Vol. 52 (2021) *Acta Physica Polonica B* No 8

SIMPLIFIED DIFFERENTIAL EQUATIONS FOR MASTER INTEGRALS AT N^{3}LO*

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(Received March 23, 2021; accepted April 7, 2021)

We present the application of the Simplified Differential Equations approach for the computation of three-loop families of master integrals. More specifically, we apply our method to compute the ladder-box-type integrals with up to one massive leg.

DOI:10.5506/APhysPolB.52.879

1. Introduction

For the theoretical predictions to reach the precision of HL-LHC and future collider experiments and to be able to indicate or not the existence of new physics, the calculation of N^{3}LO corrections becomes a necessity for scattering processes with two particles in the final state. The virtual corrections of the latter demand the calculation of three-loop *Feynman integrals* (FI). Within this direction, the 9 families of FI for all the internal and external massless particles (relevant for di-jet and di-photon productions) have been recently calculated [1–5], while from the families with one external massive particle (relevant for Higgs-jet production) only one has been computed [5, 6]. Currently, none of the families with two external massive particles, relevant for di-boson productions, has been computed.

Most of the aforementioned calculations have been done working within the framework of dimensional regularization ($d = 4 - 2\varepsilon$) and using the method of *differential equations* (DE) [7], which utilizes the *integration-by-parts relations* (IBP) [8] that relate any FI of a family to a minimal finite

* Presented by D.D. Canko at XXVII Cracow Epiphany Conference on *Future of Particle Physics*, Cracow, Poland, January 7–10, 2021.

basis of integrals, called *master integrals* (MI). The DE method combined with the method of properly choosing the basis of MI so that it consists only *pure functions* (UT basis) [9], has been proven to be a powerful tool for this kind of calculations. The latter leads to a DE of the so-called *canonical form* [9], which allows for an iterative solution of the DE in a Laurent expansion of the basis elements on ε.

In the following, we use a variant of the DE method, the so-called Simplified Differential Equations approach (SDE) [10–12], combined with the method of *canonical form* [5, 13, 14]. One can solve a family of FI using the SDE approach by applying the following steps:

- Parametrize the external momenta in terms of a dimensional parameter, x, in a way that captures the off-shellness of an external leg. The parametrization is not unique, x can be introduced to more than one of the external momenta, and different parametrizations are optimal for different problems.
- Take derivatives of MI with respect to x and derive a system of DE on x, using IBP relations.
- Find the boundaries of MI for $x \to 0$ and use them to solve the DE.

This method has plenty of advantages compared to the standard DE method. Some of them are:

- The quick derivation of DE, due to the fact that from the differentiation fewer FI are produced for reduction to MI.
- The existence of only one DE to solve.
- The rationalization of some of the square roots with respect to x.
- When a UT basis is provided and an analytic reduction is a bottleneck, a semi-numerical reduction can be applied by putting prime numbers to all the invariants except from x, and afterwards determine the letters of the DE using other methods[1].

2. Three-loop ladder-box with one external massive leg

2.1. The family

The family of the three-loop ladder-box with one external massive leg, Fig. 1, was first studied in [6]. In our computation, we adopt the notation for the kinematics and the UT basis presented therein. Performing the reduction

[1] For example, studying the maximally cut DE.

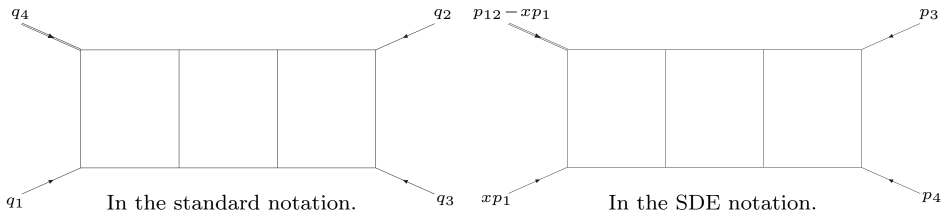

Fig. 1. The Feynman graph of the three-loop ladder-box family with one external massive leg, assuming all the external particles as incoming.

