ENERGY AND RADIATION OF A HIGHLY RELATIVISTIC SPINNING PARTICLE IN SCHWARZSCHILD'S BACKGROUND

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The Mathisson–Papapetrou equations are used for investigations of influence of the spin-gravity coupling on a highly relativistic spinning particle in Schwarzschild's field. It is established that interaction of the particle spin with the gravitomagnetic components of the field, estimated in the proper frame of the particle, causes the large acceleration of the spinning particle relative to geodesic free fall. As a result, the accelerated charged spinning particle can generate intensive electromagnetic radiation when its velocity is highly relativistic. The significant contribution of the highly relativistic spin-gravity coupling to the energy of the spinning particle is analyzed.

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1. Introduction

Known important properties of gravitational interaction in General Relativity were discovered through investigations of motions of small test bodies (particles) in the gravitational field of a massive body. For example, the physics of black holes was studied by consideration of world lines and trajectories of simple test particles which follow the geodesic lines in the Schwarzschild and Kerr metrics [1, 2]. Here, "simple" means that the particle does not possess inner structure, with inner rotation or higher multipoles. In the classical picture of the gravitational collapse of a massive object, quantum properties of the particles are not taken into account.

Electrons, protons and other particles with nonzero spin, which in some classical approximation can be considered as particles with inner rotation, do not follow geodesic trajectories exactly. However, as it is emphasized in [1], in the usual situations, deviations of motions of the spinning test body (particles) from the corresponding geodesic motions are very small: this conclusion follows from the Mathisson–Papapetrou (MP) equations [3, 4]. Just these equations, which are the generalization of the geodesic equations for description of motions of a test rotating body in General Relativity, were derived for the first time in [3]. (This paper is absent in Ref. [1], in contrast to paper [4] which was published much later than [3].) An extensive bibliography on various applications of the MP equations is presented in [5]. Many recent papers are devoted to study of the spin-gravity effects both for small test spinning particles and rotating macroscopic bodies, for example, black holes [6–15].

Unusual situations with spinning particles arise when their orbital velocity in the Schwarzschild or Kerr field becomes very high, close to the speed of light. Then the influence of the spin-gravity coupling on the particles orbits can be significant [16–30]. The physical reason for this situation is connected with the fact that in a frame which moves relative to Schwarzschild's or Kerr's source with the very high velocity, the values of components of the gravitational field are much greater than in frames with low velocities. For example, as a result of strong spin-gravity action on the particle, the space regions of existence of the highly relativistic circular orbits for spinning particles in the Schwarzschild and Kerr backgrounds are much wider than for spinless particles [18, 21, 23, 24, 29, 30]. This fact is interesting for the analysis of a possible mechanism of generation of synchrotron radiation for charged spinning particles near compact astrophysical objects.

The purpose of this paper is to investigate the contribution of the spin-gravity coupling to the energy of a spinning particle moving with high velocity in Schwarzschild's field, and to obtain some estimation for electromagnetic radiation of a highly relativistic charged spinning particle. These investigations are based on the analysis of solutions of the exact MP equations.

The paper is organized in the following way. In Section 2, the MP equations and their physical meaning are discussed. Section 3 is devoted to the analysis of the relations following from these equations in the comoving tetrads representation for Schwarzschild's metric. The dependence of the spinning particle 3-acceleration relative to geodesic free fall as measured by the comoving observer on the particle velocity in Schwarzschild's field is evaluated. For a charged spinning particle, the expression for the intensity of its electromagnetic radiation caused by the acceleration is evaluated in Section 4. In Section 5, we investigate the difference in the values of energies of the spinning and spinless particles at their high velocities in Schwarzschild's field. We conclude in Section 6.

2. Mathisson–Papapetrou equations

The original formulation of the Mathisson–Papapetrou equations is [3]

$$\frac{D}{\mathrm{d}s}\left(mu^\lambda + u_\mu \frac{DS^{\lambda\mu}}{\mathrm{d}s}\right) = -\frac{1}{2}u^\pi S^{\rho\sigma} R^\lambda{}_{\pi\rho\sigma}\,, \tag{1}$$

$$\frac{DS^{\mu\nu}}{\mathrm{d}s} + u^\mu u_\sigma \frac{DS^{\nu\sigma}}{\mathrm{d}s} - u^\nu u_\sigma \frac{DS^{\mu\sigma}}{\mathrm{d}s} = 0\,, \tag{2}$$

$$S^{\lambda\nu} u_\nu = 0\,, \tag{3}$$

where $u^\lambda \equiv \mathrm{d}x^\lambda/\mathrm{d}s$ is the particle's 4-velocity, $S^{\mu\nu}$ is the antisymmetric tensor of spin, m and $D/\mathrm{d}s$ are the mass and the covariant derivative along u^λ, respectively. Here and in the following, greek indices run through 1, 2, 3, 4 and latin indices run through 1, 2, 3; the signature of the metric $(-, -, -, +)$ and the unites $c = G = 1$ are chosen.

