# ANOMALOUS DIMENSIONS AT LARGE CHARGE IN $d = 4$ O($N$) THEORY*

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Recently it was shown that the scaling dimension of the operator $\phi^n$ in $\lambda(\bar{\phi}\phi)^2$ theory may be computed semiclassically at the Wilson–Fisher fixed point in $d = 4 - \epsilon$, for generic values of $\lambda n$, and this was verified to two-loop order in perturbation theory at leading and subleading $n$. In subsequent work, this result was generalised to operators of fixed charge $\bar{Q}$ in O($N$) theory and verified up to three loops in perturbation theory at leading and subleading $\bar{Q}$. Here, we extend this verification to four loops in O($N$) theory, once again at leading and subleading $\bar{Q}$. We also investigate the strong-coupling regime.

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## 1. Tini Veltman (by Tim Jones)

I first encountered Tini when, as a second-year graduate student, I attended a summer school in Louvain (Leuven) in the summer of 1973. The gauge theory revolution was just beginning, and a number of lecture courses at the school were on topics which would seem now very old-fashioned. My interest, however, was firmly on quantum field theory (QFT), and I was working on a paper in the area [1]. Tini's lectures, delivered with enthusiasm and style, were on the Yang–Mills theory and based on the legendary CERN yellow report DIAGRAMMAR. I cannot say I followed them completely, but the appendices of DIAGRAMMAR were a revelation. They provided a concise summary on how to extract the Feynman rules from a QFT, and how to calculate the diagrams using dimensional regularisation. Little details such as an algorithm for calculation of symmetry factors were included. On my return to Oxford, I soon embarked on the calculation for

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which I remain most well-known, the two-loop QCD $\beta$-function. Several more senior people to whom I spoke about the calculation opined that it was not feasible because of the issue of overlapping divergences, but thanks to DIAGRAMMAR I encountered no technical obstacles and the calculation was published in 1974 [2]. The calculation was also done by William Caswell (a student of Curtis Callan), who sadly died on September 11, 2001.

From 1980 to 1983, I was a postdoc at the University of Michigan. Tini arrived in 1981, and played a very active role in group activities. Group lunch before a seminar was always a pleasure, and visiting speakers had to be very much on top of their subject in the face of Tini's penetrating inquiries. He collaborated with Marty Einhorn and me on a project on the nature of radiative corrections to the $\rho$-parameter. We were struck by the fact that in the Standard Model (SM) the corrections have a sign which is independent of the fermion masses. We wondered if that might be true of an arbitrary heavy sector, and succeeded in generalising the SM case of fermions of isospin $(\frac{1}{2}, 0)$ to $(j, j \pm \frac{1}{2})$ for arbitrary $j$. (Such multiplets admit a fermion mass term like the SM one). Had this held true in full generality, it would have placed a very interesting constraint on Beyond the Standard Model Physics. However, we were disappointed to find that in the presence of additional scalars, contributions of the opposite sign could be obtained. Nevertheless, I think it was a nice paper [3].

I left Michigan in 1983. Over the years I would see Tini on occasion at conferences and CERN. The last time we really spoke was when he came to Liverpool in 2007 to deliver a Barkla lecture, an annual event here (instituted by Alon Faraggi). Other Barkla lecturers have included other Nobel Laureates Frank Wilczek, Gerard 't Hooft, Francois Englert and Didier Queloz.

Tini and I were never exactly friends, but I held him in the highest respect. He was a major figure in 20$^{\text{th}}$-century physics, and as I indicated above, he had a decisive influence on my own career. I like to think he would have approved of the calculations presented in this paper. He preferred explicit calculations to hand-waving.

## 2. Introduction

Renormalizable theories with scale invariant scalar self-interactions have been subjects of enduring interest. In particular, the study of theories with quartic ($\phi^4$) interactions in $d = 4 - \epsilon$ dimensions has played a central role in the development of the theory of critical phenomena, since the pioneering work of Wilson [4, 5] and Wilson and Fisher [6] in 1971. Study of the renormalisation group flow of the coupling or couplings of the theories facilitates the determination of the order of phase transitions and the associated critical indices. For example, the theory with a single scalar field exhibits a

Wilson–Fisher fixed point (FP) where the coupling constant $\lambda$ is $O(\epsilon)$, and this infra-red (IR) attractive FP is associated with a second-order phase transition.

Historically, the majority of work in renormalisable quantum field theories has involved the weak coupling expansion, in other words, the Feynman diagram loop expansion. However, this expansion fails or becomes ponderous at either strong coupling or (less obviously) for $\phi^n$ amplitudes at large $n$. The latter has obviously developed in importance as collider energies have increased. Remarkable progress [7–15] here came with the use of a semi-classical expansion in the path integral formulation of the theory[1].

In Ref. [11]. the anomalous dimension of the $\phi^n$ operator was considered in the O($N$)-invariant $g(\phi^2)^2$ theory with an $N$-dimensional scalar multiplet $\phi$, for large $n$ and fixed $gn^2$. In Ref. [12], the scaling dimension of the same operator in the U(1)-invariant $\lambda(\bar{\phi}\phi)^2$ theory (corresponding to the special case $N = 2$) was computed at the Wilson–Fisher fixed point $\lambda_*$ as a semiclassical expansion in $\lambda_*$, for fixed $\lambda_* n$. Subsequently, this was generalised in Ref. [13] to the case of an operator of charge $\bar{Q}$ in the O($N$)-invariant theory. In Ref. [12], the U(1) result was compared with perturbation theory up to two loops, and in Ref. [13], the check was performed for the O($N$) theory up to three loops. Here we proceed directly with the O($N$) case, since, at least for our purposes, many salient features of the analysis are very similar in both cases; and the results for U(1) may be recovered from those for O($N$), essentially by setting $N = 2$. We extend the comparison with perturbation theory up to four loops, and also discuss the large $(g\bar{Q})$ case, generalising the large $\lambda n$ analysis of Ref. [12].

