# A THEORETICAL DESCRIPTION OF $2\nu\beta^-\beta^-$ DECAY TO THE EXCITED $2^+$ STATES

S. Ünlü

Department of Physics, Burdur Mehmet Akif Ersoy University, Burdur, Turkey

N. Çakmak

Department of Physics, Karabuk University, Karabuk, Turkey

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We try to give a theoretical analysis of $2\nu\beta^-\beta^-$ decay to the final excited $2^+$ states by considering SU(4) symmetry restoration within the framework of quasi-particle random phase approximation (QRPA). Pyatov's method is used to restore the symmetry violations stemming from the mean-field approximation. A comparison of the calculated decay rates with other calculations and the corresponding experimental data is given.

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## 1. Introduction

The predictions related to the existence of the Majorana neutrino mass, the existence of a right-handed leptonic current, the existence of a massless Goldstone boson, the super-symmetry [1–8] depend on the nuclear structure calculations containing the nuclear matrix elements. Hence, it is very important to improve the accuracy of these matrix elements by developing corrected or new approaches. The theoretical description of the nuclear matrix elements for the $2\nu\beta\beta$ decay process plays a crucial role in testing the accuracy of the models.

The proton–neutron quasi-particle random phase approximation ($pn$QRPA) has been considered the most powerful method for single- and double-beta transition calculations of nuclear systems which are far away from closed shells. According to this method, the nuclear matrix element consists of the contributions coming from the virtual intermediate excitations. As known, the dominant contribution to the nuclear matrix element comes from the spin–isospin excitations (Gamow–Teller excitations) in intermediate nuclei. Hence, it is very important to restore SU(4) symmetry violations stemming from the mean-field approximation in order to give a reliable prediction of the spin–isospin ($1^+$) excitations. The restoration of the symmetry violations leads to a remarkable quenching on the nuclear matrix element for $2\nu\beta^-\beta^-$ decay to the final ground state [9–11]. The difference of the present work from our recent works can be summarized as follows: The $2\nu\beta^-\beta^-$ decay to the final excited $2^+$ states are described within the SU(4) symmetry restoration. The present results are compared with other calculations in the literature and the corresponding experimental data. Thus, the effect of the symmetry restoration on the decay to the final excited states can be clearly understood.

## 2. Theoretical formalism

Let us consider a system of nucleons in a spherical symmetric average field with pairing forces. The single quasi-particle Hamiltonian of the system is given by

$$H_{\rm sqp} = \sum_{jm} \varepsilon_j(a)\alpha^\dagger_{jm}(a)\alpha_{jm}(a)\,, \qquad (a=n,p)\,, \tag{1}$$

where $\varepsilon_j(a)$ is the single quasi-particle (sqp) energy, and $\alpha^\dagger_{jm}(a)$ ($\alpha_{jm}(a)$) is the quasi-particle creation (annihilation) operator.

The effective interaction potential for $2^+$ excitations in final nuclei consists of isoscalar and isovector parts and is defined as follows:

$$h = G\sum_{a,\eta} K^\dagger_\eta(a)K_\eta(a) + G_{pn}\sum_\eta \left(K^\dagger_\eta(p)K_\eta(n) + K^\dagger_\eta(n)K_\eta(p)\right)\,, \tag{2}$$

where $K_\eta(a)$ ($\eta = 0,\pm1,\pm2$) is a electric quadrupole transition operator as defined in [12]. The electric quadrupole excitations in final nuclei are represented by a phonon creation operator in the following form:

$$Q^\dagger_k(\eta)|0\rangle = \sum_{a'a}\left[Z^k_{a'a}A^\dagger_{a'a}(\eta) - W^k_{a'a}A_{a'a}(\eta)\right]|0\rangle\,, \tag{3}$$

where $A^\dagger_{a'a}(\eta)$ and $A_{a'a}(\eta)$ are quasi-boson creation and annihilation operators for charge-conserving transitions, respectively.

