

RELATIVISTIC VARIABLES FOR COVARIANT,
RETARDLESS WAVE EQUATIONS*

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published online 21 June 2022*

A reduced form of the Dirac equation has been previously introduced and studied in the center-of-mass reference frame. In this work, we show that this equation can be written in a covariant form in a generic reference frame by using specific momentum variables. These variables are also consistent with the retardless form of the interaction of the model.

DOI:10.5506/APhysPolB.53.7-A2

1. Introduction

In a previous work [1], we developed a local reduction of many-body relativistic equations (more precisely, the Dirac-like equation (DLE) and the Mandelzweig–Wallace equation (MW) [2]) for studying the spectroscopy of quark composed systems. An accurate calculation of the charmonium spectrum was performed using a small number of free parameters [3]. In that work, a specific form of the regularized vector interaction was used [4].

In general, for the theoretical formulation of the model, we used the center-of-mass reference frame (CMRF) where the hadronic bound system is at rest.

This choice is perfectly legitimate in the sense that the internal dynamics of the bound system can be studied completely in that frame.

However, in order to understand in more detail the relativistic character of the model, it is useful to develop its covariant version in a generic reference frame (GRF). This covariant version of the relativistic equation can be also used to study the scattering processes of the hadronic systems.

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The derivation of the wave equation in covariant form, for a two-body hadronic system, represents the main objective of the present work. In this context, we show that the retardless and local character of the interaction used in Ref. [1] is fully compatible with the relativistic covariance properties of the model.

The technique used in this work to obtain the covariant form of the relativistic equation is not completely new: similar procedures can be found, for example, in Refs. [5–7]. Here, we highlight the specific role of the relativistic variables assuming that the total energy of each particle represents the time component of its four-momentum. As a consequence, the particles are not on-shell. Furthermore, the particle energy is considered a “fixed” quantity, determined in the CMRF by the internal dynamics of the bound system. Due to this choice, the time component of the momentum transfer in the CMRF is vanishing giving a retardless interaction operator.

In classical words, each particle of the *bound state* produces a static field with which the other particle interacts.

For clarity, we point out that the model assumptions introduced above significantly differ from the relativistic scheme in which one particle is considered on-shell, as, for example, in the *relativistic spectator formalism* developed in Refs. [8–11].

Our choice of the relativistic variables and the procedure for obtaining a two-body covariant relativistic equation are applied to different cases of CMRF relativistic equations. Finally, we put in a covariant form also the reduction operators introduced in [1]. In this way, the covariant form of the *correlated Dirac wave functions* is determined. In this regard, we note that our choice of relativistic variables is fully consistent with the definition of the reduction operators.

For a thorough description of the CMRF relativistic model and for a comparison with other relativistic equations, the reader is referred to Ref. [1]. In the present work, we focus our attention on its “covariantization” by means of suitable variables.

The remainder of the paper is organized as follows. In Subsection 1.1, we introduce the symbols and notations used in the work. In Section 2, we discuss the relativistic variables of the model and analyze their Lorentz transformations. In particular, in Subsection 2.1, we introduce the basic variables; in Subsection 2.2, we study the transformations of the four-vectors and of the Dirac wave functions of the model; in Subsection 2.3, the momentum variables for a GRF are determined. In Section 3, the covariant form of the wave equation is obtained focusing the attention on the Dirac-like case; the covariant generalization of other forms of the relativistic equation is analyzed in Subsection 3.1. The covariant expression of the correlated Dirac wave functions is studied in Section 4. Finally, in Appendix A, we briefly discuss the method to fix the value of the particle energy.

1.1. Symbols and notation

In this work, the quantities defined as four-vectors will be denoted as in the following example: $V = (V^0, \vec{V})$. The Lorentz indices will be written only when strictly necessary and in the invariant products; for example, $V^\mu U_\mu = V^0 U^0 - \vec{V} \cdot \vec{U}$.

The superscript c , not used in [1], denotes here a four-vector (and also a wave function) related to the CMRF. No specific symbol is used for the same quantity in a GRF.

The lower index $i(j) = 1, 2$ represents the *particle index*. The particle index is never summed in this work.