Note that equations (1), (2) and (3) have an important unusual feature as compare to equations in classical (nonrelativistic) mechanics which describe the propagation of the center of mass of a rotating body and possible changes of its angular velocity. Namely, in classical mechanics, the motion of such a body is fully determined by the given initial values of the coordinates and velocity of the center of mass, and the value of the angular velocity. The situation is different with equations (1), (2) and (3). Indeed, then the left-hand side of equation (1) contains the terms

$$\frac{Du_\mu}{\mathrm{d}s}\frac{DS^{\lambda\mu}}{\mathrm{d}s} + u_\mu \frac{D^2 S^{\lambda\mu}}{\mathrm{d}s^2}\,. \tag{4}$$

Note that the second term in (4) is proportional to the second derivative of the angular velocity. As a result, according to the theory of differential equations, the fixed initial values of the coordinates, linear velocity and angular velocity without a given initial value of the angular acceleration, in general, are insufficient for determination of a single solution of equations (1), (2) and (3). There is a similar situation if after the differentiation of (3), instead of (4) one deals with the expression

$$-S^{\lambda\mu}\frac{D^2 u_\mu}{\mathrm{d}s^2} - \frac{DS^{\lambda\mu}}{\mathrm{d}s}\frac{Du_\mu}{\mathrm{d}s}\,. \tag{5}$$

Indeed, (5) contains the second derivative of the linear velocity and then to determine some unique solution, it is not sufficient to point out only initial values of coordinates and linear velocity, without acceleration.

There is a more simple case, without the second derivatives in the MP equations, when one considers the deviation of the particle motions from geodesics in the linear spin approximation. Then it is sufficient instead of equations (1) and (2) to deal with equations

$$m\frac{Du^\lambda}{ds} = -\frac{1}{2}u^\pi S^{\rho\sigma} R^\lambda{}_{\pi\rho\sigma}\,, \tag{6}$$

$$\frac{DS^{\mu\nu}}{ds} = 0 \tag{7}$$

(at relation (3), it follows from (1) and (2) that $m = \mathrm{const}$).

To avoid the terms with too high derivatives in the exact MP equations, it was proposed to consider instead of (1), (2) and (3) some modified equations [31, 32]

$$\frac{DP^\lambda}{ds} = -\frac{1}{2}u^\pi S^{\rho\sigma} R^\lambda{}_{\pi\rho\sigma}\,, \tag{8}$$

$$\frac{DS^{\mu\nu}}{ds} = 2P^{[\mu}u^{\nu]}\,, \tag{9}$$

$$S^{\lambda\nu}P_\nu = 0\,, \tag{10}$$

where

$$P^\nu = mu^\nu + u_\lambda \frac{DS^{\nu\lambda}}{ds} \tag{11}$$

is the particle 4-momentum. An important difference in equations (1), (2), (3) and (8), (9), (10) consists in the form of relations (3) and (10): due to the second term in the right-hand side of expression (11), in general, vector P^ν is not parallel to u^ν, and relation (3) does not follow from (10). (By the way, from equations (8), (9) and (10), some explicit expression for the components of u^λ through P^μ are obtained [33].)

Often relations (3) and (10) are treated as supplementary conditions for the MP equations. Without any supplementary condition, these equations describe some wide range of the representative points [4] which can be in different connection with a rotating body. To describe just the inner rotation of the body, it is necessary to fix the concrete corresponding representative point. In the Newtonian mechanics, the inner angular momentum of a rotating body is defined relative to its center of mass and just the motion of this center represents the propagation of the body in space. In relativity, the position of the center of mass of a rotating body depends on the frame [34, 35]. Then condition (3), which follows from the usual definition of the center-of-mass position [36], is common for the so-called proper and non-proper centers of mass. (We use the terminology when the proper frame for a spinning body is determined as the frame where the axis of the body rotation is at rest; correspondingly, the proper center of mass is calculated in the proper frame.) The usual solutions of the MP equations at condition (3) in the Minkowski spacetime describe the motion of the proper center of mass of a spinning body, whereas the helical solutions describe the motions of the family of nonproper centers of mass [34, 35].

