ANALYTIC REPRESENTATION OF ALL PLANAR TWO-LOOP FIVE-POINT MASTER INTEGRALS WITH ONE OFF-SHELL LEG*

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(Received March 23, 2021; accepted April 8, 2021)

In this contribution, we present analytic expressions in terms of polylogarithmic functions for all three families of planar two-loop five-point master integrals with one off-shell leg, recently published in `arXiv:2009.13917 [hep-ph]`. The calculation is based on the Simplified Differential Equations approach. The results are relevant to the study of many $2 \to 3$ scattering processes of interest at the LHC, especially for the leading-color $W+2$ jets production.

DOI:10.5506/APhysPolB.52.1091

1. Introduction

As we advance to the third decade of the 21st century, the established Standard Model of particle physics faces serious and interesting challenges from the domains of cosmology and astrophysics. One of those challenges for example is the particle nature of dark matter, and whether its dynamics can be described through the introduction of one or several new particles, thus imposing the need to extend our understanding of particle physics. The major experiments of particle physics however, spearheaded by the LHC program at CERN, have yet to reveal any clear signs of new physics that would require the extension of the established Standard Model.

To make progress in the current situation, a *precision* [1] program has been initiated, in part because it is clear by now that any new physics at the LHC data will appear in the form of small deviations from theoretical predictions, but also due to the increased precision of the accumulated

* Presented by N. Syrrakos at XXVII Cracow Epiphany Conference on *Future of Particle Physics*, Cracow, Poland, January 7–10, 2021.

experimental data. Thus, the need arises for equally precise theoretical predictions, in order to be able to exploit the full discovery potential of the LHC and its future High Luminosity upgrade.

From a theoretical standpoint, it is estimated that the LHC Run 3 and the High Luminosity Run, scheduled to commence after it, will require at least Next-to-Next-to-Leading-Order (NNLO) corrections for the QCD dominated processes [2–4]. A major ingredient of these higher-order perturbative corrections is the calculation of the relevant scattering amplitudes for specific scattering processes, and within these amplitudes, complicated two-loop Feynman diagrams need to be computed. Through the Feynman rules of quantum field theory, we can relate these two-loop Feynman diagrams to two-loop Feynman integrals, which are the topic of this contribution.

The current frontier in two-loop calculations is in $2 \to 3$ scattering processes. For massless external particles and massless internal propagators, all planar [5] and non-planar Feynman integrals have been calculated [6]. First results for $2 \to 3$ scattering processes involving one massive external particle for planar topologies were presented in [5] a few years ago, with the full list of all two-loop planar Feynman integrals relevant to $2 \to 3$ scattering processes with one off-shell leg appearing recently using a numerical approach [7]. Recently, some results on non-planar five-point Feynman integrals with one off-shell leg have also appeared using a new approach [8]. Here, we will present analytic results for all planar two-loop Feynman integrals with one off-shell leg in terms of polylogarithmic functions up to transcendental weight four [9].

2. Method

For a typical calculation of a two-loop Feynman integral, one starts by defining a *family* of Feynman integrals, relevant to the specific scattering process that one wants to study. In this particular case, we are interested in $2 \to 3$ scattering processes, therefore we have

$$\begin{aligned} F_{a_1,\ldots,a_{11}}(\{p_j\},\epsilon) &= \int \left(\prod_{r=1}^{2} \frac{\mathrm{d}^d k_r}{i\pi^{d/2}} \right) \frac{\mathrm{e}^{2\epsilon\gamma_E}}{D_1^{a_1} \ldots D_{11}^{a_{11}}} \,, \\ D_i &= \left(c_{ij}k_j + f_{ij}p_j\right)^2 \,, \qquad d = 4-2\epsilon \,. \end{aligned} \tag{1}$$

To calculate this kind of integrals, we rely on the two following properties:

1. Integrals of total derivatives with respect to loop momenta vanish within dimensional regularisation ($d = 4 - 2\epsilon$).

2. Feynman integrals satisfy differential equations (DE) derived with respect to kinematic invariants.

The first property allows us to derive the so-called Integration-By-Part (IBP) identities [10–12], which give rise to linear relations among Feynman integrals of the same family. This leads to the determination of a minimal set of integrals, the so-called master integrals, which form the basis $\mathbf{G}$ of the vector space spanned by all Feynman integrals of a specific family. So instead of having to compute all Feynman integrals of a given family, we only need to compute the master integrals and we can relate every other Feynman integral to them through IBP identities.