For the covariant form of the Dirac equation, we use the gamma matrices of the i^{th} particle γ_i^μ in the standard representation. For the Hamiltonian form of the Dirac operators, we also introduce $\beta_i = \gamma_i^0$ and the matrices $\gamma_i^0 \gamma_i^\mu = (\mathcal{I}_i, \vec{\alpha}_i)$. The symbol $\mathcal{I}_i$, that represents the identity 4×4 matrix for the i^{th} particle, will be omitted when not strictly necessary; for example, $V_i^0 \mathcal{I}_i - \vec{V} \cdot \vec{\alpha}_i$ will be written as $V_i^0 - \vec{V} \cdot \vec{\alpha}_i$. In the same way, $V_\mu \gamma_i^\mu + b \mathcal{I}_i$ will be written as $V_\mu \gamma_i^\mu + b$.

The Dirac wave functions will be represented by the letter Ψ ; the spinorial wave functions by the letter Φ .

As customary, throughout the work, we use the so-called natural units, that is $\hbar = c = 1$.

2. The variables of the covariant model. Lorentz and Dirac Boost transformations

2.1. The basic variables of the model

We consider a hadronic *bound system*, composed of two spin 1/2 particles (a quark and an antiquark) with masses m_1 and m_2 .

We assume that in the CMRF, the four-momenta of the two particles are

$$\begin{aligned} p_1^c &= (E_1^c, -\vec{p}^c), \\ p_2^c &= (E_2^c, +\vec{p}^c), \end{aligned} \quad (1)$$

where E_1^c and E_2^c represent the “fixed”, *constant* energy values of the two particles. This point is studied in more detail in Appendix A, where in Eq. (A.1), a standard auxiliary prescription is recalled to determine their values, shown in Eq. (A.2). The sign of the internal three-momentum $\vec{p}^c$ is defined as in Refs. [1, 3].

Note that the total four-momentum eigenvalue in the CMRF is

$$P^c = p_1^c + p_2^c = (M, \vec{0}), \quad (2)$$

where M represents the mass of the bound system.

For the relevant case of two equal mass particles, as given in Eq. (A.3) of Appendix A, we have

$$E_1^c = E_2^c = \frac{M}{2}. \quad (3)$$

Also, in a GRF, the bound system is an eigenstate of total four-momentum with eigenvalue

$$P = (E, \vec{P}), \quad (4)$$

where $\vec{P}$ is the total three-momentum of the bound system and the total energy E has the standard on-shell expression

$$E = \sqrt{M^2 + \vec{P}^2}. \quad (5)$$

In this work, for simplicity, we shall not write explicitly the total momentum eigenfunction that will not be used in the calculations.

2.2. Lorentz and Dirac Boost transformations

The content of this subsection is completely standard. Here, it is reported and applied to our model in order to improve the self-consistency of the paper.

Using the definitions of Eqs. (4), (5) for the total four-momentum of the system and recalling that the speed of the bound system in a GRF is $\vec{\beta} = \vec{P}/E$, we can write the Lorentz transformations of any CMRF four-vector V^c to the same four-vector V in a GRF. These transformations have the following standard form:

$$\begin{aligned} V^0 &= \frac{1}{M} (EV^{c0} + \vec{P}\vec{V}^c), \\ \vec{V} &= \vec{V}^c + \frac{\vec{P}}{M} \left(\frac{\vec{P}\vec{V}^c}{E + M} + V^{c0} \right). \end{aligned} \quad (6)$$

The inverse Lorentz transformations are

$$\begin{aligned} V^{c0} &= \frac{1}{M} (EV^0 - \vec{P}\vec{V}), \\ \vec{V}^c &= \vec{V} + \frac{\vec{P}}{M} \left(\frac{\vec{P}\vec{V}}{E + M} - V^0 \right). \end{aligned} \quad (7)$$

Some words of comment about the first expression of Eq. (7): the time component of a four-vector “seen” in the CMRF is a Lorentz-invariant quantity.