The Gamow–Teller (GT) transition operator is described as a combination of $\beta^-$ and $\beta^+$ decay operators

$$F^\rho_\mu = \tfrac{1}{2}\left(T^+_\mu + \rho(-1)^\mu T^-_{-\mu}\right)\,, \qquad (\rho=\pm, \mu=0,\pm1)\,. \tag{4}$$

$\beta^+$ and $\beta^-$ decay operators are given as $T^+_\mu = \sum_{i=1}^A \sigma_\mu(i)t_+(i)$ and $T^-_\mu = (T^+_\mu)^\dagger$, respectively. Here, $\sigma_\mu$ and $t_+$ ($t_-$) are the Pauli spin and isospin raising (lowering) operators, respectively. The charge-exchange effective interaction in particle–hole channel is defined within Pyatov's restoration method. As known, SU(4) is not a good symmetry due to Coulomb ($V_{\mathrm{c}}$) and spin–orbit terms ($V_{\vec{l}\vec{s}}$) in total Hamiltonian

$$\left[H - \left(V_{\mathrm{c}} + V_{\vec{l}\vec{s}}\right), F_{\mu}^{\rho}\right] = 0\,. \tag{5}$$

This commutativity is broken in the mean-field approximation

$$\left[H_{\mathrm{sqp}} - \left(V_{\mathrm{c}} + V_{\vec{l}\vec{s}}\right), F_{\mu}^{\rho}\right] \neq 0\,. \tag{6}$$

The SU(4) symmetry violations in the mean-field level of approximation are restored by including an effective interaction potential in the following form:

$$h_{\mathrm{GT}} = \sum_{\rho=\pm} \frac{1}{4\gamma_{\rho}} \sum_{\mu=0,\pm1} \left[H_{\mathrm{sqp}} - V_{\mathrm{c}} - V_{\vec{l}\vec{s}}, F_{\mu}^{\rho}\right]^{\dagger} \left[H_{\mathrm{sqp}} - V_{\mathrm{c}} - V_{\vec{l}\vec{s}}, F_{\mu}^{\rho}\right]\,. \tag{7}$$

The strength parameter of the residual interaction can be found from the following condition:

$$\left[H_{\mathrm{sqp}} + h_{\mathrm{GT}} - V_{\mathrm{c}} - V_{\vec{l}\vec{s}}, F_{\mu}^{\rho}\right] = 0 \tag{8}$$

and taken out to be a free parameter

$$\gamma_{\rho} = \frac{\rho(-1)^{\mu}}{2} \langle 0| \left[\left[H_{\mathrm{sqp}} - \left(V_{\mathrm{c}} + V_{\vec{l}\vec{s}}\right), F_{\mu}^{\rho}\right], F_{\mu}^{\rho}\right] |0\rangle\,. \tag{9}$$

The spin–isospin effective interaction in the particle–particle channel is defined in the following form:

$$h_{pp} = -2g_{pp} \sum_{\mu} (P_{\mu})^{\dagger} P_{\mu}\,. \tag{10}$$

The spin–isospin excitations in intermediate nuclei are described as follows:

$$\Gamma_{m}^{\dagger}(\mu)|0\rangle = \sum_{pn} \left[X_{pn}^{m} C_{pn}^{\dagger}(\mu) - Y_{pn}^{m} C_{pn}(\mu)\right] |0\rangle\,, \tag{11}$$

where $C_{pn}^{\dagger}(\mu)$ and $C_{pn}(\mu)$ are quasi-boson creation and annihilation operators for charge-exchange transitions, respectively.

The nuclear matrix elements for $2\nu\beta^-\beta^-$ decay to the final ground and excited $2^+$ states are defined as follows:

$$M_{\mathrm{GT}}\left(0_{\mathrm{gs}}^{+}\right) = \sum_{m} \frac{\left\langle 0_{\mathrm{gs}}^{+}\right| \sigma\tau^{-} \left|1_{m}^{+}\right\rangle \left\langle 1_{m}^{+}\right| \sigma\tau^{-} \left|0_{\mathrm{gs}}^{+}\right\rangle}{\left(W\left(0_{\mathrm{gs}}^{+}\right)/2 + E\left(1_{m}^{+}\right) - Mc^{2}\right)/\left(m_{e}c^{2}\right)}\,, \tag{12}$$

$$M_{\mathrm{GT}}\left(2_k^+\right)=\frac{1}{\sqrt{3}}\sum_m\frac{\left\langle 2_k^+\right|\sigma\tau^-\left|1_m^+\right\rangle\left\langle 1_m^+\right|\sigma\tau^-\left|0_{\mathrm{gs}}^+\right\rangle}{\left[\left(W\left(2_k^+\right)/2+E\left(1_m^+\right)-Mc^2\right)/\left(m_ec^2\right)\right]^3}\,. \tag{13}$$

