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GOLDSTONE BOSON DECAYS AND CHIRAL ANOMALIES* **

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Martinus Veltman was the first to point out the inconsistency of the experimental value for the decay rate of $\pi^0 \to \gamma\gamma$ and its calculation by J. Steinberger with the very successful concept of the pion as the (pseudo) Nambu–Goldstone boson of the spontaneously broken global axial symmetry of strong interactions. That inconsistency has been resolved by J. Bell and R. Jackiw in their famous paper on the chiral anomalies. We review the connection between the decay amplitudes of an axion into two gauge bosons in Abelian vector-like and chiral gauge theories. The axion is the Nambu–Goldstone boson of a spontaneously broken axial global symmetry of the theory. Similarly as for the vector-like gauge theory, also in the chiral one, the axion decay amplitude is uniquely determined by the anomaly of the current of *that* global symmetry. Certain subtlety in the calculation of the anomaly in chiral gauge theories is emphasised.

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1. Introduction

In 1999, Martinus Veltman shared with Gerard t'Hooft the Nobel Prize in physics for their contribution to the proof of renormalisability of non-Abelian gauge theories. It is less remembered that he also was the first, together with Sutherland [1, 2], to point out the inconsistency of the experimental value for the decay rate of $\pi^0 \to \gamma\gamma$ and its direct calculation by Steinberger [3] with the very successful concept of the pion as the (pseudo)Nambu–Goldstone boson (PNGB) of the spontaneously broken global axial symmetry of strong

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interactions. That inconsistency has been resolved by Bell and Jackiw in their famous paper on the chiral anomalies [4]. In beyond the Standard Model theories there may be new PNGBs that play important roles in particle physics and cosmology. The most famous example is the QCD axion that can solve the strong CP problem [5–7] and/or explain the origin of dark matter [8–10] (for a review, see [11]). Axion-like particles (ALPs) may also drive inflation [12, 13] or make dark matter dynamical [14–16]. The important aspect of the ALPs physics is the link of their properties to the chiral anomalies. The PNGB playing the role of the QCD axion must have anomalous couplings to gluons, similarly as the pion to photons to explain the $\pi^0 \to \gamma\gamma$ decay. Such couplings are not needed for the ALPs that play the other roles mentioned above but their experimental signatures depend on whether the anomalous couplings are present or not.

Extensions of the Standard Model with ALPs in the particle spectrum have been under continuous research for various reasons. Some of them are: a global symmetry as a remnant of gauge symmetries to protect the axion potential against gravitational corrections [17–20], the potential link of ALPs to the fermion mass theories [21], ALPs in chiral gauge theories [22, 23], and the experimental signatures of ALPs.

In this brief review, we recall some selected topics and subtleties related to the link between the properties of ALPs and the global chiral anomalies. For simplicity (and capturing the main points), we work with global U(1) and Abelian gauge symmetries.

2. Axion decay in gauge theories

2.1. Vector-like gauge theories

The model we consider first is defined by the Lagrangian with a local U(1) symmetry

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}_{\mathrm{L}}\not{D}\psi_{\mathrm{L}} + \bar{\psi}_{\mathrm{R}}\not{D}\psi_{\mathrm{R}} + |\partial_\mu\phi|^2 - V\left(|\phi|^2\right) - \left(y\phi\bar{\psi}_{\mathrm{L}}\psi_{\mathrm{R}} + \mathrm{h.c.}\right) , \tag{2.1}$$

where $D_\mu = \partial_\mu - iqgA_\mu$ and the gauge symmetry is vector-like, that is the gauge charges of the left- and right-handed Weyl fermions are: $q_{\mathrm{L}} = q_{\mathrm{R}} \equiv q$. Without loss of generality, one can normalise the gauge charge as $q = 1$. The scalar field ϕ is a singlet of the gauge symmetry. The Lagrangian is classically invariant under two orthogonal vector and axial global symmetries, $\mathrm{U}(1)_{\mathrm{V}}$ and $\mathrm{U}(1)_{\mathrm{A}}$, respectively, defined by the transformations

$$\psi_{\mathrm{L,R}} \to \mathrm{e}^{iQ^{\mathrm{V,A}}_{\mathrm{L,R}}\theta}\psi_{\mathrm{L,R}}\,, \qquad \phi \to \mathrm{e}^{iQ^{\mathrm{V,A}}_{\phi}\theta}\phi, \tag{2.2}$$

with the charges

$$\begin{aligned}\mathbf{U}(1)_{\mathrm{V}}: \quad Q^{\mathrm{V}}_{\mathrm{R}} &= Q^{\mathrm{V}}_{\mathrm{L}} \equiv Q^{\mathrm{V}}\,, \qquad Q^{\mathrm{V}}_{\phi} = 0\,,\\ \mathbf{U}(1)_{\mathrm{A}}: \quad Q^{\mathrm{A}}_{\mathrm{R}} &= -Q^{\mathrm{A}}_{\mathrm{L}} \equiv Q^{\mathrm{A}}/2\,, \qquad Q^{\mathrm{A}}_{\phi} = -Q^{\mathrm{A}}\,. \end{aligned} \tag{2.3}$$

Without loss of generality, the global charges are normalised as $Q^{\mathrm{V}} = Q^{\mathrm{A}} = 1$[1]. In the Dirac fermion notation, $\psi = (\psi_{\mathrm{R}}, \psi_{\mathrm{L}})^{\mathrm{T}}$, and $\mathrm{U}(1)_{\mathrm{V}}$ and $\mathrm{U}(1)_{\mathrm{A}}$ transformations are written as $\psi \to \mathrm{e}^{iQ^{\mathrm{V}}\theta}\psi$ and $\psi \to \mathrm{e}^{iQ^{\mathrm{A}}\gamma_5\theta}\psi$, respectively, where $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \mathrm{diag}(1,-1)$.

