POSITIVITY OF THE INVARIANT KERNEL UNDERLYING QUANTUM THEORY OF THE COULOMB FIELD

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We present a proof of positivity of an invariant kernel, which is of basic importance for the Staruszkiewicz theory of the quantum Coulomb field. Presented proof of positivity is independent of the Staruszkiewicz theory and is based on the classical Schoenberg's theorem for conditionally negative definite functions, as well as on the generalized Bochner's theorem.

1. Introduction

Let us consider the following function:

$$u \times v \mapsto \langle u|v \rangle = \exp\left\{ -\frac{e^2}{\pi} \left(\lambda \coth \lambda - 1 \right) \right\} \tag{1}$$

for u, v ranging over the Lobachevsky space $u \cdot u = 1$, $v \cdot v = 1$. Here, λ is the hyperbolic angle between u and v: $\cosh \lambda = u \cdot v$. Although it is not evident, (1) is equal to an invariant positive definite kernel on the Lobachevsky space. Reference [9] gives a decomposition of (1) into the Fourier integral (here, $z = e^2/\pi$ and the second term below is absent for $z > 1$)

$$\langle f|f \rangle = \int \mathrm{d}u \, \mathrm{d}v \overline{f(u)} f(v) \langle u|v \rangle = \frac{1}{(2\pi)^3} \int\limits_0^\infty \mathrm{d}\nu \, \nu^2 \, K(\nu; z) \int\limits_{\mathbb{S}^2} \mathrm{d}^2 p \, |\mathcal{F}f(p; \nu)|^2$$

$$+ \frac{(1-z)^2 (2\mathrm{e})^z}{16\pi^2} \int\limits_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{\mathrm{d}^2 p \, \mathrm{d}^2 k}{(p \cdot k)^z} \, \overline{\mathcal{F}f(p; i(1-z))} \, \mathcal{F}f(k; i(1-z)) \tag{2}$$

for $|f\rangle = \int \mathrm{d}u f(u)|u\rangle$ with smooth f of compact support on the Lobachevsky space $u \cdot u = 1$, with the invariant measure $\mathrm{d}u$ on the Lobachevsky space.

In formula (2), the constant e stands for the basis of natural logarithms and e in $z = e^2/\pi$ stands for the elementary charge, experimental value of which is approximately equal to $\frac{1}{\sqrt{137}}$ in units in which $\hbar = c = 1$. Here, the Gel'fand–Graev–Vilenkin inverse Fourier transform of f on the Lobachevsky space is used

$$f(u) = \frac{1}{(2\pi)^3} \int\limits_0^\infty \mathrm{d}\nu \, \nu^2 \int\limits_{\mathbb{S}^2} \mathrm{d}^2 p \, \mathcal{F}f(p; \nu) \, (p \cdot u)^{-i\nu - 1}$$

together with the Gelfand–Graev–Vilenkin Fourier transform $\mathcal{F}f$ of f on the Lobachevsky space, equal

$$\mathcal{F}f(p; \nu) = \int \mathrm{d}u \, f(u) \, (p \cdot u)^{i\nu - 1} \,,$$

which is a homogeneous of degree $i\nu - 1$ function of p on the positive sheet of the cone (and thus with $\mathcal{F}f(p; i(1 - z))$ homogeneous of degree $z - 2$ in p). Decomposition (2) can be computed as in [9] without the assumption of positive definiteness of (1) (the invariance of (1) is evident). Positive definiteness of (1) is equivalent (in terms of [9]) to the positivity of the weight function $K(\nu; z = e^2/\pi)$

$$K(\nu; z) = -\frac{4\pi}{\nu} z^2 \mathrm{e}^z \sum_{n=-\infty}^{+\infty} \frac{[\nu + i(2n + 1 - z)]^{n-1}}{[\nu + i(2n + 1 + z)]^{n+2}}$$

in (2) for each positive real ν. However, positivity of the weight function $K(\nu; e^2/\pi)$ is not evident, compare [12]. In fact, a proof of positivity of the kernel (1), independent of the Staruszkiewicz theory [8], would give us a proof of (relative) consistency of his theory. This is the task of the present paper.

2. Positivity of a Hermitian form on a space of homogeneous states

In this section, we define a linear space $(E^*)^e_{\mathrm{tr}}$ of generalized homogeneous states and a Hermitian form $(\cdot, \cdot)_{\mathfrak{z}}$ on this space. We then prove its positivity. In the next section, we use positivity of $(\cdot, \cdot)_{\mathfrak{z}}$ in the proof of positivity of the kernel (1). Although the space $(E^*)^e_{\mathrm{tr}}$ and the form $(\cdot, \cdot)_{\mathfrak{z}}$ have some deeper relation with the Staruszkiewicz theory, we do not enter into this relation here, and treat the space $(E^*)^e_{\mathrm{tr}}$ and the form $(\cdot, \cdot)_{\mathfrak{z}}$ on $(E^*)^e_{\mathrm{tr}}$ as an intermediate auxiliary construction in the proof of positivity of the kernel (1).

We define the linear space $(E^*)^e_{\text{tr}}$ of electric-type transversal homogeneous of degree $\chi = -1$ states as the space of states spanned (over $\mathbb{C}$) by the following states:

$$\tilde{f}_\mu(p) = \sum_i^N \alpha_i \frac{u_{i\mu}}{u_i \cdot p}, \qquad \sum_i^N \alpha_i = 0, \tag{3}$$

where u_i runs over a finite set of time-like unit $(u_i \cdot u_i = 1)$ four-vectors, and p runs over the positive energy sheet of the cone $p \cdot p = 0$ in momentum space, regarded as distribution supported at the cone $p \cdot p = 0$ in the momentum p. Note that if we allow in this definition only real $\tilde{f}$ and α_i and both energy sheets of the light cone in the momentum space, and finally discard the condition $\sum \alpha_i = 0$, then we obtain the space of (Fourier transforms of) homogeneous of degree -1 solutions of the d'Alembert equation — the *electric-type* solutions generated by the Dirac solution

$$\tilde{f}(p) = \frac{u}{u \cdot p}, \qquad u = (1,0,0,0), \qquad f(x) = \left(\theta(-x \cdot x)\frac{1}{|\boldsymbol{x}|}, 0, 0, 0\right).$$

Note that the condition

$$\sum_i \alpha_i = 0$$

is equivalent to the transversality condition

$$p^\mu \tilde{f}_\mu = 0,$$

which, together with the assumption that $\operatorname{supp}\tilde{f}_\mu \subset \{p; p \cdot p = 0\}$, assures the inverse Fourier transform of $\tilde{f}_\mu$, regarded as distribution concentrated on the cone, to be a solution of the vacuous Maxwell equations.