The mathematical expressions of the reduced matrix elements of single $\beta$ virtual transitions are given in Ref. [9, 13]. The overlap integral is not considered in the calculation of the decay amplitudes. Instead, the $2\nu\beta^-\beta^-$ decay amplitude is determined as an average of two different amplitudes which are obtained according to the initial and final basis. The energy denominator depends on the intermediate state energies and decay energies. $W(0_{\mathrm{gs}}^+)$ corresponds to the energy for $2\nu\beta^-\beta^-$ decay to the final ground state. The energy for $2\nu\beta^-\beta^-$ decay to the final excited states is represented by $W(2_k^+)=W(0_{\mathrm{gs}}^+)-E(2_k^+)$ $(k=1,2)$. $E(1_m^+)$ represents intermediate state energies with respect to the initial ground state. $M$ corresponds to the mass difference which is adjusted so that the calculated energy for the first intermediate state becomes equal to the measured one. The $2\nu\beta^-\beta^-$ decay half-lives are determined by using the following expression:

$$\frac{1}{T_{1/2}}=f_{2\nu}|M_{\mathrm{GT}}|^2\,, \tag{14}$$

where $f_{2\nu}$ is a phase-space factor for $2\nu\beta^-\beta^-$ decay transitions.

## 3. Results and discussions

The calculated results related to the $2\nu\beta^-\beta^-$ decay to the final excited $2^+$ states are given in this section. The Woods–Saxon potential with the Chepurnov parametrization [12] is used as a mean-field basis. The proton and neutron pairing gaps are defined as $\Delta_p=C_p/\sqrt{A}$ and $\Delta_n=C_n/\sqrt{A}$, respectively [14]. The pairing strength parameters ($C_p$ and $C_n$) are chosen according to the corresponding experimental pairing gaps [15].

The $G$ and $G_{pn}$ strength parameters for electric quadrupole interactions are proportional to $A^{-7/3}$ as given in [12]. The $G_{pn}$ interaction parameter is 10 times lower than $G$ parameter due to the dominance of isoscalar ($pp$ or $nn$) configurations. The fixed value of $G$ interaction constant is determined from the agreement of the calculated first $2^+$ energy with the experimental one (see Fig. 1). The corresponding quadrupole transition probabilities for the fixed $G$ values are given in Table 1. The quadrupole transition probabilities in Table 1 are computed according to the bare charges ($e_p=1$, $e_n=0$). As seen, the transition probabilities obtained with bare charges are close to the experimental probabilities except for the $^{128}$Xe isotope. The probability for this isotope can be obtained using the effective charges as $e_p=1+\delta, e_n=\delta$. In this case, the experimental quadrupole excitation probability can be reproduced by a reasonable effective charge as $\delta=0.28$.