to MI using Kira 2.0 [15] and FIRE6 [16], we found a set of 83 MI. Any FI of this family can be expressed via a proper choice of the integer indices a in the following expression:

$$G_{a_1,\dots,a_{15}}(\{q_j\},\varepsilon) = \int \left(\prod_{r=1}^{3} \frac{\mathrm{d}^d l_r}{i\pi^{d/2}}\right) \frac{\mathrm{e}^{3\varepsilon\gamma_E}}{D_1^{a_1}\dots D_{15}^{a_{15}}} \quad \text{with} \quad d = 4-2\varepsilon\,, \tag{1}$$

where $D_{11},\dots,D_{15}$ are propagators coming from *irreducible scalar products* ($\{a_{11},a_{12},a_{13},a_{14},a_{15}\} \leq 0$), and the chosen parametrization for the propagators is[2]

$$\begin{aligned} D_1 &= l_1^2\,, \quad D_2 = l_2^2\,, \quad D_3 = l_3^2\,, \quad D_4 = (l_1-l_2)^2\,, \quad D_5 = (l_2-l_3)^2\,,\\ D_6 &= (l_3+q_2)^2\,, \quad D_7 = (l_1+q_{23})^2\,, \quad D_8 = (l_2+q_{23})^2\,, \quad D_9 = (l_3+q_{23})^2\,,\\ D_{10} &= (l_1+q_{123})^2\,, \quad D_{11} = (l_1+q_2)^2\,, \quad D_{12} = (l_2+q_2)^2\,,\\ D_{13} &= (l_2+q_{123})^2\,, \quad D_{14} = (l_3+q_{123})^2\,, \quad D_{15} = (l_1-l_3)^2\,. \end{aligned} \tag{2}$$

The external momenta ($q_1^2 = q_2^2 = q_3^2 = 0$ and $q_4^2 = m^2$) are expressed in terms of Mandelstam variables using the notation

$$q_2 \cdot q_3 = s/2\,, \qquad q_1 \cdot q_3 = t/2\,, \qquad q_1 \cdot q_2 = \left(m^2 - s - t\right)/2\,. \tag{3}$$

Moving on to the SDE approach, we chose a one-x parametrization

$$q_1 \to xp_1\,,\ q_2 \to p_3\,,\ q_3 \to p_4\,,\ q_4 \to p_{12} - xp_1 \text{ with } p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0\,,$$

where the Mandelstam variables and the mass of (3) are rephrased in terms of x and the Mandelstam variables of the light-like momenta ($s_{12} = p_{12}^2$ and $s_{23} = p_{23}^2$)

$$s = s_{12}\,, \qquad t = xs_{23}\,, \qquad m^2 = (1-x)s_{12}\,. \tag{4}$$

[2] From now on, we use the abbreviation $q_{i\dots j} = q_i + \dots + q_j$ and $p_{i\dots j} = p_i + \dots + p_j$.

As regards the propagators in (2), after making the transformations ($l_1 \to k_1 - q_{23}$, $l_2 \to -k_2 - q_{23}$, $l_3 \to k_3 - q_{23}$) and applying the SDE approach, they take the form of

$$
\begin{aligned}
&D_1 = (k_1+p_{12})^2\,, \quad D_2 = (k_2-p_{12})^2\,, \quad D_3 = (k_3+p_{12})^2\,, \quad D_4 = (k_1+k_2)^2\,,\\
&D_5 = (k_2+k_3)^2\,, \quad D_6 = (k_3+p_{123})^2\,, \quad D_7 = k_1^2\,, \quad D_8 = k_2^2\,, \quad D_9 = k_3^2\,,\\
&D_{10} = (k_1+xp_1)^2\,, \quad D_{11} = (k_1+p_{123})^2\,, \quad D_{12} = (k_2-p_{123})^2\,,\\
&D_{13} = (k_2-xp_1)^2\,, \quad D_{14} = (k_3+xp_1)^2\,, \quad D_{15} = (k_1-k_3)^2\,.
\end{aligned}
$$

It is worth commenting here on the fact that in this parametrization, x is introduced in 3 propagators. Taking derivatives of the MI to create the DE, we obtain around 160 FI for reduction, while on the other hand, if we had used a two-x parametrization, we would had 6 x-dependent propagators and around 800 FI to reduce. Thus, in this case, the one-x parametrization is better than the two-x one[3].

