CASIMIR EFFECTS FOR SCALAR FIELDS UNDER ELECTRIC FIELDS IN 1+1 DIMENSIONS

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In the present study, we examine Casimir effects of the charged massless scalar field in 1+1 dimensions in the external background potential which includes linear and non-linear electrostatic fields. We calculate the Casimir energy for Dirichlet, Neumann, and mixed boundary conditions using the perturbation theory. We find that the Casimir energy is strengthened in the Neumann boundary condition and is lowered in other cases.

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1. Introduction

The Casimir effect, a proof for the existence of vacuum fluctuations, was proposed by Casimir in 1948 [1] in an attempt to explain the interaction between two polarizable atoms [2]. In the original formulation, the Casimir effect is understood as an emerging attraction between two neutral superconducting parallel plates separated by a distance. The attractive force between these plates is derived by differentiating vacuum energy densities with and without boundaries. The effect was confirmed by Lamoreaux's measurements in 1997 [3].

The influence of the external magnetic field on Casimir effect, based on the formulation of Landau quantization, is an interesting topic. In Ref. [4], the Casimir energy of the scalar field is calculated as a function of the magnetic field. The fields considered in Ref. [4] include both bosonic and fermionic degrees of freedom. The results in that study are in agreement with those in Ref. [5], which state that the magnetic field inhibits the Casimir energy for a bosonic field and enhances the energy for a fermionic field [6].

The scalar Casimir effects influenced by an external electric field were first studied by Ambjørn and Wolfram [7] who perform the calculation of vacuum polarizations of the charged scalar field by summing over scalar modes.

We also mention the research by Elizalde and Romeo [8] who computed Casimir energy for both chargeless and charged scalar field under electric field using numerical estimation based on the zeta function regularization, the Dirichlet boundary condition is applied in the paper. Recently, Taya [9] has studied the relation between the dynamic Casimir effect and Schwinger mechanism of a massive charged scalar field under a strong electric field, the results of which suggest that the electric field enhances the dynamical Casimir effect.

The present paper investigates the Casimir effect of the massless charged scalar field under the electrostatic fields using perturbation theory in 1+1 dimensions. The concrete boundary conditions include: Dirichlet, Neumann and mixed (hybrid) boundary conditions. Unlike the study in Ref. [7] which focuses on the calculation of the vacuum polarization of the scalar field, in our work, we figure out the calculation of Casimir energy and the behavior of its perturbative contribution. In [8], the authors used the zeta function regularization to estimate the Casimir energy for every value of the external potential for both closed and open boundaries. The open boundary in the paper assumed that boundary condition is applied at one point and the second one at infinity. In contrast to the study in Ref. [8] which numerically estimated the total Casimir energy as a function of external potential for concrete values, in the present paper, we assume that the external field is a perturbation. Hence, we obtain perturbative contributions to Casimir energies. Furthermore, we extend the external potential to the non-linear form conceived from the beta function of massless QED [10].

Our strategy is as follows: Section 2 is devoted to the description of solving the Klein–Gordon equation with the external field to get a general solution. Next, the linear electrostatic is assumed as an electrostatic perturbation. By the perturbative method, the Casimir energies are determined by three kinds of boundaries: Dirichlet, Neumann, and mixed, respectively. The beta-function form of the electrostatic field is used to calculate the Casimir energies in Section 3. Section 4 is devoted to the summary and outlook.

2. Casimir effect of the scalar field under a linear electrostatic potential

2.1. Dirichlet boundary condition

For the sake of simplicity, let us consider a massless charged scalar field confined in a finite spatial interval $0 \le x \le L$. The field is coupled with an electrostatic field via the following Lagrangian:

$$\mathcal{L} = (D_\mu \phi)^* (D^\mu \phi) \,, \tag{1}$$

and the corresponding action

$$S = \int \mathrm{d}^2 x \, (D_\mu \phi)^* \, (D^\mu \phi) \ . \tag{2}$$

Here, $D_\mu = \partial_\mu + iqA_\mu$ is a covariant derivative expressed via the time-independent external field A_μ which has the form of

$$A_\mu = (A_0, A_1) = \Delta\phi \left(\frac{x}{L} - \frac{1}{2}, 0 \right) \ . \tag{3}$$

Note that in a $1+1$ dimensional system, the coupling constant q has a dimension of mass, while the scalar field is of dimensionless quantity.

The scalar field can be represented in the form: $\psi(x,t) = \mathrm{e}^{-i\omega t}\phi(x)$. Hence, the Klein–Gordon equation with the external potential (3), after separating the time variable, is reduced to the following equation for spatial mode $\phi(x)$:

$$\partial_x^2 \phi(x) + \left(\omega - qL\Delta\phi \left(\frac{x}{L} - \frac{1}{2} \right) \right)^2 \phi(x) = 0 \, . \tag{4}$$

In order to simplify the above equation, it is convenient to introduce the dimensionless parameters: $\xi = \frac{x}{L}$, $\Omega = \omega L$, $\epsilon = qL\Delta\phi$. The differential equation can be rewritten in its dimensionless form

$$\partial_\xi^2 \phi(\xi) + \left(\Omega - \epsilon \left(\xi - \frac{1}{2} \right) \right)^2 \phi(\xi) = 0 \, . \tag{5}$$

Solving this equation, one gets a general solution

$$\phi(y) = \sqrt{y} \left(C_1 J_{1/4} \left(\frac{y^2}{2\epsilon} \right) + C_2 J_{-1/4} \left(\frac{y^2}{2\epsilon} \right) \right) \, , \tag{6}$$

with a new variable $y = \Omega - \epsilon \left(\xi - \frac{1}{2} \right)$ and the Bessel function $J_\nu(x)$. The coefficients C_1, C_2 can be determined by boundary conditions.

