

HOSOTANI MECHANISM  
IN THE “COLOR”-SINGLET PLASMA\*

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By projecting the partition function on the “color”-singlet state, we investigate the Hosotani mechanism in the fermion-gauge boson plasma. The present toy-model analysis of the one-loop effective potential at finite temperature shows that the critical temperature of gauge symmetry breaking increases at higher temperature in the smaller volume.

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**1. Introduction**

Phase transition phenomena in finite temperature systems are found in various aspects of physics research. In the unified theory of particle physics, the analysis of finite-temperature field theory suggests [1] that the broken symmetry by the Brout–Englert–Higgs (BEH) mechanism was restored in the hot early universe. In other words, it can be said that the division of the present fundamental interactions is the result of the phase transition in the early universe.

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The theory of the fundamental scalar field, which is indispensable for the BEH mechanism, is unnatural in the sense of suffering from huge quantum corrections. Accordingly, the gauge-Higgs unified model has been proposed as a candidate giving the solutions to the naturalness [2–8]. In the model, the extra-dimensional component of the gauge field plays the role of the Higgs field. Thus, there is no potential term that corresponds to the Higgs potential at the tree level in the theory; this is due to the gauge symmetry, which guarantees naturalness of the theory. Therefore, in the gauge-Higgs unified model, it is expected that the quantum effect at one (or more) loop order will bring about a non-trivial vacuum gauge field. This mechanism is dubbed as the Hosotani mechanism (or Wilson loop mechanism) [9]. It is known that fermionic matter fields with certain suitable representations are needed for such symmetry breaking at zero temperature. The characteristics of the Hosotani mechanism at finite temperature have already been studied [10–12]. It has been pointed out that phase transitions at high temperature always occur [11] (and the transition is first order for matter fields with the periodic boundary condition in the extra dimension) in the Hosotani mechanism.

There is another type of phase transition in the non-Abelian gauge theory. It is the quark–hadron phase transition, which is known as hadronization from quark–gluon plasma (QGP) [13]. Although the similar confinement is not assumed in the gauge-Higgs scenario, non-perturbative effects are expected in the early phase of the hot universe<sup>1</sup>. It is also well known that the AdS/CFT method illustrates the reduction of the degrees of freedom in QGP in the non-perturbative region [18] in a certain non-Abelian gauge theory.

Being motivated by the reduction in the number of degrees of freedom in QGP, we revisit the general behavior of the Hosotani mechanism at finite temperature through a toy model in this paper. We adopt the color-singlet hypothesis in the present analysis<sup>2</sup>. This is a hypothesis that can be considered in the hadron phase transition from quark–gluon plasma, and it is a hypothesis that QGP whitens globally owing to the nature of the strong interaction. There are some debates as to whether this holds true in the analysis of actual QGPs, *etc.* However, since it is followed by a well-defined mathematical operation that expresses the reduction of degrees of freedom using the group integration technique, it is considered to be a powerful analytical method that can approximate the aimed aspect of the effective behavior at least “phenomenologically”.

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<sup>1</sup> In a different context, the idea of vacuum selection at finite temperature in supersymmetric grand unified theories, which is attributed to the difference in degree of freedom of particles, is discussed by the authors of the papers on “supercosmology” [14–17].

<sup>2</sup> The projection onto the singlet sector is also utilized in recent studies on large- $N$  Yang–Mills theories at finite temperature [19, 20].

In the following sections of this paper, we use a toy model with SU(2) symmetry (instead of a large group of unified models) to find a one-loop effective potential at finite temperature under the hypothesis of the color singlet.

This paper is organized as follows. In Sec. 2, we review the color-singlet hypothesis and associated technique in analytical formulations. In Sec. 3, we define the effective potential for the toy model of SU(2) gauge theory in the background space  $R^{D-2} \otimes S^1$ . Numerical calculations of the effective potential in five dimensions ( $D = 5$ ) are shown in Sec. 4 and the possible phase transition is studied. The discussion is presented in the last section. In Appendix, an interesting and useful analysis of the effective potential with approximations is examined.

## 2. The projected partition function for the color-singlet states

In this section, we review the argument of the global color symmetry in QGP under the color-singlet hypothesis. Using the technique shown in the present section, we will obtain the effective potential in a toy model for the Hosotani mechanism in the “color”-neutral plasma in the next section.