The second property means that instead of performing a direct integration of the loop momenta, we may derive and solve differential equations for the basis $\mathbf{G}$. In the standard approach [13, 14], these differential equations are derived in terms of all kinematic variables that are involved in the scattering processes that we are studying

$$\frac{\partial}{\partial s_{ij}}\mathbf{G} = \mathbf{A}(\{s_{ij}\},\epsilon)\mathbf{G}\,. \tag{2}$$

In general, the resulting differential equations can be quite complicated. In recent years, several ideas have been proposed to simplify the derivation and solution of these equations. In [15], the Simplified Differential Equations (SDE) approach was proposed, which introduces an external dimensionless parameter x through a re-parametrisation of the external momenta, and derives the differential equations by differentiating only with respect to that parameter, regardless of the number of kinematic scales involved

$$\partial_x\mathbf{G} = \mathbf{A}(\{s_{ij}\},x,\epsilon)\mathbf{G}\,. \tag{3}$$

In [16], it was proposed that instead of working directly with the basis $\mathbf{G}$, one can find a special basis of master integrals, $\mathbf{g}$, which can be related to the original basis through the transformation matrix $\mathbf{T}$, $\mathbf{g} = \mathbf{TG}$ and that satisfies a so-called canonical differential equation. Combining these two ideas yields a canonical SDE of the following form:

$$\partial_x\mathbf{g} = \epsilon\,\mathbf{M}(\{s_{ij}\},x)\mathbf{g}\,. \tag{4}$$

The resulting canonical differential equation has two very important features which greatly simplify its solution. First of all, the ϵ dependence is fully factorised out of the differential equation matrix and secondly, the differential equation matrix $\mathbf{M}(\{s_{ij}\},x)$ is Fuchsian, *i.e.* it has only simple poles in x (for the cases considered here)[1]. These properties allow us to solve these differential equations in terms of a class of iterated integrals, known as Multiple

[1] In general, more complicated structures may arise leading to solutions in terms of more complicated functions, *e.g.* elliptic integrals.

or Goncharov Polylogarithms (GPLs) [17]

$$\mathcal{G}(a_1, a_2, \ldots, a_n; x) = \int_0^x \frac{\mathrm{d}t}{t - a_1} \mathcal{G}(a_2, \ldots, a_n; t) \,, \tag{5}$$

$$\mathcal{G}(0, \ldots, 0; x) = \frac{1}{n!} \log^n(x) \,. \tag{6}$$

3. Results

The top-sector diagrams for the three planar two-loop families of master integrals that we solved in [9] are presented in Fig. 1, where in parenthesis we indicate the number of master integrals for each individual family. With single solid lines, we represent the massless propagators, as well as the massless external momenta, whereas the double line depicts the off-shell leg.

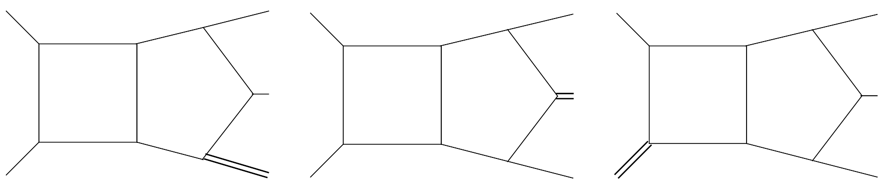

Fig. 1. The two-loop diagrams representing the top-sector of the planar pentabox family P_1(74 MI), P_2(75 MI) and P_3(86 MI). All external momenta are incoming.

The kinematic configuration for this scattering process can be described through four independent external momenta due to momentum conservation, $\sum_1^5 q_i = 0$, one of which is massive, $q_1^2 \equiv p_{1s}$. Thus, we have six independent kinematic variables, $\vec{s} = \{q_1^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}\}$, with $s_{ij} := (q_i + q_j)^2$.