For the infrared fields having the form of (3), we define the invariant inner product

$$\left(\tilde{f}, \tilde{f}\right)_\mathfrak{J} = \int_{\mathbb{S}^2} \left(\tilde{f}(p), \mathfrak{J}_{\bar{p}}\tilde{f}(p)\right)_{\mathbb{C}^4} \mathrm{d}^2 p = -\int_{\mathbb{S}^2} \overline{\tilde{f}_\mu(p)}\,\tilde{f}^\mu(p)\,\mathrm{d}^2 p, \tag{4}$$

$$\mathfrak{J}_{\bar{p}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad p = (p_0, p_1, p_2, p_3) = (p_0, \boldsymbol{p}), \quad p{\cdot}p = 0, \quad p_0 > 0.$$

Recall, that by the homogeneity condition, $\tilde{f}$ is determined by its values on the unit two sphere $\mathbb{S}^2$, $|\boldsymbol{p}| = 1$, in the cone $\mathscr{O}_{\bar{p}} = \{p : p \cdot p = 0, p_0 > 0\}$, and $\mathrm{d}^2 p$ can be identified with the ordinary measure on $\mathbb{S}^2$.

In the case when both sheets of the light cone in momentum space are allowed, and $\tilde{f}$ as well as α_i are real, then the sum (3) can be realized physically as the electromagnetic potential of the infrared radiation field produced in the scattering process of point charges α_i, with some four-velocities p_i coming in (which have, say, the corresponding α_i positive) and with the four-velocities p_i coming out which have the corresponding α_i with the opposite sign, compare [7]. In particular, for the potential

$$\tilde{f}_\mu(p) = \frac{e}{2\pi} \left(\frac{u_\mu}{u \cdot p} - \frac{v_\mu}{v \cdot p} \right)$$

corresponding to the infrared field produced by a point charge e scattered at the origin such that u^μ, v^μ are the time-like four-velocities of the point charge before and after the scattering, respectively, the inner product (4) is equal to

$$\left(\tilde{f}, \tilde{f} \right)_{\mathfrak{J}} = 2\frac{e^2}{\pi} \left(\lambda \coth \lambda - 1 \right) ,$$

where λ is the hyperbolic angle between u and v, i.e. $\cosh \lambda = u \cdot v$, compare [7].

In the investigation of the Hermitian form (4), the operator B standing in the formula for the inner product $(\tilde{\varphi}, \tilde{\varphi}') = (\tilde{\varphi}, B\tilde{\varphi}')_{L^2(\mathbb{R}^3;\mathbb{C}^4)}$ and in the formula for the Krein-inner product

$$\left(\tilde{\varphi}, \mathfrak{J}'\tilde{\varphi}' \right) = \left(\tilde{\varphi}, B\mathfrak{J}'\tilde{\varphi}' \right)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} = \left(\tilde{\varphi}, \mathfrak{J}_{\bar{p}}\tilde{\varphi}' \right)_{L^2\left(\mathbb{R}^3, d\mu_{\mathscr{O}_{\bar{p}}};\mathbb{C}^4\right)} ,$$

in the single-particle Krein–Hilbert space $\mathcal{H}'$ of the free e.m. potential field in the Gupta–Bleuler gauge, will be useful. Here, $d\mu_{\mathscr{O}_{\bar{p}}}(\boldsymbol{p}) = \frac{d^3\boldsymbol{p}}{2|\boldsymbol{p}|}$ is the standard invariant measure on the cone $\mathscr{O}_{\bar{p}}$. Recall that B is the operator of pointwise multiplication by the matrix

$$\frac{1}{2r}B(p), \qquad p \in \mathscr{O}_{\bar{p}};$$

which is strictly positive and self-adjoint in $\mathbb{C}^4$, with

$$B(p) = \begin{pmatrix} \frac{r^{-2}+r^2}{2} & \frac{r^{-2}-r^2}{2r}p^1 & \frac{r^{-2}-r^2}{2r}p^2 & \frac{r^{-2}-r^2}{2r}p^3 \\ \frac{r^{-2}-r^2}{2r}p^1 & \frac{r^{-2}+r^2-2}{2r^2}p^1p^1 + 1 & \frac{r^{-2}+r^2-2}{2r^2}p^1p^2 & \frac{r^{-2}+r^2-2}{2r^2}p^1p^3 \\ \frac{r^{-2}-r^2}{2r}p^2 & \frac{r^{-2}+r^2-2}{2r^2}p^2p^1 & \frac{r^{-2}+r^2-2}{2r^2}p^2p^2 + 1 & \frac{r^{-2}+r^2-2}{2r^2}p^2p^3 \\ \frac{r^{-2}-r^2}{2r}p^3 & \frac{r^{-2}+r^2-2}{2r^2}p^3p^1 & \frac{r^{-2}+r^2-2}{2r^2}p^3p^2 & \frac{r^{-2}+r^2-2}{2r^2}p^3p^3 + 1 \end{pmatrix},$$

$$(5)$$

again strictly positive self-adjoint on $\mathbb{C}^4$. Here, $r = p^0(\boldsymbol{p}) = |\boldsymbol{p}|$. For each $p \in \mathcal{O}_{\bar{p}}$,

$$
w_1^{+}(p) = \begin{pmatrix} 0 \\ \dfrac{p^2}{\sqrt{(p^1)^2+(p^2)^2}} \\ \dfrac{-p^1}{\sqrt{(p^1)^2+(p^2)^2}} \\ 0 \end{pmatrix} , \qquad
w_1^{-}(p) = \begin{pmatrix} 0 \\ \dfrac{p^1 p^3}{\sqrt{(p^1)^2+(p^2)^2}\, r} \\ \dfrac{p^2 p^3}{\sqrt{(p^1)^2+(p^2)^2}\, r} \\ -\dfrac{\sqrt{(p^1)^2+(p^2)^2}}{r} \end{pmatrix} ,
$$