In contrast to condition (3), relation (10) picks out a unique world line of a spinning particle in the gravitational field. However, from a physical point of view, equation (10) has an explicit restriction for its applications in the region of the highly relativistic motions of a spinning particle relative to the source of the gravitational field [22, 37].

Taking into account condition (3) or (10), one finds that the MP equations have the constant of motion

$$S^2 = \frac{1}{2} S_{\mu\nu} S^{\mu\nu}\,, \tag{12}$$

where $|S|$ is the absolute value of spin. When dealing with the MP equations, the condition for a spinning test particle

$$\frac{|S|}{mr} \equiv \varepsilon \ll 1 \tag{13}$$

must be taken into account [38], where r is the characteristic length scale of the background space-time (in particular, for the Schwarzschild metric r is the radial coordinate). For a macroscopic spinning test particle, relation (13) is a direct consequence of the physical property that for this particle $|S|$ is of the order of $mvr_{\rm part}$, where v is the linear velocity of a point at the surface of the rotating particle and $r_{\rm part}$ is the radius of the particle, with the clear conditions $v < 1$ and $r_{\rm part}/r \ll 1$.

Equations (1) and (2) with condition (3) can be presented through the 3-component value S_i [25], where by definition

$$S_i = \frac{1}{2u_4} \sqrt{-g}\, \varepsilon_{ikl} S^{kl}\,, \tag{14}$$

and ε_{ikl} is the spatial Levi-Civitá symbol. Then Eq. (2) takes the form [25] of

$$u_4 \dot{S}_i + 2\left(\dot{u}_{[4}u_{i]} - u^\pi u_\rho \Gamma^\rho_{\pi[4}u_{i]}\right) S_k u^k + 2S_n \Gamma^n_{\pi[4}u_{i]}u^\pi = 0\,, \tag{15}$$

where a dot denotes differentiation with respect to the proper time s, and square brackets denote antisymmetrization of indices; $\Gamma^n_{\pi 4}$ are the Christoffel symbols. The form of equation (1) in the terms of S_i is presented in [25].

3. Some relations following from the Mathisson–Papapetrou equations for a highly relativistic spinning particle in the Schwarzschild field

Let us consider equations (1), (2) and (3) in the linear spin approximation. Then according to (6) the deviation of the spinning particle world line from the geodesic line, for which $Du^\lambda/\mathrm{d}s = 0$, is determined by the term

$$-\frac{1}{2}u^\pi \frac{S^{\rho\sigma}}{m} R^\lambda{}_{\pi\rho\sigma}\,. \tag{16}$$

Since the components $S^{\rho\sigma}$ are proportional to S, according to (13), expression (16) is proportional to the small value ε. At the same time, in general, the absolute values of the particle's 4-velocity components $|u^\pi|$ can be within the wide region, from $|u^\pi| \ll 1$ to $|u^\pi| \gg 1$. (In this aspect, the situation is similar to the known one in Special Relativity, where according to the dynamics of a moving particle, its 4-velocity components are proportional to the relativistic Lorentz factor γ, and $\gamma \ll 1$ corresponds to the slow motions, whereas for the particle's velocities very close to the speed of light, the relation $\gamma \gg 1$ is satisfied.) It means that when the particle velocity is not very high, *i.e.* when the relation $|u^\pi| \gg 1$ is not satisfied, it is possible to search the solutions of the MP equations in the form of some small corrections to the corresponding solutions of the geodesic equations (at the condition that the values of the Riemann tensor components are not very high). Concerning the case $|u^\pi| \gg 1$, more detailed analysis is necessary. Indeed, at first glance, even when the relation $|u^\pi| \gg 1$ is satisfied and the absolute value of expression (16) becomes much greater than at the low velocity, one can suppose that this situation is a result of the kinematic effect only, when the value of the proper time of the highly relativistic particle is much less than for a slow particle. To verify this supposition, it is appropriate to consider the value of expression (16) in the frame comoving with the particle.