The paper is organised as follows: In Section 3 we describe the semiclassical calculation in the O($N$) case, following Ref. [13]. Then in Section 4, we compare the result of this calculation with perturbative calculations up to and including 4 loops. This represents a significant extension of previous calculations. In Section 5, we address the large $(g\bar{Q})$ limit and compare in detail with earlier work.

## 3. The O($N$) case

In the O($N$) case, we have a multiplet of fields $\phi_i$, $i = 1 \ldots N$, and the Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi_i\partial_\mu\phi_i + \frac{g}{4!}(\phi_i\phi_i)^2 \,. \tag{3.1}$$

[1] An analogous analysis was pursued for $\phi^6$ theories for $d = 3 - \epsilon$ and $\phi^3$ theories for $d = 6 - \epsilon$ in Refs. [16–18].

The $\beta$-function for this theory is well-known [21]

$$16\pi^2\beta(g) = -\epsilon g + \frac{g^2}{3}(N+8) - \frac{g^3}{3}(3N+14) + \mathcal{O}\left(g^4\right) , \tag{3.2}$$

and leads to an infra-red conformal fixed point

$$g_* = \frac{3\epsilon}{N+8} + \frac{9(3N+14)}{(N+8)^3}\epsilon^2 + \mathcal{O}\left(\epsilon^3\right) . \tag{3.3}$$

As shown in Ref. [13], the fixed-charge operator of charge $\bar{Q}$ may be taken to be

$$T_{\bar{Q}} = T_{i_1 i_2 \dots i_{\bar{Q}}} \phi_{i_1}\phi_{i_2}\dots\phi_{i_{\bar{Q}}} , \tag{3.4}$$

where $T_{i_1 i_2 \dots i_{\bar{Q}}}$ is symmetric, and traceless on any pair of indices. The scaling dimension $\Delta_{T_{\bar{Q}}}$ is expanded as

$$\Delta_{T_{\bar{Q}}} = \bar{Q}\left(\frac{d}{2}-1\right) + \gamma_{T_{\bar{Q}}} = \sum_{\kappa=-1} g^\kappa \Delta_\kappa\left(g\bar{Q}\right) . \tag{3.5}$$

We initially work in general $d$. The semiclassical computation of $\Delta_{-1}$ and $\Delta_0$ is performed by mapping the theory via a Weyl transformation to a cylinder $\mathbb{R}\times S^{d-1}$, where $S^{d-1}$ is a sphere of radius $R$; where the $\mathcal{R}\phi^*\phi$ term ($\mathcal{R}$ being the Ricci curvature) generates an effective $m^2\phi^*\phi$ mass term with $m = \frac{d-2}{2R}$. This mapping process along with other technical simplifications [12] relies on conformal invariance and, therefore, we now assume that we are at the conformal fixed point in Eq. (3.3). It was shown in Ref. [12] that stationary configurations of the action are characterised by a chemical potential $\mu$, related to the cylinder radius $R$ by

$$R\mu_* = \frac{3^{\frac{1}{3}} + \left[6g_*\bar{Q} + \sqrt{36\left(g_*\bar{Q}\right)^2 - 3}\right]^{\frac{2}{3}}}{3^{\frac{2}{3}}\left[6g_*\bar{Q} + \sqrt{36\left(g_*\bar{Q}\right)^2 - 3}\right]^{\frac{1}{3}}} . \tag{3.6}$$

The computation of the leading contribution $\Delta_{-1}$ is entirely analogous to the U(1) case and is given by

$$\frac{4\Delta_{-1}\left(g_*\bar{Q}\right)}{g_*\bar{Q}} = \frac{3^{\frac{2}{3}}\left[x+\sqrt{x^2-3}\right]^{\frac{1}{3}}}{3^{\frac{1}{3}} + \left[x+\sqrt{x^2-3}\right]^{\frac{2}{3}}} + \frac{3^{\frac{1}{3}}\left\{3^{\frac{1}{3}} + \left[x+\sqrt{x^2-3}\right]^{\frac{2}{3}}\right\}}{\left[x+\sqrt{x^2-3}\right]^{\frac{1}{3}}} , \tag{3.7}$$

where $x = 6g_*\bar{Q}$. Its expansion for small $g_*\bar{Q}$ takes the form

$$\frac{\Delta_{-1}\left(g_*\bar{Q}\right)}{g_*} = \bar{Q}\left[1 + \frac{1}{3}g_*\bar{Q} - \frac{2}{9}\left(g_*\bar{Q}\right)^2 + \frac{8}{27}\left(g_*\bar{Q}\right)^3 - \frac{14}{27}\left(g_*\bar{Q}\right)^4 + \mathcal{O}\left\{\left(g_*\bar{Q}\right)^5\right\}\right]. \tag{3.8}$$