$$
w_{r^{-2}}(p) = \begin{pmatrix} \dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}}\dfrac{p^1}{r} \\ \dfrac{1}{\sqrt{2}}\dfrac{p^2}{r} \\ \dfrac{1}{\sqrt{2}}\dfrac{p^3}{r} \end{pmatrix} , \qquad
w_{r^2}(p) = \begin{pmatrix} \dfrac{1}{\sqrt{2}} \\ -\dfrac{1}{\sqrt{2}}\dfrac{p^1}{r} \\ -\dfrac{1}{\sqrt{2}}\dfrac{p^2}{r} \\ -\dfrac{1}{\sqrt{2}}\dfrac{p^3}{r} \end{pmatrix} ,
$$

are the eigenvectors of the matrix $B(p)$ which are orthonormal in $\mathbb{C}^4$, where $w_1^{+}(p), w_1^{-}(p)$ correspond to the eigenvalue equal to $+1$, and $w_{r^{-2}}(p), w_{r^2}(p)$ correspond to the eigenvalues r^{-2}, r^2, respectively, compare [13].

There is a canonical decomposition of the one-particle Krein–Hilbert space $\mathcal{H}'$ of the field A_μ associated to the operator B, which allows the construction of the subspace $\mathcal{H}'_{\mathrm{tr}} \subset \mathcal{H}'$ of physical transversal states. The decomposition of $\mathcal{H}'$ associated to B can, in principle, be extended over the space of homogeneous functions on the cone. If, in addition, restrictions of these functions to the unit sphere $\mathbb{S}^2$ belong to $L^2(\mathbb{S}^2)$, then this decomposition will allow us to make some statements concerning positivity of the form (4) as defined on homogeneous of degree -1 four-vector functions summable on $\mathbb{S}^2$.

Namely, recall that the ordinary one particle state, *i.e.* a four-component function $\tilde{\varphi}^\mu$ on the cone — an element of the Hilbert space $\mathcal{H}'$, has the unique decomposition

$$
\tilde{\varphi} = w_1^{+}\tilde{f}_{+} + w_1^{-}\tilde{f}_{-} + w_{r^{-2}}\tilde{f}_{0+} + w_{r^2}\tilde{f}_{0-} .
$$

Here, the four-component functions w, are given above and are at each point p of the cone $\mathcal{O}_{1,0,0,1}$ equal to the eigenvectors of the 4×4 matrix $B(p)$ given above. The complex valued functions $\tilde{f}_{+}, \tilde{f}_{-}$ are square integrable on the cone with respect to the invariant measure $\frac{\mathrm{d}^3\boldsymbol{p}}{2|\boldsymbol{p}|}$ on the cone, and the scalar function $\tilde{f}_{0+}$ is square integrable with respect to the measure $\frac{\mathrm{d}^3\boldsymbol{p}}{|\boldsymbol{p}|^3}$. Finally, the complex valued function $\tilde{f}_{0-}$ is square integrable on the cone with respect

to the measure $|\boldsymbol{p}|\,\mathrm{d}^3\boldsymbol{p}$. The subspace $\mathcal{H}'_{\mathrm{tr}}$ of physical (one particle) states consists precisely of all those functions $\tilde{\varphi}$ which have the decomposition

$$\tilde{\varphi} = w_1{}^+\tilde{f}_+ + w_1{}^-\tilde{f}_- \,.$$

Note, in particular, that the elements of $\mathcal{H}'_{\mathrm{tr}}$ are transversal in the stronger sense, *i.e.* not only $p^\mu\tilde{\varphi}_\mu = 0$ but $p_1\tilde{\varphi}_1 + p_1\tilde{\varphi}_2 + p_3\tilde{\varphi}_3 = 0$. Let now the (four-component) function $\tilde{\varphi}$ be replaced with a function $\tilde{f}$ on the cone, homogeneous of degree $\chi = -1 + i\nu$, $\nu \in \mathbb{R}$. In this case, $\tilde{f}$ likewise has the unique decomposition

$$\tilde{f} = w_1{}^+\tilde{f}_+ + w_1{}^-\tilde{f}_- + w_{r^{-2}}\tilde{f}_{0+} + w_{r^2}\tilde{f}_{0-} \,,$$

where in this decomposition the functions $\tilde{f}_+, \tilde{f}_-, \tilde{f}_{0+}, \tilde{f}_{0-}$ are homogeneous of degree $\chi = -1+i\nu$, as the functions $w_1{}^+, w_1{}^-, w_{r^{-2}}, w_{r^2}$ are homogeneous of degree zero functions on the light cone. We assume that the functions $\tilde{f}$ are regular enough in having the restrictions to the unit sphere $\mathbb{S}^2$ which belong to $L^2(\mathbb{S}^2)$. In this case, the decomposition of $\tilde{f}$ can be used to the analysis of the positivity of (4) on the linear space of homogeneous of degree $\chi = -1 + i\nu$ states, or in particular on homogeneous of degree -1 states of the form (3) which can be transversal, *i.e.*

$$\sum_i^N \alpha_i = 0$$

or not necessary transversal, *i.e.*

$$\sum_i^N \alpha_i \neq 0 \,.$$

In particular, we can consistently define the physical subspace $(E^*)_{\mathrm{tr}}$ of homogeneous states as the space of all those functions on the cone which can be represented as the linear combination

$$w_1{}^+\tilde{f}_+ + w_1{}^-\tilde{f}_- + w_{r^{-2}}\tilde{f}_{0+}$$

with $\tilde{f}_+, \tilde{f}_-, \tilde{f}_{0+}$ homogeneous of degree $\chi = -1 + i\nu$, and restrictions to $\mathbb{S}^2$ belonging to $L^2(\mathbb{S}^2)$. Note, in particular, that the elements of $(E^*)_{\mathrm{tr}}$ are transversal: $p^\mu\tilde{f}_\mu = 0$.