The QCD is known as a theory describing the strong force and causes the confinement of quarks and gluons. The transition from hadronic matter to quark–gluon plasma is considered to be a transition from local color confinement to global color confinement at finite temperature [13]. We should consider the restricted partition function of the color-singlet state to realize the global color symmetry [13, 21–27]. To this end, first, we define a generalized partition function which includes the generators of the Cartan subalgebra  $\hat{C}_\alpha$  in the gauge group as follows:

$$Z_C(\psi_\alpha) = \text{Tr} \left[ e^{-\beta \hat{H} + i\psi_\alpha \hat{C}_\alpha} \right], \quad (2.1)$$

where, as usual,  $\beta$  is the inverse of the temperature  $T$  and  $\hat{H}$  denotes the Hamiltonian.

Next, we introduce the characteristic functions. The function of the parameter  $\psi_\alpha$  specified by the representation  $j$  of the group (SU(3) for QCD) is called the characteristic function  $\chi_j(\psi_\alpha)$ . The characteristic functions satisfy

$$\int d\mu(\psi_\alpha) \chi_j^*(\psi_\alpha) \chi_{j'}(\psi_\alpha) = \delta_{jj'}, \quad (2.2)$$

where  $d\mu(\psi_\alpha)$  is the invariant measure of the group. For SU(2), *i.e.*, the two-color case, it is known that

$$d\mu(\psi) = \frac{\sin^2 \frac{\psi}{2}}{2\pi} d\psi \quad (-2\pi \leq \psi < 2\pi), \quad (2.3)$$

where  $\psi$  is the single variable for the center of SU(2).

We assume that the generalized partition function can be expanded by the characteristic functions and the characteristic function for the singlet is known to be unity. Thus, the restricted partition function  $Z$  is finally obtained by projection onto the color singlet as

$$Z = \int d\mu(\psi_\alpha) Z_C(\psi_\alpha). \quad (2.4)$$

Here, we notice that Jacobi's imaginary transformation [28]

$$\frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} e^{\frac{-\beta^2 n^2}{4t}} e^{i2\pi n y} = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} e^{-\left[\frac{2\pi}{\beta}(n+y)\right]^2 t} \quad (2.5)$$

and from the formula in Ref. [1], we find that the equalities

$$\begin{aligned} & \sum_n \ln \left[ \frac{4\pi^2(n+y)^2}{\beta^2} + \omega^2 \right] \\ &= - \int_0^\infty \frac{dt}{t} \sum_n \exp \left[ - \left( \frac{4\pi^2(n+y)^2}{\beta^2} + \omega^2 \right) t \right] \\ &= - \frac{\beta}{\sqrt{4\pi}} \int_0^\infty \frac{dt}{t^{3/2}} \sum_n e^{i2\pi n y} \exp \left[ -\omega^2 t - \frac{\beta^2 n^2}{4t} \right] \\ &= - \frac{\beta\omega\Gamma(-\frac{1}{2})}{\sqrt{4\pi}} - \frac{2\beta\omega}{\sqrt{4\pi}} \sum_{n=1}^{\infty} 2e^{i2\pi n y} \left( \frac{2}{n\beta\omega} \right)^{1/2} K_{1/2}(n\beta\omega) \\ &= \beta\omega - \sum_{n=1}^{\infty} \frac{2}{n} e^{-n\beta\omega} e^{i2\pi n y} \\ &= \beta\omega + 2 \ln \left( 1 - e^{-\beta\omega + i2\pi y} \right) \end{aligned} \quad (2.6)$$

hold, up to the terms independent of an arbitrary constant  $\omega$ . Therefore, we can express  $\ln Z_C(\psi_\alpha)$  as

$$\ln Z_C(\psi_\alpha) = -\frac{1}{2} \text{Tr} \sum_n \ln \left[ \frac{\left( 2\pi n + \psi_\alpha \hat{C}_\alpha \right)^2}{\beta^2} + \omega^2 \right] \quad (2.7)$$

for bosonic fields, and

$$\ln Z_C(\psi_\alpha) = \frac{1}{2} \text{Tr} \sum_n \ln \left[ \frac{\left[ 2\pi \left( n + \frac{1}{2} \right) + \psi_\alpha \hat{C}_\alpha \right]^2}{\beta^2} + \omega^2 \right] \quad (2.8)$$

for fermionic fields, where each trace indicates the sum over possible energy eigenvalue,  $\omega$ , and all degrees of freedom. Precisely speaking, the expressions for the generalized partition function here involve the contribution of the vacuum energy. Incidentally, the expression is convenient for evaluation of the effective potential for the Hosotani mechanism.