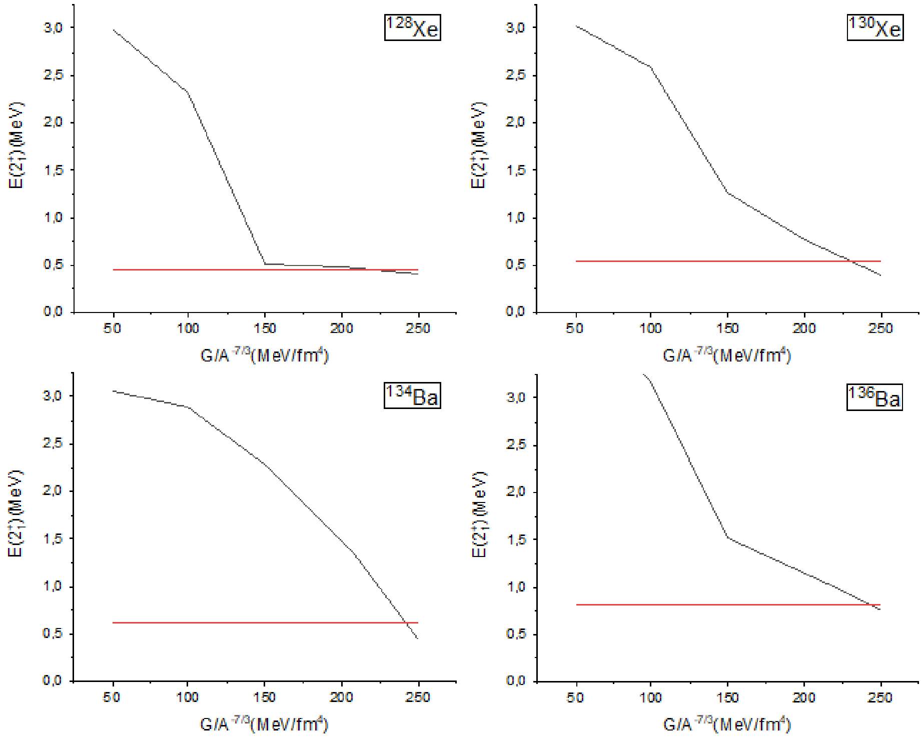


Fig. 1. The determination of strength parameter in $(pp{+}nn)$QRPA calculations. The horizontal lines indicate the experimental energies [16].

Table 1. The calculated and experimental quadrupole transition probabilities.

| Nuclei | $B\left(\mathrm{E2}; 0^+_{\mathrm{gs}} \to 2^+_1\right)_{\mathrm{cal}}$ (W.u.) | $B\left(\mathrm{E2}; 0^+_{\mathrm{gs}} \to 2^+_1\right)_{\mathrm{exp}}$ (W.u.) [16] |
|---|---|---|
| $^{128}$Xe | 14.95 | 40.20 |
| $^{130}$Xe | 33.48 | 30.80 |
| $^{134}$Ba | 27.58 | 32.63 |
| $^{136}$Ba | 20.31 | 19.87 |

The spin–isospin excitations in intermediate nuclei are obtained within $pn$QRPA method. The spin–isospin effective interaction in the particle–hole channel is included in such a way that SU(4) symmetry violations stemming from mean-field approximation are restored. Thus, the mathematical formalism related to the charge-exchange excitations becomes free of the particle–hole strength parameter. As known, the existence of $2\nu\beta^-\beta^-$ decay transitions is attributed to the fact that SU(4) is not an exact symmetry of the total Hamiltonian. However, the extra-symmetry violations in the mean-field level of approximation lead to a serious increment of the decay amplitude. Hence, it is important to restore these symmetry violations in order to make a reliable prediction of the decay amplitude. Nevertheless, the restoration of symmetry violations leads to a significant quenching on the decay amplitude to the final ground state. In other words, the calculated amplitudes for the decay to the final ground state get closer to the experimental amplitudes due to the quenching effect of restoration. The charge-exchange effective interaction potential in the particle–particle channel can be added to the restored Hamiltonian in order to reproduce the experimental data related to the $2\nu\beta^-\beta^-$ decay to the final ground state. However, it can be said that the decay amplitudes obtained for a zero value of the particle–particle strength parameter are very close to the corresponding experimental amplitudes (see Fig. 2). For $A = 128$ and 136 decay systems, a small contribution of particle–particle interaction is enough to reproduce the experimental decay amplitude. The experimental amplitude for $A = 130$ decay system can be reproduced by a repulsive contribution of particle–particle interaction.