Observe that for homogeneous transversal states $\tilde{f}_\mu$ of homogeneity degree of $\chi = -1 + i\nu$, or for any $\tilde{f}_\mu \in (E^*)^e_{\mathrm{tr}}$ of the general form (3) with

$$\sum_i^N \alpha_i = 0 \,,$$

the bilinear form (4) is non-negatively defined. Indeed, any such element can be decomposed into the three components

$$\tilde{f}_\mu = w_1{}^+{}_\mu \tilde{f}_+ + w_1{}^-{}_\mu \tilde{f}_- + w_{r^{-2}}{}_\mu \tilde{f}_{0+} \,,$$

(the fourth component of the general decomposition is lacking due to the transversality). On the other hand, the components $w_1{}^+{}_\mu \tilde{f}_+$, $w_1{}^-{}_\mu \tilde{f}_-$, $w_{r^{-2}}{}_\mu \tilde{f}_{0+}$ are orthogonal with respect to (4), the bilinear form (4) is positive (for the first two components $w_1{}^+{}_\mu \tilde{f}_+$, $w_1{}^-{}_\mu \tilde{f}_-$) or zero (for the last $w_{r^{-2}}{}_\mu \tilde{f}_{0+}$). Thus, non-negativity on transversal homogeneous of degree $\chi = -1 + i\nu$ as well as positivity on the states (3) follows whenever

$$\sum_i^N \alpha_i = 0 \,.$$

Thus, we may summarize the results in the following:

Lemma 1. *The invariant Hermitian bilinear form (4)*

$$\left(\tilde{f}, \tilde{f} \right)_{\mathfrak{J}} = \int_{\mathbb{S}^2} \left(\tilde{f}(p), \mathfrak{J}_{\bar{p}} \tilde{f}(p) \right)_{\mathbb{C}^4} \mathrm{d}^2 p = - \int_{\mathbb{S}^2} \overline{\tilde{f}_\mu(p)} \tilde{f}^\mu(p) \, \mathrm{d}^2 p$$

is non-negatively definite on the linear space $(E^*)_{\mathrm{tr}}$ of transversal homogeneous of degree $\chi = -1+i\nu$ states as well as on the space $(E^*)^e_{\mathrm{tr}}$ of transversal electric-type states

$$\tilde{f}_\mu(p) = \sum_i^N \alpha_i \frac{u_{i\mu}}{u_i \cdot p} \,, \qquad \sum_i^N \alpha_i = 0 \,, \qquad u_i \cdot u_i = 1 \,, \qquad i = 1, \ldots, N \,.$$

3. Positivity of the invariant kernel $\langle \cdot | \cdot \rangle$ on the Lobachevsky space

Now, we construct a continuous and invariant kernel $\langle \cdot | \cdot \rangle$ on the Lobachevsky space $\mathscr{L}_3$, which is of considerable importance for the Staruszkiewicz theory. We give a proof of its positivity. In order to achieve this result, we use the linear subspace $L[\mathfrak{F}_{\chi=1}]$ of states homogeneous of degree -1.

For this reason, consider now the specific homogeneous of degree -1 state (Fourier transform of the Dirac homogeneous of degree -1 solution restricted to the positive-energy sheet of the cone) of the form of

$$\tilde{f}_\mu^{|u\rangle}(p) = \frac{u_\mu}{u \cdot p} \tag{6}$$

with a fixed unit time-like vector u in the Lobachevsky space. Then construct the linear span $L[\mathfrak{F}_{\chi=1}]$ of all such (6) with u ranging over the Lobachevsky space $\mathscr{L}_3$ of unit time-like vectors u, $u \cdot u = 1$. In other words, we consider the space $L[\mathfrak{F}_{\chi=1}]$ spanned by all Lorentz transforms

$$\Lambda(g)^{-1}\tilde{f}^{\,|u\rangle}(\Lambda(g)p) \;=\; \tilde{f}^{\,|u'\rangle}(p) = \frac{u'}{u' \cdot p}\,, \qquad u' = \Lambda(g)^{-1}u\,,$$
$$g \;\in\; \mathrm{SL}(2,\mathbb{C})$$

of one single state of the form of $\tilde{f}^{\,|u\rangle}$. Here, we have used the natural antihomomorphism $g \to \Lambda(g)$ from the $\mathrm{SL}(2,\mathbb{C})$ into the group of Lorentz transformations.

Note that this Lorentz transformation is induced by the linear dual of the (conjugated) Łopuszański transformation acting in the Fock space of the quantum field A_μ. Namely, we put the natural formula for the invariant pairing

$$\left(\tilde{f}, \tilde{\varphi}\right)_{\mathrm{pairing}} \;=\; \left(\tilde{f}, \mathfrak{J}'\tilde{\varphi}\right) = -\int\limits_{\mathscr{O}_{\pm 1,0,0,1}} \tilde{f}^\mu(p)\tilde{\varphi}_\mu(p)\,\mathrm{d}\mu_{\mathscr{O}_{\pm 1,0,0,1}}(p)$$

$$= \int\limits_{\mathscr{O}_{\pm 1,0,0,1}} \left(\tilde{f}(p), \mathfrak{J}_{\bar{p}}\tilde{\varphi}(p)\right)_{\mathbb{C}^4} \mathrm{d}\mu_{\mathscr{O}_{\pm 1,0,0,1}}(p)\,. \qquad (7)$$

The space $L[\mathfrak{F}_{\chi=1}]$ contains the transversal electric-type homogeneous states (respectively, homogeneous of degree -1 solutions of the d'Alembert equation) of the form of (3) with

$$\sum_i^N \alpha_i = 0$$

along with the longitudinal solutions of the form of (3) with

$$\sum_i \alpha_j \neq 0\,.$$

The space of invariant kernels $\langle \cdot | \cdot \rangle$ on $\mathscr{L}_3$ is rather reach. However, in this particular case of positive definite kernels on the Lobachevsky space $\mathscr{L}_3 \cong \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2,\mathbb{C})$ acted on by $\mathrm{SL}(2,\mathbb{C})$ the invariant kernels are fully classified, compare *e.g.* [1]. The manifold $\mathscr{L}_3$ can also be realized by 2×2 Hermitian complex matrices $\widehat{u} = \sigma_\mu u^\mu$, where $\sigma_0 = \mathbf{1}_2$ and σ_i, $i = 1,2,3$, are the Pauli matrices and $u \cdot u = 1$. Next we consider the smooth left action $\mathrm{SL}(2,\mathbb{C}) \times \mathscr{L}_3 \ni g \times \widehat{u} \to g \cdot \widehat{u} \in \mathscr{L}_3$ of $\mathrm{SL}(2,\mathbb{C})$ on $\mathscr{L}_3$ defined by the formula

$$g \cdot \widehat{u} = g\widehat{u}g^* = \widehat{\Lambda(g^{-1})u}\,, \qquad g \in \mathrm{SL}(2,\mathbb{C})\,, \qquad \widehat{u} \in \mathscr{L}_3\,.$$