In [7], pure bases for the three planar families were constructed and canonical differential equations were derived using the standard approach of differentiating with respect to all kinematic variables $\vec{s}$. The resulting differential equations are of the form of

$$\mathrm{d}\,\mathbf{g} = \epsilon \sum_a \mathrm{d} \log(W_a) \, \tilde{\mathbf{M}}_a \mathbf{g} \,. \tag{7}$$

Note that (7) is understood as a multi-variable differential equation, with W_a being functions of the kinematics and $\tilde{\mathbf{M}}_a$ matrices independent of the kinematic variables. In this particular case, the algebraic structure of W_a is such that a direct integration and solution of (7) in terms of GPLs is an insurmountable task and, indeed, in [7], a numerical method was used to provide solutions.

The situation can be substantially simplified if one utilises the SDE approach, as was recently presented in [9]. In this approach, we re-parametrize external momenta in terms of a dimensionless parameter x

$$q_1 \to p_{123} - x p_{12}\,, \qquad q_2 \to p_4\,, \qquad q_3 \to -p_{1234}\,, \qquad q_4 \to x p_1\,. \tag{8}$$

The top-sector diagrams using the new momentum parametrization (8) are presented in Fig. 2. Due to (8), the kinematic configuration now consists of five new external momenta, p_i, $i = 1 \dots 5$, four of which are independent, $\sum_1^5 p_i = 0$, and all of which are now massless, $p_i^2 = 0$, $i = 1 \dots 5$, with $p_{i\dots j} := p_i + \cdots + p_j$. Thus, we have six new independent kinematic variables, $\{S_{12}, S_{23}, S_{34}, S_{45}, S_{51}, x\}$, with $S_{ij} := (p_i + p_j)^2$. The canonical SDE now reads

$$\partial_x \mathbf{g} = \epsilon \sum_b \frac{1}{x - l_b} \mathbf{M}_b\, \mathbf{g}\,, \tag{9}$$

where again, $\mathbf{M}_b$ are rational matrices independent of the kinematics, and l_b are the so-called letters, which depend only on the five Mandelstam invariants $\{S_{12}, S_{23}, S_{34}, S_{45}, S_{51}\}$. Notice that in general the number of letters in (9) is smaller than the number of letters in (7). The structure of (9) is that of a Fuchsian system of ordinary differential equations, thus a fully analytic solution in terms of GPLs can be realised, once appropriate boundary terms at $x \to 0$ are provided.

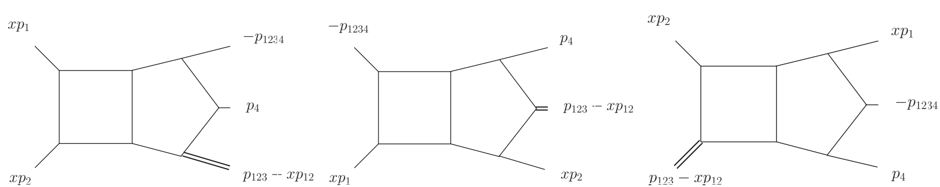

Fig. 2. Top-sector diagrams in SDE parametrisation.

In order to obtain the necessary boundary terms, we employ the Expansion-By-Regions method [18] which allows us to find the asymptotic behaviour of Feynman integrals for the $x \to 0$ limit. It is convenient for the calculation of boundary terms to cast the pure bases, $\mathbf{g}$, in terms of a specific basis of master integrals, $\mathbf{G}$. Via IBP reduction using FIRE6 [19], we can relate the two bases through $\mathbf{g} = \mathbf{T}\,\mathbf{G}$. Having done that, we can relate the $x \to 0$ limit of the basis $\mathbf{G}$ to that of $\mathbf{g}$ and obtain the relevant boundary terms[2].

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[2] We gratefully acknowledge the help of Chris Wever and Adam Kardos in this step, especially for the use of A. Kardos' Mathematica package Gsuite.

Now that all necessary ingredients are in place, we can write the solution of (9) in the following compact, pure and universally transcendental form:

$$\begin{aligned}\mathbf{g} \;=\; & \epsilon^0\mathbf{b}_0^{(0)} + \epsilon\left(\sum \mathcal{G}_a\mathbf{M}_a\mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)}\right) \\ & +\epsilon^2\left(\sum \mathcal{G}_{ab}\mathbf{M}_a\mathbf{M}_b\mathbf{b}_0^{(0)} + \sum \mathcal{G}_a\mathbf{M}_a\mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)}\right) \\ & +\epsilon^3\left(\sum \mathcal{G}_{abc}\mathbf{M}_a\mathbf{M}_b\mathbf{M}_c\mathbf{b}_0^{(0)} + \sum \mathcal{G}_{ab}\mathbf{M}_a\mathbf{M}_b\mathbf{b}_0^{(1)} + \sum \mathcal{G}_a\mathbf{M}_a\mathbf{b}_0^{(2)} + \mathbf{b}_0^{(3)}\right) \\ & +\epsilon^4\left(\sum \mathcal{G}_{abcd}\mathbf{M}_a\mathbf{M}_b\mathbf{M}_c\mathbf{M}_d\mathbf{b}_0^{(0)} + \sum \mathcal{G}_{abc}\mathbf{M}_a\mathbf{M}_b\mathbf{M}_c\mathbf{b}_0^{(1)} \right.\\ & \left. + \sum \mathcal{G}_{ab}\mathbf{M}_a\mathbf{M}_b\mathbf{b}_0^{(2)} + \sum \mathcal{G}_a\mathbf{M}_a\mathbf{b}_0^{(3)} + \mathbf{b}_0^{(4)}\right)\,, \end{aligned} \tag{10}$$

where $\mathbf{M}$ are the residue matrices coming from (9) and $\mathbf{b}_0^{(i)}$ are the boundary terms from the $x \to 0$ limit. The terms $\mathcal{G}_{ab\ldots}$ are a shorthand notation for the GPLs $\mathcal{G}(l_a, l_b, \ldots; x)$.

Having the solution in this form allows us to obtain numerical values in a straightforward way using the public program Ginac [20]. For a Euclidean point, we can obtain numerical values with 32 significant digits for each of the three non-zero top-sector basis elements within a few seconds (1.9, 3.3, 2 sec respectively), as presented in Table I. We refer to Section 4 of [9] for a more detailed discussion on numerical results for physical regions and timings.

TABLE I

Numerical results for the non-zero top sector element of each family with 32 significant digits.

P_1	g_{72}	$\epsilon^0 : 3/2$ $\epsilon^1 : -2.2514604753379400332169314784961$ $\epsilon^2 : -17.910593443812320786572184851867$ $\epsilon^3 : -26.429770706459534336624681550003$ $\epsilon^4 : 21.437938934510558345847354772412$
P_2	g_{73}	$\epsilon^1 : 2.8124788185742741402751457351382$ $\epsilon^2 : 5.4813042746593704203645729908938$ $\epsilon^3 : 11.590234540689191439870956817546$ $\epsilon^4 : -5.9962816226829136730734255754596$
P_3	g_{84}	$\epsilon^0 : 1/2$ $\epsilon^1 : 3.2780415861887284967738281876762$ $\epsilon^2 : 0.11455863130537720411162743574627$ $\epsilon^3 : -16.979642659429606120982671925458$ $\epsilon^4 : -48.101985355625914648042310964575$

4. Summary and future work

During the last decade, we have witnessed an explosion in our ability to compute multi-loop multi-scale Feynman integrals, allowing us to perform high precision phenomenological studies for scattering processes relevant to the LHC physics. These advances have been facilitated mainly by our better understanding of the mathematical properties of Feynman integrals, as well as the class of special functions in which we express them.

The current frontier in two-loop Feynman integral calculations is marked by $2 \to 3$ master integrals involving one massive external leg. Recently, results for all planar families using a numerical approach were presented in [7]. In this contribution, we described the analytic calculation of the aforementioned planar families, which were presented in [9]. As a next step, we see the extension of this work to the remaining non-planar families. First results have recently appeared for the top-row left family of Fig. 3 using a numerical approach [8], but it is hoped that the goal of obtaining fully analytical results for all five-point non-planar families with one off-shell leg will be realised in the near future.

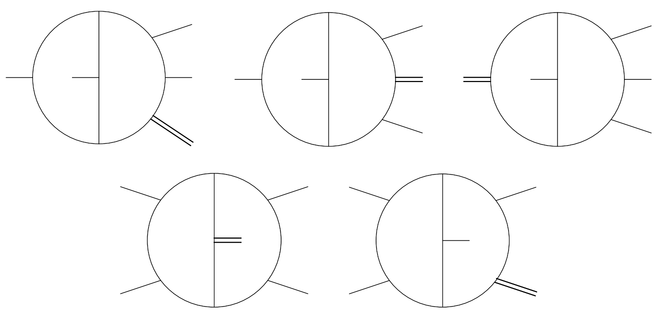

Fig. 3. The two-loop diagrams representing the top-sector of the non-planar pentabox families.