Associated with those global symmetries, one can find the Noether currents

$$\begin{aligned} J^{\mathrm{V}}_{\mu} &= iQ^{\mathrm{V}}\left(\bar{\psi}_{\mathrm{L}}\gamma_{\mu}\psi_{\mathrm{L}} + \bar{\psi}_{\mathrm{R}}\gamma_{\mu}\psi_{\mathrm{R}}\right),\\ J^{\mathrm{A}}_{\mu} &= iQ^{\mathrm{A}}\left[\left(\bar{\psi}_{\mathrm{L}}\gamma_{\mu}\psi_{\mathrm{L}} - \bar{\psi}_{\mathrm{R}}\gamma_{\mu}\psi_{\mathrm{R}}\right) + i\left(\phi^*\partial_{\mu}\phi - \phi\partial_{\mu}\phi^*\right)\right]. \end{aligned} \tag{2.4}$$

Classically, these currents are conserved; $\partial^{\mu}J^{\mathrm{V}}_{\mu} = \partial^{\mu}J^{\mathrm{A}}_{\mu} = 0$ (classically).

Note that since there are two orthogonal $\mathrm{U}(1)$ symmetries, any linear combinations of them are also classical symmetries of the Lagrangian. For example, one can define the two symmetry axes as $Q^1_i = \cos\varphi Q^{\mathrm{V}}_i - \sin\varphi Q^{\mathrm{A}}_i$ and $Q^2_i = \sin\varphi Q^{\mathrm{V}}_i + \cos\varphi Q^{\mathrm{A}}_i$ with $i = \mathrm{L}, \mathrm{R}, \phi$. The corresponding symmetry currents $J^1_{\mu} = \cos\varphi J^{\mathrm{V}}_{\mu} - \sin\varphi J^{\mathrm{A}}_{\mu}$ and $J^2_{\mu} = \sin\varphi J^{\mathrm{V}}_{\mu} + \cos\varphi J^{\mathrm{A}}_{\mu}$ are also conserved classically.

Among infinitely many choices of global symmetry axes, $\mathrm{U}(1)_{\mathrm{V}}$ and $\mathrm{U}(1)_{\mathrm{A}}$ directions are special since a non-zero vacuum expectation value of the field ϕ

$$\phi = \frac{1}{\sqrt{2}}(f+\sigma)\,\mathrm{e}^{ia(x)/f} \tag{2.5}$$

breaks spontaneously the $\mathrm{U}(1)_{\mathrm{A}}$, while its orthogonal one, $\mathrm{U}(1)_{\mathrm{V}}$, remains unbroken[2]. The physical spectrum of the theory below the scale f contains then the Nambu–Goldstone boson $a(x)$ of the spontaneously broken $\mathrm{U}(1)_{\mathrm{A}}$ symmetry, which we also call the axion, the massive Dirac fermion and the massless gauge boson, γ. The Lagrangian for these fields takes the form of (for definiteness, we put $Q^{\mathrm{A}} = -1$)

$$\mathcal{L} \supset -\frac{1}{4}F^2_{\mu\nu} + \bar{\psi}\left(\gamma^{\mu}D_{\mu} - \frac{yf}{\sqrt{2}}\right)\psi + \frac{1}{2}(\partial_{\mu}a)^2 - i\frac{y}{\sqrt{2}}a\bar{\psi}\gamma_5\psi + (\cdots)\,, \tag{2.6}$$

where $(\cdots)$ corresponds to the higher-order terms of the axion field. One can give a small mass to the axion by introducing a term that explicitly breaks $\mathrm{U}(1)_{\mathrm{A}}$. For instance, with the term $-\sqrt{2}\epsilon\phi$, the axion acquires the mass $m_a^2 = \frac{\epsilon}{f}$.

[1] In this example, $\mathrm{U}(1)_{\mathrm{V}}$ transformation is a special case of the gauge transformation with the constant gauge transformation parameter. Still, it is useful to talk about the $\mathrm{U}(1)_{\mathrm{V}}$ global symmetry here for later discussions.

[2] The more general case is discussed in detail in [23].

The axion decay rate into two gauge bosons can be calculated in the standard way. From Lorentz and CP invariance, we see that the amplitude must be proportional to $\epsilon_{\mu\nu\rho\sigma}k_1^\mu k_2^\nu \epsilon_1^\rho \epsilon_2^\sigma$, where $k_{1,2}$ are the photon momenta and $\epsilon_{1,2}$ are their polarisation vectors. Since there is no direct coupling between the axion and gauge bosons, the leading contribution to the amplitude is given by triangle diagrams with fermions with mass $M = yf/\sqrt{2}$ running in the loop. The coupling between the axion and fermions is given by $-iy/\sqrt{2}$, as can be seen in Eq. (2.6), and the amplitude picks up this coupling. The result reads

$$i\mathcal{M}(a \to \gamma\gamma) = q^2 \frac{ig^2}{4\pi^2}\left(\frac{y}{\sqrt{2}}\right)\frac{1}{M}\left[1 + \mathcal{O}\left(\frac{m_a^2}{M^2}\right)\right]\epsilon_{\mu\nu\rho\sigma}k_1^\mu k_2^\sigma \epsilon_1^\rho \epsilon_2^\sigma\,. \tag{2.7}$$

Note that the leading order term is independent of the Yukawa coupling y, since y in the numerator cancels the one in the fermion mass $M = yf/\sqrt{2}$. We observe that this result can be obtained at tree level by the effective Lagrangian with a term

$$\mathcal{L}_{\text{eff}} \ni q^2 \frac{g^2}{16\pi^2 f} a F_{\mu\nu}\tilde{F}^{\mu\nu}\,. \tag{2.8}$$

Under the global $\mathrm{U}(1)_\mathrm{A}$ transformation, the axion field gets shifted as $a(x) \to a(x) + f\theta$ (with $Q^\mathrm{A} = -1$). This shows that at the quantum level, $\mathrm{U}(1)_\mathrm{A}$ is no longer a symmetry of the model. We see that the leading contribution to the $a \to \gamma\gamma$ amplitude is directly related to this anomaly and that link will be reviewed in more detail in the next section. This anomalous violation of a global axial symmetry reconciles an apparent inconsistency of the decay rate for $\pi^0 \to \gamma\gamma$ with the concept of the pion as a (pseudo)Nambu–Goldstone boson of the spontaneously broken approximate axial symmetry of strong interactions of the light quarks [4].

