# SECTOR DECOMPOSITION SCHEME FOR N$^3$LO BEAM FUNCTION*

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This article presents preliminary results for the calculation of quark-to-quark N$^3$LO beam function. To this end, we employ the techniques of sector decomposition and selector functions, along with other techniques to supplement the calculation. Our results show agreement with available predictions from the renormalization group.

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## 1. Introduction

The beam function $\mathcal{B}(z, x_\mathrm{T}^2)$ is an object which appears in the small-$q_\mathrm{T}$ factorization of cross sections of processes such as Drell–Yan production of a particle with invariant mass $Q$. In the small transverse momenta region ($\Lambda_\mathrm{QCD} \ll q_\mathrm{T} \ll Q$), the cross section can be factorized in the form [1–3] of

$$\sigma \simeq \mathcal{B}_{q/N}\left(z_1, x_\mathrm{T}^2\right) \otimes \bar{\mathcal{B}}_{q/N}\left(z_2, x_\mathrm{T}^2\right) \otimes H\left(Q^2\right)\,, \tag{1}$$

in which $Q$ and $q_\mathrm{T}$ dependences are separated. The beam function $\mathcal{B}_{q/N}(z, x_\mathrm{T}^2)$ describes initial collinear emissions from a parton $q$ of hadron $N$, which goes into the hard process with fraction $z$ of the collinear momentum.

In soft-collinear effective theory (SCET), *cf.* [4, 5], the beam function is defined as

$$\mathcal{B}_{q/N}\left(z, x_\mathrm{T}^2\right) = \frac{1}{2\pi}\int \mathrm{d}t\, \mathrm{e}^{-izt\bar{n}\cdot p} \sum_X \langle N(p)|\bar{\chi}(t\bar{n}+x_\perp)|X\rangle \frac{\bar{\not{n}}}{2}\langle X|\chi(0)|N(p)\rangle\,, \tag{2}$$

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where $\chi$ is a collinear quark field of SCET, and $n \equiv \hat{p}$ and $\bar{n}$ are two unit, light-like vectors, which satisfy $n \cdot \bar{n} = 2$. The anti-collinear beam function $\bar{\mathcal{B}}_{q/N}(z, x_{\mathrm{T}}^2)$ is defined similarly except for $\bar{n}$ and $p$ replaced with $n$ and $\bar{p}$. Using $n$ and $\bar{n}$, one expresses momenta in light-cone coordinates as follows:

$$l = \left(\frac{n \cdot l}{2}, \frac{\bar{n} \cdot l}{2}, l_\perp\right) = (l_+, l_-, l_\perp)\,, \tag{3}$$

where $n \cdot l_\perp = \bar{n} \cdot l_\perp = 0$ and $l_{\mathrm{T}}^2 \equiv -l_\perp^2$.

The hadron-to-parton beam function is not a perturbatively calculable object. The parton-to-parton beam function, however, can be calculated perturbatively, and hence can be given as a series with respect to the strong coupling constant $\alpha_{\mathrm{S}}$[1]. One, then, matches the parton-to-parton beam functions with collinear PDFs to obtain the hadron-to-parton beam function [6].

For the real emission case, the evaluation of Eq. (2) involves the phase space integral

$$\begin{aligned}&\int\limits_0^\infty \mathrm{d}^d k \left[\prod_{i=1}^{n} \int\limits_0^\infty \frac{\mathrm{d}^d l_i}{(2\pi)^{d-1}} \delta^+\left(l_i^2\right)\right] \\ &\times \delta^d\left(k - \sum_j^n l_j\right) \delta(\bar{n} \cdot (k - (1-z)p)) \mathrm{e}^{-ik_\perp \cdot x_\perp} \left|M\left(q \to q' + X\right)\right|^2 .\end{aligned} \tag{4}$$