This research is co-financed by Greece and the European Union (European Social Fund- ESF) through the Operational Program Human Resources Development, Education and Lifelong Learning 2014–2020 in the context of the project “Higher order corrections in QCD with applications to High Energy experiments at LHC” — MIS 5047812.

REFERENCES

[1] G. Heinrich, «Collider Physics at the Precision Frontier», arXiv:2009.00516 [hep-ph].

[2] S. Amoroso *et al.*, «Les Houches 2019: Physics at TeV Colliders: Standard Model Working Group Report», arXiv:2003.01700 [hep-ph].

[3] G. Heinrich, «QCD calculations for the LHC: status and prospects», arXiv:1710.04998 [hep-ph].

[4] P. Azzi *et al.*, «Standard Model physics at the HL-LHC and HE-LHC», *CERN Yellow Rep. Monogr.* **7**, 1 (2019), arXiv:1902.04070 [hep-ph].

[5] C.G. Papadopoulos, D. Tommasini, C. Wever, «The pentabox Master Integrals with the Simplified Differential Equations approach», *J. High Energy Phys.* **1604**, 078 (2016), arXiv:1511.09404 [hep-ph].

[6] D. Chicherin *et al.*, «All Master Integrals for Three-Jet Production at Next-to-Next-to-Leading Order», *Phys. Rev. Lett.* **123**, 041603 (2019), arXiv:1812.11160 [hep-ph].

[7] S. Abreu *et al.*, «Two-loop integrals for planar five-point one-mass processes», *J. High Energy Phys.* **2011**, 117 (2020), arXiv:2005.04195 [hep-ph].

[8] C.G. Papadopoulos, C. Wever, «Internal reduction method for computing Feynman integrals», *J. High Energy Phys.* **2002**, 112 (2020), arXiv:1910.06275 [hep-ph].

[9] D.D. Canko, C.G. Papadopoulos, N. Syrrakos, «Analytic representation of all planar two-loop five-point Master Integrals with one off-shell leg», arXiv:2009.13917 [hep-ph].

[10] K.G. Chetyrkin, F.V. Tkachov, «Integration by parts: The algorithm to calculate β-functions in 4 loops», *Nucl. Phys. B* **192**, 159 (1981).

[11] F.V. Tkachov, «A theorem on analytical calculability of 4-loop renormalization group functions», *Phys. Lett. B* **100**, 65 (1981).

[12] S. Laporta, «High-precision calculation of multi-loop Feynman integrals by difference equations», *Int. J. Mod. Phys. A* **15**, 5087 (2000), arXiv:hep-ph/0102033.

[13] A.V. Kotikov, «Differential equations method. New technique for massive Feynman diagram calculation», *Phys. Lett. B* **254**, 158 (1991).

[14] A.V. Kotikov, «Differential equation method. The calculation of N-point Feynman diagrams», *Phys. Lett. B* **267**, 123 (1991); *Erratum*, *ibid.* **295**, 409 (1992).

[15] C.G. Papadopoulos, «Simplified differential equations approach for Master Integrals», *J. High Energy Phys.* **1407**, 088 (2014), arXiv:1401.6057 [hep-ph].

[16] J.M. Henn, «Multiloop Integrals in Dimensional Regularization Made Simple», *Phys. Rev. Lett.* **110**, 251601 (2013), arXiv:1304.1806 [hep-th].

[17] A.B. Goncharov, «Multiple polylogarithms, cyclotomy and modular complexes», *Math. Res. Lett.* **5**, 497 (1998), arXiv:1105.2076 [math.AG].

[18] B. Jantzen, A.V. Smirnov, V.A. Smirnov, «Expansion by regions: revealing potential and Glauber regions automatically», *Eur. Phys. J. C* **72**, 2139 (2012), arXiv:1206.0546 [hep-ph].

[19] A.V. Smirnov, F.S. Chuharev, «FIRE6: Feynman Integral REduction with modular arithmetic», *Comput. Phys. Commun.* **247**, 106877 (2020), arXiv:1901.07808 [hep-ph].

[20] J. Vollinga, S. Weinzierl, «Numerical evaluation of multiple polylogarithms», *Comput. Phys. Commun.* **167**, 177 (2005), arXiv:hep-ph/0410259.