### 3. The SU(2) toy model

Here, we consider the Hosotani mechanism at finite temperature in the SU(2) gauge theory with massless fermions in the adjoint representation. We consider  $D$ -dimensional spacetime and assume that the topology of space is  $R^{D-2} \otimes S^1$  and the circumference of the compact dimension is set to  $L$ . All the fields obey the periodic boundary condition with respect to the compact dimension<sup>3</sup>.

Although the several conditions towards asymptotic freedom *etc.* may be a necessary condition for strong non-perturbative effects implicitly assumed, we will temporarily ignore the conditions in the present toy model.

The vacuum expectation value of the extra-dimensional component of the SU(2) gauge field  $\mathbf{A}_y$  is now parametrized as

$$gL\langle\mathbf{A}_y\rangle = \frac{\theta}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (3.1)$$

where  $g$  is the SU(2) gauge coupling. Note that the trivial vacuum is associated with  $\theta = 0$ , where the gauge-invariant Wilson loop over  $S^1$  becomes  $\exp igL\langle\mathbf{A}_y\rangle = I$ , where  $I$  is the identity matrix. The residual large gauge symmetry tells the identification  $\theta \sim \theta + 4\pi$ . Moreover, when  $\theta = 2\pi$ , it gives  $\exp igL\langle\mathbf{A}_y\rangle = -I$ ; then the SU(2) symmetry is unbroken. This is due to the  $Z_2$  symmetry in SU(2) and, consequently, we expect that the partition function is periodic in  $\theta$  with a period  $2\pi$ .

The restricted partition function after projections is written as [13, 21–27]

$$Z(\theta) = \int_{-2\pi}^{2\pi} \frac{d\psi}{2\pi} \sin^2 \frac{\psi}{2} Z_G(\theta, \psi) [Z_a(\theta, \psi)]^{N_a}, \quad (3.2)$$

where  $N_a$  is the number of the adjoint fermion fields. In this expression, the logarithm of the generalized partition function  $Z_G$  for the gauge bosons and the ghosts is formally given by

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<sup>3</sup> Although various bizarre boundary conditions are investigated by many papers including Ref. [29], we take the simplest condition in the present paper.

$$\ln Z_G(\theta, \psi) = -\frac{D-2}{2}V \sum_{A=1}^3 \sum_n \sum_k \int \frac{d^{D-2}\mathbf{p}}{(2\pi)^{D-2}} \times \ln \left[ \frac{(2\pi n + C_A \psi)^2}{\beta^2} + \mathbf{p}^2 + M_A^2(k) \right], \quad (3.3)$$

where

$$C_1 = +1, \quad C_2 = -1, \quad C_3 = 0, \quad M_A^2 = \left( \frac{2\pi k + C_A \theta}{L} \right)^2, \quad (3.4)$$

and the generalized partition function for the adjoint fermions are written by

$$\ln Z_a(\theta, \psi) = \frac{2^{[D/2]}}{4}V \sum_{A=1}^3 \sum_n \sum_k \int \frac{d^{D-2}\mathbf{p}}{(2\pi)^{D-2}} \times \ln \left[ \frac{[2\pi(n + \frac{1}{2}) + C_A \psi]^2}{\beta^2} + \mathbf{p}^2 + M_A^2(k) \right], \quad (3.5)$$

where  $[D/2]$  is the Gauss' symbol such that  $[4/2] = [5/2] = 2$ .

In these expressions, the constant  $V$  denotes the  $(D-2)$ -dimensional volume of the hot-plasma system. The definition of the volume of the system is a crucial problem. We can consider the small objects, such as false vacuum bubbles, or the cores of the exotic stars, or the fermion droplets in the universe.

