim N_\mathrm{c}\sqrt{(k_N^\mathrm{F})^2+M_Q^2} \sim \sqrt{(N_\mathrm{c}k_N^\mathrm{F})^2+M_N^2} > M_N$. At nuclear matter densities, where $k_N^\mathrm{F} \sim \Lambda_\mathrm{QCD}$, at equal densities, the quark would be relativistic and the energy per particle very large. Therefore, the nucleon gas is preferred. In such a description, the effect of confining interactions would make a gas of free quarks have even higher energy per baryon, so this conclusion is quite robust.

What happens as the density is increased? At larger baryon density, the hard core nucleon interactions become important. These will temper the increase in baryon density by increasing the number of nucleons. The baryon density cannot exceed the density in hard nucleon cores. Of course, the interaction strength is of the order of N_c for the cores, not infinite strength for finite N_c. This strength is sufficient to shift the nucleon energy by a value of the order of the nucleon mass. If the nucleon Fermi energy is shifted by this amount, this would naively correspond to a very large shift in the baryon density, because $\rho_N \sim (k_N^\mathrm{F})^3 \sim M_N^3 \sim N_\mathrm{c}^3 \Lambda_\mathrm{QCD}^3$.

How can this be avoided? If the nucleons sit on a shell of decreasing thickness when the nucleon Fermi energy of the nucleons increases, then the density, which is the integral over the momenta of particles in the shell, need not increase. A finite value of baryon density may be maintained for the contribution from nucleons. The density of the nucleons should be expected to saturate at the density of matter corresponding to the hard cores of nucleons, which is of the order of $\rho_\text{hard cores} \sim \Lambda_\mathrm{QCD}^3$. The increase in the baryon number density comes from increasing the quark density. For quark Fermi momentum $k_q^\mathrm{F} \ll \Lambda_\mathrm{QCD}$, this increase associated with the quarks is quite small. As the baryon density associated with quarks slowly increases, the quark and nucleon Fermi energies rapidly rise until there is a quark Fermi sea with a Fermi energy of the order of $E_Q^\mathrm{F} \sim \Lambda_\mathrm{QCD}$, and the nucleons become relativistic in the Fermi shell with $E_N^\mathrm{F} \sim N_\mathrm{c}\Lambda_\mathrm{QCD} \sim M_N$. This rapid increase in the Fermi energies at a slowly varying density leads to a hard equation of state with sound velocities of the order of one [4].

We will construct a description of Quarkyonic Matter that combines both nucleonic and quark degrees of freedom using a field theory. Such a field theoretical description of nucleons will be limited to those nucleons with momentum close to the Fermi surface. Inside the Fermi sea of quarks, nucleonic degrees of freedom will be Pauli blocked, because the nucleons and states with nucleons are Pauli blocked. A nucleon cannot propagate when its momentum is $k_N < N_\mathrm{c} k_q$, since the state of the nucleon is already occupied.

Of course, this argument of total blocking of the nucleon states only holds when the system is at zero temperature. We also want a theory of Quarkyonic Matter that is useful at finite temperature [6]. Such a description must have pion degrees of freedom, and be capable of phenomenologically including the effects of meson–nucleon exchanges. Massive vector mesons are ultimately responsible for generating hard core repulsive nucleon interactions. A good theory should be a theory of nucleons and mesons at low densities, and also be capable of describing quarks and gluons at high density and temperature. Quarkyonic Matter should be a possibility, although one needs to verify dynamically that Quarkyonic Matter is the energetically preferred form of matter.

The central purpose of this paper is to present a construction of such a model, and is an expanded tutorial exposition of that presented in Ref. [7]. Our model properly resolves the duality between nucleonic and quark descriptions. The nucleon can be thought of as a nucleonic state or as an ensemble of quarks. This means that if quarks occupy states, the quarks composing a nucleon cannot occupy the same states. In a field theoretical model, we can have a field that corresponds to a nucleon, and a field that corresponds to quarks, so long as it is constrained so that the quark fields associated with these states do not overlap the same states in the nucleon as are occupied by the quarks. This can be accomplished by an unconstrained nucleon field, an unconstrained quark field, and a negative metric nucleon ghost field that fills the same states occupied by the quarks. The only purpose of such ghost fields is to cancel away the degrees of freedom of the unconstrained nucleon field composed of quark states that might occupy states already occupied by quarks.

The presentation consists of a slightly expanded version of the results of Ref. [7].