As in the U(1) case, for simplicity, we give in Eq. (3.6) the result for $d = 3$. The non-leading corrections $\Delta_0$ are once more given by the determinant of small fluctuations. There are two modes corresponding to those in the Abelian case, with the dispersion relation

$$\omega_{\pm}^2(l) = J_l^2 + 3\mu^2 - m^2 \pm \sqrt{4J_l^2\mu^2 + \left(3\mu^2 - m^2\right)^2}\,, \tag{3.9}$$

where

$$J_l^2 = \frac{l(l+d-2)}{R^2} \tag{3.10}$$

is the eigenvalue of the Laplacian on the sphere. In addition there are $\frac{N}{2} - 1$ "Type II" (non-relativistic) [19] Goldstone modes and $\frac{N}{2} - 1$ massive states with dispersion relation

$$\omega_{\pm\pm}(l) = \sqrt{J_l^2 + \mu^2} \pm \mu\,, \tag{3.11}$$

with $J_l$ as defined in Eq. (3.10). We then find that $\Delta_0$ is given by

$$\Delta_0\left(g_*\bar{Q}\right) = \frac{1}{2}\sum_{l=0}^{\infty}\sigma_l\,, \tag{3.12}$$

where

$$\sigma_l = Rn_l\left\{\omega_+^*(l) + \omega_-^*(l) + \left(\frac{N}{2} - 1\right)\left[\omega_{++}^*(l) + \omega_{--}^*(l)\right]\right\}. \tag{3.13}$$

Here

$$n_l = \frac{(2l+d-2)\Gamma(l+d-2)}{\Gamma(l+1)\Gamma(d-1)} \tag{3.14}$$

is the multiplicity of the Laplacian on the $d$-dimensional sphere, and $\omega_{\pm}^*$, $\omega_{++}^*$, $\omega_{--}^*$ are defined as in Eqs. (3.9), (3.11), respectively, evaluated at the fixed point with $R$, $\mu_*$ related by Eq. (3.6). For the small $(g_*\bar{Q})$ computation, we need to isolate the divergent contribution in the sum in Eq. (3.12). We use the large-$l$ expansion of $\sigma_l$

$$\sigma_l = \sum_{n=1}^{\infty} c_n l^{d-n} \tag{3.15}$$

with

$$
\begin{aligned}
c_1 = N\,, \qquad c_2 &= 3N\,,\\
c_3 &= \frac{1}{2}\left[5N-2+(N+2)(R\mu_*)^2\right]\,,\\
c_4 &= \frac{1}{2}\left[N-2+(N+2)(R\mu_*)^2\right]\,,\\
c_5 &= \frac{N+8}{8}\left(R^2\mu_*^2-1\right)^2\left[-1+\left(\gamma-\frac{3}{2}\right)\epsilon\right]\\
&\quad -\frac{29}{12}\left(R^2\mu_*^2-1\right)\epsilon-\left(\frac{11}{24}R^2\mu_*^2-\frac{1}{5}\right)N\epsilon\,. \qquad (3.16)
\end{aligned}
$$

We can write

$$
\begin{aligned}
\Delta_0\left(g_*\bar{Q}\right) &= -\frac{15\mu_*^4R^4+6\mu_*^2R^2-5}{16}+\frac{1}{2}\sum_{l=1}^{\infty}\bar{\sigma}_l+\sqrt{\frac{3\mu_*^2R^2-1}{2}}\\
&\quad -\frac{1}{16}\left(\frac{N}{2}-1\right)\left[7-16R\mu_*+6R^2\mu_*^2+3R^4\mu_*^4\right]\,, \qquad (3.17)
\end{aligned}
$$

where

$$
\bar{\sigma}_l=\sigma_l-c_1l^3-c_2l^2-c_3l-c_4-c_5\frac{1}{l}\,. \qquad (3.18)
$$

Here, the divergent parts have been isolated and the sums over $l$ performed, as explained in Refs. [12] and [13]. The sum over $\frac{1}{l^{d-n}}$ for $n=5$ leads to a pole in $\epsilon$ which cancels against the pole in the bare coupling. The sum over $\overline{\sigma}_l$ is then finite and setting $d=4$, and expanding in small $g_*\bar{Q}$ can be performed analytically. We obtain

$$
\begin{aligned}
\Delta_0 &= -\frac{1}{6}(10+N)g_*\bar{Q}+\frac{1}{18}(6-N)\left(g_*\bar{Q}\right)^2\\
&\quad +\frac{1}{27}\left[N-36+2(14+N)\zeta_3\right]\left(g_*\bar{Q}\right)^3\\
&\quad -\frac{1}{81}\left[4(N-73)+2(6N+65)\zeta_3+5(N+30)\zeta_5\right]\left(g_*\bar{Q}\right)^4+\ldots \qquad (3.19)
\end{aligned}
$$

Adding Eqs. (3.8) and (3.19), we find [13]

$$
\begin{aligned}
\frac{\Delta_{-1}\left(g_*\bar{Q}\right)}{g_*}+\Delta_0\left(g_*\bar{Q}\right) &= \bar{Q}+\frac{1}{6}\left[2\bar{Q}-(N+10)\right]g_*\bar{Q}\\
&\quad -\frac{1}{18}\left[4\bar{Q}+(N-6)\right]\left(g_*\bar{Q}\right)^2\\
&\quad +\frac{1}{27}\left[8\bar{Q}+N-36+2(N+14)\zeta_3\right]\left(g_*\bar{Q}\right)^3\\
&\quad +\left\{-\frac{14}{27}\bar{Q}-\frac{1}{81}[4(N-73)+2(6N+65)\zeta_3\right.\\
&\quad \left.+5(N+30)\zeta_5]\right\}\left(g_*\bar{Q}\right)^4+\ldots \qquad (3.20)
\end{aligned}
$$

## 4. The diagrammatic calculation

In this section, we carry out the perturbative calculation to confirm the semiclassical result at leading and next-to-leading order in $\bar{Q}$ up to four-loop level, as displayed in Eq. (3.20).