For description of the comoving frame of reference, we use the set of orthogonal tetrads $\lambda^\mu_{(\nu)}$, where $\lambda^\mu_{(4)} = u^\mu$ and the relations

$$\lambda^\mu_{(\nu)}\lambda^\pi_{(\rho)} g_{\mu\pi} = \eta_{(\nu)(\rho)}\,, \qquad g_{\mu\nu} = \lambda^{(\pi)}_\mu \lambda^{(\rho)}_\nu \eta_{(\pi)(\rho)} \tag{17}$$

take place (here, in contrast to the indices of the global coordinates, the local indices are placed in the parenthesis; $g_{\mu\nu}$ and $\eta_{(\nu)(\rho)}$ are the metric tensor of the curved spacetime and the Minkowski tensor, respectively). Without loss in generality, we direct the first space local vector (1) along the direction of spin. Then from equation (6), we have [25]

$$a_{(i)} = \frac{S_{(1)}}{m} R_{(i)(4)(2)(3)}\,, \tag{18}$$

where $a_{(i)}$ are the local components of the particle 3-acceleration relative to geodesic free fall as measured by the comoving observer; $S_{(1)}$ is the single nonzero component of the particle spin. Note that the right-hand side of equation (18) is the direct consequence of expression (16).

We take into account the definition of the gravitomagnetic components $B^{(i)}_{(k)}$ of the gravitational field for a moving observer in General Relativity according to [39]

$$B^{(i)}_{(k)} = -\frac{1}{2} R^{(i)(4)}{}_{(m)(n)} \varepsilon^{(m)(n)}{}_{(k)} \,. \tag{19}$$

Then equation (18) can be written in the form of

$$a_{(i)} = \frac{S_{(1)}}{m} B^{(i)}_{(1)} \,. \tag{20}$$

Let us analyze equation (20) in the case when the spinning particle is moving in the gravitational field of Schwarzschild's mass. We use the standard Schwarzschild coordinates $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$, $x^4 = t$ when the nonzero components of the metric tensor $g_{\mu\nu}$ are

$$\begin{aligned} g_{11} &= -\left(1 - \frac{2M}{r}\right)^{-1}, \qquad & g_{22} = -r^2\,, \\ g_{33} &= -r^2 \sin^2\theta\,, & g_{44} = 1 - \frac{2M}{r}\,, \end{aligned} \tag{21}$$

where M is the mass of Schwarzschild's source of the gravitational field. For a particle which is moving with any velocity (less than the velocity of light) in Schwarzschild's field, it is appropriate to take into account the Lorentz factor γ as estimated by an observer which is at rest relative to the source of the gravitational field. Then, for γ we write

$$\gamma = \frac{1}{\sqrt{1 - v^2}}\,, \tag{22}$$

where v^2 is the second power of the particle's 3-velocity relative to this observer. According to the general procedure which is described in [40], in the case of metric (21), we have the expression for the 3-velocity components v^i

$$v^i = \frac{\mathrm{d}x^i}{\sqrt{g_{44}}\,\mathrm{d}t}\,. \tag{23}$$

Then for v^2, we write

$$v^2 = v_i v^i = \gamma_{ik} v^i v^k\,, \tag{24}$$

where γ_{ik} is the 3-space metric tensor, with the following relationship between γ_{ik} and $g_{\mu\nu}$ for the diagonal metric: $\gamma_{ik} = -g_{ik}$. It follows from (22)–(24) with $u_\mu u^\mu = 1$ that

$$\gamma = \sqrt{u_4 u^4}\,. \tag{25}$$

By (25) and general relation for the particle 4-velocity $u_\mu u^\mu = 1$, we obtain

$$\gamma^2 = 1 + u_\perp^2 + \left(1 - \frac{2M}{r}\right)^{-1} u_\parallel^2\,, \tag{26}$$

where $u_\perp = r\mathrm{d}\varphi/\mathrm{d}s$ and $u_\parallel = \mathrm{d}r/\mathrm{d}s$ are the tangential and radial components of the particle's 4-velocity, respectively.