Then, $\mathscr{L}_3$ is equal to the orbit $\mathscr{O}_{\widehat{u}} = \{g \cdot \widehat{u}, g \in \mathrm{SL}(2,\mathbb{C})\}$ of the point $\widehat{u}$ with $u = (1,0,0,0)$. The isotropy group at the point $\widehat{u}$ with $u = (1,0,0,0)$ is equal to the maximal compact subgroup $K = \mathrm{SU}(2,\mathbb{C})$ of $\mathrm{SL}(2,\mathbb{C})$. Therefore, $\mathscr{L}_3 = \mathscr{O}_{\widehat{u}}$, $u = (1,0,0,0)$, is diffeomorphic to the space $\mathscr{L}_3 \cong \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2,\mathbb{C})$ of left cosets qK, $q \in \mathrm{SL}(2,\mathbb{C})$, with the above left action, which coincides, under this diffeomorphism, with the ordinary left action $g \times qK \to gqK$ on the left cosets qK. To the coset $\mathfrak{e}K$ of the identity element $\mathfrak{e} \in \mathrm{SL}(2,\mathbb{C})$, there corresponds the point $\widehat{u}$, $u = (1,0,0,0)$, invariant under $\mathrm{SU}(2,\mathbb{C})$. Choosing various invariant positive definite kernels $\langle \cdot | \cdot \rangle$ on $\mathscr{L}_3$, we achieve in this way various cyclic spherical unitary representations $\mathbb{U}$ of the $\mathrm{SL}(2,\mathbb{C})$ group on the completion of $L[\mathfrak{F}_{\chi=1}]$ with respect to the inner product defined by the invariant kernel. Let $K = \mathrm{SU}(2,\mathbb{C})$ be the maximal compact subgroup of $\mathrm{SL}(2,\mathbb{C})$. Let a unitary representation U of $\mathrm{SL}(2,\mathbb{C})$ be called K-spherical (or merely spherical) if the decomposition of the restriction of U to K contains the trivial representation $k \to 1$ of K. Equivalently, U is spherical whenever there is a unit vector $v \in H_U$ such that $U_k v = v$ for all $k \in K$. Then, in particular, it follows by the classification results (or Gel'fand's theory of spherical functions and his generalization of Bochner's theorem for semi-simple Lie groups, in particular for $\mathrm{SL}(2,\mathbb{C})$ group) [1], that each unitary cyclic and spherical representation $\mathbb{U}$ of $\mathrm{SL}(2,\mathbb{C})$ can be reached by the respective choice of the invariant kernel on the Lobachevksy space, or to each such representation there exists the corresponding invariant kernel.

It follows that the most general representation $\mathbb{U}$ which can be achieved in this way has the general form [1] of

$$\mathbb{U} = \int_{\mathbb{R}} \mathfrak{S}(m=0,\rho)\,\mathrm{d}\rho \oplus \int_{[0,1]\subset\mathbb{R}} \mathfrak{D}(\nu)\,\mathrm{d}\nu\,, \tag{8}$$

where $\mathfrak{S}(m,\rho)$ is the irreducible representation of the principal series denoted by the pair $(l_0 = \frac{m}{2}, l_1 = \frac{i\rho}{2})$, with $m \in \mathbb{Z}$ and $\rho \in \mathbb{R}$ in the notation of the book [2], and correspond to the characters $\chi = (n_1, n_2) = \left(\frac{m}{2} + \frac{i\rho}{2}, -\frac{m}{2} + \frac{i\rho}{2}\right)$ in the notation of the book [3]. Here, $\mathfrak{D}(\nu)$ are the irreducible unitary representations of the supplementary series denoted by the pair $(l_0 = 0, l_1 = \nu)$ in the notation of the book [2], and correspond to the character $\chi = (n_1, n_2) = (\nu, \nu)$ in the notation of the book [3] with[1] the real parameter $\nu \in (0,1)$. Finally, $\mathrm{d}\rho$ and $\mathrm{d}\nu$ are the arbitrary σ-measures on the reals $\mathbb{R}$ and on the interval $[0,1] \subset \mathbb{R}$, respectively.

However, the classification of positive definite invariant kernels on the Lobachevsky space, as presented *e.g.* in [1], requires a considerable work

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[1] In the notation of Gel'fand and Neumark works, the parameter ν numbering the supplementary series $\mathfrak{D}(\nu)$ is twice as ours ν and ranges over the interval $(0,2)$.

in each particular case, needed to give a more concrete form to the possible kernels, compare *e.g.* the example of positive definite kernels on the Lobachevsky plane

$$\mathscr{L}_2 = \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)}$$

invariant under $\mathrm{SL}(2,\mathbb{R})$. Unfortunately, the case of

$$\mathscr{L}_3 = \frac{\mathrm{SL}(2,\mathbb{C})}{\mathrm{SU}(2)}$$

has not been worked out in [1] in explicit form. Therefore, we prefer to construct the required kernel, which is of particular importance, with the help of the Hermitian form (4).

Recall that for two points u, v of the Lobachevsky space, we have

$$\left(\tilde{f}^{\,|u\rangle}, \tilde{f}^{\,|v\rangle}\right)_{\mathfrak{J}} = -4\pi\lambda \coth\lambda\,,$$

where λ is the hyperbolic angle between u and v: $\cosh\lambda = u\cdot v$, compare [7]. The Hermitian bilinear invariant form (4) is not positive definite on the linear space $L[\mathfrak{F}_{\chi=1}]$ of states spanned by the states $\tilde{f}^{\,|u'\rangle}$ of the form (6) with u' ranging over the Lobachevsky space $\mathscr{L}_3$. Nonetheless, it defines (after addition of the constant term 4π and changing the sign) the "polarization" of a Lévy–Schoenberg kernel on the Lobachevsky space $\mathscr{L}_3 = \mathrm{SL}(2,\mathbb{C})/K = \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2,\mathbb{C})$ (we are using the terminology of [1]). Namely, the kernel

$$u \times v \mapsto -\left(\left(\tilde{f}^{\,|u\rangle}, \tilde{f}^{\,|v\rangle}\right)_{\mathfrak{J}} + 4\pi\right)$$

on $\mathscr{L}_3 = \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2,\mathbb{C})$ preserves the conditions (2.16)–(2.19) of [1]. In particular, (2.19) of [1] means in our case that for each positive real number t,

$$u \times v \mapsto \langle u|v\rangle_t = \mathrm{e}^{t\left(\left(\tilde{f}^{\,|u\rangle}, \tilde{f}^{\,|v\rangle}\right)_{\mathfrak{J}} + 4\pi\right)} = \mathrm{e}^{-t4\pi(\lambda\coth\lambda - 1)} \tag{9}$$

is an invariant positive definite kernel on the Lobachevsky space and thus defines positive definite and invariant inner product on the linear space S spanned by $\tilde{f}^{\,|u\rangle}$ and all its Lorentz transforms $\tilde{f}^{\,|u'\rangle}$ defined by (6) with u' ranging over the Lobachevsky space. Here, λ is the hyperbolic angle between u and v.