Imposing the Dirichlet boundary condition at boundaries $\xi = \{0\,; 1\}$, one realizes that the energy density of the system will be given by solving the following equation:

$$J_{\frac{1}{4}} \left(\kappa_{\Omega\epsilon}^- \right) J_{-\frac{1}{4}} \left(\kappa_{\Omega\epsilon}^+ \right) - J_{\frac{1}{4}} \left(\kappa_{\Omega\epsilon}^+ \right) J_{-\frac{1}{4}} \left(\kappa_{\Omega\epsilon}^- \right) = 0 \, , \tag{7}$$

where

$$\kappa_{\Omega\epsilon}^\pm \equiv \frac{1}{2\epsilon} \left(\Omega \pm \frac{\epsilon}{2} \right)^2 \, . \tag{8}$$

Assuming the external potential is weak $\epsilon \ll 1$, one can apply the asymptotic expansion for a large argument of the Bessel function. Consequently, Eq. (7) can be written as follows (after neglecting irrelevant factors and higher corrections):

$$\left\{\left(8\Omega^2+3\right)\epsilon^2-8\Omega^4\right\}\sin\Omega-3\Omega\epsilon^2\cos\Omega=0\,. \tag{9}$$

It is not difficult to find the solutions of this equation with the form of

$$\Omega_n=n\pi-\frac{3\epsilon^2}{8(n\pi)^3}\,;\qquad n=1\,,2,\,\dots \tag{10}$$

In this paper, we would like to solve the Klein–Gordon equation by the perturbative method. We assume the external fields to be weak potentials $\epsilon \ll 1$. Therefore, we can use the perturbative method to solve the differential equation by the following perturbative expansion of the wave function and the energy density up to the second-order correction:

$$\phi_n(\xi) \;=\; \phi_n^{(0)}(\xi)+\epsilon\,\phi_n^{(1)}(\xi)+\epsilon^2\,\phi_n^{(2)}(\xi)+\mathcal{O}\left(\epsilon^3\right)\,, \tag{11}$$

$$\Omega_n \;=\; \Omega_n^{(0)}+\epsilon\,\Omega_n^{(1)}+\epsilon^2\,\Omega_n^{(2)}+\mathcal{O}\left(\epsilon^3\right)\,. \tag{12}$$

Substituting expansions (11) and (12) into the differential equation (5) and solving this equation up to the second-order correction, one gets the perturbative solution of the Klein–Gordon equation satisfying in Dirichlet boundary condition

$$\phi_n^{(0)}(\xi) \;=\; \sqrt{2}\sin w_n\xi\,, \tag{13a}$$

$$\phi_n^{(1)}(\xi) \;=\; \frac{1}{2\sqrt{2}w_n}\left[(2\xi-1)\sin w_n\xi+2\,(1-\xi)\,w_n\xi\cos w_n\xi\right]\,, \tag{13b}$$

$$\phi_n^{(2)}(\xi) \;=\; \frac{1}{8\sqrt{2}w_n^2}\Big[\left\{6\xi(\xi-1)-2\xi^2w_n^2(\xi-1)^2+1\right\}\sin w_n\xi -2\xi w_n(\xi-1)(2\xi-1)\cos w_n\xi\Big]\,, \tag{13c}$$

where $w_n=n\pi\,, n\in 1\,,2\,,\dots$ In addition, the components of the energy density in expression (12) can be derived as

$$\Omega_n^{(0)}=w_n\,,\qquad \Omega_n^{(1)}=0\,,\qquad \Omega_n^{(2)}=-\frac{3}{8w_n^3}\,,\qquad w_n=n\pi\,. \tag{14}$$

Let us rewrite the dimensionless results (13) and (14) into the following dimensionful expressions. The scalar field

$$\phi_n(x)=\phi_n^{(0)}(x)+\epsilon\,\phi_n^{(1)}(x)+\epsilon^2\phi_n^{(2)}(x)\,, \tag{15}$$

and the energy density (10)

$$\omega_n = \omega_n^{(0)} + \epsilon^2\,\omega_n^{(2)} \tag{16}$$

are written in the dimensionful forms by using the transformation from dimensionless parameters to dimensionful one $\omega_n = \frac{\Omega_n}{L}$, $x = L\xi$ in Eqs. (11) and (12). From this point on, we denote ω_n^X with the subscription n which indicates the discrete spectrum, and the superscription X which denotes the kind of boundary conditions: Dirichlet, Neumann and mixed boundary conditions, respectively.