In consequence, we can rewrite that expression in explicitly invariant form, that is,

$$V^{c0} = \frac{P_\mu V^\mu}{M}. \quad (8)$$

The Dirac Boost operator, for the i^{th} quark, has the following standard form:

$$B_i = F_B \left[(E + M) + \vec{\alpha}_i \cdot \vec{P} \right]; \quad (9)$$

the inverse operator is

$$B_i^{-1} = F_B \left[(E + M) - \vec{\alpha}_i \cdot \vec{P} \right] \quad (10)$$

with

$$F_B = [2M(E + M)]^{-1/2}. \quad (11)$$

We recall that the B_i are used to transform CMRF Dirac wave function Ψ^c to a GRF Dirac wave function Ψ ; furthermore, $\bar{\Psi}^c$ is transformed with the inverse Boost operators, that is,

$$\begin{aligned} B_1 B_2 \Psi^c &= \Psi, \\ \bar{\Psi}^c B_1^{-1} B_2^{-1} &= \bar{\Psi}. \end{aligned} \quad (12)$$

Due to the properties of the Dirac Boost, one has

$$B_i V_\mu^c \gamma_i^\mu B_i^{-1} = V_\mu \gamma_i^\mu, \quad (13)$$

where, for a given four-vector V^c , V is given by Eq. (6). In particular, taking the unit four-vector $u^c = (1, \vec{0})$ and, consequently, $u = P/M$, one has

$$B_i \gamma_i^0 B_i^{-1} = \frac{P_\mu \gamma_i^\mu}{M}. \quad (14)$$

2.3. The momentum variables in a GRF

We can now construct the momentum variables in a GRF. Starting from Eq. (1) with the transformations of Eq. (6), one can obtain the four-momenta of the two particles p_1 , p_2 in a GRF.

However, in order to write the wave equation, it is necessary to introduce, in a GRF the total and the relative four-momenta denoted as P and p , respectively.

The total four-momentum is simply related to the particle momenta by the standard expression

$$P = p_1 + p_2 \quad (15)$$

consistently with the CMRF definition of Eq. (2), with Eq. (4), and with the Lorentz transformations of Eq. (6).

The relative four-momentum can be defined as in Ref. [12]

$$p = -\eta_2 p_1 + \eta_1 p_2. \quad (16)$$

The *constants* η_1 and η_2 must satisfy the condition

$$\eta_1 + \eta_2 = 1, \quad (17)$$

and can be *conventionally* chosen as in the nonrelativistic case

$$\eta_i = \frac{m_i}{m_1 + m_2}. \quad (18)$$

For bound systems of two equal mass particles, one simply has $\eta_1 = \eta_2 = 1/2$.

From the previous equations, one can express the particle momenta p_i by means of P^μ and p^μ in the following way:

$$p_i = p_i(P; p) = \eta_i P - \tau_i p \quad (19)$$

with $\tau_1 = +1$ and $\tau_2 = -1$. In the remainder of the work, the four-momenta p_i will be always considered as functions of the total and relative four-momenta, as given by the previous equation.

For convenience, we introduce the following definition for the time component of the relative four-momentum in the CMRF:

$$p^{c0} = \Delta. \quad (20)$$

Equations (1) and (16), referred to the CMRF, can be used to determine the value of Δ by means of E_1^c and E_2^c . For the case of equal mass particles (see Eq. (3)), one simply has $\Delta = 0$. In Eq. (A.4) of Appendix A, we give the explicit value of Δ for two particles of a different mass. That value is obtained by using the standard prescription of Eq. (A.1) for determining the particle energies in the CMRF.

In any case, Δ is a *fixed* quantity of the bound system; as a consequence, we anticipate that the time component of the momentum transfer in the interaction operator is always vanishing in the CMRF as it will be shown in Eq. (31) of Section 3. We can write

$$p^c = (\Delta, \vec{p}^c) \quad (21)$$

and apply the transformation equations (6) to determine p in a GRF.

For further developments, by means of Eq. (8), we can also write

$$\Delta = \frac{P_\mu p^\mu}{M} \quad (22)$$

from which we obtain the expression of p^0 in a GRF

$$p^0 = \frac{M\Delta}{E} + \frac{\vec{P} \cdot \vec{p}}{E}. \quad (23)$$

In Subsection 3.1. we shall also use explicitly the CMRF particle energies E_i^c . These *invariant* quantities can be written taking, in the CMRF, the time component of Eq. (19). By using the definition of Eq. (20), one obtains

$$E_i^c = \eta_i M - \tau_i \Delta. \quad (24)$$

Finally, in order to determine the invariant relative momentum integration element, we set $V = p$ and $V^c = p^c$ in the second relation of Eq. (6) and calculate the Jacobian determinant. Taking into account that $p^c{}^0 = \Delta$ is a *constant* (and, for simplicity, choosing $\vec{P}$ along a coordinate axis), one finally finds

$$d^3 p^c = \frac{M}{E} d^3 p \quad (25)$$

that represents the covariant integration element in a GRF.