We should recall that the above expressions include vacuum contributions and should be regularized by getting rid of divergences, which are irrelevant to physical quantities. With the aid of Jacobi's imaginary transformation [28]

$$\sum_k \exp \left[ - \left( \frac{2\pi k + \Theta}{L} \right)^2 t \right] = \frac{L}{\sqrt{4\pi t}} \sum_k \exp \left[ - \frac{L^2 k^2}{4t} \right] e^{-ik\Theta}, \quad (3.6)$$

we find that the similar manipulation as in Sec. 2 leads to

$$\ln Z_G(\theta, \psi) = \frac{(D-2)\Gamma(D/2)}{2\pi^{D/2}} \beta L V \sum'_{n,k} \frac{1 + 2 \cos[k\theta + n\psi]}{[\beta^2 n^2 + L^2 k^2]^{D/2}}, \quad (3.7)$$

$$\ln Z_a(\theta, \psi) = \frac{2^{[D/2]}\Gamma(D/2)}{2\pi^{D/2}} \beta L V \sum'_{n,k} \frac{(-1)^{n-1} \{1 + 2 \cos[k\theta + n\psi]\}}{[\beta^2 n^2 + L^2 k^2]^{D/2}}, \quad (3.8)$$

where the primes on sums indicate the omission of  $n = k = 0$  in the summations. By this omission, the expressions have become finite and the regularization has been accomplished.

Each partition function can be divided into two parts, say,

$$\begin{aligned}\ln Z_G(\theta, \psi) &= \ln Z_{G0}(\theta) + \ln Z_{GT}(\theta, \psi), \\ \ln Z_a(\theta, \psi) &= \ln Z_{a0}(\theta) + \ln Z_{aT}(\theta, \psi).\end{aligned}\quad (3.9)$$

Here, contributions of vacuum fluctuations are given, with the use of the Riemann’s zeta function  $\zeta_R(z)$  and the  $D^{\text{th}}$  polylogarithm function  $\text{Li}_D(z)$ , by

$$\begin{aligned}\ln Z_{G0}(\theta) &= \frac{(D-2)\Gamma(D/2)}{\pi^{D/2}L^D}\beta LV \sum_{k=1}^{\infty} \frac{1+2\cos k\theta}{k^D} \\ &= \frac{(D-2)\Gamma(D/2)}{\pi^{D/2}L^D}\beta LV \left[ \zeta_R(D) + \text{Li}_D(e^{ik\theta}) + \text{Li}_D(e^{-ik\theta}) \right],\end{aligned}\quad (3.10)$$

$$\begin{aligned}\ln Z_{a0}(\theta) &= -\frac{2^{[D/2]}\Gamma(D/2)}{\pi^{D/2}L^D}\beta LV \sum_{k=1}^{\infty} \frac{1+2\cos k\theta}{k^D} \\ &= -\frac{2^{[D/2]}\Gamma(D/2)}{\pi^{D/2}L^D}\beta LV \left[ \zeta_R(D) + \text{Li}_D(e^{ik\theta}) + \text{Li}_D(e^{-ik\theta}) \right],\end{aligned}\quad (3.11)$$

while the finite-temperature parts can be written by

$$\begin{aligned}\ln Z_{GT}(\theta, \psi) &= (D-2)\beta LV [w_b(0, 0) + 2w_b(\theta, \psi)], \\ \ln Z_{aT}(\theta, \psi) &= 2^{[D/2]}\beta LV [w_f(0, 0) + 2w_f(\theta, \psi)],\end{aligned}\quad (3.12)$$

where

$$w_b(\theta, \psi) \equiv \frac{\Gamma(D/2)}{\pi^{D/2}} \sum_{n=1}^{\infty} \sum_{k=-\infty}^{\infty} \frac{\cos k\theta \cos n\psi}{[\beta^2 n^2 + L^2 k^2]^{D/2}},\quad (3.13)$$

and

$$w_f(\theta, \psi) \equiv \frac{\Gamma(D/2)}{\pi^{D/2}} \sum_{n=1}^{\infty} (-1)^{n-1} \sum_{k=-\infty}^{\infty} \frac{\cos k\theta \cos n\psi}{[\beta^2 n^2 + L^2 k^2]^{D/2}}.\quad (3.14)$$

Incidentally,  $w_b$  and  $w_f$  coincide with the effective potential for the SU(2) gauge theory with torus ( $T^2$ ) compactification with appropriate boundary conditions [30]. In our case, however, the parameter  $\psi$  is to be integrated for the “color”-singlet projection.