In the canonical quantization perspective, it is possible to introduce the positive- and negative-frequency solutions of the Klein–Gordon equation by a complete form

$$\psi_n^{(+)}(t,x) = \frac{C}{\sqrt{2}}\mathrm{e}^{-i\omega_n t}\phi_n(x)\,, \qquad \psi_n^{(-)}(t,x) = \left[\psi_n^{(+)}(t,x)\right]^*. \tag{17}$$

To determine the normalization coefficient C, one can use the following normalization conditions:

$$\left(\psi_n^{(\pm)}(t,x),\, \psi_m^{(\pm)}(t,x)\right) = \delta_{nm}\,, \qquad \left(\psi_n^{(\pm)}(t,x),\, \psi_m^{(\mp)}(t,x)\right) = 0\,, \tag{18}$$

where the scalar product is defined by [15]

$$(\psi_1\,, \psi_2) = i\int_0^L \mathrm{d}x\,(\psi_1^* D_t\psi_2 - \psi_2 D_t\psi_1^*)\;. \tag{19}$$

Inserting the explicit expression (15) into the normalization condition (18) and neglecting higher corrections, one gets the following formula to determine the normalization coefficient:

$$C^2\int_0^L \mathrm{d}x\,(\omega_n - qA_t)\,\phi_n^*(x)\phi_m(x) = \delta_{nm}\,. \tag{20}$$

The normalization coefficient can be determined by

$$C = \left[w_n - \frac{\epsilon^2(w_n^2+18)}{16w_n^3}\right]^{-1/2}, \tag{21}$$

where the dimensionless notation w_n is defined in Eq. (14). We would like to note that this notation is different from the dimensionful energy density ω_n defined in Eq. (16).

According to the procedure of canonical quantization, the field operator can be represented by summing the modes as

$$\psi(x,t) = \sum_n \left[\widehat{a}_n \psi_n^{(+)}(t,x) + \widehat{a}_n^{\,+} \psi_n^{(-)}(t,x)\right] . \tag{22}$$

The annihilation and creation operators $\widehat{a}_n$, $\widehat{a}_n^{\,+}$ of the field satisfy the following commutation relations:

$$\left[\widehat{a}_n \, , \widehat{a}_{n'}^{\,+}\right] = \delta_{nn'} \, , \qquad \left[\widehat{a}_n \, , \widehat{a}_{n'}\right] = \left[\widehat{a}_n^{\,+} \, , \widehat{a}_{n'}^{\,+}\right] = 0 \, . \tag{23}$$

The vacuum state of the scalar field in this case is

$$\widehat{a}_n|0\rangle = 0 \, , \tag{24}$$

and the scalar field states can be obtained by applying the creation operators to the vacuum state.

According to the Noether theorem, the canonical energy-momentum tensor of the charged scalar field can be determined by the following formula:

$$T^{\mu\nu} = D^\mu \psi D^\nu \psi - g^{\mu\nu} \mathcal{L} \, . \tag{25}$$

Therefore, the Casimir energy can be obtained by integrating the considered volume of the mean value of the 00-component of the energy-momentum tensor in the vacuum state

$$E_{\mathrm{C}} \equiv \int_0^L \mathrm{d}x \, \langle 0|T^{00}(x)|0\rangle = \sum_{n=1}^{\infty} \int_0^L \mathrm{d}x \, C^2 \left(\omega_n - qA_t\right)^2 \phi_n^*(x) \phi_n(x) \, . \tag{26}$$

Inserting the solution of Klein–Gordon equation (15) and the normalization coefficient C in Eq. (21) into formula (26), neglecting higher corrections, one gets

$$E_{\mathrm{C}} = \frac{1}{L} \sum_{n=1}^{\infty} \left(n\pi + \frac{3\epsilon^2}{8(n\pi)^3} \right) = -\frac{\pi}{12L} + \frac{3\zeta(3)\epsilon^2}{8\pi^3 L} \, . \tag{27}$$

The first term in Eq. (27) is the regularized Casimir energy respective to the non-perturbative case, without an external field. It corresponds to the Casimir energy of the scalar field in 1+1 dimensions. The second term is a perturbative contribution under the Dirichlet boundary condition.

In conclusion, we have just derived the Casimir energy of the scalar field under the Dirichlet boundary condition with an external potential using the perturbative method in Eq. (27). The opposite signs of the non-perturbative and perturbative terms reflect the fact that the electrostatic field lowers the Casimir energy. Moreover, the absence of a linear term ϵ in (27) shows that the Casimir energy does not depend on the alignment of the external electrostatic field.

2.2. Neumann boundary condition

In this section, we consider the Casimir effect under an external field by the Neumann boundary condition. In particular, the scalar field satisfies the following boundary condition:

$$\partial_x \phi(x)\Big|_{x=0} = \partial_x \phi(x)\Big|_{x=L} = 0\,. \tag{28}$$

In this boundary condition, general solution (6) is written with notation (8)

$$J_{\frac{3}{4}}\left(\kappa^+_{\Omega\epsilon}\right) J_{-\frac{3}{4}}\left(\kappa^-_{\Omega\epsilon}\right) - J_{-\frac{3}{4}}\left(\kappa^+_{\Omega\epsilon}\right) J_{\frac{3}{4}}\left(\kappa^-_{\Omega\epsilon}\right) = 0\,. \tag{29}$$