3. The covariant wave equation in a GRF

The objective of this section is to write in a GRF the retardless wave equation of our model. We discuss here the general procedure, referring, *for definiteness*, to the DLE. Other forms of equation will be examined in Subsection 3.1. We anticipate that the final result will be written as an integral wave equation.

As a first step, we recall the standard CMRF wave equation, as it was written in the previous work [1]. We have

$$[D_1 O_2 + D_2 O_1 + W] |\Psi^c\rangle = 0. \quad (26)$$

In the *Hamiltonian form* that was used in [1], the Dirac operators D_i are

$$D_i = -(E_i^c - \vec{\alpha}_i \cdot \vec{p}_i^c) + \beta_i m_i \quad (27)$$

and, for the case DLE equation,

$$O_i^{\text{DLE}} = \mathcal{I}_i. \quad (28)$$

We write the same relativistic equation as an integral equation in the momentum space

$$\begin{aligned} & [[-(E_1^c - \vec{\alpha}_1 \cdot \vec{p}_1^c) + \beta_1 m_1] O_2 + [-(E_2^c - \vec{\alpha}_2 \cdot \vec{p}_2^c) + \beta_2 m_2] O_1] \Psi^c(\vec{p}^c) \\ & + \int d^3 p'^c W(\vec{q}^c) \Psi^c(\vec{p}'^c) = 0. \end{aligned} \quad (29)$$

The interaction term has been written in the momentum space as

$$\langle \vec{p}^c | W | \Psi^c \rangle = \int d^3 p'^c W(\vec{q}^c) \Psi^c(\vec{p}'^c), \quad (30)$$

where $\vec{q}^c = \vec{p}^c - \vec{p}'^c$ represents the three-momentum transfer in the CMRF. Note that being $p^{0c} = \Delta$ a fixed quantity, the time component of the momentum transfer q^{c0} is always *vanishing*

$$q^c = (0, \vec{q}^c) \quad (31)$$

that gives rise to a retardless (or instantaneous) interaction. Furthermore, the dependence of the interaction on $\vec{q}^c$ corresponds to a local form of the interaction in the coordinate space. This form was used in our previous works [1, 3].

The next step consists in multiplying the DLE of Eq. (29) by $\gamma_1^0 \gamma_2^0$ from the left in order to write the Dirac operators in the so-called *covariant form* that is traditionally used to study the Lorentz transformation properties in the Dirac theory. The result is

$$\begin{aligned} & [(-p_{1\mu}^c \gamma_1^\mu + m_1) \gamma_2^0 O_2 + (-p_{2\mu}^c \gamma_2^\mu + m_2) \gamma_1^0 O_1] \Psi^c(\vec{p}^c) \\ & + \int d^3 p'^c V(\vec{q}^c) \Psi^c(\vec{p}'^c) = 0, \end{aligned} \quad (32)$$

where the *invariant* interaction, written in the covariant form, is

$$V(\vec{q}^c) = \gamma_1^0 \gamma_2^0 W(\vec{q}^c). \quad (33)$$

In the last step, recalling that Eq. (32) is still written in the CMRF, we shall replace in that equation the GRF covariant Dirac expressions.

In more detail, from Eq. (12), we make the replacement $\Psi^c = B_1^{-1} B_2^{-1} \Psi$.

Furthermore, we express the particle four-momenta p_i by means of the total (P) and relative (p) four-momenta by using the definition of Eq. (19).

We also premultiply the equation by $B_1 B_2$ and use Eq. (13) to transform the terms with $p_{i\mu}^c \gamma_i^\mu$ and Eq. (14) to transform the operators $\gamma_i^0 O_i$ (we are considering here the O_i of Eq. (28) for the DLE).