The partition function restricted to SU(2) “color”-singlet states for the present model is then given by

$$Z(\theta) = Z_{G0}(\theta)[Z_{a0}(\theta)]^{N_a} \times \frac{1}{2\pi} \int_{-2\pi}^{2\pi} d\psi \left[ \sin^2 \frac{\psi}{2} \right] Z_{GT}(\theta, \psi)[Z_{aT}(\theta, \psi)]^{N_a}.\quad (3.15)$$

Now, the effective potential  $\mathcal{V}(\theta)$  is defined by

$$\mathcal{V}(\theta) \equiv -\frac{1}{\beta V L} \ln Z(\theta). \quad (3.16)$$

One can find that the effective potential at zero temperature becomes

$$\mathcal{V}_0(\theta) \equiv -\frac{1}{\beta V L} [\ln Z_{G0}(\theta) + N_a \ln Z_{a0}(\theta)], \quad (3.17)$$

where  $\ln Z_{G0}$  and  $\ln Z_{a0}$  have been given by (3.10) and (3.11), which is the well-known effective potential in the Hosotani mechanism.

We should also notice that the limit of the infinite volume gives

$$\mathcal{V}(\theta) \rightarrow \mathcal{V}_0(\theta) - \frac{1}{\beta V L} [\ln Z_{GT}(\theta, 0) + N_a \ln Z_{aT}(\theta, 0)], \quad (3.18)$$

because for a large volume,  $\psi = 0$  is a stationary point in the integrand of (3.15).

If  $\theta = 0$  gives a global minimum of  $\mathcal{V}(\theta)$ , the gauge symmetry is unbroken, while if the global minimum is located at  $\theta \neq 0 \pmod{2\pi}$ , the SU(2) symmetry is reduced to be U(1).

#### 4. Numerical calculations for $D = 5$

Now, we shall evaluate  $\mathcal{V}(\theta)$  in this model. To handle the numerical function, we use the following integral expression:

$$w_b(\theta, \psi) = \frac{1}{2\pi^{D/2}} \int_0^\infty dt t^{D/2-1} \vartheta_3\left(\frac{\theta}{2\pi}, i\frac{L^2 t}{\pi}\right) \left[ \vartheta_3\left(\frac{\theta}{2\pi}, i\frac{\beta^2 t}{\pi}\right) - 1 \right], \quad (4.1)$$

and

$$w_f(\theta, \psi) = -\frac{1}{2\pi^{D/2}} \int_0^\infty dt t^{D/2-1} \vartheta_3\left(\frac{\theta}{2\pi}, i\frac{L^2 t}{\pi}\right) \left[ \vartheta_4\left(\frac{\theta}{2\pi}, i\frac{\beta^2 t}{\pi}\right) - 1 \right], \quad (4.2)$$

where  $\vartheta_n(v, \tau)$  is Jacobi's theta function [28].

Hereafter, we shall concentrate on the case of  $D = 5$ . We show typical shapes of functions  $w_b$  and  $w_f$  in Fig. 1<sup>4</sup>. One can find that both functions  $w_b$  and  $w_f$  have a period  $2\pi$  for both variables  $\theta$  and  $\psi$ , the maximum of which is at 0 for each variable and the minimum at  $\pi$ . Therefore, the extrema of the effective potential  $\mathcal{V}$  should be found at  $\theta = 0$  and  $\pi$  (modulo  $2\pi$ ).

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<sup>4</sup> We used the command `FunctionInterpolation` in Mathematica [31].



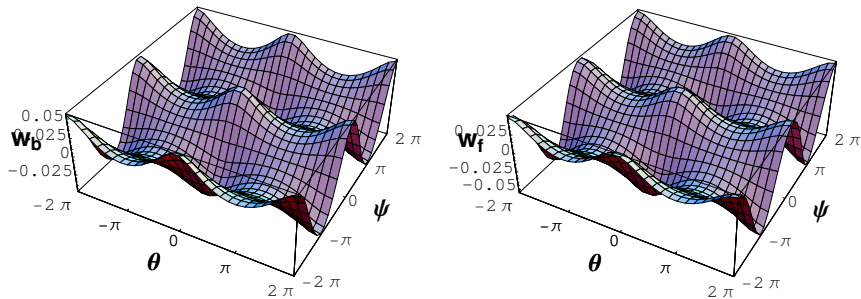


Fig. 1. The functions for  $\beta = 1.2$  and  $L = 1$ : (left)  $w_b$  and (right)  $w_f$ .