One can highlight this point by considering a model with another fermion pair $(\psi'_\mathrm{L}, \psi'_\mathrm{R})$ with the same gauge charge, $q'_\mathrm{L} = q'_\mathrm{R} = q$, and the opposite $\mathrm{U}(1)_\mathrm{A}$ charge compared to those of the original pair, $(\psi_\mathrm{L}, \psi_\mathrm{R})$. Classical symmetries allow the Lagrangian to have the Yukawa term

$$-\left(y'\phi^\dagger \bar{\psi}'_\mathrm{L}\psi'_\mathrm{R} + \text{h.c.}\right)\,. \tag{2.9}$$

After ϕ acquires the v.e.v. in Eq. (2.5), the new fermions obtain the mass $M' = y'f/\sqrt{2}$. Since they couple to $\phi^\dagger$ rather than ϕ (due to the opposite $\mathrm{U}(1)_\mathrm{A}$ charge), the coupling to the axion has the opposite sign, $i(y'/\sqrt{2})a\bar{\psi}'\gamma_5\psi'$, compared to the previous case. The new fermions give the same contribution to $i\mathcal{M}(a \to \gamma\gamma)$ as Eq. (2.7) but with the opposite sign.

The leading contributions to the $i\mathcal{M}(a \to \gamma\gamma)$ from ψ and from ψ' cancel out. This is consistent with the fact that the theory with the new fermion pair is free from the axial anomaly (see Sec. 3). The next-to-leading terms in this case do not cancel and give

$$i\mathcal{M}(a \to \gamma\gamma) = q^2 \frac{ig^2}{4\pi^2 f} \left[\frac{m_a^2}{24} \left(\frac{1}{M^2} - \frac{1}{M'^2} \right) + \mathcal{O}\left(\frac{1}{M^4} \right) \right] \epsilon_{\mu\nu\rho\sigma} k_1^\mu k_2^\sigma \epsilon_1^\rho \epsilon_2^\sigma \,. \tag{2.10}$$

Before closing this subsection, we comment on the case where the vector-like U(1) gauge symmetry is broken by the Brout–Englert–Higgs mechanism. This can easily be realised by adding to the above model (2.1) a new scalar, ϕ', with a non-vanishing gauge charge $q' \neq 0$ and assume that ϕ' gets a v.e.v. The Yukawa term for ϕ' is forbidden due to the non-zero gauge charge and the previous calculation of the axion decay is not modified except that the gauge bosons (we call them Z in this case) are now massive. We have

$$i\mathcal{M}(a \to ZZ) = q^2 \frac{ig^2}{4\pi^2 f} (1 + \varDelta)\, \epsilon_{\mu\nu\rho\sigma} k_1^\mu k_2^\sigma \epsilon_1^\rho \epsilon_2^\sigma \tag{2.11}$$

with

$$\varDelta = \frac{m_a^2 + 2m_Z^2}{24M^2} + \mathcal{O}\left(\frac{1}{M^4} \right) . \tag{2.12}$$

It is somewhat amusing that the expression of the leading term of the axion decay amplitude is unchanged from the previous case with the unbroken U(1) despite the fact that gauge bosons in this case have a longitudinal component. The latter effect is encapsulated in the polarization vectors $\epsilon_1(k_1)$ and $\epsilon_2(k_2)$, which are different from the ones for massless gauge bosons in Eq. (2.7).

2.2. Chiral gauge theories

When the gauge theory is chiral, the model of (2.1) needs extensions. First of all, when the gauge charges of left- and right-handed Weyl fermions that couple to a scalar, ϕ, are chiral ($q_{\mathrm{L}} \neq q_{\mathrm{R}}$), the guage invariance of the Yukawa term requires that the scalar necessarily carries a non-zero gauge charge, $q_\phi = q_{\mathrm{L}} - q_{\mathrm{R}} \neq 0$. Therefore, in this case, the v.e.v. of ϕ breaks a global $\mathrm{U}(1)_{\mathrm{A}}$ spontaneously and also breaks the local U(1). Secondly, since the gauge boson acquires a mass, for the axion (the pseudo-Nambu–Goldstone boson of the $\mathrm{U}(1)_{\mathrm{A}}$ breaking) to remain in the physical spectrum, one needs at least two scalars (or two phases) because one combination of them is eaten up by the Brout–Englert–Higgs mechanism.

We illustrate these points in an explicit model. Our model contains two scalars (ϕ_1, ϕ_2) and one pair of fermions ($\psi_{\rm L}$, $\psi_{\rm R}$)[3]. We assume ϕ_1 and ϕ_2 have non-zero but different gauge charges $q_{\phi_1} \neq q_{\phi_2}$ and $q_{\phi_1} = q_{\rm L} - q_{\rm R} \neq 0$. In this case, only ϕ_1 can have a gauge-invariant Yukawa term with the fermions

$$\mathcal{L} \ni -y\phi_1 \bar{\psi}_{\rm L}\psi_{\rm R} + \text{h.c.}\,. \tag{2.13}$$

We assume both ϕ_1 and ϕ_2 develop non-zero v.e.v.s; $\langle\phi_i\rangle = f_i \neq 0$ ($i = 1, 2$). Writing $\phi_i = \frac{1}{\sqrt{2}}(f_i + \sigma_i(x))\,{\rm e}^{ia_i(x)/f_i}$, the phase degrees of freedom transform as $a_i(x) \to a_i(x) + q_{\phi_i} f_i \alpha(x)$ under the gauge transformation. Therefore, defining

$$\begin{aligned} a(x) &= \cos\varphi a_1(x) - \sin\varphi a_2(x)\,, \\ \tilde{a}(x) &= \sin\varphi a_1(x) + \cos\varphi a_2(x)\,, \end{aligned} \tag{2.14}$$

with

$$\cos\varphi = \frac{q_{\phi_2} f_2}{\tilde{f}}\,, \qquad \sin\varphi = \frac{q_{\phi_1} f_1}{\tilde{f}}\,, \qquad \tilde{f} = \sqrt{(q_{\phi_1} f_1)^2 + (q_{\phi_2} f_2)^2}\,, \tag{2.15}$$

$\tilde{a}(x)$ transforms as $\tilde{a}(x) \to \tilde{a}(x) + \tilde{f}\alpha(x)$, while $a(x)$ is invariant under the gauge transformation. We can thus identify $\tilde{a}(x)$ as the would-be Nambu–Goldstone boson to be eaten by the gauge boson and $a(x)$ remains physical in the low-energy spectrum.