The one-loop contribution to $\gamma_{T_{\bar{Q}}}$ comes solely from the diagram depicted in Fig. 1 (a) and is given by

$$\gamma^{(1)}_{T_{\bar{Q}}} = -\tfrac{1}{3} g\bar{Q}\left(1-\bar{Q}\right) . \tag{4.1}$$

As mentioned before, the derivation of the semiclassical result relied on working at the conformal fixed point $g_*$. However, surprisingly, at two, three and four loops we will see that the functional forms of the semiclassical and perturbative results agree for general $g$ and not just on substitution of $g = g_*$ with $g_*$ as given in Eq. (3.3). It is only at one loop where the agreement only holds at the fixed point. Specifically, the leading terms $\bar{Q}\left(\frac{d}{2}-1\right)+\gamma^{(1)}_{T_{\bar{Q}}}$ on the left-hand side of Eq. (3.5) (as given in Eq. (4.1)) only agree with the $\mathcal{O}(g^0)$ and $\mathcal{O}(g)$ terms in $\frac{\Delta_{-1}(g\bar{Q})}{g} + \Delta_0(g\bar{Q})$ on the right-hand side of Eq. (3.5) (as obtained from Eq. (3.20)) after substituting $g = g_* \approx \frac{3\epsilon}{N+8}$. In this case, specialising to the fixed point has induced a mixing between the classical and one-loop $\mathcal{O}(\bar{Q})$ terms.

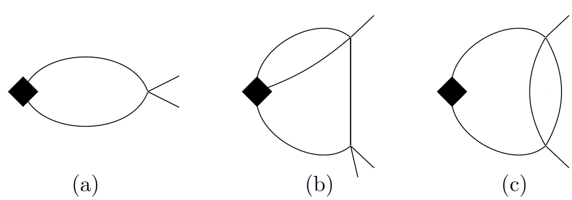


Fig. 1. One- and two-loop diagrams for $\gamma_{T_{\bar{Q}}}$ contributing at leading $n$.

The leading $\mathcal{O}(\bar{Q}^3)$ two-loop contribution to $\gamma_{T_{\bar{Q}}}$ comes purely from the diagram depicted in Fig. 1 (b) (with three lines emerging from the $T_{\bar{Q}}$ vertex), while the next-to-leading $\mathcal{O}(\bar{Q}^2)$ contributions are generated by this diagram together with those in Fig. 1 (c) (with two lines emerging from the $T_{\bar{Q}}$ vertex). The contributions are given by

$$\gamma^{(2)}_{(b)} = -\tfrac{2}{9} g^2 \bar{Q}\left(\bar{Q}-1\right)\left(\bar{Q}-2\right) , \tag{4.2}$$

$$\gamma^{(2)}_{(c)} = -\tfrac{1}{9} g^2 \left(3+\tfrac{1}{2}N\right)\bar{Q}\left(\bar{Q}-1\right) , \tag{4.3}$$

producing leading and next-to-leading terms given by

$$\gamma^{(2)}_{T_{\bar{Q}}} = -\tfrac{1}{18}\left(g\bar{Q}\right)^2\left(4\bar{Q}-6+N\right) , \tag{4.4}$$

in accord with the semiclassical results in Eq. (3.20). As emphasised earlier, this agreement holds for general $g$ and not just at the conformal fixed point. This is because at two and higher loops, in contrast to what we saw at one loop, specialising to the fixed point $g = g_*$ as given in Eq. (3.3) does not induce any mixing between leading or next-to-leading terms at different loop orders. Therefore, if Eq. (3.5) holds at the fixed point, it must also hold in general. In fact, the agreement was already checked at the fixed point in Ref. [13] in the general O($N$) case, and in the U(1) case in Ref. [12].

The leading $\mathcal{O}(\bar{Q}^4)$ three-loop contributions to $\gamma_{T_{\bar{Q}}}$ come purely from the diagrams depicted in Fig. 2 (with four lines emerging from the $T_{\bar{Q}}$ vertex), while the next-to-leading $\mathcal{O}(\bar{Q}^3)$ contributions are generated by these diagrams together with those in Fig. 3 (with three lines emerging from the $T_{\bar{Q}}$ vertex).

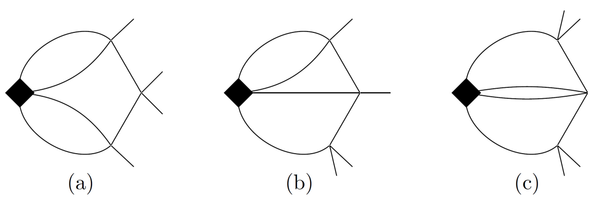


Fig. 2. Three-loop diagrams for $\gamma_{T_{\bar{Q}}}$ contributing at leading $n$.

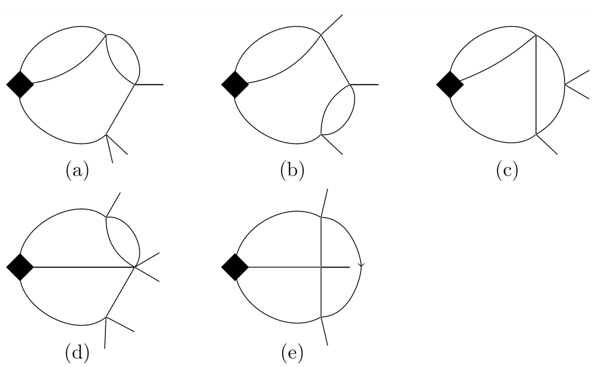


Fig. 3. Three-loop diagrams for $\gamma_{T_{\bar{Q}}}$ contributing at next-to-leading $n$.