2. Ghost and removing un-physical states

We consider a theory at finite quark density so that there is a quark chemical potential, μ_Q. As discussed in the previous section, we want to ensure that nucleon states that have quarks as constituents do not overlap filled states associated with quarks. Such quarks can block nucleons with chemical potential less than $\mu_G \sim N_{\rm c}\mu_Q$. The density of such states is

$$\rho^G = \frac{1}{1+{\rm e}^{\beta(N_C E_Q - \mu_G)}}\,. \tag{5}$$

When we consider the constituent quark model, then $E_N = N_{\rm c} E_Q$. A color singlet state in the quark Fermi sea may be considered to be a nucleon, so that the energy of this $N_{\rm c}$ quark state is also a nucleon energy. Therefore,

$$\rho^G = \frac{1}{1+{\rm e}^{\beta(E_N - \mu_G)}}\,. \tag{6}$$

The density of states for an ideal gas of quarks is

$$\rho^Q = \frac{1}{1+{\rm e}^{\beta(E_Q - \mu_Q)}}\,. \tag{7}$$

The phase-space density of nucleons which are not overlapping with states that are occupied is

$$\rho^N_{\rm const} = \rho^n = \frac{1}{1+\mathrm{e}^{\beta(E_N-\mu_N)}} - \frac{1}{1+\mathrm{e}^{\beta(E_N-\mu_G)}}\,. \tag{8}$$

This equation describes a shell of nucleons sitting above a Fermi sea of quarks when the temperature is low. As the temperature increases, the shell becomes more diffuse.

How can this shell distribution be implemented in a field theoretical model? We have an unconstrained nucleon field with chemical potential μ_N and mass M_N, a ghost nucleon field with chemical potential $\mu_G \sim N_{\rm c}\mu_Q$ and mass M_N, and quark field with mass $m_Q = M_N/N_{\rm c}$ and chemical potential μ_Q. The nucleon field will be denoted by N, the quark field will be Q and the ghost field will be G. The ghost field will have the same Lorentz structure as the nucleonic field. It will satisfy anti-periodic boundary conditions in imaginary time, like the nucleons. It will however be a commuting and not an anti-commuting field. In a path integral, it will be represented by a c-number integration variable rather than a Grassman algebra variable. The action for such a theory in Euclidean time is

$$\begin{aligned} S_{\rm E} \;=\; & \int\limits_0^\beta \mathrm{d}t \int\limits_V \mathrm{d}^3x \left\{ \bar{N}\left(\frac{1}{i}\gamma\cdot\partial - i\mu_N\gamma^0 + M_N\right)N \right. \\ & +\bar{G}\left(\frac{1}{i}\gamma\cdot\partial - i\mu_G\gamma^0 + M_N\right)G \\ & \left. +\bar{Q}\left(\frac{1}{i}\gamma\cdot\partial - i\gamma^0\mu_Q + M_Q\right)Q \right\}. \end{aligned} \tag{9}$$

It is useful to define the propagator

$$S(\mu_N, M) = \frac{1}{\frac{1}{i}\gamma\cdot\partial - i\mu_N\gamma^0 + M_N}\,. \tag{10}$$

Now, if we integrate over a Grassman variable, the path integral for the partition function will give

$$Z_N = \det{}^{-1} S(\mu_N, M_N)\,, \tag{11}$$

where the integration over the ghost c-number field yields

$$Z_G = \det\, S(\mu_G, M_N)\,. \tag{12}$$

The formula for the grand potential is obtained from

$$\Omega = g\frac{1}{\beta V}\,\mathrm{Tr}\,\left\{\ln(S(\mu_N, M_N)) + N_\mathrm{c}\ln(S(\mu_Q, M_Q)) - \ln(S(\mu_G, M_N))\right\}\,. \tag{13}$$