Similarly as for the vector-like gauge theory, classically, the theory has two global symmetries: ${\rm U(1)_V}$ and ${\rm U(1)_A}$. The ${\rm U(1)_A}$ symmetry ($Q^{\rm A}(\psi_{\rm R}) = -Q^{\rm A}(\psi_{\rm L}) = \frac{1}{2}$, $Q^{\rm A}(\phi_1) = -1$) is spontaneously broken by $\langle\phi_1\rangle = f_1$. The Nambu–Goldstone mode of this broken symmetry is $a_1(x)$, which can be expressed in terms of the physical field $a(x)$ and the would-be Nambu–Goldstone boson $\tilde{a}(x)$ as $a_1(x) = \cos\varphi a(x) + \sin\varphi\tilde{a}(x)$. At the leading order, the interaction between the physical axion $a(x)$ and the fermions is given by

$$\mathcal{L} \ni i\frac{y\cos\varphi}{\sqrt{2}} a(x)\bar{\psi}_{\rm L}\psi_{\rm R} + \text{h.c.} = i\frac{y\cos\varphi}{\sqrt{2}} a(x)\bar{\psi}\gamma_5\psi\,. \tag{2.16}$$

In the last expression, we combine the Weyl fermions into the four-component Dirac spinor field as $\psi = (\psi_{\rm R}, \psi_{\rm L})^{\rm T}$. In the Dirac spinor notation, the fermion kinetic term is organised as

$$i\bar{\psi}\gamma^\mu\left(\partial_\mu - ig\left[\alpha - \beta\gamma_5\right]A_\mu - M\right)\psi\,, \tag{2.17}$$

where $\alpha = (q_{\rm L} + q_{\rm R})/2$, $\beta = (q_{\rm L} - q_{\rm R})/2$ and $M = yf_1/\sqrt{2}$.

[3] We assume the existence of additional fermions that cancel the $[{\rm U(1)}]^3$ gauge anomaly. Such fermions can always be introduced so that they do not couple to the scalars and do not modify the axion decay.

Understanding the axion–fermion and gauge boson–fermion interactions in Eqs. (2.16) and (2.17), respectively, we are ready to compute the axion decay amplitude, $a \to ZZ$, in this scenario. A diagramatic calculation of the amplitude is performed in Appendix and the result reads [23]

$$i\mathcal{M}(a \to ZZ) = \frac{i\cos\varphi g^2}{4\pi^2 f_1}\left[\left(\alpha^2 + \frac{1}{3}\beta^2\right) + \Delta\right]\epsilon^{\mu\nu\rho\sigma}\epsilon^1_\mu\epsilon^2_\nu q_1^\rho q_2^\sigma\,, \quad (2.18)$$

where Δ are higher-order terms in m_a^2/M^2 and m_Z^2/M^2 given in Eq. (A.23).

It is not obvious how the leading term with the factor $(\alpha^2 + \beta^2/3)$ is related to the chiral anomaly. In the next section, we provide a general argument for the relation between the axion decay and divergences of three-current-correlator and interpret the factor $(\alpha^2 + \beta^2/3)$ from the anomaly view point.

3. Axion decays and chiral anomaly

Our starting point is the LSZ formula for the S-matrix element of the axion decay $a(p) \to Z(k_1)Z(k_2)$

$$\begin{aligned}\langle\epsilon_1(k_1)\epsilon_2(k_2)|S|a(p)\rangle &= \epsilon_1^\mu\epsilon_2^\nu\left[i\int \mathrm{d}x\,\mathrm{e}^{-ipx}\right]\left[i\int \mathrm{d}y_2\,\mathrm{e}^{ik_1y_1}\right]\left[i\int \mathrm{d}y_1\,\mathrm{e}^{ik_2y_2}\right]\\ &\times\left(\Box_x + m_a^2\right)\left(\Box_{y_1} + m_Z^2\right)\left(\Box_{y_2} + m_Z^2\right)\langle\Omega|T\{a(x)A_\mu(y_1)A_\nu(y_2)\}|\Omega\rangle\,. \quad (3.1)\end{aligned}$$

The next step is to use the Schwinger–Dyson equations [24]

$$\begin{aligned}&\left(\Box_x + m_{\phi_x}^2\right)\langle\phi_x\phi_1\cdots\phi_n\rangle = \left\langle\mathcal{L}'_{\text{int}}[\phi_x]\phi_1\cdots\phi_n\right\rangle\\ &-i\hbar\sum_j\delta^4(x - x_j)\langle\phi_1\cdots\phi_{j-1}\phi_{j+1}\cdots\phi_n\rangle\,, \quad (3.2)\end{aligned}$$

and remove three $(\Box + m^2)$ from the second line of Eq. (3.1). In the above equation and hereafter, we use a shorthand notation for a Green function of a time-ordered product $\langle\cdots\rangle \equiv \langle\Omega|T\{\cdots\}|\Omega\rangle$. In Eq. (3.2), $\phi_i \equiv \phi(x_i)$ represents a general field, which will be $a(x)$, $A_\mu(y_1)$ and $A_\nu(y_2)$ in our case, and $\mathcal{L}'_{\text{int}}[\phi_x] \equiv \partial\mathcal{L}_{\text{int}}[\phi_x]/\partial\phi_x$, where the Lagrangian is assumed to have a form of $\mathcal{L} \ni -\frac{1}{2}\phi_i(\Box_i + m_{\phi_i}^2)\phi_i + \mathcal{L}_{\text{int}}[\phi_i]$. Generally, we observe

$$\mathcal{L}_{\text{int}} = -\,A^\mu(x)J_\mu^{\text{gauge}}(x) - \frac{\cos\varphi}{2f_1}a(x)\left[\partial^\mu J_\mu^{\text{A}}(x)\right], \quad (3.3)$$

where J_μ^{gauge} is the U(1) gauge current and $J_\mu^{\text{A}} = \bar\psi\gamma_5\gamma_\mu\psi$ is the fermionic part of the U(1)_A current. Here, we have assumed a scalar field ϕ_1 has a Yukawa term with fermions and its phase $a_1(x)$ in $\phi_1 = f_1\,\mathrm{e}^{ia_1(x)/f_1}$ has the physical axion component as $a_1(x) = \cos\varphi a(x) \ldots$ This is the exact situation we have encountered in the model discussed in Section 2.2 for chiral gauge theories. For the vector-like gauge theories discussed in Section 2.1, one can simply take $f_1 = f$ and $\cos\varphi = 1$.