The simple pole contributions from individual three-loop diagrams may be extracted from Ref. [20] and are listed in Table I, together with the corresponding symmetry factor. A factor of $g^3$ is understood in each case. The $N$-dependent factors $A$ and $B$ are given by

$$A = \tfrac{1}{8}(N+6)\,, \qquad B = \tfrac{1}{16}(N+14)\,. \tag{4.5}$$

When added and multiplied by a loop factor of 3, the leading and non-leading three-loop contributions to $\gamma_{T_{\bar{Q}}}$ are found to be

$$\gamma^{(3)}_{T_{\bar{Q}}} = \tfrac{1}{27}\left(g\bar{Q}\right)^3 \left[8\bar{Q} + N - 36 + 2(14+N)\zeta_3\right] , \tag{4.6}$$

once again in accord with the semiclassical results in Eqs. (3.20), for general $g$. Equivalently, this agreement was already checked at the fixed point in Ref. [13].

TABLE I

Three-loop results from Figs. 2 and 3.

| Graph | Symmetry factor | Simple pole |
|---|---|---|
| 2(a) | $\frac{1}{54}\frac{\bar{Q}!}{(\bar{Q}-4)!}$ | $-\frac{2}{3}$ |
| 2(b) | $\frac{2}{27}\frac{\bar{Q}!}{(\bar{Q}-4)!}$ | $\frac{4}{3}$ |
| 2(c) | $\frac{1}{54}\frac{\bar{Q}!}{(\bar{Q}-4)!}$ | $\frac{2}{3}$ |
| 3(a) | $\frac{1}{27}\frac{\bar{Q}!}{(\bar{Q}-3)!}$ | $-\frac{2}{3}$ |
| 3(b) | $\frac{4}{27}\frac{\bar{Q}!}{(\bar{Q}-3)!}A$ | $-\frac{2}{3}$ |
| 3(c) | $\frac{2}{27}\frac{\bar{Q}!}{(\bar{Q}-3)!}$ | $\frac{4}{3}$ |
| 3(d) | $\frac{4}{27}\frac{\bar{Q}!}{(\bar{Q}-3)!}A$ | $\frac{4}{3}$ |
| 3(e) | $\frac{8}{27}\frac{\bar{Q}!}{(\bar{Q}-3)!}B$ | $4\zeta_3$ |

The leading $\mathcal{O}(\bar{Q}^5)$ four-loop contributions to $\gamma_{T_{\bar{Q}}}$ come purely from the diagrams depicted in Fig. 4 (with five lines emerging from the $T_{\bar{Q}}$ vertex), while the next-to-leading $\mathcal{O}(\bar{Q}^4)$ contributions are generated by these diagrams together with those in Figs. 5 and 6 (with four lines emerging from the $T_{\bar{Q}}$ vertex).

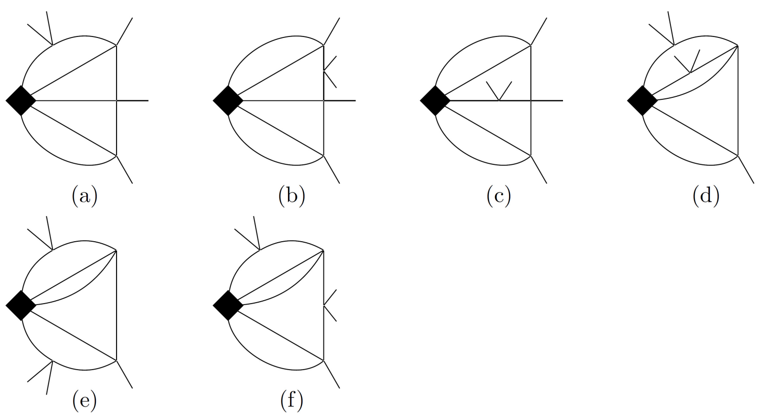


Fig. 4. Four-loop diagrams for $\gamma_{T_{\bar{Q}}}$ contributing at leading $n$.

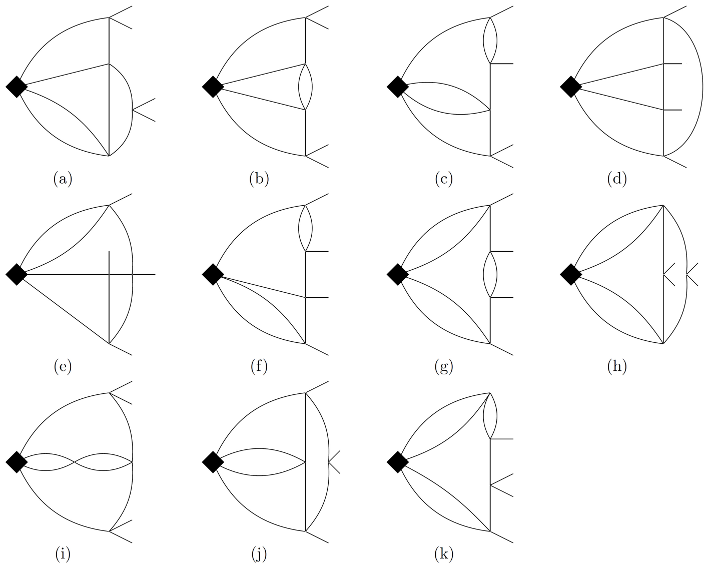


Fig. 5. Four-loop diagrams for $\gamma_{T_{\bar{Q}}}$ contributing at next-to-leading $n$.