The factor of N_c for the quarks comes from the fact that there are N_c quark fields. If our quarks and nucleons are isodoublets, the degeneracy factor is $g = 2$. As expected, the ghost contribution to the action has the opposite sign from that of the nucleons, and is present to precisely to cancel out the contribution of modes of the nucleon where the quark states of the nucleon are already occupied by the quark states. Since we have chosen the boundary condition in Euclidean time to be precisely the same between the ghost fields and the quark and nucleon fields, except for mass and chemical potential, the determinant is the same. It is straightforward to evaluate these determinants by standard methods of diagonalizing in momentum space, and performing a contour integral representation for the Matsubara frequency sum. The result is that the grand potential is

$$\begin{aligned}\Omega \;=\; & -gT\int\frac{\mathrm{d}^3p}{(2\pi)^3}\left\{\ln\left(1+\mathrm{e}^{-\beta(E_N(\vec{p})-\mu)}\right) - \ln\left(1+\mathrm{e}^{-\beta(E_N(\vec{p})-\mu_G)}\right)\right. \\ & \left. + N_\mathrm{c}\ln\left(1+\mathrm{e}^{-\beta\left(E_Q(\vec{p})-\mu_Q\right)}\right)\right\}\,. \end{aligned} \tag{14}$$

Let us notice that the ghosts are present to subtract the contribution of the pressure of the nucleons due to the Pauli blocking. Since the entropy and number density follow by the ordinary thermodynamic relations term by term in the expression above, except for an overall minus sign for the ghost contribution, all of the expression for the pressure, energy density, entropy, and number density are simply the nucleon and quark contributions minus that of the ghost nucleons.

When interactions are included, one can explore various theories to see if one can obtain a reasonable shell structure for Quarkyonic Matter. One can compute the pressure and energy density as a function of quark, ghost, and nucleon chemical potentials. The ghost chemical potential is determined in terms of the quark number density. Extremizing the energy density with respect to quark number density at fixed total baryon number density will determine the chemical potentials. This gives an expression for the pressure in terms of the net nucleon density and quark density, and the temperature.

3. Various effective theories of Quarkyonic Matter

We can now use our effective field theory to solve many different problems.

3.1. Constructing an excluded-volume model at $T \to 0$ limit

The excluded-volume model [5] can be derived from action (9). First, one can check the shell-like distribution of nucleons in the ideal gas limit by assuming $\mu_G = N_{\rm c}\mu_Q$. The grand potential (14) of the symmetric two-flavor system can be written as follows:

$$\Omega = -p = \epsilon_n - \mu_n n_n - (\mu_n - \mu_G) n_G + \epsilon_Q - \mu_{\tilde{Q}} n_{\tilde{Q}} \,, \tag{15}$$

where $\tilde{Q}$ represents the quantity counted in a baryon number unit and the relations $\mu_G = \mu_{\tilde{Q}} = N_{\rm c}\mu_Q$ ($n_G = N_{\rm c}^3 n_{\tilde{Q}}$), $\mu_n = \mu_N$, and $n_n = n_N - n_G$ are understood. The energy densities are obtained as

$$\begin{aligned} \epsilon_n &= \epsilon_N - \epsilon_G \\ &= \frac{2}{\pi^2} \int\limits_{\left(\mu_G^2 - m_N^2\right)^{\frac{1}{2}}}^{\left(\mu_N^2 - m_N^2\right)^{\frac{1}{2}}} \mathrm{d}p p^2 \left(\vec{p}^{\,2} + M_N^2\right)^{\frac{1}{2}} \,, \end{aligned} \tag{16}$$

$$\epsilon_Q = \frac{2N_{\rm c}}{\pi^2} \int\limits_0^{\left(\mu_Q^2 - m_Q^2\right)^{\frac{1}{2}}} \mathrm{d}p p^2 \left(\vec{p}^{\,2} + M_Q^2\right)^{\frac{1}{2}} \,. \tag{17}$$

To include the excluded-volume effect ($n_0 \neq \infty$), one should apply the enhanced chemical potential of the nucleon ($\mu_N^{\rm ex}$) and ghost ($\mu_G^{\rm ex}$) which satisfy the following relations:

$$n_n^{\rm ex} = \frac{n_n}{1 - \frac{n_n}{n_0}} = \frac{n_N - n_G}{1 - \frac{n_N - n_G}{n_0}} = \frac{2}{\pi^2} \int\limits_{\sqrt{\left(\mu_G^{\rm ex}\right)^2 - m_N^2}}^{\sqrt{\left(\mu_N^{\rm ex}\right)^2 - m_N^2}} \mathrm{d}p p^2 \,, \tag{18}$$

$$n_N^{\rm ex} = \frac{n_N}{1 - \frac{n_n}{n_0}} = \frac{2}{\pi^2} \int\limits_0^{\sqrt{\left(\mu_N^{\rm ex}\right)^2 - m_N^2}} \mathrm{d}p p^2 \,, \tag{19}$$

$$n_G^{\rm ex} = \frac{n_G}{1 - \frac{n_n}{n_0}} = \frac{2}{\pi^2} \int\limits_0^{\sqrt{\left(\mu_G^{\rm ex}\right)^2 - m_N^2}} \mathrm{d}p p^2 \,. \tag{20}$$