In the following, we will consider the case of the particle motion in the plane $\theta = \pi/2$ when its spin is orthogonal to this plane. It is convenient to orient the first space local axis (1) along the spin and the second axis (2) along the direction of the particle's motion. By the definition of the orthonormal tetrads, the third space local axis (3) is orthogonal to axis (1) and (2). Then, by direct calculation according to (17), (19) and (21), we obtain

$$B^{(1)}_{(2)} = B^{(2)}_{(1)} = \frac{3M}{r^3}\frac{u_\parallel u_\perp}{\sqrt{\gamma^2-1}}\left(1 - \frac{2M}{r}\right)^{-1/2}\,, \tag{27}$$

$$B^{(1)}_{(3)} = B^{(3)}_{(1)} = \frac{3M}{r^3}\frac{u_\perp^2 \gamma}{\sqrt{\gamma^2-1}}\,. \tag{28}$$

Let us compare the values from (27) and (28) at low and high velocities. When the velocity is low with $u_\parallel = \delta_1$, $u_\perp = \delta_2$, $|\delta_1| \ll 1$, $|\delta_2| \ll 1$, and $\gamma^2 - 1 = \varDelta^2 \ll 1$, where

$$\varDelta^2 = \left(1 - \frac{2M}{r}\right)^{-1} \delta_1^2 + \delta_2^2\,, \tag{29}$$

it follows from (27) and (28) that

$$B^{(1)}_{(2)} = B^{(2)}_{(1)} \approx \frac{3M}{r^3}\frac{\delta_1\delta_2}{\varDelta}\left(1 - \frac{2M}{r}\right)^{-1/2}\,, \tag{30}$$

$$B^{(1)}_{(3)} = B^{(3)}_{(1)} \approx \frac{3M}{r^3}\frac{\delta_2^2}{\varDelta}\,. \tag{31}$$

That is, at low velocity, the common term $3M/r^3$ in the expressions for the gravitomagnetic components (30) and (31) is multiplied by corresponding small factors

$$\left|\frac{\delta_1\delta_2}{\Delta}\right| \ll 1\,, \qquad \left|\frac{\delta_2^2}{\Delta}\right| \ll 1\,.$$

In the highly relativistic region, when $\gamma^2 \gg 1$ and both $u_\parallel^2$ and $u_\perp^2$ have order γ^2, it follows from (27) and (28) that

$$B^{(1)}_{(2)} = B^{(2)}_{(1)} \sim \frac{3M}{r^3}\left(1-\frac{2M}{r}\right)^{-1/2}\gamma\,, \tag{32}$$

$$B^{(1)}_{(3)} = B^{(3)}_{(1)} \sim \frac{3M}{r^3}\gamma^2\,. \tag{33}$$

When only $u_\perp^2 \gg 1$, with $u_\parallel^2 \ll u_\perp^2$, the values from (27) are proportional to $u_\parallel$, and the values from (28) are proportional to γ^2. In the case when $u_\parallel^2 \gg 1$ and $u_\perp^2 \ll u_\parallel^2$, the values from (27) and (28) are proportional to $u_\perp$ and $u_\perp^2$, respectively. Thus, according to (20), (32), (33), the absolute values of $a_{(i)}$ become much greater at the highly relativistic velocities of the spinning particle. It means that the smallness of ε from (13) does not lead to the conclusion about the small influence of the particle spin on its acceleration as estimated by the comoving observer. (Note that in the above considered partial case of the particle motion in Schwarzschild's field, the relation $|S_{(1)}| = |S|$ takes place.)

4. Acceleration and electromagnetic radiation of a highly relativistic spinning particle

It follows from (20), (27), (28) that the absolute value of the spinning particle acceleration

$$|\vec{a}| = \sqrt{a_{(1)}^2 + a_{(2)}^2 + a_{(3)}^2}$$

is determined by the expression

$$|\vec{a}| = \frac{3M}{r^2}\frac{|S|}{mr}|u_\perp|\sqrt{1+u_\perp^2}\,. \tag{34}$$

According to (34), $|\vec{a}|$ does not depend on the radial component of the particle velocity and essentially depends on its tangential velocity. In the case of the highly relativistic motion with $u_\perp^2 \gg 1$ by (34), we have

$$|\vec{a}| = \frac{3M}{r^2}\varepsilon\gamma^2\,, \tag{35}$$

where γ is the Lorentz factor calculated by the tangential velocity $u_\perp$, and ε is determined in (13).