For large arguments of the Bessel function approximation, the above equation is equivalent to

$$\left\{\left(8\Omega^2 - 5\right)\epsilon^2 - 8\Omega^4\right\}\sin\Omega + 5\Omega\epsilon^2\cos\Omega = 0\,, \tag{30}$$

and yields solutions as follows:

$$\Omega_n = n\pi + \frac{5\epsilon^2}{8(n\pi)^3}\,. \tag{31}$$

Hence, the energy density of the field can be obtained as the solution of the general equation (29) in a small ϵ approximation. Next, let us solve the Klein–Gordon equation which satisfies the Neumann boundary condition. Considering the external electrostatic field as a perturbation, we can find the solution for the Klein–Gordon equation of the form as in (11) with the following components:

$$\phi_n^{(0)}(\xi) = \sqrt{2}\cos w_n\xi\,, \tag{32a}$$

$$\phi_n^{(1)}(\xi) = \frac{1}{2\sqrt{2}w_n^2}\left[2\left\{(\xi-1)\xi w_n^2 - 1\right\}\sin w_n\xi + (2\xi-1)w_n\cos w_n\xi\right], \tag{32b}$$

$$\begin{aligned}\phi_n^{(2)}(\xi) = \frac{1}{8\sqrt{2}w_n^4}\Big[&2(2\xi-1)w_n\left\{(\xi-1)\xi w_n^2 - 5\right\}\sin w_n\xi \\ &+ \left(w_n^2\left\{2(\xi-1)\xi\left[5-(\xi-1)\xi w_n^2\right]+1\right\} - 10\right)\cos w_n\xi\Big],\end{aligned} \tag{32c}$$

where $w_n = n\pi$. The components of energy density (12) have the same form as in Eq. (31), especially

$$\Omega_n^{(0)} = w_n\,, \qquad \Omega_n^{(1)} = 0\,, \qquad \Omega_n^{(2)} = \frac{5}{8w_n^3}\,, \qquad w_n = n\pi\,. \tag{33}$$

It follows that, in the perturbative method, the energy density of the scalar field, which satisfies the Neumann boundary condition, is re-examined in the perturbative form (12) with the components in (33).

We apply similar procedures to the Dirichlet boundary condition case. First, deforming the classical solution in (32) to a quantum form and normalizing using (20) and applying formula (26), we can find the Casimir energy

$$E_{\mathrm{C}}^{\mathrm{N}} = \frac{1}{L}\sum_{n=1}^{\infty}\left(n\pi - \frac{5\epsilon^2}{8\pi^3 n^3}\right) = -\frac{\pi}{12L} - \frac{5\zeta(3)\epsilon^2}{8\pi^3 L}\,. \tag{34}$$

Result (34) suggests that the external potential enhances the Casimir energy under the Neumann boundary condition.

Figure 1 shows Casimir energies as the functions of the linear electrostatic field for both Dirichlet and Neumann boundary conditions. The right inset illustrates the lowest mode for energy densities $\omega_0^{\mathrm{D}}, \omega_0^{\mathrm{N}}$ as a function of ϵ. When dividing both results (27) and (34) by the difference of electrostatic potential between two boundaries $\Delta\phi$, we can find the dependence of $\frac{E_{\mathrm{C}}}{\Delta\phi}$ and ϵ in the left inset.

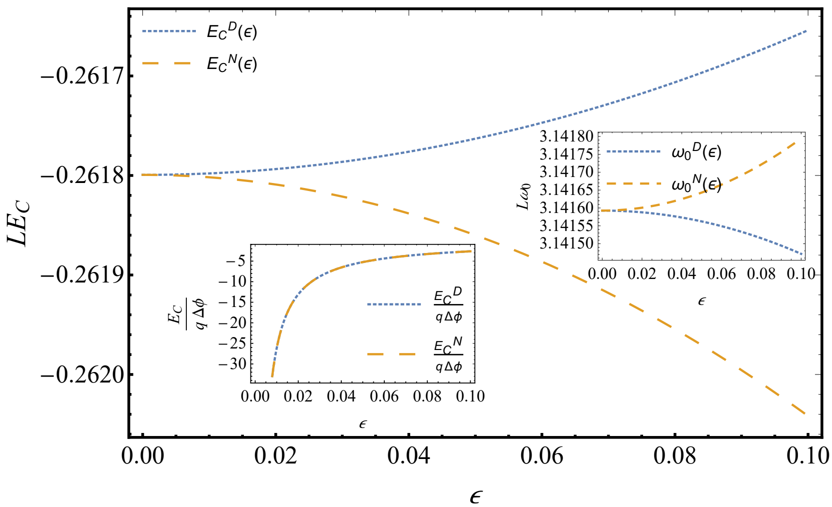

Fig. 1. Casimir energies of the scalar field as a function of external potential in (27) and (34). The right inset demonstrates the lowest energy densities, and the left one illustrates the quantity of ratio between the Casimir energy per variation of electrostatic field $\frac{E_{\mathrm{C}}}{\Delta\phi}$ as a function of ϵ.

2.3. Mixed boundary condition

In this section, let us consider another type of boundary condition which is sometimes called the mixed or hybrid boundary condition. This boundary condition consists of the Dirichlet boundary condition at $x = 0$ and the

Neumann boundary condition at $x = L$

$$\psi(t,x)\Big|_{x=0} = \frac{\partial\psi(t,x)}{\partial x}\Big|_{x=L} = 0\,. \tag{35}$$