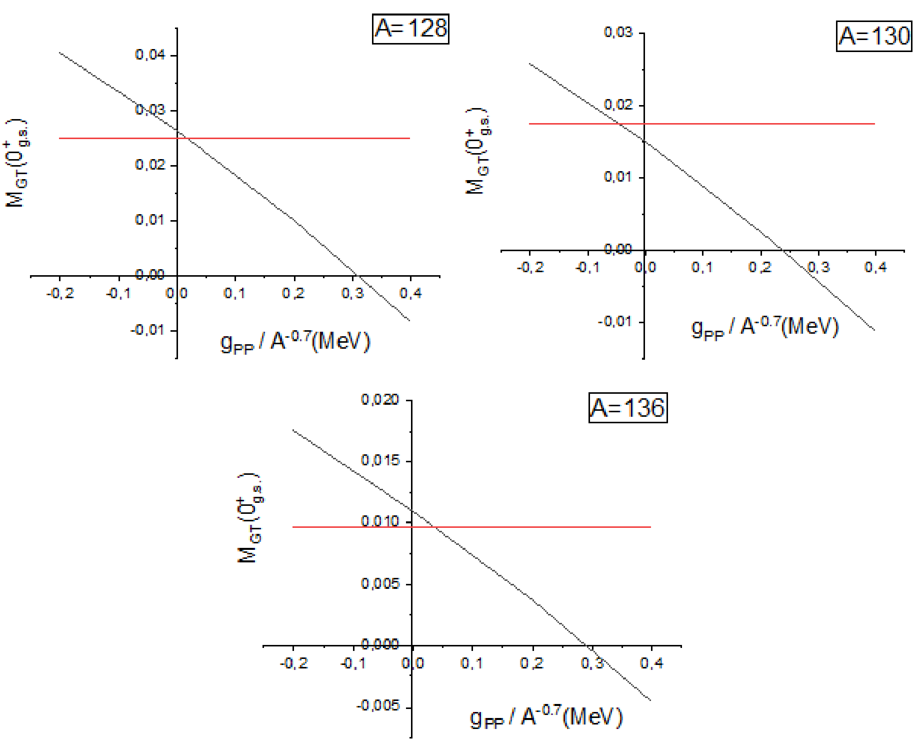


Fig. 2. The determination of the particle–particle strength parameter in $pn$QRPA calculations. The horizontal lines indicate the experimental amplitudes [17, 18].

In this case, the nuclear matrix element for the decay to the final excited $2^+$ states of $^{130}$Xe should be computed for a zero value of the particle–particle strength parameter. This is not a serious problem because of good agreement with the experimental data. For $A = 134$ decay system, the particle–particle constant is taken equal to zero due to lack of the experimental data.

The calculated values of the nuclear matrix elements for the decays to the final excited $2^+$ states are presented in Table 2. The calculations give us a remarkable amplitude for the decay to the excited state of $^{128}$Xe. For a detailed analysis, the energy spectrum of the decay amplitudes for $A = 128$ system is presented in Fig. 3. Let us note that the above and below graphs represent the calculated distributions according to the initial and final basis, respectively. It can be clearly seen that the first intermediate state plays a key role in the determination of the decay amplitude. The higher intermediate states than the first one make a negligible contribution to the decay amplitude for the final excited state. It is possible to check the wave function of the first $1^+$ state for $^{128}$I by making a comparison of the calculated $\log(ft)$ values with the corresponding experimental values [19].

Table 2. The nuclear matrix elements for $2\nu\beta^-\beta^-$ decay to the final $2^+$ states.

| Transition | $M_{\rm GT}\ (2_1^+)$ | $M_{\rm GT}\ (2_2^+)$ |
|---|---|---|
| $^{128}\mathrm{Te} \to {}^{128}\mathrm{Xe}$ | 0.0458 | — |
| $^{130}\mathrm{Te} \to {}^{130}\mathrm{Xe}$ | 0.00013 | 0.00081 |
| $^{134}\mathrm{Xe} \to {}^{134}\mathrm{Ba}$ | 0.0030 | — |
| $^{136}\mathrm{Xe} \to {}^{136}\mathrm{Ba}$ | −0.00040 | 0.00083 |

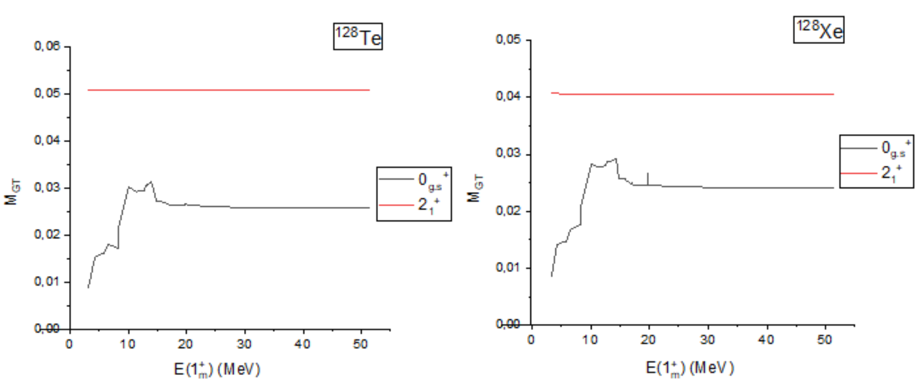


Fig. 3. The energy spectrum of nuclear matrix element for the $^{128}\mathrm{Te} \to {}^{128}\mathrm{Xe}$ system.

The $\beta^+$ decay $\log(ft)$ value for $^{128}$I is 5.16 and very close to the corresponding experimental value (5.05). The same quantity for $\beta^-$ decay is 6.84 and not far away from the experimental value (6.09).