One obstacle in the evaluation of Eq. (4) is the Fourier transform $\mathrm{e}^{-ik_\perp \cdot x_\perp}$, which we replace in our calculation by $\mathrm{e}^{-k_{\mathrm{T}}^2 k_{\mathrm{T}}^2}$. The consequent integral differs from the original integral by a simple factor. The collinear matrix element $M(q \to q' + X)$ contains singularities which have to be regularized. The main interest of this paper is the infrared divergences which can be classified into collinear, anti-collinear and soft divergences. On top of them, the integration of Eq. (4) contains additional divergences called "rapidity divergences" [1, 2], which correspond to the region where the emissions scale like $(\frac{1}{\lambda}, \lambda, 1)$ in the light-cone coordinates (Eq. (3)), where $\lambda \ll 1$. These unphysical divergences are not regularized by dimensional regularization, and hence additional regulator is required. In our calculation, we choose the prescription of [7], and include the factor $\left(\frac{\nu}{l_{i+}}\right)^\alpha$ for a momentum $l_i$ of every unresolved final state particle[2]. This regulator breaks the symmetry $p \leftrightarrow \bar{p}$, $\bar{n} \leftrightarrow n$ between $\mathcal{B}(z, x_{\mathrm{T}}^2)$ and $\bar{\mathcal{B}}(z, x_{\mathrm{T}}^2)$. Nonetheless, such unphysical divergences vanish in the product $\mathcal{B}(z_1, x_{\mathrm{T}}^2)\bar{\mathcal{B}}(z_2, x_{\mathrm{T}}^2)$ [2]. This product, in

[1] Throughout this paper, we refer to the coefficient of $\alpha_{\mathrm{S}}^3$ as N$^3$LO beam function and denote it by $\mathcal{B}^{(3)}(z, x_{\mathrm{T}}^2)$.

[2] The scale $\nu$ plays similar role to $\mu$ in dimensional regularization.

fact, contains the dependence on the invariant mass $Q$, which means that the factorization of the collinear and hard degrees of freedom is not fully achieved. However, the dependence on $Q$ can be factorized into the simple form

$$\lim_{\alpha\to 0}\left[\mathcal{B}_{i/j}(z_1,x_{\mathrm{T}}^2)\,\bar{\mathcal{B}}_{\bar{i}/k}(z_2,x_{\mathrm{T}}^2)\right]_Q = \left(\frac{x_{\mathrm{T}}^2 Q^2}{4\mathrm{e}^{-2\gamma_{\mathrm{E}}}}\right)^{F_{i\bar{i}}^{\mathrm{b}}(x_{\mathrm{T}}^2)} B_{i/j}^{\mathrm{b}}(z_1,x_{\mathrm{T}}^2)\,B_{\bar{i}/k}^{\mathrm{b}}(z_2,x_{\mathrm{T}}^2)\;. \tag{5}$$

This procedure, called "refactorization", produces the beam function $B_{i/j}^{\mathrm{b}}(z,x_{\mathrm{T}}^2)$ which is truly independent of $Q$ [2]. The refactorized beam function of Eq. (5) contains UV and IR divergences (hence the superscript b for bare).

## 2. The strategy of calculation

In order to extract the aforementioned divergences, one needs to factorize them into simple monomials. Our strategy is to divide the phase space into sectors such that in each sector, a subset of divergences are all factorized as monomials and the remaining divergences are dealt with in other sectors.

We extend the method of [8–12] which consists of two key elements, namely "selector function" and "sector decomposition".

### *2.1. Selector functions*

A selector function is a function which "selects" certain singular points and suppresses the rest of divergences. A maximal set of divergent limits[3] $C_i$ and the corresponding selector function $S_i$ satisfies the following conditions:

$$\sum_i S_i = 1\,, \qquad\qquad S_i\Big|_{C_j} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}\;. \tag{6}$$

Therefore, the integral is decomposed as

$$\int \mathrm{d}x_1\mathrm{d}x_2\ldots\mathcal{I}(x_1,x_2,\ldots) = \int \mathrm{d}x_1\mathrm{d}x_2\ldots S_1(x_1,x_2,\ldots)\mathcal{I}(x_1,x_2,\ldots)$$
$$+\int \mathrm{d}x_1\mathrm{d}x_2\ldots S_2(x_1,x_2,\ldots)\mathcal{I}(x_1,x_2,\ldots)+\ldots\,, \tag{7}$$

where each term has only a subset of divergences. We refer to the terms in such decomposition as "primary sectors".

[3] Defined such that addition of another divergent limit to such a set does not contribute to the divergence due to the exponent or the $\delta$ function in Eq. (4).

A set of selectors can be constructed as

$$S_i = \frac{1}{d_i\left(\sum_j \frac{1}{d_j}\right)}\,, \qquad \text{where} \qquad d_i\Big|_c \begin{cases} \neq 0, & \text{if } c \in C_i \\ = 0, & \text{if } c \notin C_i \end{cases}. \tag{8}$$

Hence, $d_i$ can, for example, be a product of functions $1-\cos\theta$ and $E$ for a relative angle $\theta$ and an energy $E$. A crucial modification of the selector function for our scheme is inclusion of the transverse momenta $l_{i\mathrm{T}}$, which is necessary in order to handle the rapidity divergences.