Indeed, that the conditions (2.16)–(2.18) of [1] are preserved is immediate. We need only show that (2.19) of [1] is preserved, *i.e.* that the kernel (9) is positive definite. However, in order to see this, note that

$$\left(\sum_i \alpha_i \tilde{f}^{\,|u_i\rangle}, \sum_j \alpha_j \tilde{f}^{\,|u_j\rangle}\right)_{\mathfrak{J}} \geq 0$$

whenever

$$\sum_i \alpha_i = 0$$

for $\tilde{f}^{|u\rangle}$ defined by (6), as we have already shown that the bilinear form $(\cdot, \cdot)_{\mathfrak{I}}$ is positive definite on the linear space of electric-type transversal states (3), compare the Lemma of Section 2. This means that the function

$$u \times v \mapsto -\left(\left(\tilde{f}^{|u\rangle}, \tilde{f}^{|v\rangle}\right)_{\mathfrak{I}} + 4\pi\right)$$

is a conditionally-negative definite kernel on the Lobachevsky space in the sense of Schoenberg [6], compare also [5], §9.1. Thus by the classical result of Schoenberg [6] (compare *e.g.* also [5], §9.1, Theorem 9.7)

$$u \times v \mapsto \langle u|v\rangle_t = \mathrm{e}^{t\left(\left(\tilde{f}^{|u\rangle}, \tilde{f}^{|v\rangle}\right)_{\mathfrak{I}} + 4\pi\right)} = \mathrm{e}^{-t4\pi(\lambda\coth\lambda - 1)}$$

is a positive definite kernel on the Lobachevsky space for all positive t. Its invariance follows from the invariance of the bilinear form $(\cdot, \cdot)_{\mathfrak{I}}$ and the transformation rule for $\tilde{f}^{|u\rangle}$ defined by (6).

This positivity result is of particular importance in the theory of Starusz-kiewicz, so we state it as a separate

Theorem 1. *For each positive real number t, the function*

$$u \times v \mapsto \langle u|v\rangle_t = \mathrm{e}^{t\left(\left(\tilde{f}^{|u\rangle}, \tilde{f}^{|v\rangle}\right)_{\mathfrak{I}} + 4\pi\right)} = \mathrm{e}^{-t4\pi(\lambda\coth\lambda - 1)}$$

defines a positive definite invariant kernel on the Lobachevsky space $\mathscr{L}_3$ of unit time-like vectors u. Here, λ is the hyperbolic angle between u and v in $\mathscr{L}_3$.

We can choose $u \times v \mapsto \langle u|v\rangle_t$ as the invariant kernel defining the inner product $\langle \cdot|\cdot\rangle_{\mathfrak{I},t}$ on the linear space $L[\mathfrak{F}_{\chi=1}]$ of states spanned by $\tilde{f}^{|u'\rangle}$ defined by (6), with u' ranging over the Lobachevsky space, by the formula

$$\left\langle \sum_{i=1}^{m} \alpha_i \tilde{f}^{|u_i\rangle} \,\middle|\, \sum_{i=1}^{m} \beta_j \tilde{f}^{|v_j\rangle} \right\rangle_{\mathfrak{I},t} = \sum_{i,j=1}^{m} \bar{\alpha}_i \beta_j \langle u_i|v_j\rangle_t, \qquad (10)$$

and define the Hilbert space completion $\mathcal{H}_t \nsubseteq E^*$ of it. Then, we recover the unitary representation $\mathbb{U}^t$ of the $\mathrm{SL}(2,\mathbb{C})$ group induced by the action the dual of the (conjugate) of the Łopuszański representation on the linear space $L[\mathfrak{F}_{\chi=1}]$ of states and its Hilbert space completion $\mathcal{H}_t$. Indeed, by

comparing this construction with the result of [9] and [11], we obtain the following formula:

$$
\mathbb{U}^t = \begin{cases} \mathfrak{D}(\nu_0) \bigoplus\limits_{\rho>0} \int \mathfrak{S}(m=0,\rho)\mathrm{d}\rho\,, & \nu_0 = 1-4\pi t\,, \quad \text{if}\quad 0 < 4\pi t < 1\,, \\[2mm] \int\limits_{\rho>0} \mathfrak{S}(m=0,\rho)\,\mathrm{d}\rho\,, & \text{if}\quad 1 < 4\pi t\,, \end{cases}
$$

(11)

where $\mathrm{d}\rho$ is the ordinary Lebesgue measure on $\mathbb{R}_+$, compare (2).

4. Positivity of the weight function $K(\nu; z)$

Now let us go back to the consistency of the axioms of the Staruszkiewicz theory [8]. This consistency is equivalent to the existence (proved by explicit construction) of the unitary representation U of the $\mathrm{SL}(2,\mathbb{C})$ which makes the phase field $S(x)$ a scalar field on the de Sitter hyperplane. Construction of this U is based (compare [9, 10, 14]) on the fact that the mapping (1) is equal to an invariant positive definite kernel on the Lobachevsky space. Positivity of (1) would, of course, immediately follow from the consistency of the Staruszkiewicz theory [8], but of course to show the consistency we should prove that (1) is an invariant positive definite kernel independently of the consistency assumption. However, as we have shown in Section 3, the function (1) defines indeed an invariant positive definite kernel on the Lobachevsky space, using Schoenberg's theorem on conditionally negative definite functions. Thus, the consistency of the axioms of [8], is thereby proved.