The equation of energy density that satisfies this boundary condition from general solution (6) is as follows:

$$J_{\frac{1}{4}}\left(\kappa^{+}_{\Omega\epsilon}\right) J_{\frac{3}{4}}\left(\kappa^{-}_{\Omega\epsilon}\right) + J_{-\frac{3}{4}}\left(\kappa^{-}_{\Omega\epsilon}\right) J_{-\frac{1}{4}}\left(\kappa^{+}_{\Omega\epsilon}\right) = 0\,, \tag{36}$$

with $\kappa^{\pm}_{\Omega\epsilon}$ defined in Eq. (8). Hence, the simplified form of Eq. (36) at the large argument approximation of Bessel function is given as

$$8\Omega\left(\Omega^2 - \epsilon^2\right)\cos\Omega - \epsilon\left(4\Omega + \epsilon\right)\sin\Omega = 0\,. \tag{37}$$

This equation gives a solution in a discretized form of $\Omega = \Omega_n$ as

$$\Omega_n = w_n + \frac{\epsilon}{2w_n} + \epsilon^2\left(\frac{1}{8w_n^3} - \frac{1}{2w_n^5}\right)\,, \tag{38}$$

with $w_n = \pi\left(n + \frac{1}{2}\right)$, $n = 0, 1, \ldots$

The energy density for the mixed boundary condition depends on the direction of the electrostatic field via the linear-dependent term in (38). This does not occur in the cases of Dirichlet and Neumann boundary condition.

With small values of ϵ, the perturbative method gives the solution for the Klein–Gordon equation, which has the same form as (11) and (12). First, the wave function (11) has the following components:

$$\phi_n^{(0)} = \sqrt{2}\sin w_n\xi\,, \tag{39a}$$

$$\phi_n^{(1)}(\xi) = \frac{1}{2\sqrt{2}w_n^3}\left[\left((2\xi-1)w_n^2-1\right)\sin w_n\xi + 2\xi w_n\left(1-(\xi-1)w_n^2\right)\cos w_n\xi\right]\,, \tag{39b}$$

$$\phi_n^{(2)}(\xi) = \frac{\left[\left(2\xi\left(2\xi^2+\xi-3\right)+1\right)w_n^4 - 2(\xi-1)^2\xi^2 w_n^6 - 2(\xi(\xi+3)+1)w_n^2 + 5\right]\sin w_n\xi}{8\sqrt{2}w_n^6} - \frac{\xi\left[(\xi-1)(2\xi-1)w_n^4 - (3\xi+2)w_n^2 + 5\right]\cos w_n\xi}{4\sqrt{2}w_n^5}\,. \tag{39c}$$

Let us recall that in mixed boundary condition, we used the notation $w_n = \pi\left(n+\frac{1}{2}\right)$, $n = 0\,, 1\,, 2, \ldots$ The energy density in Eq. (12) yields

$$\Omega_n^{(0)} = w_n\,, \qquad \Omega_n^{(1)} = \frac{1}{2w_n^2}\,, \qquad \Omega_n^{(2)} = \frac{1}{8w_n^3} - \frac{1}{2w_n^5}\,. \tag{40}$$

Repeating procedure with the previous cases, we obtain the Casimir energy as

$$\begin{aligned} E_{\mathrm{C}} &= \frac{1}{L}\sum_n \left(w_n - \frac{\left(w_n^2 - 4\right)\epsilon^2}{8w_n^5} \right) \\ &= \sum_{n=0}^{\infty} \frac{w_n}{L} + \frac{\epsilon^2}{L}\sum_{n=0}^{\infty}\left(\frac{1}{w_n^5} - \frac{1}{8w_n^3} \right) \equiv E_0 + E_\epsilon \,. \end{aligned} \tag{41}$$

Here, we have just split the Casimir energy into two parts: non-perturbative energy E_0

$$E_0 = \sum_{n=0}^{\infty} \frac{\pi}{L}\left(n + \frac{1}{2} \right) , \tag{42}$$

and perturbative term E_ϵ

$$E_\epsilon = \frac{\epsilon^2}{L}\sum_{n=0}^{\infty}\left(\frac{1}{w_n^5} - \frac{1}{8w_n^3} \right) = -\left(\frac{7\zeta(3)}{8\pi^3} - \frac{18\zeta(5)}{\pi^5} \right)\frac{\epsilon^2}{L} \,. \tag{43}$$

The perturbative term is convergent. However, the non-perturbative term E_0 is divergent, which therefore should be regularized. Applying the modification of the Abel–Plana formula [17]

$$\sum_{n=0}^{\infty} F\left(n + \frac{1}{2} \right) - \int_0^{\infty} F(t)\mathrm{d}t = -i\int_0^{\infty} \frac{\mathrm{d}t}{\mathrm{e}^{2\pi t}+1}\left[F(it) - F(-it) \right] , \tag{44}$$

we can take the sum over half-integer numbers for E_0 to get

$$E_0 = \frac{L\Lambda_{\mathrm{UV}}^2}{2\pi} + \frac{\pi}{24L} \,. \tag{45}$$

The divergent term in (45) has the same form as those in the Dirichlet or Neumann boundary conditions. Therefore, it is equal to the contribution of the free space, without boundary. The Casimir energy of the scalar field under the mixed boundary condition is positive. It is obvious that the respective Casimir force is the repulsive [12]

$$E_0 = \frac{\pi}{24L} \,. \tag{46}$$

In short, the Casimir energy for the scalar field under the mixed boundary condition has the form of

$$E_{\mathrm{C}}^{\mathrm{M}} = \frac{\pi}{24L} - \left(\frac{7\zeta(3)}{8\pi^3} - \frac{18\zeta(5)}{\pi^5} \right)\frac{\epsilon^2}{L} \,. \tag{47}$$

Figure 2 illustrates the Casimir energy in (47) as a function of ϵ. The inset describes the lowest level of the energy density in (38). Although the energy density in (38) depends on the direction of the external field via the sign of ϵ, the total Casimir energy in (47) does not.