### *2.2. Sector decomposition*

The integrals in each primary sectors have a complicated structure of "overlapping singularities", which require sector decomposition.

Consider a toy model

$$I_{\mathrm{toy}} = \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 x_1^{-1+\epsilon} x_2^{\epsilon} \frac{1}{(x_1+x_2)}\,. \tag{9}$$

The integral of Eq. (9) contains an overlap, namely the denominator $x_1+x_2$, which diverges only when both $x_1$ and $x_2$ vanish. By multiplying Eq. (9) by $1=\theta(x_1-x_2)+\theta(x_2-x_1)$, one divides the space into two sectors. Applying the change of variables $x_1 \to x_1x_2$ and $x_2 \to x_1x_2$, respectively, restores the unit cube and yields

$$I_{\mathrm{toy}} = \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 x_1^{-1+2\epsilon} x_2^{\epsilon} \frac{1}{(1+x_2)} + \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 x_1^{-1+\epsilon} x_2^{-1+2\epsilon} \frac{1}{(x_1+1)}\,. \tag{10}$$

Hence, the divergences factorize and the above integrals can be evaluated with help of Laurent expansion

$$\int_0^1 \mathrm{d}x\, x^{-1+\epsilon} f(x) = \frac{1}{\epsilon} f(0) + \int_0^1 \frac{f(x)-f(0)}{x^{1-\epsilon}}\,, \tag{11}$$

where $\lim\limits_{x\to 0} f(x)$ is well defined. Thus, the coefficients in the expansions in $\epsilon$ can be integrated numerically.

## 3. Numerical results

The algorithm described above has been implemented for the NNLO case whose results have shown a good agreement with [6]. At N$^3$LO, the quark-to-quark beam function with a colour factor $n_f C_F^2 T_F$ has been calculated.

The cancellation of $\alpha$ poles mentioned earlier serves as a test for the method. From our direct calculation, we obtain

$$\mathcal{B}^{(3)}_{q/q,\, n_f C_F^2 T_F}\left(\frac{1}{2}, x_\mathrm{T}^2\right) = \frac{1}{\alpha}\left(-\frac{20}{\epsilon^3} - \frac{23}{\epsilon^2} - \frac{97}{\epsilon} + \mathcal{O}\left(\epsilon^0\right)\right) + \mathcal{O}\left(\alpha^0\right)\,, \quad (12)$$

$$\bar{\mathcal{B}}^{(3)}_{q/q,\, n_f C_F^2 T_F}\left(\frac{1}{2}, x_\mathrm{T}^2\right) = \frac{1}{\alpha}\left(\frac{20}{\epsilon^3} + \frac{23}{\epsilon^2} + \frac{97}{\epsilon} + \mathcal{O}\left(\epsilon^0\right)\right) + \mathcal{O}\left(\alpha^0\right)\,, \quad (13)$$

where the numerical values are obtained up to $\sim \pm 1\%$ accuracy.

The cancellations of the $\alpha$ poles happen order by order in $\alpha_\mathrm{S}$ in the product $\mathcal{B}(z_1, x_\mathrm{T}^2)\bar{\mathcal{B}}(z_2, x_\mathrm{T}^2)$. Thus, the N$^3$LO term satisfies

$$\begin{aligned} &\mathcal{B}^{(3)}\left(z_1, x_\mathrm{T}^2, \mu\right)\bar{\mathcal{B}}^{(0)}\left(z_2, x_\mathrm{T}^2, \mu\right) + \mathcal{B}^{(0)}\left(z_1, x_\mathrm{T}^2, \mu\right)\bar{\mathcal{B}}^{(3)}\left(z_2, x_\mathrm{T}^2, \mu\right) \\ &+\mathcal{B}^{(2)}\left(z_1, x_\mathrm{T}^2, \mu\right)\bar{\mathcal{B}}^{(1)}\left(z_2, x_\mathrm{T}^2, \mu\right) + \mathcal{B}^{(1)}\left(z_1, x_\mathrm{T}^2, \mu\right)\bar{\mathcal{B}}^{(2)}\left(z_2, x_\mathrm{T}^2, \mu\right) \\ &= 0 + \mathcal{O}\left(\alpha^0\right)\,. \end{aligned} \quad (14)$$

The cancellation of the $\alpha$ poles among Eqs. (12) and (13) and the components from lower order quark-to-quark beam functions has been verified explicitly for the $n_f C_F^2 T_F$ term.

Higher order terms in $\epsilon$ and $\alpha$, and terms with other colour factors can be calculated in the same manner.

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