At the critical temperature, the values of  $\mathcal{V}(0)$  and  $\mathcal{V}(\pi)$  are equal in the present model<sup>5</sup>. In Fig. 2, we show the critical lines in the parameter space spanned by  $r/L$  and  $\beta/L$ , where,  $r$  is the radius of a ball with volume  $V$  such that  $V = \frac{4\pi}{3}r^3$ <sup>6</sup>. In the region above the line, the gauge symmetry is broken. One can find that the SU(2) symmetry is broken for any value of  $r$  for a sufficiently high temperature. For a smaller  $r$ , the critical temperature becomes higher; the symmetry restoration is suppressed by the color-singletness. The suppression is larger for a larger number of fermions, which are introduced to break the symmetry, though it works in a much smaller volume.

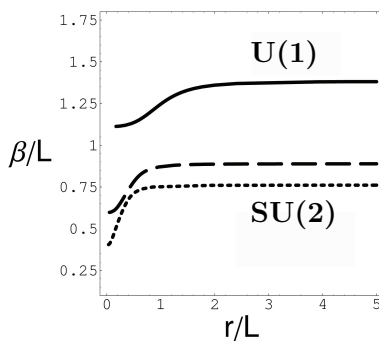


Fig. 2. Phase structure of the model. The solid line indicates the boundary of the two phases for  $N_a = 1$ , the broken line indicates that for  $N_a = 2$ , and the dotted line indicates that for  $N_a = 3$ .

The typical change in shape of the potential is exhibited in Fig. 3, which shows the minima of the effective potential in the present model can appear

<sup>5</sup> This is not the case if the matter field with a weird boundary condition exists in the model.

<sup>6</sup> As it is well known,  $V = \frac{\pi^{D/2-1}}{\Gamma(\frac{D}{2})} r^{D-2}$  in the case of  $D$ -dimensional spacetime.

at  $\theta = 0$  or  $\theta = \pi$  (modulo  $2\pi$ ).

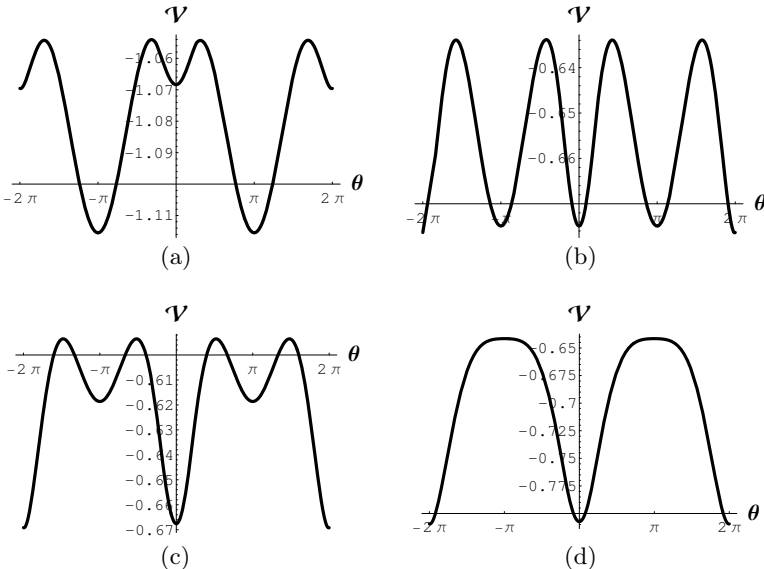


Fig. 3. The effective potential  $\mathcal{V}$  for  $N_a = 1$ ,  $\beta = 1.2$  and  $L = 1$ : (a)  $r = 0.5$ , (b)  $r = 0.814$ , (c)  $r = 1$ , (d)  $r = \infty$ .

## 5. Discussion

In this paper, the effective potential at finite temperature was obtained for the SU(2) toy model based on the color-singlet hypothesis. For a smaller volume, the critical temperature of the SU(2)–U(1) phase transition becomes higher.

A future task to be considered is the analysis of a more realistic general gauge-Higgs unified model with a larger symmetry group at finite temperature. The symmetry breaking in the SU(3) gauge theory with various fermions has been studied by lattice calculations [32] and it is reported that the SU(3)-confined phase exists at the strong coupling regime. We should continue to pursue the strong coupling effect and the finite size effect with various analytical and numerical methods in order to deepen our understanding of the Hosotani mechanism at finite temperature. We should also consider general gauge theories in higher dimensional extra space, orbifold, and warped spacetimes.