We use expression (35) to estimate the electromagnetic radiation of a spinning particle which posses the electric charge q. Indeed, according to the known result of the classical electrodynamics, the intensity I of the electromagnetic radiation in the frame where the velocity of the charged particle is equal to 0 with nonzero acceleration w is given by expression [40]

$$I = \frac{2q^2w^2}{3c^3}\,, \tag{36}$$

where c is the speed of light. Inserting into (36) expression (35) as w in units where $c = 1$, we get

$$I = 6q^2\frac{M^2}{r^4}\varepsilon^2\gamma^4\,. \tag{37}$$

This equation shows that due to the term γ^4, the value of I can be significant for some high tangential velocities even for small values of ε and far from Schwarzschild's horizon ($r \gg 2M$).

Equation (37) is valid in the linear spin approximation for any particle trajectory in the equatorial plane of Schwarzschild's background. In the important partial case of the circular orbits in this background, we can write the generalization of Eq. (37) in the exact consideration by the particle spin. For this purpose, we use the representation of the MP equations in the terms of the comoving tetrads [25]. The exact form of these equations is

$$\begin{aligned}
a_{(1)} &= \frac{S_{(1)}}{m}R_{(1)(4)(2)(3)}\,,\\
a_{(2)} &= \frac{S_{(1)}}{m}\left(R_{(2)(4)(2)(3)} - a_{(2)}\gamma_{(2)(3)(4)} - \dot{a}_{(3)}\right)\,,\\
a_{(3)} &= \frac{S_{(1)}}{m}\left(R_{(3)(4)(2)(3)} - a_{(3)}\gamma_{(2)(3)(4)} - \dot{a}_{(2)}\right)\,,
\end{aligned} \tag{38}$$

where $\gamma_{(2)(3)(4)}$ are the Ricci coefficients of rotation calculated by the comoving tetrads. In the linear spin approximation, equations (38) coincide with equation (18).

In the case of Schwarzschild's metric, the standard coordinates $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$, $x^4 = t$, for the equatorial plane with $\theta = \pi/2$, we have $R_{(1)(4)(2)(3)} = 0$. Then according to (38), $a_{(1)} = 0$. In addition, for the circular orbits in this plane when $u^1 = 0$, $u^2 = 0$, $u^3 = \mathrm{const} \neq 0$, $u^4 = \mathrm{const} \neq 0$, $a_{(2)} = 0$, and $a_{(3)} = \mathrm{const} \neq 0$, it follows from (38):

$$a_{(3)} = \frac{S_{(1)}}{m}R_{(3)(4)(2)(3)}\left(1 - \frac{S_{(1)}}{m}\gamma_{(2)(3)(4)}\right)^{-1}\,. \tag{39}$$

After the direct calculation, we obtain

$$R_{(3)(4)(2)(3)} = -\frac{3M}{r}\left(u^3\right)^2 u^4 \left(1-\frac{2M}{r}\right)^{1/2}\left(u_4 u^4 - 1\right)^{-1/2}, \tag{40}$$

$$\gamma_{(2)(3)(4)} = -\left(1-\frac{3M}{r}\right) u^3 u^4 . \tag{41}$$

(Note that for the circular orbit with $r = 3M$, we have $\gamma_{(2)(3)(4)} = 0$, *i.e.* in this partial case, the contribution of the nonlinear spin terms in (39) is equal to 0.) The explicit expressions for the components of the particles 4-velocity u^3 and u^4 on the circular orbits in Schwarzschild's background follow directly from the exact MP equations (1)–(3) [23]. As a result, after (41) for the circular orbits with $r \neq 3M$, we obtain

$$1 - \frac{S_{(1)}}{m}\gamma_{(2)(3)(4)} = 2 + O(\varepsilon)\,, \tag{42}$$

where the small value ε is determined in (13). Then, according to (39)

$$a_{(3)} = \frac{S_{(1)}}{2m} R_{(3)(4)(2)(3)}\,. \tag{43}$$

Taking into account (40) and (43), we write

$$|\vec{a}| = \frac{3M}{2r^2}\frac{|S_{(1)}|}{mr}|u_\perp|\sqrt{1+u_\perp^2}\,. \tag{44}$$

Since $S_{(2)} = 0$ and $S_{(3)} = 0$, we note that $|S_{(1)}| = |S|$. That is, the right-hand side of (44) differs from the right-hand side of (34) only in the numerical factor 1/2. As a result, according to (36) for the circular orbits of a spinning particle, we have

$$I = \frac{3}{2} q^2 \frac{M^2}{r^4}\varepsilon^2\gamma^4\,. \tag{45}$$

It means that the intensity of the electromagnetic radiation of a spinning particle on the circular orbits by the strict MP equations is proportional to γ^4.