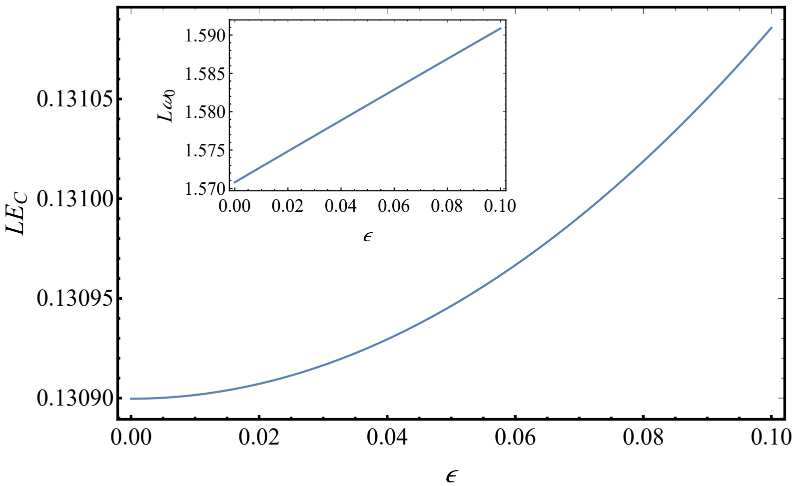

Fig. 2. Casimir energy for scalar field with the electrostatic perturbation by mixed boundary conditions from result (47), while the inserted plot shows the lowest energy density of the system in (38).

3. Casimir effect of a scalar field under a non-linear potential

In the previous section, we have studied the Casimir effect under a linear perturbation electric field. In this section, we consider that the system resides in a thermodynamic equilibrium with the effective local chemical potential $\mu(x) = \phi(x)$ [10].

Our configuration is assumed to be a semi-metal in the form of a slab of a finite length L in the x direction $(0 \le x \le L)$. The electrostatic potential $\Delta\phi \equiv \phi(L) - \phi(0)$ is applied to the opposite boundaries $x = 0, L$ of the slab. For the sake of simplification in the later calculation, we set $\phi(L) = \Delta\phi\,,\ \phi(0) = 0$. In this assumption, the external field has a form of

$$A_0 = \Delta\phi\, h(\nu) \left(B\left(\frac{x}{L}; 1{-}\nu, 1{-}\nu\right) - B\left(\frac{1}{2}; 1{-}\nu, 1{-}\nu\right) \right) , \tag{48}$$

where $\Delta\phi$ is the electrostatic potential, which is also a perturbation,

$$B(z; a, b) = \int_0^z \frac{t^{a-1} \mathrm{d}t}{(1-t)^{b-1}} \tag{49}$$

is the Euler incomplete beta function, and

$$h(\nu) = \frac{\Gamma(2-2\nu)}{\Gamma^2(1-\nu)} \equiv \frac{1}{B(1-\nu, 1-\nu)} \tag{50}$$

is the normalization coefficient expressed via the gamma function $\Gamma(x)$ and the beta function $B(a,b) \equiv B(1;a,b)$. The field of interest is considered inside the region of $0 \le x \le L$, therefore, the electrostatic potential in (48) can be approximated as

$$\begin{aligned} A_0(\xi) &= \Delta\phi \left[\frac{2^{2\nu}\left(\xi - \frac{1}{2}\right)}{B(1-\nu, 1-\nu)} + \frac{2^{2\nu+2}\nu\left(\xi - \frac{1}{2}\right)^3}{3B(1-\nu, 1-\nu)} + \mathcal{O}\left(\xi - \frac{1}{2}\right)^5 \right] \\ &\equiv \Delta\phi\, k\left(\xi, \nu\right) + \Delta\phi\, \mathcal{O}\left(\xi - \frac{1}{2}\right)^5 , \end{aligned} \tag{51}$$

where

$$k\left(\xi, \nu\right) \equiv \frac{2^{2\nu}\left(\xi - \frac{1}{2}\right)}{B(1-\nu, 1-\nu)} + \frac{2^{2\nu+2}\nu\left(\xi - \frac{1}{2}\right)^3}{3B(1-\nu, 1-\nu)} . \tag{52}$$

The Klein–Gordon equation under external potential (51) can be represented in a dimensionless form

$$\partial_\xi^2 \phi\left(\xi\right) + \left(\Omega - \epsilon\, k\left(\xi, \nu\right)\right)^2 \phi\left(\xi\right) = 0 . \tag{53}$$

In an assumption that the electrostatic potential is a perturbation, we have $\Delta\phi \ll 1$, therefore, $\epsilon \ll 1$. We can solve this equation by perturbative expansion (11) and its energy density (12). Next, after putting our solution into quantum representation, we can normalize the field by condition (20) to find the coefficient C. Finally, we can obtain the Casimir energy from Eq. (26).

In the remainder of this section, we provide energy densities and Casimir energies for three kinds of boundary conditions as follows.