Having a UT basis, we obtained a DE which is of canonical form

$$
\partial_x \mathbf{g} = \varepsilon \left(\sum_{i=1}^{4} \frac{\mathbf{M}_i}{x - l_i} \right) \mathbf{g}\,, \tag{5}
$$

with $l_i = \{0,\, 1,\, s_{12}/(s_{12}+s_{23}), -s_{12}/s_{23}\}$ the letters of the alphabet and $\mathbf{M}_i$ being purely numerical matrices. We solved the DE up to weight six on ε and in the Euclidean region $\{0 < x < 1,\ s_{12} < 0,\ s_{12} < s_{23} < 0\}$. The solution has the form of

$$
\begin{aligned}
\mathbf{g} \;=\; & \varepsilon^0 \mathbf{b}_0^{(0)} + \varepsilon \left(\sum \mathcal{G}_i \mathbf{M}_i \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)} \right)\\
& + \varepsilon^2 \left(\sum \mathcal{G}_{ij} \mathbf{M}_i \mathbf{M}_j \mathbf{b}_0^{(0)} + \sum \mathcal{G}_i \mathbf{M}_i \mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)} \right) + \ldots\\
& + \varepsilon^6 \Big(\mathbf{b}_0^{(6)} + \sum \mathcal{G}_{ijklmn} \mathbf{M}_i \mathbf{M}_j \mathbf{M}_k \mathbf{M}_l \mathbf{M}_m \mathbf{M}_n \mathbf{b}_0^{(0)}\\
& + \sum \mathcal{G}_{ijklm} \mathbf{M}_i \mathbf{M}_j \mathbf{M}_k \mathbf{M}_l \mathbf{M}_m \mathbf{b}_0^{(1)}\\
& + \sum \mathcal{G}_{ijkl} \mathbf{M}_i \mathbf{M}_j \mathbf{M}_k \mathbf{M}_l \mathbf{b}_0^{(2)} + \sum \mathcal{G}_{ijk} \mathbf{M}_i \mathbf{M}_j \mathrm{M}_k \mathbf{b}_0^{(3)}\\
& + \sum \mathcal{G}_{ij} \mathbf{M}_i \mathbf{M}_j \mathbf{b}_0^{(4)} + \sum \mathcal{G}_i \mathbf{M}_i \mathbf{b}_0^{(5)} \Big)\,,
\end{aligned}
$$

where the matrices $\mathbf{b}_0^{(i)}$ are the boundaries and $\mathcal{G}_i, \ldots, \mathcal{G}_{ijklmn}$ are the Goncharov poly-logarithms [17] of weight $1, \ldots, 6$, respectively, with argument x and letters from the set l_i. Our results were crossed-checked numerically with the results from [6] and perfect agreement was found in all cases.

[3] In problems where roots appear, like in [11–13], the two-x parametrization is optimal for their rationalization with respect to x.

2.2. Calculation of boundaries

As a starting point, we have some boundaries that are already known because some of the MI appearing in this family are known in close form. Thus, we can directly obtain boundary conditions for

$$\{gb_1, gb_2, gb_3, gb_4, gb_5, gb_6, gb_7, gb_{17}, gb_{18}, gb_{19}, gb_{44}\}\,. \tag{6}$$

Afterwards, we take advantage of the fact that if for a basis element its leading regions contributing to its asymptotic limit $x \to 0$ are of the form of $x^{\alpha+\beta\varepsilon}$ with $\alpha \geq 1$, then its boundary term should vanish. Using this observation, we set the following boundaries to zero:

$$\begin{aligned}&\{gb_{10}, gb_{11}, gb_{14}, gb_{15}, gb_{21}, gb_{22}, gb_{23}, gb_{24}, gb_{25}, gb_{26}, gb_{28},\\ &gb_{31}, gb_{37}, gb_{38}, gb_{45}, gb_{46}, gb_{47}, gb_{48}, gb_{50}, gb_{53}, gb_{55}, gb_{58},\\ &gb_{59}, gb_{63}, gb_{64}, gb_{66}, gb_{68}, gb_{70}, gb_{80}, gb_{82}, gb_{83}\} = 0\,.\end{aligned} \tag{7}$$