As a natural extension of the present analysis, we come to the idea that no fermion number condition on the matter field in the model should be taken into account. In addition, the Kaluza–Klein charge (originated from the momentum in the extra dimension) is also a conserved quantity [33–37]. Thus, we can also assume the no net Kaluza–Klein charge in the

closed system. For the case of the Hosotani mechanism, however, there appears a problematic issue that in the present model, the projection onto a state of a definite Kaluza–Klein charge breaks the residual large gauge symmetry, such as  $\theta \sim \theta + 4\pi$  (or  $\theta \sim \theta + 2\pi$ , due to  $Z_2$ ). The treatment of these conserved charges will be challenged in the future with more elaborate investigations.

## Appendix A

### *An approximation scheme by finite sums*

The defined functions  $w_b$  and  $w_f$  are expressed in the summation forms (3.13) and (3.14). We try to approximate these by finite sums. We define the following function:

$$\begin{aligned} w_{\text{app}}(\theta, \psi) &\equiv \frac{\Gamma(D/2)}{\pi^{D/2}} \sum_{n=1}^1 \sum_{k=-2}^2 \frac{\cos k\theta \cos n\psi}{[\beta^2 n^2 + L^2 k^2]^{D/2}} \\ &= \frac{\Gamma(D/2)}{\pi^{D/2}} \cos \psi \left[ \frac{1}{\beta^D} + 2 \frac{\cos \theta}{(\beta^2 + L^2)^{D/2}} + 2 \frac{\cos 2\theta}{(\beta^2 + 4L^2)^{D/2}} \right]. \end{aligned} \quad (\text{A.1})$$

As an approximation, we replace both  $w_b$  and  $w_f$  with  $w_{\text{app}}$ . This approximation is justified for a large  $D$  and also for  $\beta/L \ll 1$ . Figure 4 shows the function  $w_{\text{app}}$  for  $D = 5$ ,  $\beta = 1.2$ , and  $L = 1$ . The approximation looks fine, at least for the values of parameters around this assumption. Owing to the simplification, we can use the following formula to evaluate the projection integral:

$$\frac{1}{2\pi} \int_{-2\pi}^{2\pi} \sin^2 \frac{\psi}{2} e^{z \cos \psi} d\psi = I_0(z) - I_1(z), \quad (\text{A.2})$$

where  $I_n(z)$  is the modified Bessel function of the first kind.

According to this approximation, the sum included in the vacuum contribution in  $Z$  is also approximated by a finite sum

$$\sum_{k=1}^{\infty} \frac{\cos k\theta}{k^D} \approx \cos \theta + \frac{\cos 2\theta}{2^D}. \quad (\text{A.3})$$

Figure 5 shows the reposted phase diagram (Fig. 2) and the gray lines obtained from the above approximation in addition.

For a large volume, the integral over  $\psi$  are dominant in the region around the stationary points  $\psi \approx 0$ . Therefore, the approximation of taking  $n = 1$  only is good for cases with large volumes. One can see that the bending location on the lines are well approximated.

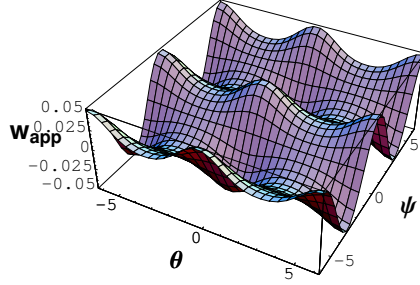


Fig. 4. The function  $w_{\text{app}}$  for  $\beta = 1.2$  and  $L = 1$ .

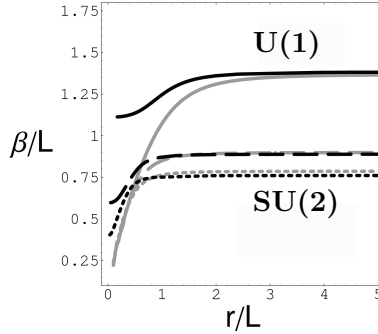


Fig. 5. Phase structure of the model: The lines in Fig. 2 are reposted here and corresponding gray lines are drawn according to the present approximation scheme.

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