ON THE Θ-TERM IN ELECTRODYNAMICS*

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Received 9 November 2022, accepted 21 November 2022, published online 28 November 2022

The term $\theta\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$, when added to the electromagnetic Lagrangian $-\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu}$, does not change the signature of the Lagrangian. Actually, it increases the part with negative kinetic energy term at the spatial infinity. For this reason' it does not change the conclusion that at the spatial infinity the magnetic part of the electromagnetic field should be absent.

DOI:10.5506/APhysPolB.53.11-A1

The best established conservation law in physics is the electric charge conservation. The same is true for the universality of electric charge, *i.e.*, the equality of the absolute magnitude of electric charge of an electron and a proton. The most remarkable fact in physics is the quantum nature of an electric charge. This fact was well established even before the discovery of quanta of energy. In 1909, Einstein brought to the broader audience another remarkable fact, observed earlier by Jeans, that the magnitude of the electric charge squared e^2 has the same physical dimension as hc, where h is the Planck constant introduced just few years before. The magnitude of the ratio $\frac{e^2}{hc}$ was then estimated to be of an order of 10^{-3}. Einstein has proposed that the same theoretical framework which will have a constant e^2 included in its mathematical structure will have as a consequence the quantum theory of radiation, and, therefore, the Planck constant h will have been 'explained'. In other words, once e^2 and the fine structure constant

$$\alpha = \frac{e^2}{\hbar c} \tag{1}$$

are given then h would have been established as a secondary constant of Nature. This was not what have happened historically, as we well know now. Meanwhile, the quantum theory of radiation was established but it was also recognized as an incomplete theoretical scheme. This is precisely due to the remarkable experimental fact of the electric charge quantization $Q = Ne$.

Since Gauss, we know that the electric charge 'resides at spatial infinity', as we would describe it in our modern language. On the other hand, special theory of relativity tells us that spatial infinity where electric charge resides is a dynamic concept as an electric charge exists for the eternity of time allowed to it at spatial infinity. Physically, a signal propagating from or to spatial infinity takes an infinite duration of time. This can be formally understood from the observation that the Gauss law is valid in every Lorentz frame. The phenomenological theory of the electric charge proposed by one of us [1–3] contains the only constant of Nature which is relevant to the problem of the quantum nature of electric charge, which is e, the magnitude of the electronic charge. Staruszkiewicz proposed some time ago [1, 2] that the closed dynamical system which contains electric charge must necessarily contain infrared photons which carry information about electric charges emitting them so they could be observed at spatial infinity.

It is well known that spatial infinity of the Minkowski spacetime is the timelike $2+1$-dimensional de Sitter hyperboloid. This is what is needed for the purposes of doing quantum field theory because such a manifold has a well defined Cauchy surface. Quantum mechanics of the electric charge is the quantum field theory of the phase field $S(x)$ defined on the de Sitter spatial infinity [1, 2]. In [3] and below, the phase field is denoted $e(x)$ for obvious reasons.

Electromagnetic field at the spatial infinity is described completely by two homogeneous of degree zero solutions of the d'Alembert equation [1–3]. They are defined as follows. At the spatial infinity, the potential $A_\mu(x)$ must be homogeneous of degree -1:

$$A_\mu(\lambda x) = \lambda^{-1} A_\mu(x)\,, \tag{2}$$

for all $\lambda > 0$ [3, 4].

Using the Maxwell equations and the above homogeneity condition, one finds that

$$x^\mu F_{\mu\nu}(x) = \partial_\nu e(x)\,, \tag{3}$$

$$\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma} x_\nu F_{\rho\sigma}(x) = \partial^\mu m(x)\,. \tag{4}$$

These equations can be solved with respect to $F_{\mu\nu}$, which shows that the functions $e(x)$ and $m(x)$ determine $F_{\mu\nu}(x)$ completely. $e(x)$ is the electric part of the field, while $m(x)$ is the magnetic part. It was shown [1–3] that

$$-\mathrm{d}^4x F_{\mu\nu}F^{\mu\nu} = 2\frac{\mathrm{d}\zeta^0}{\zeta^0}\sqrt{g}\,\mathrm{d}^3\zeta\left(g^{ik}\partial_i e\partial_k e - g^{ik}\partial_i m\partial_k m\right). \quad (5)$$

The metric on spatial infinity g_{ik} is defined in an obvious way

$$g_{ik} = \left(\zeta^0\right)^{-2} g_{\mu\nu}\frac{\partial x^\mu}{\partial\zeta^i}\frac{\partial x^\nu}{\partial\zeta^k}, \qquad i,k = 1,2,3. \quad (6)$$

The coordinates covering spatial infinity,

$$\zeta^0 = \sqrt{-xx} \to +\infty, \quad (7)$$

are the hyperspherical coordinates [1–3]:

$$x^0 = \zeta^0 \sinh\zeta^1, \quad (8)$$
$$x^1 = \zeta^0 \cosh\zeta^1 \sin\zeta^2 \cos\zeta^3, \quad (9)$$
$$x^2 = \zeta^0 \cosh\zeta^1 \sin\zeta^2 \sin\zeta^3, \quad (10)$$
$$x^3 = \zeta^0 \cosh\zeta^1 \cos\zeta^2. \quad (11)$$

The Lagrangian density is seen to be a difference of two identical Lagrangian densities. The part with the right sign, giving rise upon quantization to a positive definite inner product, is called electric. The part with the wrong sign is called magnetic. It is seen that the magnetic part enters the total Lagrangian with the negative sign. This is unphysical and probably explains nonexistence of magnetic monopoles [3, 7].

We hold it self-evident that the sign of the Lagrangian is physically important and that the wrong sign implies the existence of negative norm states. One may keep them, but then one is not working in the framework of quantum mechanics [6].

We wish to note that the addition of the term

$$\theta\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \quad (12)$$

to the Lagrangian [5] does not change this conclusion, simply because it does not change the signature of the Lagrangian treated as a quadratic form:

$$\mathrm{d}^4x\left(-\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} + \Theta\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right) =$$
$$\frac{1}{8\pi}\frac{\mathrm{d}\zeta^0}{\zeta^0}\sqrt{g}\,\mathrm{d}^3\zeta\left(g^{ik}\partial_i e'\partial_k e' - g^{ik}\partial_i m'\partial_k m'\right), \quad (13)$$

where

$$e' = e\cosh\gamma - m\sinh\gamma\,, \tag{14}$$
$$m' = e\sinh\gamma + m\cosh\gamma\,. \tag{15}$$

The parameter γ is defined by the relation

$$\sinh 2\gamma = 8\pi\Theta\,. \tag{16}$$

Therefore, the asymptotic Lagrangian at the spatial infinity, calculated as above, but including the θ term, will also be a difference of two identical Lagrangians, one having necessarily the 'wrong' sign. This means, that the argument against the existence of magnetic monopoles [7] given at [3] is not affected by the θ term.

It was also shown [1–3] that the electric charge Q is always quantized in the units of electronic charge e. Magnetic monopoles (if they existed) would possibly carry a fractional electric charge [5]. Hence, nonexistence of magnetic monopoles is compatible with the quantization of electric charge [1–3].

This research was partially supported by NSF grant to University of South Carolina (P.O.M.), and by the National Science Center, Poland (NCN) grant and University of South Carolina (A.S.). We would also like to acknowledge the warm hospitality of the Jagiellonian University (P.O.M.) and University of South Carolina (A.S.) during the time this work was first written up (September 1996).

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