Finally, the covariant momentum integration is performed by means of Eq. (25). For convenience, we also multiply by -1 the whole equation, obtaining

$$\begin{aligned} & [(p_{1\mu} \gamma_1^\mu - m_1) \Omega_2 + (p_{2\mu} \gamma_2^\mu - m_2) \Omega_1] \Psi(p; P) \\ & - \frac{M}{E} \int d^3 p' V(p - p') \Psi(p'; P) = 0, \end{aligned} \quad (34)$$

where $p - p'$ represents the four-momentum transfer in a GRF obtained by transforming q^c of Eq. (31) by means of Eq. (6) in a standard way. The covariant operators Ω_i , obtained transforming the γ_i^0 as explained before, for the DLE equation, take the form of

$$\Omega_i^{\text{DLE}} = \frac{P_\mu \gamma_i^\mu}{M}. \quad (35)$$

Equation (34) represents the covariant, retardless equation of our model.

In order to discuss a different procedure for its derivation, we can write it as an equation for invariant matrix elements, multiplying by $\bar{\Psi}(p; P)$ and performing the covariant integration

$$\begin{aligned} & \frac{M}{E} \int d^3p \bar{\Psi}(p; P) [(p_{1\mu} \gamma_1^\mu - m_1) \Omega_2 + (p_{2\mu} \gamma_2^\mu - m_2) \Omega_1] \Psi(p; P) \\ & - \left(\frac{M}{E} \right)^2 \int d^3p_b \int d^3p_a \bar{\Psi}(p_b; P) V(p_b - p_a) \Psi(p_a; P) = 0. \end{aligned} \quad (36)$$

One factor M/E is obviously redundant and can be canceled; it has been written explicitly in the previous equation in order to highlight the covariant character of the integrations.

We shall now derive Eq. (36) in a slightly different way, in order to analyze in more detail the properties of the relative four-momentum and the corresponding covariant integration procedure of our model.

The relative four-momentum p (and p_a , p_b) of Eq. (36) is a *constrained* quantity, because in the CMRF its time component is *fixed*, as given in Eqs. (20) and (21).

We consider *only for this derivation*, that is for the next Eq. (37), an *unconstrained* relative four-momentum p and the unconstrained particle four-momenta p_i that are expressed by means of p and P with the same relation of Eq. (19). However, to recover Eq. (36), we have to introduce a *constraint function* that will be discussed below.

We can write an explicitly four-dimensional matrix-element equation in the form of

$$\begin{aligned} & \int d^4p \theta(p; P) \bar{\Psi}(p; P) [(p_{1\mu} \gamma_1^\mu - m_1) \Omega_2 + (p_{2\mu} \gamma_2^\mu - m_2) \Omega_1] \Psi(p; P) \\ & - \int d^4p_b \theta(p_b; P) \int d^4p_a \theta(p_a; P) \bar{\Psi}(p_b; P) V(p_b - p_a) \Psi(p_a; P) = 0, \end{aligned} \quad (37)$$

where $\theta(p; P)$ represents the *constraint function*.

In our model, the covariant constraint function is written by means of the Dirac delta function in the form of

$$\theta(p; P) = \delta \left(\frac{1}{M} P_\mu p^\mu - \Delta \right) = \frac{M}{E} \delta \left[p^0 - \left(\frac{M\Delta}{E} + \frac{\vec{P} \cdot \vec{p}}{E} \right) \right]. \quad (38)$$

We find that, using this constraint function, Eq. (36) is immediately recovered with the same value of p^0 , given in Eq. (23) for a GRF.

The previous derivation can be also used as a starting point to study different forms of constraint function $\theta(p; P)$ for reproducing the physical spectroscopy of the hadronic systems.

3.1. Other forms of the wave equation

We study here the covariant form of other wave equations that are obtained by replacing the operators of Eq. (28) with other expressions.

We have shown that, in the case of the DLE, taking the O_i^{DLE} of Eq. (28) for the CMRF operators in the Hamiltonian form, we obtain the Ω_i^{DLE} of Eq. (35) for the GRF operators in the covariant form.

In our previous works, we have also considered the MW equation. In that model, the operators O_i^{MW} have the form of

$$O_i^{\text{MW}} = \frac{\vec{\alpha}_i \cdot \vec{p}_i^c + \beta_i m_i}{\sqrt{m_i^2 + (\vec{p}_i^c)^2}}. \quad (39)$$

These operators were denoted as S_i in [1] because they represent the energy sign of the free particle in the CMRF.