Then, the pressure can be written in terms of the quantities obtained in the reduced system volume

$$\begin{aligned} p &= -\epsilon_n + \mu_n n_n + (\mu_n - \mu_G) n_G - \epsilon_Q + \mu_{\tilde{Q}} n_{\tilde{Q}} \\ &= -\epsilon_n^{\rm ex} + \mu_n^{\rm ex} n_n^{\rm ex} + (\mu_n^{\rm ex} - \mu_G^{\rm ex}) n_G^{\rm ex} - \epsilon_Q + \mu_{\tilde{Q}} n_{\tilde{Q}} \,, \end{aligned} \tag{21}$$

which is equivalent to the Fermi–Dirac statistics of van der Waals gas without attractive contribution [8, 9]. Here the modified version of the constraint $\mu_G = N_{\rm c}\mu_Q$ is applied

$$k_{\rm F}^{G{\rm ex}} = \left((\mu_G^{\rm ex})^2 - m_N^2\right)^{\frac{1}{2}} = N_{\rm c} k_{\rm F}^Q = N_{\rm c}\left(\mu_Q^2 - m_Q^2\right)^{\frac{1}{2}} \,, \tag{22}$$

which leads to $n_G^{\rm ex} = N_{\rm c}^3 n_{\tilde{Q}}$. The energy densities are given by

$$\epsilon_n^{\rm ex} = \frac{2}{\pi^2} \int\limits_{\sqrt{\left(\mu_G^{\rm ex}\right)^2 - m_N^2}}^{\sqrt{\left(\mu_N^{\rm ex}\right)^2 - m_N^2}} {\rm d}p p^2 \left(\vec{p}^{\,2} + M_N^2\right)^{\frac{1}{2}} \,, \tag{23}$$

$$\epsilon_Q = \frac{2N_{\rm c}}{\pi^2} \int\limits_0^{\sqrt{\mu_Q^2 - m_Q^2}} {\rm d}p p^2 \left(\vec{p}^{\,2} + M_Q^2\right)^{\frac{1}{2}} \,. \tag{24}$$

From the trivial relations

$$\frac{\partial \epsilon^{\rm ex}}{\partial n^{\rm ex}} = \mu^{\rm ex} = \left(1 - \frac{n}{n_0}\right)\mu + \frac{\epsilon}{n_0} \,, \tag{25}$$

$$\epsilon^{\rm ex} = \frac{\epsilon}{1 - \frac{n}{n_0}} \,, \tag{26}$$

all the quantities for the total system (denoted without the superscript 'ex') can be obtained from potential (14) expressed in terms of the quantities calculated in the reduced system volume (denoted by the superscript 'ex')

$$\mu_n = \frac{\partial \epsilon}{\partial n_n} = \frac{\mu_n^{\rm ex}}{\left(1 - \frac{n_n}{n_0}\right)} - \frac{\epsilon_n^{\rm ex}}{n_0} \,, \tag{27}$$

$$\mu_Q = \frac{\partial \epsilon}{\partial n_{\tilde{Q}}} = \mu_{\tilde{Q}} + N_{\rm c}^3 \left(1 - \frac{n_n}{n_0}\right)\left(\mu_n^{\rm ex} - \mu_{\tilde{Q}}\right) \,, \tag{28}$$

where $\epsilon = \epsilon_n + \epsilon_Q$. In a practical calculation, the quark chemical potential (28) needs an additional cut-off factor $N_{\rm c}^3 \to N_{\rm c}^3 (\frac{3\pi^2}{2} n_{\tilde{Q}})^{\frac{1}{3}} / \sqrt{(\frac{3\pi^2}{2} n_{\tilde{Q}})^{\frac{2}{3}} + \Lambda^2}$

to satisfy the physical baryon number conservation ($n_B = n_n + n_{\tilde{Q}}$ with $n_n, n_{\tilde{Q}} \geq 0$) [5]. Although action (9) leads to the grand potential $\Omega(T \to 0,\ V,\ \mu) = -pV$, it is easier to work within the free energy $F(T \to 0,\ V,\ N) = \epsilon = -p + \mu n$ since the equilibrium will be determined by an additional constraint which requires the extremum of $F(T \to 0,\ V,\ N) = \epsilon$ at the fixed total baryon number density ($n_B = n_n + n_{\tilde{Q}}$)