Dirichlet boundary condition

The solution of the Klein–Gordon equation (53) which satisfies the Dirichlet boundary condition gives the energy density

$$\omega_n^{\mathrm{D}} = \frac{w_n}{L} + \frac{\epsilon^2}{L} f_n^{\mathrm{D}}(\nu) , \tag{54}$$

with

$$f_n^{\mathrm{D}}(\nu) \equiv \frac{20(23\nu + 30)\nu w_n^2 - 2940\nu^2 - \left((17\nu + 70)\nu + 45\right) w_n^4}{15B^2(1-\nu, 1-\nu)2^{3-4\nu} w_n^7} , \tag{55}$$

and $w_n = n\pi\,, \left(n = 1\,, 2\,, \dots\right)$.

The Casimir energy of the scalar field can be represented by

$$E_{\mathrm{C}}^{\mathrm{D}}(\epsilon,\nu) = -\frac{\pi}{12L} + \frac{\epsilon^2}{L} g^{\mathrm{D}}(\nu)\,, \tag{56}$$

with

$$g^{\mathrm{D}}(\nu) \equiv \frac{3\zeta(3)\,2^{4\nu}}{8\pi^3 B^2(1-\nu,1-\nu)} + \frac{\nu\left\{\pi^4(17\nu+70)\zeta(3)-20\pi^2(23\nu+30)\zeta(5)+2940\nu\zeta(7)\right\}}{30\pi^8\Gamma^2(1-\nu)\Gamma^{-2}\left(\frac{3}{2}-\nu\right)}. \tag{57}$$

The result in expression (56) reflects that the Casimir energy for the scalar field under the perturbation theory will be weakened when imposing the Dirichlet boundary condition. As $\nu \to 0$, we get the result for the linear potential case (27).

Neumann boundary condition

As with the Dirichlet case, the result for the Neumann boundary can be summarized as follows. The energy density is

$$\omega_n^{\mathrm{N}} = \frac{w_n}{L} + \frac{\epsilon^2}{L} f_n^{\mathrm{N}}(\nu)\,, \tag{58}$$

with

$$f_n^{\mathrm{N}}(\nu) \equiv \frac{3060\nu^2 + \left(\nu\left(23\nu+90\right)+75\right)w_n^4 - 180\nu\left(3\nu+4\right)w_n^2}{30\pi w_n^7 \Gamma^2\left(1-\nu\right)\Gamma^{-2}\left(\frac{3}{2}-\nu\right)}\,, \tag{59}$$

and $w_n = n\pi$. Furthermore, the Casimir energy of the field in this case can be summarized as

$$E_{\mathrm{C}}^{\mathrm{N}}(\epsilon,\nu) = -\frac{\pi}{12L} - \frac{\epsilon^2}{L} g^{\mathrm{N}}(\nu)\,, \tag{60}$$

with

$$g^{\mathrm{N}}(\nu) \equiv \frac{(23\nu^2+90\nu+75)\pi^4\zeta(3) - 180\pi^2(3\nu^2+4\nu)\zeta(5) + 3060\nu^2\zeta(7)}{30\pi^8\Gamma^2(1-\nu)\Gamma^{-2}\left(\frac{3}{2}-\nu\right)}. \tag{61}$$

This result indicates that the Casimir energy for the scalar field is enhanced as the external potential is applied. The intuitive behaviors of results in (56) and (60) for several values of ν, in particular, $\nu = \{0.1\,;0.2\}$, are illustrated in Fig. 3. The amplitude of the Casimir energy for two boundary conditions shows the difference: the Dirichlet boundary condition depresses, while the Neumann boundary condition boosts the Casimir energy of field. The inset demonstrates the dependence of ν for functions $\frac{E_{\mathrm{C}}}{q\Delta\phi}$ in unit of ϵ. As $\nu \to 1$, the perturbative contribution vanishes, the system becomes the normal Casimir effect.

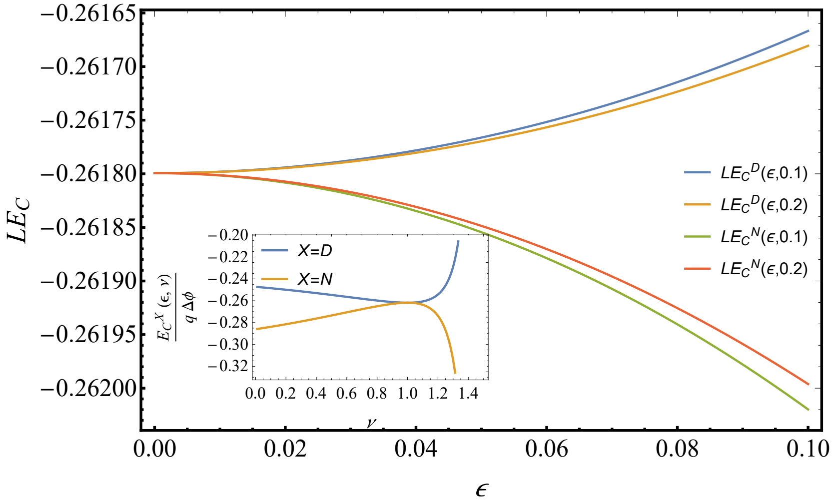

Fig. 3. Casimir energy of the scalar field under the Dirichlet and Neumann boundary conditions under the beta-form potential for concrete values of $\nu = \{0.1\,; 0.2\}$. For the limit $\nu = 0$, one gets the results in Section 2. The inserted graph shows that quantity $\frac{E_{\rm C}^X}{\Delta\phi}$ varies as ν. X is the abbreviation for Dirichlet and Neumann, respectively, $X = \{{\rm D}; {\rm N}\}$.