In order to find the corresponding Ω_i^{MW} , one has to multiply the operators of Eq. (39) by γ_i^0 from the left and then determine their covariant GRF expression. With standard handling, one obtains

$$\Omega_i^{\text{MW}} = \frac{\tau_i \cdot (p_\mu - \frac{\Delta}{M} P_\mu) \gamma_i^\mu + m_i}{\sqrt{m_i^2 + \Delta^2 - p_\nu p^\nu}} \quad (40)$$

with τ_i defined just after Eq. (19). Note that the previous expression takes a simple form when $\Delta = 0$, corresponding to the case of equal mass particles.

Other forms for the operators O_i and Ω_i can be studied. As in work [3], we have analyzed numerically some specific expressions for the charmonium spectrum. In particular, we obtained very similar results as those published in Ref. [3] by using the following operators. In “Model A”, we used

$$O_i^{\text{A}} = \frac{E_i^c - \vec{\alpha}_i \cdot \vec{p}_i^c}{E_i^c}. \quad (41)$$

The corresponding covariant GRF operators are

$$\Omega_i^{\text{A}} = \frac{p_{i\mu} \gamma_i^\mu}{E_i^c}. \quad (42)$$

In “Model B”, we have replaced in Eqs. (39) and (40) the particle masses m_i with the CMRF energies E_i^c . We have

$$O_i^B = \frac{\vec{\alpha}_i \cdot \vec{p}_i^c + \beta_i E_i^c}{\sqrt{(E_i^c)^2 + (\vec{p}_i^c)^2}} \quad (43)$$

and

$$\Omega_i^B = \frac{\tau_i \cdot (p_\mu - \frac{\Delta}{M} P_\mu) \gamma_i^\mu + E_i^c}{\sqrt{(E_i^c)^2 + \Delta^2 - p_\nu p^\nu}}. \quad (44)$$

In the previous Eqs. (41)–(44), the CMRF particle energies E_i^c are given by Eq. (24).

4. The covariant form for the correlated Dirac wave function

In this section, we study the problem of constructing the correlated Dirac wave function in a GRF consistently with the covariant form of the model. In [1], the correlated wave function was determined only in the CMRF. Now we have to boost this wave function to a GRF. In more detail, we can write

$$\Psi_{\text{corr}}(p; P) = B_1 B_2 \Psi_{\text{corr}}^c(\vec{p}^c), \quad (45)$$

where $\Psi_{\text{corr}}^c(\vec{p}^c)$ represents the CMRF correlated Dirac wave function introduced in Ref. [1]. Furthermore, the Dirac Boost operators B_1 and B_2 of Eq. (9) are standardly used here.

In principle, the previous expression of $\Psi_{\text{corr}}(p; P)$ would be sufficient to study the bound state problem in any GRF. However, we prefer to express $\Psi_{\text{corr}}(p; P)$ by means of explicitly covariant quantities.

To this aim, one has to recall, in the first place, that the relative momentum $\vec{p}^c$ can be written, with the Lorentz transformations (7), in terms of the GRF momenta p and P ; these quantities are the arguments of Ψ_{corr} in the l.h.s. of Eq. (45).

Then, we elaborate on the boosted correlated wave function.

In the CMRF, the correlated Dirac wave function was obtained by means of the reduction operator K_i^c , of the form of

$$K_i^c = \left(\frac{1}{\frac{\vec{\sigma}_i \cdot \vec{p}_i^c}{m_i + E_i^c}} \right). \quad (46)$$

In more detail, for a two-body system, one has to apply the reduction operators of the two particles

$$\Psi_{\text{corr}}^c = K_1^c K_2^c \Phi^c(\vec{p}^c), \quad (47)$$

where $\Phi^c(\vec{p}^c)$ represents the spinorial wave function. The numerical normalization constant is calculated apart, as in Eq. (38) of Ref. [1] and omitted in the following.

For the present procedure of “covariantization”, we preliminarily introduce the following operator:

$$A_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (48)$$

that, applied to Φ^c , simply constructs a Dirac spinor with vanishing lower components.

Then, by means of A_i , we can express with standard calculations the operator K_i^c in the following form:

$$K_i^c = F_i^c (p_{i\mu}^c \gamma_i^\mu + m_i) A_i \quad (49)$$

with the invariant factor

$$F_i^c = (E_i^c + m_i)^{-1}. \quad (50)$$

The expression of Eq. (49) is equivalent to Eq. (46) but is more suitable to be transformed into a covariant expression.