Moreover, also positivity of $K(\nu; e^2/\pi)$ follows from the positive definiteness of the invariant kernel (1) *i.e.* from the theorem of Section 3. Indeed, it follows on application of the generalization of Bochner's theorem[2], extended to the relation between positive measures on the set of irreducible K-spherical unitary representations of semi-simple Lie groups G and the corresponding positive definite kernels on $G/K \times G/K$ (or positive definite functions on G, corresponding to positive definite kernels on $G/K \times G/K$), compare *e.g.* [1], with $G = \mathrm{SL}(2,\mathbb{C})$, $K = \mathrm{SU}(2,\mathbb{C})$ and with the Lobachevsky space $G/K = \mathscr{L}_3$ as the homogeneous Riemannian manifold. Recall, please, that a continuous function φ on G is called positive definite iff

$$
\sum_{i,j} \varphi\left(g_i^{-1} g_j\right) \alpha_i \bar{\alpha}_j \geq 0
$$

for any finite set of complex numbers α_i, and any finite set of $g_i \in G$; and that φ is K-spherical iff $\varphi(k_1 g k_2) = \varphi(g)$, for all $g \in G$, $k_1, k_2 \in K \subset G$.

---

[2] Due to Krein, Naimark, and Gel'fand [4], §31.10.

φ is called normalized iff $\varphi(\mathfrak{e}) = 1$. Recall that a unitary representation U of G is called K-spherical if there exists a unit vector $v \in H(U)$ such that $U(k)v = v$ for each $k \in K$. Recall further that to any K-spherical unitary representation U of G, there corresponds the K-spherical continuous normalized and positive definite function $\varphi(g) = (U(g)v, v)$, where $(\cdot, \cdot)$ is the inner product in the Hilbert space $H(U)$ of the representation U. If the K-spherical representation U is cyclic, with the cyclic vector v which is invariant under $U(k)$, $k \in K$, then the correspondence between the unitary equivalence class of U and the spherical function $\varphi(g) = (U(g)v, v)$ is bi-unique. This is, in particular, the case for irreducible K-spherical U. If the K-spherical representation U is irreducible, then the corresponding K-spherical normalized and positive definite function φ is called *elementary*. Recall that each positive definite kernel κ on $G/K \times G/K$ can be lifted to the bi-uniquely corresponding positive definite function φ on G through the formula $\varphi(g) = \kappa(gK, \mathfrak{e}K)$ (compare *e.g.* [1]). Let K be a maximal compact subgroup of a semisimple G. Finally, let $\nu \in \mathscr{M}$ be all equivalence classes of all irreducible K-spherical representations U_ν of a semisimple Lie group G and let φ_ν be the elementary normalized positive definite K-spherical functions corresponding to the irreducible representants U_ν, one for each unitary equivalence class. Then we have the following generalization of Bochner's theorem (compare §31.10, Eq. (4), p. 426 of [4], or Theorem 3.23 in [1]):

Theorem 2 (Generalized Bochner's theorem). *Let φ be a continuous positive definite K-spherical function on G. Then, there exists a unique nonnegative measure μ on $\mathscr{M}$ such that*

$$\varphi(g) = \int_{\mathscr{M}} \varphi_\nu(g)\, \mathrm{d}\mu(\nu)\,, \qquad g \in G\,.$$

The Fourier transform of the kernel (1) found in [9] (or decomposition (8) of the representation corresponding to the kernel (1) or equivalently to the kernel (9) with $4\pi t$ put equal e^2/π) applied to the spherical positive definite function φ corresponding to the kernel (1) gives decomposition of φ into the elementary spherical functions φ_ν with the same measure μ on the set of equivalence classes of spherical unitary irreducible representations of $G = \mathrm{SL}(2, \mathbb{C})$ as in (2). On the set of spherical representations of the principal series, it is given by the Lebesgue measure $\mathrm{d}\nu$ on $\mathbb{R}_+$ with the weight function equal to $(2\pi^2)^{-1}\nu^2 K(\nu; e^2/\pi)$: $\mathrm{d}\mu(\nu) = (2\pi^2)^{-1}\nu^2 K(\nu; e^2/\pi)\mathrm{d}\nu$. Therefore, the generalized Bochner's theorem (Eq. (4), p. 426 of [4] or Theorem 3.23 of [1]) implies positivity of the measure $\mathrm{d}\mu(\nu) = (2\pi^2)^{-1}\nu^2 K(\nu; e^2/\pi)\mathrm{d}\nu$. Thus, positivity of the weight function $K(\nu; e^2/\pi)$ in (2) follows for almost all ν. By the analyticity of $K(\nu; e^2/\pi)$ in both arguments (compare [9]) positivity of $K(\nu; e^2/\pi)$ in the ordinary sense follows. Let us explain it in

more detail. Please, recall that the unitary irreducible representations of the principal series $\mathfrak{S}(m = 0, \nu)$, $\nu \in \mathbb{R}$, and of the supplementary series $\mathfrak{D}(\nu_0)$, $\nu_0 \in [0, 1]$ exhaust all irreducible equivalence classes of all spherical unitary irreducible representations of $G = \mathrm{SL}(2, \mathbb{C})$. Next, recall that the Fourier decomposition of the kernel $\kappa(u, v) = \langle u | v \rangle$, given by (1) and found in [9], is equal to

$$
\begin{aligned}
\langle u | v \rangle &= \frac{1}{(2\pi)^3} \int\limits_0^\infty \mathrm{d}\nu \, \nu^2 \, K(\nu; z) \int\limits_{\mathbb{S}^2} \mathrm{d}^2 p \, \overline{(p \cdot u)^{i\nu - 1}} (p \cdot v)^{i\nu - 1} \\
&\quad + \frac{(1 - z)^2 (2e)^z}{16\pi^2} \int\limits_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{\mathrm{d}^2 p \, \mathrm{d}^2 k}{(p \cdot k)^z} \, \overline{(p \cdot u)^{z - 2}} (k \cdot v)^{z - 2} \\
&= \kappa(u, v) = \kappa(gK, \mathfrak{e}K) = \varphi(g), \\
u &= gv = \Lambda\left(g^{-1}\right) v, \qquad v = (1, 0, 0, 0), \qquad g \in \mathrm{SL}(2, \mathbb{C}), \quad (12)
\end{aligned}
$$

here with e in front of the second integral equal to the basis of natural logarithms and with $\mathfrak{e}$ equal to the unit in $\mathrm{SL}(2, \mathbb{C})$. Recall that the $\mathrm{d}^2 p$ integral over $\mathbb{S}^2$ in the first summand is equal to the inner product of two functions