Being left with 41 unknown boundaries, we used a method [5, 13] that finds relations between the boundaries of a family of FI. More specifically, through the Jordan-decomposition of $\mathbf{M}_0 = \mathbf{S}_0\mathbf{D}_0\mathbf{S}_0^{-1}$, we define the resummation matrix at $x = 0$, $\mathbf{R}_0 = \mathbf{S}_0 \mathrm{e}^{\varepsilon \mathbf{D}_0 \log(x)} \mathbf{S}_0^{-1}$, which correctly resumms the logarithms of x from the basis elements. Thus, we can write $\mathbf{g} = \mathbf{R}_0 \mathbf{g}_{\mathrm{reg}0}$, where $\mathbf{g}_{\mathrm{reg}0}$ is the regular part of the basis element at $x = 0$, via which are defined the asymptotic boundaries $\mathbf{g}_{\mathrm{bound}} = \mathbf{g}_{\mathrm{reg}0}\big|_{x=0}$. Multiplying $\mathbf{R}_0$ from the right with $\mathbf{g}_{\mathrm{bound}}$ and from the left with $\mathbf{T}^{-1}$ (the transformation matrix that takes us from the UT basis to the MI), we obtain the asymptotic limit at $x \to 0$ of the MI

$$\mathbf{F}_{x\to 0} = \mathbf{T}^{-1}\mathbf{R}_0 \mathbf{g}_{\mathrm{bound}}\,. \tag{8}$$

This should be equal to the asymptotic limit found for the MI by expansion-by-regions [18] found by `asy` [19]. Thus, by comparing the regions found by `asy` with that found by the resummation matrix method, we obtain relations between different boundaries. In fact, we obtain two kinds of relations. We call the first of them *pure relations* because they contain only boundaries of the basis elements, *e.g.*

$$gb_{71} = \left(-12 gb_2 + 4 gb_{13} + 32 gb_{16} + 48 gb_{41} + 36 gb_{42} - 45 gb_{43}\right)/30\,,$$

while the second of them we call *impure* due to the fact that there are relations between boundaries and asymptotic limits, *e.g.*

$$gb_{41} = F^{\mathrm{s}}_{41} s_{12}\varepsilon^5 + gb_2/9 - gb_{13}/12 - 2gb_{16}/3\,.$$

By applying this method, we obtained 28 *pure* relations and we were left with 13 asymptotic regions

$$\left\{F_8^{\mathrm{h}},\, F_9^{\mathrm{h}},\, F_{12}^{\mathrm{h}},\, F_{13}^{\mathrm{h}},\, F_{16}^{\mathrm{h}},\, F_{20}^{\mathrm{h}},\, F_{27}^{\mathrm{h}},\, F_{29}^{\mathrm{h}},\, F_{32}^{\mathrm{s}},\, F_{39}^{\mathrm{s}},\, F_{41}^{\mathrm{s}},\, F_{51}^{\mathrm{h}},\, F_{56}^{\mathrm{h}}\right\},$$

where with F_i^{h} we denote the x^0 region and with F_i^{s} the $x^{-3\varepsilon}$. For the calculation of the F_i^{h} limits, we used the method of expansion-by-regions in the momentum space and IBP reduction. The F_i^{s} limits were calculated using their Feynman-parameter representation provided by `asy`, together with a technique of integrating out bubble subintegrals inspired by [2].

3. Three-loop massless ladder-box

Having the solution for the family of the three-loop ladder-box with one external massive leg, it is easy within the SDE approach to obtain also the solution for a UT basis of the massless three-loop ladder-box family, Fig. 2 taking the $x \to 1$ limit. For one to obtain the solution taking the $x \to 1$ limit [5, 12], one needs to apply the following steps:

1. Rewrite the solution as an expansion in $\log(1-x)$:

$$\mathbf{g} = \sum_{n\geq 0} \epsilon^n \sum_{i=0}^{n} \frac{1}{i!}\mathbf{c}_i^{(n)} \log^i(1-x)\,.$$

2. Define the regular part of $\mathbf{g}$ at $x=1$ and from it the truncated part:

$$\mathbf{g}_{\mathrm{reg}} = \sum \epsilon^n \mathbf{c}_0^{(n)} \quad \text{and} \quad \mathbf{g}_{\mathrm{trunc}} = \mathbf{g}_{\mathrm{reg}}\big|_{x=1}\,.$$

3. Define the resummation matrix $\mathbf{R}_1$ and the numerical matrix $\mathbf{R}_{10}$:

$$\mathbf{R}_1 = \mathrm{e}^{\epsilon \mathbf{M}_1 \log(1-x)} = \mathbf{S}_1 \mathrm{e}^{\epsilon \mathbf{D}_1 \log(1-x)} \mathbf{S}_1^{-1} \quad \text{and} \quad \mathbf{R}_1 \xrightarrow{(1-x)^{a_i\epsilon}\to 0} \mathbf{R}_{10}\,.$$