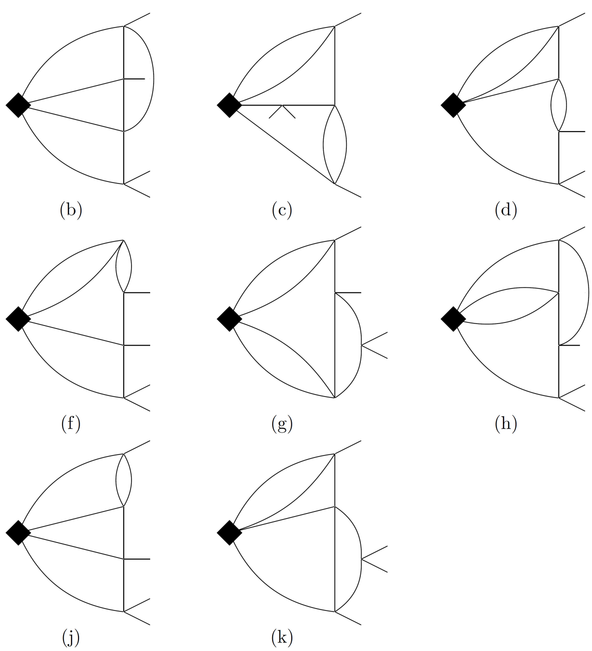


Fig. 6. Four-loop diagrams for $\gamma_{T_{\bar{Q}}}$ contributing at next-to-leading $n$ (continued).

The simple pole contributions from the four-loop diagrams in Fig. 4 were readily evaluated using standard techniques (see, for instance, Ref. [21]). Those from Figs. 5, 6 may be extracted from Ref. [20]. The contributions from each four-loop diagram are listed in Tables II, III and IV, respectively, together with the corresponding symmetry factor. A factor of $g^4$ is understood in each case, and the $N$-dependent factor $C$ is given by

$$C = \tfrac{1}{32}(N+30)\,. \tag{4.7}$$

When added and multiplied by a loop factor of 4, the leading and non-leading four-loop contributions to $\gamma_{T_{\bar{Q}}}$ are found to be

$$\gamma^{(4)}_{T_{\bar{Q}}} = -\tfrac{1}{81}\left(g\bar{Q}\right)^4\left[42\bar{Q} + 4(N-73) + 2(6N+65)\zeta_3 + 5(N+30)\zeta_5\right]\,, \tag{4.8}$$

once again in accord with the semiclassical results in Eqs. (3.20), for general $g$.

TABLE II

Four-loop results from Fig. 4.

| Graph | Symmetry factor | Simple pole |
|---|---|---|
| 4(a) | $\frac{4}{81}\frac{\bar{Q}!}{(\bar{Q}-5)!}$ | $\frac{5}{2}$ |
| 4(b) | $\frac{2}{81}\frac{\bar{Q}!}{(\bar{Q}-5)!}$ | $-\frac{2}{3}$ |
| 4(c) | $\frac{1}{81}\frac{\bar{Q}!}{(\bar{Q}-5)!}$ | $-\frac{5}{6}$ |
| 4(d) | $\frac{1}{81}\frac{\bar{Q}!}{(\bar{Q}-5)!}$ | $\frac{11}{6}$ |
| 4(e) | $\frac{2}{81}\frac{\bar{Q}!}{(\bar{Q}-5)!}$ | $\frac{2}{3}$ |
| 4(f) | $\frac{1}{81}\frac{\bar{Q}!}{(\bar{Q}-5)!}$ | $-\frac{1}{2}$ |

TABLE III

Four-loop results from Fig. 5.

| Graph | Symmetry factor | Simple pole |
|---|---|---|
| 5(a) | $\frac{2}{81}\frac{\bar{Q}!}{(\bar{Q}-4)!}$ | $\frac{1}{6}(11-6\zeta_3)$ |
| 5(b) | $\frac{2}{81}\frac{\bar{Q}!}{(\bar{Q}-4)!}A$ | $\frac{1}{6}(11-6\zeta_3)$ |
| 5(c) | $\frac{4}{81}\frac{\bar{Q}!}{(\bar{Q}-4)!}A$ | $-\frac{1}{2}$ |
| 5(d) | $\frac{16}{81}\frac{\bar{Q}!}{(\bar{Q}-4)!}C$ | $10\zeta_5$ |
| 5(e) | $\frac{8}{81}\frac{\bar{Q}!}{(\bar{Q}-4)!}B$ | $\frac{3}{2}(2\zeta_3-\zeta_4)$ |
| 5(f) | $\frac{8}{81}\frac{\bar{Q}!}{(\bar{Q}-4)!}A$ | $-\frac{2}{3}$ |
| 5(g) | $\frac{2}{81}\frac{\bar{Q}!}{(\bar{Q}-4)!}A$ | $\frac{1}{2}(1-2\zeta_3)$ |
| 5(h) | $\frac{1}{324}\frac{\bar{Q}!}{(\bar{Q}-4)!}$ | $-2(1-\zeta_3)$ |
| 5(i) | $\frac{1}{162}\frac{\bar{Q}!}{(\bar{Q}-4)!}$ | $-2(1-\zeta_3)$ |
| 5(j) | $\frac{2}{81}\frac{\bar{Q}!}{(\bar{Q}-4)!}$ | $-(1-2\zeta_3)$ |
| 5(k) | $\frac{1}{81}\frac{\bar{Q}!}{(\bar{Q}-4)!}$ | $\frac{1}{2}(1-2\zeta_3)$ |