The corresponding half-lives for the $2\nu\beta^-\beta^-$ decays to the final $2^+$ states should be determined to understand their contributions to the total decay probability. The calculated half-lives for the decays to the final excited $2^+$ states are given in Tables 3 and 4. When the phase-space factors in [20] are used for the decay to the 1$^{\rm st}$ excited state of $^{128,130}$Xe and $^{136}$Ba, the corresponding phase-space factor for $A = 134$ system is taken from [21]. The phase-space factors in [4] are used for the decays to the 2$^{\rm nd}$ $2^+$ state. In Table 3, columns 3 and 4 show the calculated results of boson expansion formalism and another work using QRPA, respectively. The restoration of symmetry violations presents shorter decay half-lives in comparison with boson expansion formalism. Nevertheless, the decay half-lives within restoration are longer than those of other QRPA calculations except for the $^{128}$Te$\rightarrow$ $^{128}$Xe decay system. As mentioned above, the first intermediate state for $A = 128$ system makes a significant contribution to the decay amplitude. In Table 4, our calculations present longer half-lives for the decay to the 2$^{\rm nd}$ $2^+$ state. The calculated half-lives for $^{130}$Te$(0^+_{\rm gs}) \rightarrow$ $^{130}$Xe$(2^+_2)$ decay are not very far away from other QRPA results. For the $^{136}$Xe$\rightarrow$ $^{136}$Ba decay system, the calculated decay rates may be influenced by a semi-magic structure of the initial nucleus.

Table 3. A comparison of the calculated $0^+_{\rm gs} \rightarrow 2^+_1$ half-lives with other calculations and experimental data. Half-lives are given in the unit of year.

| Emitter | Present | [21] | [22] | Exp. [4, 23] |
|---|---|---|---|---|
| $^{128}$Te$\rightarrow$ $^{128}$Xe | $5.8 \times 10^{26}$ | $4.7 \times 10^{33}$ | $1.6 \times 10^{30}$ | $> 4.7 \times 10^{21}$ |
| $^{130}$Te$\rightarrow$ $^{130}$Xe | $3.0 \times 10^{26}$ | $6.9 \times 10^{26}$ | $2.7 \times 10^{23}$ | $> 4.5 \times 10^{21}$ |
| $^{134}$Xe$\rightarrow$ $^{134}$Ba | $3.8 \times 10^{34}$ | $5.3 \times 10^{35}$ | — | — |
| $^{136}$Xe$\rightarrow$ $^{136}$Ba | $1.4 \times 10^{26}$ | $3.9 \times 10^{26}$ | $2.0 \times 10^{24}$ | $> 4.6 \times 10^{23}$ |

Table 4. A comparison of the calculated $0^+_{\rm gs} \rightarrow 2^+_2$ half-lives with other calculations and experimental data. Half-lives are given in the unit of year.

| Emitter | Present | [22] | Exp. [23] |
|---|---|---|---|
| $^{130}$Te$\rightarrow$ $^{130}$Xe | $8.5 \times 10^{26}$ | $1.0 \times 10^{26}$ | — |
| $^{136}$Xe$\rightarrow$ $^{136}$Ba | $4.6 \times 10^{28}$ | $5.1 \times 10^{26}$ | $> 9.0 \times 10^{23}$ |

## 4. Conclusion

A theoretical analysis of $2\nu\beta^-\beta^-$ decay to the final excited $2^+$ states is presented by considering SU(4) symmetry violations within the framework of the QRPA method. In this respect, we have previously determined the suitable values of the effective interaction strength parameters for the final excited states. Then, the spin–isospin excitations in intermediate nuclei are obtained within the restoration of SU(4) symmetry violations. The present $pn$QRPA calculations are free of the particle–hole strength parameter and the particle–particle strength parameter has been determined from the agreement of the calculated amplitude with the experimental one for $2\nu\beta^-\beta^-$ decay to the final ground state. After obtaining the charge-conserving and charge-exchange strength parameters, the nuclear matrix elements and half-lives for $2\nu\beta^-\beta^-$ decay to the final excited states are computed by using these fixed parameters. The main contribution to the nuclear matrix elements comes from the first intermediate state. Especially, the present results for the $^{128}\mathrm{Te} \to {}^{128}\mathrm{Xe}$ decay system may ensure a good motivation for the experimental research in the future. It is well known that the theoretical description of nuclear double-beta decay is still important in the field of nuclear structure theories, and the determination of the nuclear matrix elements provides a good opportunity to test the nuclear model used. Therefore, the present calculations within the restoration of symmetry violations may make a significant contribution to the developments in this area.

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