It may be useful to comment on the origin of the second term in the above expression. In general, the interaction between the axion and fermions is originated from the Yukawa term

$$y\phi_1\bar{\psi}_{\rm L}\psi_{\rm R} \ \ni \ yf_1\,{\rm e}^{i\frac{a_1(x)}{f_1}}\,\bar{\psi}_{\rm L}\psi_{\rm R}\,. \tag{3.4}$$

It is possible to redefine the fermion fields so that the axion field is removed from the Yukawa term. This can be achieved by $\psi_{\rm L} \to {\rm e}^{i\frac{a_1(x)}{2f_1}}\psi_{\rm L}$, $\psi_{\rm R} \to {\rm e}^{-i\frac{a_1(x)}{2f_1}}\psi_{\rm R}$. However, since these field redefinitions are position-dependent, fermion kinetic terms are not invariant and give extra terms

$$\delta\mathcal{L} = \frac{1}{2f_1}\left(\partial^\mu a_1(x)\right)\left[\bar{\psi}_{\rm R}\gamma_\mu\psi_{\rm R} - \bar{\psi}_{\rm L}\gamma_\mu\psi_{\rm L}\right] = -\frac{\cos\varphi}{2f_1}a(x)\left[\partial^\mu J^{\rm A}_\mu(x)\right]\,, \tag{3.5}$$

which agrees with the second term in Eq. (3.3). In the last expression, we have used integration by parts and assumed $a_1(x)$ contains the physical axion $a(x)$ with the coefficient $\cos\varphi$. Also, we have omitted the term due to the non-invariance of the fermionic path integral measure since its addition to Eq. (3.5) contributes at two-loop level.

Now, let us use Eq. (3.2) with Eq. (3.3) and remove three $(\Box + m^2)$ operators from the right-hand side of the LSZ formula (3.1). The contact terms (the second term of Eq. (3.2)) in the Schwinger–Dyson equation do not contribute to the S-matrix element when the three momenta p, k_1 and k_2 are different. The result reads

$$\langle\epsilon_1(k_1)\epsilon_2(k_2)|S|a(p)\rangle \ = \ \frac{\cos\varphi}{2f_1}\epsilon_1^\mu\epsilon_2^\nu\,p^\alpha\left\langle J^{\rm A}_\alpha(-p)J^{\rm gauge}_\mu(k_1)J^{\rm gauge}_\nu(k_2)\right\rangle\,, \tag{3.6}$$

where $J^{\rm A}_\rho(-p)$ and $J^{\rm gauge}_\mu(k)$ are the currents in the momentum space; $J^I_\mu(k) \equiv \int {\rm d}^4x\,{\rm e}^{ikx}J^I(x)$. This equation clearly relates the axion decay, $a \to ZZ$, and the non-conservation of the axial current, $\langle[\partial^\rho J^{\rm A}_\rho]J^{\rm gauge}_\mu J^{\rm gauge}_\nu\rangle \neq 0$, *i.e.* chiral anomaly.

It is well known that the calculation of the divergence of three-current-correlator $\langle[\partial^\rho J^1_\rho]J^2_\mu J^3_\nu\rangle$ involves a subtlety that the result depends on the reparametrisation of the loop momenta. This ambiguity corresponds to a freedom to move the anomaly around amongst the three currents. We discuss the evaluation of the right-hand side of Eq. (3.6) in concrete examples.

Vector-like gauge theory

In model (2.1) discussed in Section 2.1, the gauge current is given by $J^{\text{gauge}}_\mu = iqg\,\bar\psi\gamma_\mu\psi$. If the gauge symmetry is not broken, the calculation of a divergence of three-current-correlators must be performed in such a way that the gauge current is conserved

$$k_1^\mu \left\langle J^{\text{A}}(-p)J^{\text{gauge}}_\mu(k_1)J^{\text{gauge}}_\nu(k_2)\right\rangle = k_2^\nu \left\langle J^{\text{A}}(-p)J^{\text{gauge}}_\mu(k_1)J^{\text{gauge}}_\nu(k_2)\right\rangle = 0\,. \tag{3.7}$$

Fixing the loop momentum ambiguity by the above condition, the divergence of the axial current is determined. The leading contribution is found as (the fermion mass dependent terms are omitted)

$$p^\alpha \left\langle J^{\text{A}}_\alpha(-p)J^{\text{gauge}}_\mu(k_1)J^{\text{gauge}}_\nu(k_2)\right\rangle = \frac{q^2g^2}{4\pi^2}\epsilon_{\mu\nu\rho\sigma}k_1^\rho k_2^\sigma \tag{3.8}$$

up to the $(2\pi)^2\delta^4(p-k_1-k_2)$ factor. Plugging this into Eq. (3.6) and taking $f_1 = f$ and $\cos\varphi = 1$, we reproduce the result in Eq. (2.7).

At the end of Section 2.1, we have discussed a case where the vector-like gauge symmetry is broken by a v.e.v. of additional scalar ϕ'. In this case, there seems no reason why condition (3.7) should be imposed. However, even with the non-zero $\langle\phi\rangle$ and $\langle\phi'\rangle$, the global $\text{U}(1)_\text{V}$, defined in Eqs. (2.2) and (2.3), is not broken. Thus, J^{V}_μ in Eq. (2.4) is conserved. Since the fermionic part of J^{gauge}_μ is proportional to J^{V}_μ, the conservation of J^{V}_μ implies condition (3.7). Therefore, even in the case where the vector-like gauge symmetry is broken, we have Eq. (3.8) and the expression of the amplitude is unchanged from the unbroken case. The result again agrees with Eq. (2.11) obtained in Section 2.1.