Mixed boundary condition

The energy density for the mixed boundary condition, in this case, has a form of

$$\omega_n^{\rm M} = \frac{w_n}{L} + \frac{\epsilon}{L} f_{1,n}^{\rm M}(\nu) + \frac{\epsilon^2}{L} f_{2,n}^{\rm M}(\nu)\,, \tag{62}$$

with

$$f_{1,n}^{\rm M}(\nu) \equiv \frac{2^{2\nu-1}\left((\nu+1)w_n^2 - 2\nu\right)}{w_n^4 B(1-\nu, 1-\nu)}\,, \tag{63a}$$

$$f_{2,n}^{\rm M}(\nu) \equiv \frac{(\nu(3\nu+10)+15)w_n^6 - 20(\nu(5\nu+9)+3)w_n^4 + 60\nu(7\nu+6)w_n^2 - 480\nu^2}{30\pi w_n^9 \varGamma^2(1-\nu)\,\varGamma^{-2}\left(\frac{3}{2}-\nu\right)}\,, \tag{63b}$$

and $w_n = \pi\left(n + \frac{1}{2}\right)$, $n = 0\,, 1\,, \ldots$ Hence, the Casimir energy yields

$$E_{\rm C}^{\rm M} = \frac{\pi}{24L} + \frac{\epsilon^2}{L} g^{\rm M}(\nu)\,, \tag{64}$$

with

$$g^{\mathrm{M}}(\nu) = \frac{\Gamma^2\left(\frac{3}{2}-\nu\right)\left(620(\nu(5\nu+9)+3)\zeta(5)-7\pi^2(\nu(3\nu+10)+15)\zeta(3)\right)}{30\pi^6\Gamma^2(1-\nu)} - \frac{\Gamma^2\left(\frac{3}{2}-\nu\right)2\nu\left(127\pi^2(7\nu+6)\zeta(7)-4088\nu\zeta(9)\right)}{\Gamma^2(1-\nu)\,\pi^{10}}\,. \quad (65)$$

The behavior of the Casimir energy to the external potential under the mixed boundary condition is shown in Fig. 4. The graph illustrates two interesting properties. First, similar to the linear case in the previous section, the Casimir energy is enhanced by perturbative potential for the mixed boundary condition. Second, when we increase ν, the amplitude of the Casimir energy decreases which is shown in the inset.

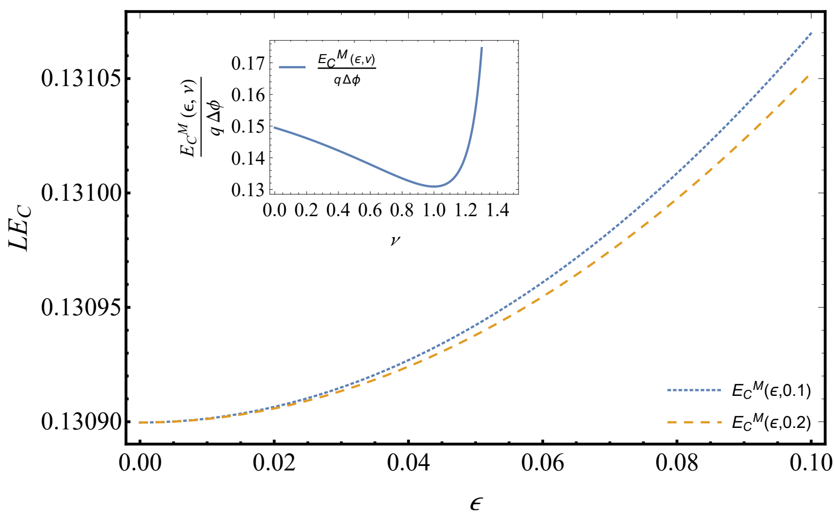

Fig. 4. Casimir energy for the scalar field satisfying mixed boundary condition is exhibited for concrete values of ν. The inserted graph is the representation of the function $\frac{E_C}{q\Delta\phi}$ with a variable ν in unit of ϵ.

4. Conclusion

We have examined Casimir effects of the charged scalar (bosonic) field influenced by external electrostatic fields using the perturbation theory. The external fields implemented in this paper include linear and non-linear electrostatic fields. There are three kinds of boundary conditions: Dirichlet, Neumann and mixed boundary conditions.

The results show that under perturbations, beyond recovering the normal Casimir energies, the perturbative energy contributions are obtained and the amplitude of the Casimir energy strongly depends on the type of boundary conditions. In particular, the external field lowers the Casimir energy under the Dirichlet and mixed boundary conditions while, on the other hand, strengthens it under the Neumann boundary one. The dependence on "conformal screening exponent" ν, to Casimir energies is also described in this study, by which it can be stated that the contribution of ν lowers the Casimir energies. The parameter ν, in this case, is taken to be the small variables.

Another point worth mentioning in this study is that the formula to determine the Casimir energy is different from the normal summation by modes $E_{\mathrm{C}} = \sum_n \omega_n$. For the presence of the external potential, the Casimir energy is determined by expression (26).

It would be interesting to extend this study to include higher dimension cases to better understand the behavior of the Casimir energy with perturbative contributions. Furthermore, one can also apply the electrostatic field into fermionic fields.

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