The results presented in Sections 3 and 4 describe the properties of the spin-gravity coupling in the proper frame of a spinning particle. In this context the question arises: can the highly relativistic spin-gravity coupling significantly deviate trajectories of the spinning particle from the geodesic trajectories by their description in the terms of the global Schwarzschild coordinates? Different cases of the essentially nongeodesic orbits of the highly relativistic spinning particle in Schwarzschild's field are investigated in [18, 22, 23].

5. Energy of a highly relativistic spinning particle in Schwarzschild's field

According to the geodesic equations, there is an expression for the energy of a spinless particle with mass m in Schwarzschild's field

$$E = mu_4 = m\left(1 - \frac{2M}{r}\right)^{1/2}\gamma\,, \tag{46}$$

where $\gamma = \sqrt{u_4 u^4}$ is the relativistic Lorentz factor calculated by the particle velocity relative to the source of the Schwarzschild field, r is the standard radial coordinate. That is, this energy is proportional to the γ factor, similar as in the case of the free particle motion in Special Relativity. Other situations arise in the case of the spinning particle motions in Schwarzschild's field. Then by the MP equations, the expression for a spinning particle can be written as [33]

$$E = mu_4 + g_{44}u_\lambda \frac{DS^{4\lambda}}{\mathrm{d}s} + \frac{1}{2}g_{4\mu,\nu}S^{\nu\mu}\,. \tag{47}$$

In contrast to (46), the value of energy (47) depends not only on the initial velocity of a spinning particle and r, but on the spin-gravity coupling as well. In the specific case of the radial motion of the spinning particle in Schwarzschild's field, the value of its energy does not depend on the absolute value and orientation of the spin and coincides exactly with the value of energy of the spinless particle. As well as in this case, the world line of the spinning particle coincides with the corresponding geodesic world line (it is easy to obtain this result after writing equations (1)–(3) at condition of $\theta = \mathrm{const}$, $\varphi = \mathrm{const}$). However, any nonzero value of the particle tangential velocity leads to some deviation of the value of the spinning particle energy from the value of the energy of the spinless particle. Naturally, when $|u_\perp| \ll 1$, *i.e.* for low values of the tangential velocity, this deviation is small. It is interesting to investigate the dependence of the spinning particle energy on the tangential velocity in the highly relativistic region, when $|u_\perp| \gg 1$. For this purpose, it is convenient to deal with the exact MP equations in the form of the first-order differential equations for the 11 dimensionless quantities y_i, where by definition

$$\begin{aligned} y_1 &= \frac{r}{M}\,, & y_2 &= \theta\,, & y_3 &= \varphi\,, & y_4 &= \frac{t}{M}\,,\\ y_5 &= u^1\,, & y_6 &= Mu^2\,, & y_7 &= Mu^3\,, & y_8 &= u^4\,,\\ y_9 &= \frac{S_1}{mM}\,, & y_{10} &= \frac{S_2}{mM^2}\,, & y_{11} &= \frac{S_3}{mM^2}\,. \end{aligned} \tag{48}$$

These equations are presented in the explicit form in [25] as

$$\begin{aligned} \dot{y}_1 &= y_5\,, & \dot{y}_2 &= y_6\,, & \dot{y}_3 &= y_7\,, & \dot{y}_4 &= y_8\,, \\ \dot{y}_5 &= A_1\,, & \dot{y}_6 &= A_2\,, & \dot{y}_7 &= A_3\,, & \dot{y}_8 &= A_4\,, \\ \dot{y}_9 &= A_5\,, & \dot{y}_{10} &= A_6\,, & \dot{y}_{11} &= A_7\,. \end{aligned} \tag{49}$$

Here, A_i are the corresponding functions of y_i and contain the two constants of motion: the energy and angular momentum (a dot denotes the usual derivative with respect to the dimensionless value $x = s/M$). At the fixed initial values of y_i, different values of these constants correspond to motions of different centers of mass of the spinning particle.