Finally, it is worth returning to our example with two pairs of Weyl fermions in Section 2.1. In that case, the leading contributions to the $i\mathcal{M}(a\to\gamma\gamma)$ from ψ and from ψ' cancel out. This is consistent with the fact that the theory with the new fermion pair with the same gauge charge, $q'_\text{L} = q'_\text{R} = q$, and the opposite $\text{U}(1)_\text{A}$ charge compared to those of the original pair, $(\psi_\text{L},\psi_\text{R})$ is free from the $\text{U}(1)_\text{A}$–$\text{U}(1)$–$\text{U}(1)$ anomaly. Indeed, the anomaly coefficient vanishes: $\text{Tr}\,[Q_\text{A}\{q,q\}] = 0$, where the trace is taken in the space of left- and right-handed fermion fields.

Chiral gauge theory

In the chiral gauge theory introduced in Section 2.2, the fermionic part of the gauge current is given by

$$J^{\text{gauge}}_\mu = g\left(\alpha J^{\text{V}}_\mu + \beta J^{\text{A}}_\mu\right)\,, \tag{3.9}$$

where $\alpha = (q_{\rm L}+q_{\rm R})/2$, $\beta = (q_{\rm L}-q_{\rm R})/2$, and $J^{\rm V}_\mu \equiv i\bar\psi\gamma_\mu\psi$ and $J^{\rm A}_\mu \equiv i\bar\psi\gamma_\mu\gamma_5\psi$ are the currents for $\mathrm{U(1)_V}$ and $\mathrm{U(1)_A}$, respectively. The divergence of three currents in question can be written as

$$p^\alpha \left\langle J^{\rm A}_\alpha(-p)J^{\rm gauge}_\mu(k_1)J^{\rm gauge}_\nu(k_2)\right\rangle = g^2\Big[\alpha^2 p^\alpha \left\langle J^{\rm A}_\alpha(-p)J^{\rm V}_\mu(k_1)J^{\rm V}_\nu(k_2)\right\rangle$$
$$+\beta^2 p^\alpha \left\langle J^{\rm A}_\alpha(-p)J^{\rm A}_\mu(k_1)J^{\rm A}_\nu(k_2)\right\rangle\Big], \tag{3.10}$$

where we have used the fact that the divergence of three-current-correlator involving even number of $J^{\rm A}$ vanishes, since there is no γ_5 in the fermionic trace.

We have assumed the gauge symmetry and $\mathrm{U(1)_A}$ symmetry are broken by the non-zero v.e.v.s of two scalars $\langle\phi_1\rangle$ and $\langle\phi_2\rangle$, and there is thus no reason to impose the conservation of $J^{\rm gauge}_\mu$ and $J^{\rm A}_\mu$. However, scalar fields are not charged under the $\mathrm{U(1)_V}$ and this symmetry remains unbroken. Therefore, the calculation of the first term must be done in such a way that $J^{\rm V}_\mu$ is conserved. As mentioned above, this gives

$$p^\alpha \left\langle J^{\rm A}_\alpha(-p)J^{\rm V}_\mu(k_1)J^{\rm V}_\nu(k_2)\right\rangle = \frac{1}{4\pi^2}\epsilon_{\mu\nu\rho\sigma}k_1^\rho k_2^\sigma\,. \tag{3.11}$$

The second term of Eq. (3.10) does not have any special currents on which we should impose conservation. On the other hand, all three currents are identical. In this case, one should fix the loop momentum ambiguity such that all three internal momenta are treated symmetrically. This results in [25]

$$p^\alpha \left\langle J^{\rm A}_\alpha(-p)J^{\rm A}_\mu(k_1)J^{\rm A}_\nu(k_2)\right\rangle = \frac{1}{3}\frac{1}{4\pi^2}\epsilon_{\mu\nu\rho\sigma}k_1^\rho k_2^\sigma\,. \tag{3.12}$$

Collecting these results, we have

$$p^\alpha \left\langle J^{\rm A}_\alpha(-p)J^{\rm gauge}_\mu(k_1)J^{\rm gauge}_\nu(k_2)\right\rangle = \frac{g^2}{4\pi^2}\left(\alpha^2+\frac{1}{3}\beta^2\right)\epsilon_{\mu\nu\rho\sigma}k_1^\rho k_2^\sigma\,, \tag{3.13}$$

up to the $(2\pi)^2\delta^4(p-k_1-k_2)$ factor. One can see that the result agrees with the axion decay formula (2.18) as expected.

The axion decay formula (2.18) suggests that the low-energy effective Lagrangian has a term

$$\frac{g^2\cos\varphi}{16\pi^2 f_1}\left(\alpha^2+\frac{1}{3}\beta^2\right)aF_{\mu\nu}\tilde F^{\mu\nu}\,. \tag{3.14}$$

In the chiral gauge theory, there is a mismatch between the axion $a(x)$ and the Goldstone mode of $\mathrm{U(1)_A}$ (*i.e.* $a_1(x)$) and their relation is given by $a_1(x) = \cos\varphi a(x) + \sin\varphi\tilde{a}(x)$, where $\tilde{a}(x)$ is the would-be Goldstone mode of the broken gauge symmetry. Under the $\mathrm{U}(1)_{\mathrm{A}}$, $a_1(x)$ transforms as $a_1(x) \to a_1(x) + f_1\theta$. To reproduce this result, the axion must shift as $a(x) \to a(x) + (f_1/\cos\varphi)\theta$. With this shift, term (3.14) reproduces the $\mathrm{U}(1)_{\mathrm{A}}$–U(1)–U(1) anomaly. We conclude that the axion decay is captured by the chiral anomaly also in the case of chiral gauge theories.