$$\mathrm{d}\epsilon = \frac{\partial \epsilon}{\partial n_n}\mathrm{d}n_n + \frac{\partial \epsilon}{\partial n_{\tilde{Q}}}\mathrm{d}n_{\tilde{Q}} = \left(\frac{\partial \epsilon}{\partial n_n} - \frac{\partial \epsilon}{\partial n_{\tilde{Q}}}\right)\mathrm{d}n_n = 0\,, \tag{29}$$

where $\mathrm{d}n_B = \mathrm{d}n_n + \mathrm{d}n_{\tilde{Q}} = 0$ is understood. In this step, the chemical potentials are not free parameters but dependent on n_n and $n_{\tilde{Q}} = n_B - n_n$. According to the shell-like distribution of the nucleons, μ_n does not decrease monotonically [5], which means that $\mu_n n_n + \mu_{\tilde{Q}} n_{\tilde{Q}} = \mu_B n_B$ (where $\mu_n = \mu_{\tilde{Q}} = \mu_B$) does not correspond to the extremum of $\mu_n n_n + \mu_{\tilde{Q}} n_{\tilde{Q}}$ exactly whenever $n_{\tilde{Q}} \neq 0$. Therefore, once the low phase space is saturated by the quasi-free quark sea ($n_{\tilde{Q}} \neq 0$), the Quarkyonic-like configuration determined at the minimum of the free energy does not always correspond to the extremum of pressure in variations of $n_{\tilde{Q}}$.

3.2. Quarkyonic sigma model

The pion–nucleon or the quark–nucleon sigma models have played an important role in QCD phenomenological studies of finite temperature and density physics. Pion nucleon interactions provide a valid description when the typical momentum exchange of pions is not too hard. We can have a theory of ghost, quark, and pions interacting by introducing a scalar sigma field and pion field as

$$\begin{aligned} M_N &= g_N\left(\phi + i\pi\gamma^5\right)\,, \\ M_Q &= g_Q\left(\phi + i\pi\gamma^5\right)\,. \end{aligned} \tag{30}$$

In the additive quark–nucleon model, $g_N = N_{\mathrm{c}} g_Q$. The ghost and nucleon couplings to the pion field are identical, to guarantee the required cancellations in the quark Fermi sea. It is also easy to generalize such considerations for a Nambu–Jona-Lasinio model.

3.3. The nucleonic interactions

Nucleon interactions involve meson fields. The coupling of meson fields to ghosts is the same as that to nucleons, since we want to preserve the cancellation of ghost and nucleon states when there are filled quark states that exclude nucleons. The nucleons and ghosts will couple to $\bar{N}\Gamma_i N + \bar{G}\Gamma_i G$, where Γ_i represent the combination of Lorentz gamma and flavor matrices coupling to the meson field denoted by i. One might replace massive meson exchange interactions with contact terms when the density is not too high.

3.4. Quark interactions

Quark interactions may be introduced with varying degrees of sophistication. One might simply introduce them through the QCD Lagrangian. Perhaps one might introduce Wilson line interactions [10]. The interactions with the nucleons are implemented in some phenomenological way since the intrinsic interactions of quarks involve gluons, but the interactions among nucleons involve meson exchange. Gluons are important at finite temperature. Such a theory at low chemical potential interpolates to a theory of quarks and gluons, where nucleon interactions no longer appear, and as such can provide a proper interpolation to the high temperature quark–gluon plasma.

4. Summary and conclusions

This theoretical framework, I believe, for Quarkyonic Matter is similar to that of the Landau–Ginsburg theory for superconductivity. It provides a rich base for extension to a variety of different types of models. Its main virtue is that it provides a theoretically motivated interpolation between low density nucleonic matter and high density quark–gluon matter.

Such a theory framework allows one to ask and provide insight about many important issues. What is the nature of chiral symmetry restoration for Quarkyonic Matter? Can one find a good matching to a theory of nucleon interactions as the density is lowered? What is the phase diagram in the μ_B, T plane and what role does Quarkyonic Matter play in this phase diagram? What is the interplay between deconfinement and Quarkyonic Matter? Can one match Quarkyonic Matter onto the lattice descriptions of matter at low baryon density and finite temperature? How might one include interactions of quarks with nucleons beyond the simple meson exchange picture?

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