Let us consider equations (49) in the partial case of the spinning particle motion in the plane $\theta = \pi/2$ with the spin orthogonal to this plane. It means that in notation (48), we put $y_2 = \pi/2$, $y_6 = 0$, $y_9 = 0$, $y_{11} = 0$. Other nonzero functions $y_i(x)$ can be found by the numerical integration of equations (49). The important point in this procedure is finding values of the energy and angular momentum which correspond just to the solutions for the proper center of mass of the particle at the fixed initial values of y_i, not to the helical solutions. There is a most simple case when $y_7 \equiv 0$, *i.e.* for the radial motion of a spinning particle: as we noted above, then the values of the energy and angular momentum are equal exactly to the corresponding values for the geodesic radial motion. Naturally, for the initial values of y_7 which satisfy condition $|y_7| \ll 1$, the corresponding values of the energy and angular momentum can be calculated as some small corrections to the values for the radial motion. For other values of y_7, we used computer searching. As a result, here we present a typical case for the spinning particle with

$$\varepsilon_0 \equiv \frac{|S|}{mM} = 10^{-2}$$

(note that in contrast to ε from (13), the value of ε_0 does not depend on r) which begins motion from the position of $r = 2.5M$ with the initial value of the radial velocity $u_\parallel = -10^{-2}$ with different initial values of the tangential velocity $u_\perp$.

Table I describes the situations when the sign of $u_\perp(0)$ is positive with the orientation of the particle spin when $S_2 \equiv S_\theta > 0$. Table II corresponds to the cases with $u_\perp(0) < 0$ and the same value of the S_2 as in Table I. Both Tables I and II show the ratio of the energy of the spinning particle $E_{\rm spin}$ to the value of the energy of the spinless particle $E_{\rm geod}$ which moves along the geodesic lines and starts with the same initial velocity as the spinning particle. For $u_\perp(0) = 0$, we have $E_{\rm spin}/E_{\rm geod} = 1$, exactly. When $|u_\perp(0)| \ll 1$, the value $E_{\rm spin}$ is almost equal to $E_{\rm geod}$ with high accuracy. Other situations arise when $|u_\perp|$ is growing up to the highly relativistic motions with

$u_\perp^2 \gg 1$. According to Tables I and II, at the highly relativistic regime, the difference between E_{spin} and E_{geod} is growing significantly with growing $|u_\perp|$. There are essential differences of the data in Tables I and II: in the first case, $E_{\mathrm{spin}}/E_{\mathrm{geod}} > 1$, whereas in the second case, $E_{\mathrm{spin}}/E_{\mathrm{geod}} < 1$. This property corresponds to the known result that at $S_2 > 0$ and $u_\perp(0) > 0$, the spin-gravity coupling acts on the particle as some attractive force, whereas at $S_2 > 0$ and $u_\perp(0) < 0$, this action is repulsive [23, 25].

TABLE I

Comparison of the energies of the spinning and spinless particles at different orbital velocity for $u_\perp(0) > 0$.

$u_\perp(0)$	$E_{\mathrm{spin}}/E_{\mathrm{geod}}$
0	1
5.88	1.06
11.75	1.24
17.67	1.63
23.50	2.00

TABLE II

Comparison of the energies of the spinning and spinless particles at different orbital velocity for $u_\perp(0) < 0$.

$u_\perp(0)$	$E_{\mathrm{spin}}/E_{\mathrm{geod}}$
−2.35	0.99
−4.70	0.96
−11.75	0.76
−17.62	0.44
−21.15	0.20

Thus, the contribution of the spin-gravity coupling to the energy of a spinning particle in Schwarzschild's field becomes large when its velocity is highly relativistic.

6. Conclusions

In addition to the known results concerning the influence of the spin-gravity coupling on world lines and trajectories of the highly relativistic spinning particle in the Schwarzschild field [23, 25], in this paper, we present the results about the effect of the highly relativistic spin-gravity coupling on the particle's energy. Depending on the correlation of the spin orientation and the particle orbital velocity, the values of the spinning particle energy can be much larger or smaller than the corresponding values for the spinless particle.

In the case of highly relativistic motions of the charged spinning particle in Schwarzschild's field, in Section 4, we considered the intensity of the energy of its electromagnetic radiation as estimated in the proper frame of the particle. It is important that this value is proportional to the γ^4, *i.e.* becomes very large for highly relativistic orbital velocities of the particle.

In further investigations, it would be of interest to apply the results of this paper to the analysis of a possible role of the highly relativistic spin-gravity coupling in the astrophysical processes with fast-spinning particles in strong gravitational fields.

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