$$
p \to (p \cdot u)^{i\nu - 1}, \qquad p \to (p \cdot v)^{i\nu - 1}
$$

homogeneous of degree $i\nu - 1$ on the positive cone $p \cdot p = 0$, $p_0 > 0$, in the Hilbert space of the unitary spherical representation of the principal series $\mathfrak{S}(m = 0, \nu)$, realized as the closure with this inner product of all homogeneous of degree $i\nu - 1$ continuous functions on the cone. The double integral $\mathrm{d}^2 p \times \mathrm{d}^2 k$ over $\mathbb{S}^2 \times \mathbb{S}^2$ of the second summand in (12) is equal to the inner product of two functions

$$
p \to (p \cdot u)^{z - 2}, \qquad p \to (p \cdot v)^{z - 2}
$$

homogeneous of degree $z - 2$ on the positive cone $p \cdot p = 0$, $p_0 > 0$, in the Hilbert space of the unitary spherical representation of the supplementary series $\mathfrak{D}(\nu_0 = 1 - z)$, realized as the closure with this inner product of all homogeneous of degree $2 - z$ continuous functions on the cone. It is easily seen that the function

$$
p \to \frac{1}{\sqrt{4\pi}} (p \cdot v)^{i\nu - 1}, \qquad v = (1, 0, 0, 0)
$$

represents the unit state invariant under the action of all elements of $K = \mathrm{SU}(2, \mathbb{C})$ in the Hilbert space of the K-spherical irreducible representation $\mathfrak{S}(m = 0, \nu)$ of the principal series, in the realization stated above. Similarly,

$$
p \to \left(\frac{(1 - z) 2^z}{16\sqrt{2}\pi^{5/2}} \right)^{1/2} (p \cdot v)^{z - 2}, \qquad v = (1, 0, 0, 0)
$$

is the function representing the unit state invariant under the action of all elements of $K = \mathrm{SU}(2, \mathbb{C})$ in the Hilbert space of the K-spherical representation $\mathfrak{D}(\nu_0 = 1 - z)$ of the supplementary series, in the realization stated above. Therefore, the elementary positive definite and normalized spherical functions φ_ν and φ_{ν_0} corresponding, respectively, to the spherical representations of the classes $\mathfrak{S}(m = 0, \nu)$ and $\mathfrak{D}(\nu_0 = 1 - z)$ are, respectively, equal to

$$\varphi_\nu(g) \;=\; \frac{1}{4\pi} \int\limits_{\mathbb{S}^2} \mathrm{d}^2 p \; \overline{(p \cdot u)^{i\nu - 1}} (p \cdot v)^{i\nu - 1}\,,$$

$$v \;=\; (1, 0, 0, 0)\,, \qquad u = gv = \Lambda\left(g^{-1}\right) v\,,$$

$$\varphi_{\nu_0}(g) \;=\; \frac{(1 - z)2^z}{16\sqrt{2}\pi^{5/2}} \int\limits_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{\mathrm{d}^2 p \, \mathrm{d}^2 k}{(p \cdot k)^z} \; \overline{(p \cdot u)^{2 - z}} (k \cdot u)^{z - 2}\,,$$

$$v \;=\; (1, 0, 0, 0)\,, \qquad u = gv = \Lambda\left(g^{-1}\right) v\,.$$

Inserting these formulas into (12), we obtain

$$\varphi(g) \;=\; \int\limits_{\mathbb{R}_+} \varphi_\nu(g) \mathrm{d}\mu(\nu) + \int\limits_{[0,1]} \varphi_{\nu_0}(g) \mathrm{d}\mu(\nu_0)$$

$$=\; \frac{1}{2\pi^2} \int\limits_0^\infty \varphi_\nu(g)\, \nu^2\, K(\nu; z) \mathrm{d}\nu + \frac{(1 - z)\mathrm{e}^z}{\sqrt{2\pi}}\, \varphi_{\nu_0}(g)\,, \qquad (13)$$

with the restriction of the measure $\mathrm{d}\mu(\nu)$ to the set $\mathbb{R}_+$ of the equivalence classes $\nu \in \mathbb{R}_+$ of the spherical representations of the principal series $\mathfrak{S}(m = 0, \nu)$ equal to

$$\mathrm{d}\mu(\nu) = \frac{1}{2\pi^2}\nu^2\, K(\nu; z)\mathrm{d}\nu\,,$$

and with the restriction of the measure $\mathrm{d}\mu(\nu_0)$ to the set $[0, 1]$ of the equivalence classes $\nu_0 \in [0, 1]$ of the spherical representations of the supplementary series $\mathfrak{D}(\nu_0)$ equal to the discrete measure concentrated at the single point $\nu_0 = 1 - z$, and giving the measure

$$\frac{(1 - z)\mathrm{e}^z}{\sqrt{2\pi}}$$

to the single point set $\{\nu_0 = 1 - z\}$. Recall that the second discrete term in (13) is present if and only if $0 < z < 1$. In (13) and in the last formula,

e is equal to the basis of natural logarithms. Thus, from the generalized Bochner theorem positivity of the weight function $\nu \to K(\nu; z)$ follows for all $z, \nu > 0$.

Thus, using the theorem of Section 3 and the generalized Bochner's theorem, we have proved

Theorem 3. *The weight function $\nu \to K(\nu; z = e^2/\pi)$ in the Fourier decomposition (2) is non-negative.*

Let us emphasize that our proof is independent of the axioms of [8], thereby proving (relative) consistency of the axioms of [8], *i.e.* we have proved that the consistency of the axioms of the Staruszkiewicz theory [8] is equivalent to the consistency of the harmonic analysis for the semisimple Lie group $\mathrm{SL}(2, \mathbb{C})$.

One can find out that there exists a small positive ϵ such that $\nu \to K(\nu; z = e^2/\pi)$ is a non-negative function for e^2/π in the interval $0 < e^2/\pi < \epsilon$, on using explicit inspection, compare [12]. Thus, in this asymptotic case $e^2/\pi \ll 1$, positivity of the weight K follows by explicit inspection, and then positivity of the kernel (1) for $e^2/\pi \ll 1$, by the generalized Bochner's theorem. Note here that the experimental value of e^2/π is ≈ 0.0023. Numerical calculations [12] confirm positivity of the weight $K(\nu; e^2/\pi)$.

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