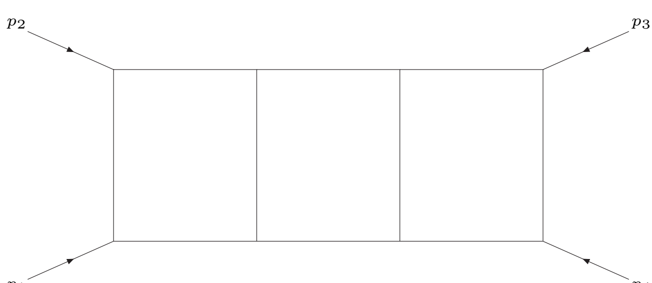

Fig. 2. The Feynman graph of the three-loop massless ladder-box family.

4. Find the $x \to 1$ limit by acting $\mathbf{R}_{10}$ to $\mathbf{g}_{\text{trunc}}$:

$$\mathbf{g}_{x\to 1} = \mathbf{R}_{10}\mathbf{g}_{\text{trunc}}\,.$$

5. Reduce the number of the basis elements to that of the MI of the massless problem using the property $\mathbf{R}_{10}^2 = \mathbf{R}_{10}$ and/or IBP.

For the FI of this family, we have chosen the following normalization:

$$G_{a_1,\dots,a_{15}}\left(\{p_j\}\,,\varepsilon\right) = \left(-s_{12}\right)^{3\varepsilon} \int \left(\prod_{l=1}^{3} \frac{\mathrm{d}^d k_l}{i\pi^{d/2}}\right) \frac{\mathrm{e}^{3\varepsilon\gamma_{\mathrm{E}}}}{D_1^{a_1}\dots D_{15}^{a_{15}}}\,, \tag{9}$$

where the propagators are obtained by setting $x=1$ to the propagators of the massive family. We compared analytically our results for the three top sector basis elements with the ones given by [2] and numerically for all basis elements with pySecDec [20] in the Euclidean region. In both cases, perfect agreement was found.

4. Discussion and outlook

We have presented the application of the SDE approach combined with the method of *canonical form* in order to calculate two families of FI, which have been solved in the past using the standard DE method.

Encouraged by the simplicity of the SDE approach at three-loop problems and the phenomenological interest of them, we are currently working with our collaborators[4] for the calculation of the rest of the planar four-point three-loop families with one external off-shell leg. These are the two tennis-court families of Fig. 3. The first of them contains 117 MI, while the

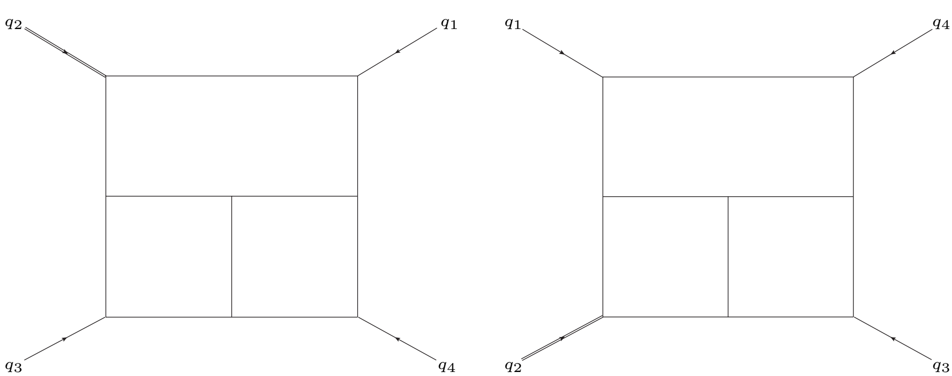

Fig. 3. The Feynman graphs of the two tennis-court families.

[4] Federico Gasparotto and Luca Mattiazzi.

second 166. Our methods together with the tools [4, 21], which we are using to find a UT basis for these families, seem to make this goal feasible in the near future.

We want to thank the organizers of XXVII Epiphany Conference for the very interesting conference. The research work was supported by the Hellenic Foundation for Research and Innovation (HFRI) under the HFRI Ph.D. Fellowship grant (Fellowship Number: 554).

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