4. Summary

We have reviewed the calculation of the axion decay amplitudes into two gauge bosons in vector-like and chiral U(1) gauge theories and its connection to the chiral anomalies. The axion is a (pseudo)Nambu–Goldstone boson (or its component invariant under gauge transformations) of the axial $\mathrm{U}(1)_{\mathrm{A}}$ global symmetry of the Lagrangian. The leading contribution to the decay amplitude depends on whether the gauge theory is vector-like or chiral. In both cases, it is directly linked to the anomalous divergence of the current of the axial global symmetry. In the case of the chiral gauge theory, the calculation of the divergence of the current–current–current Green's function requires a special attention. The vector parts of the currents coupled to the gauge bosons should be conserved, whereas symmetry conditions should be imposed on the Green's function involving their axial parts and the $\mathrm{U}(1)_{\mathrm{A}}$ current.

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Appendix

Calculation of the axion decay

We compute the $a \to ZZ$ amplitude with general interactions and a mass. The Lagrangian is given by

$$\mathcal{L} \ni i\bar{\psi}\gamma^\mu \left(\partial_\mu - ig\left[\alpha - \beta\gamma_5\right] A_\mu - m\right)\psi - i\lambda a\bar{\psi}\gamma_5\psi\,. \tag{A.1}$$

The matrix element takes a form of

$$i\mathcal{M} = (-1)(-\lambda)(ig)^2\epsilon_\mu^{*1}\epsilon_\nu^{*2}\mathcal{M}^{\mu\nu}\,, \tag{A.2}$$

where

$$\mathcal{M}^{\mu\nu} = (i)^3 \int \frac{\mathrm{d}^4k}{(2\pi)^4}$$
$$\times\left\{\mathrm{Tr}\left[\gamma_5 \frac{(\not{k}-\not{q}_1)+m}{(k-q_1)^2-m^2}\gamma^\mu(\alpha-\beta\gamma_5)\frac{\not{k}+m}{k^2-m^2}\gamma^\nu(\alpha-\beta\gamma_5)\frac{(\not{k}+\not{q}_2)+m}{(k+q_2)^2-m^2}\right]\right.$$
$$\left.+\left[(q_1,\mu)\leftrightarrow(q_2,\nu)\right]\right\}. \tag{A.3}$$

Since the matrix element should be invariant under the simultaneous exchange $(q_1,\mu)\leftrightarrow(q_2,\nu)$, it has to be proportional to $q_1^\mu q_2^\nu$ or $\epsilon^{\mu\nu\rho\sigma}q_1^\rho q_2^\sigma$. For both cases, the integral is convergent since $\sim\int \mathrm{d}^4k\frac{q^2k}{(k^2)^3}$.

First note that the numerator of the first trace can be organised as

$$\begin{aligned}&\mathrm{Tr}\left[\gamma_5[(\not{k}-\not{q}_1)+m]\gamma^\mu(\omega_+\not{k}+\omega_-m)\gamma^\nu[(\not{k}+\not{q}_2)+m]\right]\\ &-2\alpha\beta\,\mathrm{Tr}\left[[(\not{k}-\not{q}_1)-m]\gamma^\mu\not{k}\gamma^\nu[(\not{k}+\not{q}_2)+m]\right],\end{aligned} \tag{A.4}$$

where $\omega_\pm\equiv\alpha^2\pm\beta^2$. One can calculate these traces using the formulae

$$\mathrm{Tr}\left[\text{odd \# of }\gamma' s\right] = 0\,, \tag{A.5}$$
$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] = 4\left(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}\right), \tag{A.6}$$
$$\mathrm{Tr}\left[\gamma_5\gamma^\mu\gamma^\nu\right] = 0\,, \tag{A.7}$$
$$\mathrm{Tr}\left[\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] = -4i\epsilon^{\mu\nu\rho\sigma}\,. \tag{A.8}$$

In the first trace of Eq. (A.4), the m^3 term vanishes due to Eq. (A.7). The m^2 term and the mass-independent term also vanish since they have odd numbers of γ matrices. The only non-vanishing term in the first trace of Eq. (A.4) is linear in m and may be calculated in the form

$$4i\epsilon^{\mu\nu\rho\sigma}m\left[\omega_- q_\rho^1 q_\sigma^2 + 2\beta^2 k_\rho(q_1+q_2)_\sigma\right]. \tag{A.9}$$

The non-vanishing term in the second trace of Eq. (A.4) must have four γ matrices. This term can be calculated as

$$8\alpha\beta m\left[k^\mu(q_1+q_2)^\nu + k^\nu(q_1+q_2)^\mu - g^{\mu\nu}k\cdot(q_1+q_2)\right]. \tag{A.10}$$

We are left with the evaluation of the momentum integration with the denominator. One must calculate

$$\int\frac{\mathrm{d}^4k}{(2\pi)^4}(a+b^\alpha k_\alpha)\frac{1}{[(k-q_1)^2-m^2]}\frac{1}{[k^2-m^2]}\frac{1}{[(k+q_2)^2-m^2]}. \tag{A.11}$$

Using the Feynman parameter formula

$$\frac{1}{A_1A_2\cdots A_n} = \int \mathrm{d}x_1\ldots\mathrm{d}x_n\delta\left(\sum x_i - 1\right)\frac{(n-1)!}{[x_1A_1+x_2A_2+\ldots x_nA_n]^n}, \tag{A.12}$$

Eq. (A.11) becomes

$$
\begin{aligned}
&2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\\
&\times\frac{a+b^\alpha k_\alpha}{\left[x\left((k-q_1)^2-m^2\right)+y\left((k+q_2)^2-m^2\right)+(1-x-y)\left(k^2-m^2\right)\right]^3}\\
&=2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\frac{a+b^\alpha k_\alpha}{\left[k^2+2k\cdot(yq_2-xq_1)+(x+y)m_Z^2-m^2\right]^3}\,,
\end{aligned}
\tag{A.13}
$$

where $q_1^2=q_2^2\equiv m_Z^2$ has been used. The $\int\mathrm{d}k^4$ integral can be performed by using the formula [25]

$$
\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{a+b^\alpha k_\alpha}{\left[k^2+2k\cdot p+M^2\right]^n}=\frac{i}{16\pi^2}\frac{\Gamma(n-2)}{\Gamma(n)}\frac{a-b^\alpha p_\alpha}{\left[M^2-p^2\right]^{n-2}}\,. \tag{A.14}
$$