TABLE IV

Four-loop results from Fig. 6.

| Graph | Symmetry factor | Simple pole |
|---|---|---|
| 6(a) | $\frac{4}{81}\frac{\bar{Q}!}{(\bar{Q}-4)!}A$ | $-\frac{1}{6}(5-6\zeta_3)$ |
| 6(b) | $\frac{8}{81}\frac{\bar{Q}!}{(\bar{Q}-4)!}B$ | $\frac{3}{2}(2\zeta_3+\zeta_4)$ |
| 6(c) | $\frac{4}{81}\frac{\bar{Q}!}{(\bar{Q}-4)!}A$ | $-\frac{5}{6}$ |
| 6(d) | $\frac{2}{81}\frac{\bar{Q}!}{(\bar{Q}-4)!}$ | $-\frac{2}{3}$ |
| 6(e) | $\frac{4}{81}\frac{\bar{Q}!}{(\bar{Q}-4)!}A$ | $-\frac{2}{3}$ |
| 6(f) | $\frac{2}{81}\frac{\bar{Q}!}{(\bar{Q}-4)!}$ | $-\frac{1}{6}(5-6\zeta_3)$ |
| 6(g) | $\frac{2}{81}\frac{\bar{Q}!}{(\bar{Q}-4)!}$ | $-\frac{1}{6}(5-6\zeta_3)$ |
| 6(h) | $\frac{4}{81}\frac{\bar{Q}!}{(\bar{Q}-4)!}$ | $-\frac{1}{2}(5-4\zeta_3)$ |
| 6(i) | $\frac{4}{81}\frac{\bar{Q}!}{(\bar{Q}-4)!}$ | $-\frac{1}{2}(5-4\zeta_3)$ |
| 6(j) | $\frac{8}{81}\frac{\bar{Q}!}{(\bar{Q}-4)!}A$ | $-\frac{5}{2}$ |
| 6(k) | $\frac{4}{81}\frac{\bar{Q}!}{(\bar{Q}-4)!}$ | $-\frac{5}{2}$ |

## 5. The large $g_*\bar{Q}$ calculation

In this section, we discuss the large $g_*\bar{Q}$ limit of $\Delta_{T_{\bar{Q}}}$. The large $g_*\bar{Q}$ limit of $\Delta_{-1}$ as given by Eq. (3.7) is readily obtained as

$$\frac{\Delta_{-1}}{g_*} = \frac{3}{4g_*}\left[\frac{3}{4}\left(\frac{4g_*\bar{Q}}{3}\right)^{\frac{4}{3}} + \frac{1}{2}\left(\frac{4g_*\bar{Q}}{3}\right)^{\frac{2}{3}} + \mathcal{O}(1)\right]. \tag{5.1}$$

We follow the procedure described in Ref. [12] for evaluating $\Delta_0$ by means of an approximation to the sum over $l$ followed by a numerical fit. The procedure involves selecting integers $N_1$, $N_2$ and picking $A \geq 1$ such that $AR\mu_*$ is an integer (this represents a cut-off in the summation, beyond which we approximate it by an integral). The accuracy may be made as great as desired by increasing $N_1$, $N_2$ and $A$. We obtain

$$\Delta_0 = \frac{N+8}{16}\left(R^2\mu^{*2} - 1\right)^2 \ln(AR\mu_*) + F(R\mu_*), \tag{5.2}$$

where

$$F(R\mu_*) = f_{N_2,A}(R\mu_*) - \frac{1}{4}\sigma_{AR\mu_*} + \frac{1}{2}\sum_{l=0}^{AR\mu_*}\sigma_l - \frac{1}{2}\sum_{k=1}^{N_1}\frac{B_{2k}}{(2k)!}\sigma_{AR\mu_*}^{(2k-1)}, \tag{5.3}$$

and here

$$\begin{aligned} f_{N_2,A}(R\mu_*) &= \frac{1}{2}(AR\mu_*)^4 \sum_{n=1,n\neq 5}^{N_2} \frac{c_n}{(AR\mu_*)^{n-1}(n-5)} \\ &+\frac{N+8}{16}\left(R^2\mu^{*2}-1\right)^2\left(\gamma - \frac{3}{2}\right) - \frac{29}{24}\left(R^2\mu^{*2}-1\right) - \left(\frac{11}{48}R^2\mu^{*2} - \frac{1}{10}\right)N. \end{aligned} \tag{5.4}$$

With some help from one of the authors [22], we have corrected some typos in the corresponding equations in Ref. [12], which were not reflected in their final results. The function $f_{N_2,A}(R\mu_*)$ derives from replacing the sum over $l$ for $l \geq AR\mu_*$ in Eq. (3.12) by an integral over $l$. It is then appropriate to use the large $l$ expansion in Eq. (3.15). The integral over $\frac{1}{l^{1+\epsilon}}$ corresponding to the $c_5$ term leads to a pole term in $\epsilon$. The potential pole in $\Delta_0$ is cancelled by the pole in the bare coupling, but the $O(\epsilon)$ term in $c_5$ in Eq. (3.16) leads to the terms in the last line of Eq. (5.4). The details of the procedure may be found in Ref. [12]. In Eq. (5.3), we can set $d = 4$. We now evaluate $F(R\mu_*)$ in Eq. (5.3) numerically. We take $N_1 = 4$, $N_2 = 10$ and $A = 10$, using the same numbers as Ref. [12] for comparison purposes. The result is then fitted with an expansion in $(R\mu_*)^{-2}$, starting from $(R\mu_*)^4$, with 4 parameters. We find that $F(R\mu_*)$ is given by