The result reads

$$
\frac{i}{16\pi^2}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\frac{a+b^\alpha\left(xq_1-yq_2\right)_\alpha}{\left(x+y-x^2-y^2\right)m_Z^2-m^2+xy\left(m_a^2-2m_Z^2\right)}\,, \tag{A.15}
$$

where $2q_1\cdot q_2=m_a^2-2m_Z^2$ was used. Let us assume the fermion mass in the loop is much larger than the masses of the axion and the gauge boson, $m\gg m_a,m_Z$. To get the leading-order expression, we take $m_a,m_Z\to 0$. Then, we finally find Eq. (A.11) to be

$$
-\frac{i}{32\pi^2}\frac{1}{m^2}\left[a+\frac{1}{3}b^\alpha(q_1-q_2)_\alpha\right]. \tag{A.16}
$$

Now, we combine this result with numerators (A.9) and (A.10). First, we note that the fact that $\int\mathrm{d}^4k k_\alpha$ term is proportional to $(q_1-q_2)_\alpha$ implies that the pieces in Eq. (A.10) do not contribute to the amplitude. This can be seen by replacing k with (q_1-q_2) in Eq. (A.10)

$$
8\alpha\beta m\left[(q_1-q_2)^\mu(q_1+q_2)^\nu+(q_1-q_2)^\nu(q_1+q_2)^\mu-g^{\mu\nu}\left(q_1^2-q_2^2\right)\right]. \tag{A.17}
$$

The last term vanishes since $q_1^2=q_2^2=m_Z^2$. The first two terms cancel when they are contracted with the polarization tensors $\epsilon^1_\mu\epsilon^2_\nu$ and demand $\epsilon^1\cdot q_1=\epsilon^2\cdot q_2=0$.

Now, what is left is the pieces that come from Eq. (A.9). The result can be obtained by taking $a = 4i\epsilon^{\mu\nu\rho\sigma}m(\alpha^2 - \beta^2)q_1^\rho q_2^\sigma$ and $b^\alpha = 8i\epsilon^{\mu\nu\rho\sigma}m\beta^2\delta_\rho^\alpha(q_1+q_2)^\sigma$ in Eq. (A.16). This leads to

$$\begin{aligned}&\frac{1}{8\pi^2}\frac{1}{m}\epsilon_{\mu\nu\rho\sigma}\left[\left(\alpha^2-\beta^2\right)q_1^\rho q_2^\sigma + \frac{2}{3}\beta^2\left(q_1-q_2\right)^\rho\left(q_1+q_2\right)^\sigma\right]\\ &= \frac{1}{8\pi^2}\frac{1}{m}\left(\alpha^2+\frac{1}{3}\beta^2\right)\epsilon_{\mu\nu\rho\sigma}q_1^\rho q_2^\sigma\,. \end{aligned}\tag{A.18}$$

The contribution from the second trace in Eq. (A.3) can be obtained by replacing $(q_1,\mu) \leftrightarrow (q_2,\nu)$, which is identical. Therefore, the final result is obtained as

$$i\mathcal{M} = \frac{i\lambda g^2}{4\pi^2 m}\left(\alpha^2+\frac{1}{3}\beta^2\right)\epsilon^{\mu\nu\rho\sigma}\epsilon_\mu^{*1}\epsilon_\nu^{*2}q_1^\rho q_2^\sigma + \mathcal{O}\left(m_a^2, m_Z^2\right)\,. \tag{A.19}$$

Let us find out the next-to-leading terms in Eq. (A.19) that are linear in m_a^2 and m_Z^2. The next higher-order terms in the expansion of Eq. (A.15) go as

$$-\frac{i}{32\pi^2}\frac{1}{m^2}\frac{1}{24m^2}\left[a\left(m_a^2+2m_Z^2\right)+\frac{1}{5}\left(2m_a^2+3m_Z^2\right)b^\alpha\left(q_1-q_2\right)_\alpha\right]\,. \tag{A.20}$$

Due to the (q_1-q_2) structure, there is no contribution from Eq. (A.10), and the contribution from Eq. (A.9) can be obtained by taking $a = 4i\epsilon^{\mu\nu\rho\sigma}m(\alpha^2-\beta^2)q_1^\rho q_2^\sigma$ and $b^\alpha = 8i\epsilon^{\mu\nu\rho\sigma}m\beta^2\delta_\rho^\alpha(q_1+q_2)^\sigma$. This leads to

$$\begin{aligned}&\frac{1}{8\pi^2}\frac{1}{m}\frac{1}{24m^2}\epsilon_{\mu\nu\rho\sigma}\\ &\times\left[\left(\alpha^2-\beta^2\right)\left(m_a^2+2m_Z^2\right)q_1^\rho q_2^\sigma+\frac{2}{5}\beta^2\left(2m_a^2+3m_Z^2\right)\left(q_1-q_2\right)^\rho\left(q_1+q_2\right)^\sigma\right]\\ &= \frac{1}{8\pi^2}\frac{1}{m}\frac{1}{24m^2}\left[\left(m_a^2+2m_Z^2\right)\alpha^2+\frac{1}{5}\left(3m_a^2+2m_Z^2\right)\beta^2\right]\epsilon_{\mu\nu\rho\sigma}q_1^\rho q_2^\sigma\,. \end{aligned}\tag{A.21}$$

So, the final result up to the next-to-leading order is

$$i\mathcal{M} = \frac{i\lambda g^2}{4\pi^2 m}\left[\left(\alpha^2+\frac{1}{3}\beta^2\right)+\varDelta\right]\epsilon^{\mu\nu\rho\sigma}\epsilon_\mu^{*1}\epsilon_\nu^{*2}q_1^\rho q_2^\sigma \tag{A.22}$$

with

$$\begin{aligned}\varDelta &= \frac{1}{24m^2}\left[\left(m_a^2+2m_Z^2\right)\alpha^2+\frac{1}{5}\left(3m_a^2+2m_Z^2\right)\beta^2\right]\\ &\quad+\left(\text{higher order in }\frac{m_a^2}{m^2},\ \frac{m_Z^2}{m^2}\right)\,. \end{aligned}\tag{A.23}$$

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