$$\begin{aligned} F(R\mu_*) \sim & -(1.5559+0.2293N)(R\mu_*)^4+(1.8536+0.3231N)(R\mu_*)^2 \\ & -(0.4467+0.0826N)+\mathcal{O}\left((R\mu_*)^{-2}\right) , \end{aligned} \tag{5.5}$$

and this may be inserted into Eq. (5.2) to give the full result for $\varDelta_0$. Expanding $R\mu_*$ as given by Eq. (3.6) in terms of large $g_*\bar{Q}$, we find

$$R\mu_* = \left(\frac{4g_*\bar{Q}}{3}\right)^{\frac{1}{3}} + \frac{1}{3}\left(\frac{4g_*\bar{Q}}{3}\right)^{-\frac{1}{3}} + \dots , \tag{5.6}$$

and then we obtain from Eq. (5.2)

$$\begin{aligned} \varDelta_0 = & \left[\alpha + \frac{N+8}{48}\ln\left(\frac{4g_*\bar{Q}}{3}\right)\right]\left(\frac{4g_*\bar{Q}}{3}\right)^{\frac{4}{3}} \\ & + \left[\beta - \frac{N+8}{72}\ln\left(\frac{4g_*\bar{Q}}{3}\right)\right]\left(\frac{4g_*\bar{Q}}{3}\right)^{\frac{2}{3}} + \mathcal{O}(1) , \end{aligned} \tag{5.7}$$

where

$$\begin{aligned} \alpha &= -0.4046-0.0854N , \\ \beta &= -0.8218-0.0577N . \end{aligned} \tag{5.8}$$

The results for U(1) should be recovered by setting $N=2$; and indeed for $N=2$ we find Eqs. (5.5), (5.7), (5.8) agree with the corresponding results given in Ref. [12].

Following Ref. [12] and combining Eqs. (3.3), (3.5), (5.1) and (5.7), we may write the full scaling dimension in the form of

$$\begin{aligned} \varDelta_{T_{\bar{Q}}} = & \frac{1}{\epsilon}\left(\frac{4\epsilon\bar{Q}}{N+8}\right)^{\frac{d}{d-1}}\left[\frac{3(N+8)}{16} + \epsilon\left(\alpha + \frac{3(3N+14)}{16(N+8)}\right) + \mathcal{O}\left(\epsilon^2\right)\right] \\ & + \frac{1}{\epsilon}\left(\frac{4\epsilon\bar{Q}}{N+8}\right)^{\frac{d-2}{d-1}}\left[\frac{N+8}{8} + \epsilon\left(\beta - \frac{3N+14}{8(N+8)}\right) + \mathcal{O}\left(\epsilon^2\right)\right] + \mathcal{O}\left[\left(\epsilon\bar{Q}\right)^0\right] . \end{aligned} \tag{5.9}$$

In Ref. [18], we found that we could reproduce the coefficients in the large $R\mu_*$ expansion of the $N$-dependent part of $\varDelta_0$ (the terms involving $\omega^*_{++}$ and $\omega^*_{--}$ in Eq. (3.13)) by an analytic computation. This fails to work here; an analytic large-$R\mu_*$ expansion of $\omega^*_{++}$ and $\omega^*_{--}$ as given by Eq. (3.11) leads to odd negative powers of $R\mu_*$, whereas our numeric computation in Eq. (5.5) only contains even powers of $R\mu_*$. It appears that the simple properties of $\omega^*_{++}$ and $\omega^*_{--}$ identified in Ref. [18], in particular their expansion in powers of $\frac{J_l^2}{R^2\mu_*^2}$ , are not enough for our analytic computation to work in the $d = 4$ case. A little trial and error indicates that the fact that in $d = 3$, $n_l \propto \frac{d}{dl} J_l^2$, may also be crucial; but further insight is required.

## 6. Conclusions

Approaches that extend the reach of (or even transcend the need for) perturbation theory have always been challenging, and are all the more interesting now due to the increased importance attached to multi-leg amplitudes, which can present formidable calculational obstacles at higher loop orders. In this paper, we have followed Refs. [11–13] in the application of semi-classical methods to the calculation of $\phi^n$ amplitudes in $d = 4$ renormalisable scalar theories with quartic interactions. Reference [13] generalises this calculation of Ref. [12] from U(1) to an O($N$) invariant interaction. Another motivation for studying this class of theories is their (classical) scale invariance (CSI). As remarked in Ref. [13], the Standard Model (SM) is "almost" CSI. Indeed, in 1973, Coleman and Weinberg (CW) [23] had hoped to argue that the SM might indeed be viable with the omission of the Higgs (wrong-sign) (mass)$^2$ term. This attractive idea failed. Neglecting Yukawa couplings (which seemed reasonable at the time) led to a Higgs mass prediction which was too small; but including the top quark Yukawa coupling destabilised the Higgs vacuum altogether[2]. CW introduced the idea of *dimensional transmutation* as a means of generating a physical mass scale in a CSI theory. The same phenomenon has been pursued [25–27] in the CSI form of quantum gravity [28–33].

Our purpose here has been to compare the results of Ref. [13] with straightforward (albeit intricate) perturbation theory. Generally, the results have supported the validity of the semi-classical approximation in its domain of validity.

Future work might include the application of the semi-classical methods and perturbative methods used here to the remaining class of CSI theories with scalar self-interactions; that is $\phi^3$ theories in $d = 6$; or even perhaps the case of CSI quantum gravity mentioned above.

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[2] For a review of some controversy over this development, see Ref. [24].

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