We can apply the Boost operator of Eq. (9) to the K_i^c of Eq. (49) in order to determine the reduction operator K_i in a GRF. We can write

$$K_i = B_i K_i^c = F_i^c B_i (p_{i\mu}^c \gamma_i^\mu + m_i) B_i^{-1} B_i A_i. \quad (51)$$

With standard calculations, one finds

$$B_i A_i = F_B (P_\mu \gamma_i^\mu + M) A_i. \quad (52)$$

Furthermore, the factor in parenthesis of Eq. (51) can be transformed by means of Eq. (13). The result is

$$K_i = F_B F_i^c (p_{i\mu} \gamma_i^\mu + m_i) (P_\mu \gamma_i^\mu + M) A_i. \quad (53)$$

By using the reduction operators of the two particles, one obtains the correlated wave function in a GRF, in the form of

$$\begin{aligned} \Psi_{\text{corr}}(p; P) &= K_1 K_2 \Phi^c = (F_B)^2 F_1^c F_2^c (p_{1\mu} \gamma_1^\mu + m_1) (p_{2\mu} \gamma_2^\mu + m_2) \\ &\quad \times (P_\mu \gamma_1^\mu + M) (P_\mu \gamma_2^\mu + M) A_1 A_2 \Phi^c. \end{aligned} \quad (54)$$

The correlated Dirac adjoint wave function can be easily constructed by means of standard handling of the Dirac matrices. One has to use also the following property of the operator A_i :

$$A_i^\dagger \gamma_i^0 = A_i^\dagger. \quad (55)$$

The result is

$$\begin{aligned} \bar{\Psi}_{\text{corr}}(p; P) &= \Psi_{\text{corr}}^\dagger(p; P) \gamma_1^0 \gamma_2^0 = \Phi^{c\dagger} \Lambda_1^\dagger \Lambda_2^\dagger (P_\mu \gamma_1^\mu + M) (P_\mu \gamma_2^\mu + M) \\ &\times (p_{1\mu} \gamma_1^\mu + m_1) (p_{2\mu} \gamma_2^\mu + m_2) (F_B)^2 F_1^c F_2^c. \end{aligned} \quad (56)$$

Equations (54) and (56) complete the development of this section by using explicitly covariant operators for the Dirac wave function. The result shows that the use of the correlated wave function in a GRF is consistent with the whole covariant model. Finally, we note that the factor $(F_B)^2 F_1^c F_2^c$, being a *constant*, is not relevant for the covariant wave equation. Furthermore, in principle, the covariant integral wave equation of Eq. (34), with the Dirac correlated wave function of Eq. (54), could be solved in a GRF, determining *directly* the corresponding spinorial function.

The author thanks the group of “Gestión de Recursos de Computo Científico, Laboratorio de Biología Computacional, Facultad de Ciencias — Universidad Nacional de Colombia” for the access to the computation facilities that were used to perform the numerical calculations that have been necessary to develop the model discussed and generalized in this work.

Appendix A

Determination of the particle energy

Considering Eq. (1), we observe that the CMRF energy values E_1^c and E_2^c are usually determined by means of the auxiliary prescription [2, 6]

$$(E_1^c)^2 - (E_2^c)^2 = m_1^2 - m_2^2 \quad (\text{A.1})$$

that is related, in general, to the asymptotic properties of the relativistic free Hamiltonian of the two-body system.

The CMRF energies E_i^c are easily found by using Eqs. (2) and (A.1). One has

$$\begin{aligned} E_1^c &= \frac{1}{2} \left(M + \frac{m_1^2 - m_2^2}{M} \right), \\ E_2^c &= \frac{1}{2} \left(M - \frac{m_1^2 - m_2^2}{M} \right). \end{aligned} \quad (\text{A.2})$$

The validity of Eqs. (A.1) and (A.2) for a bound system in the general case of $m_1 \neq m_2$ should be carefully verified. However, for the specific case $m_1 = m_2$, that is physically very relevant for the study of the $q\bar{q}$ mesons, one simply has

$$E_1^c = E_2^c = \frac{M}{2}. \quad (\text{A.3})$$

For completeness, starting from Eq. (A.2), one can determine Δ of Eq. (20). By using Eq. (16) for the time component of the relative momentum in the CMRF and the definition of the η_i given in Eq. (18) of Subsection 2.3, one obtains

$$\Delta = \frac{M}{2} \frac{m_1 - m_2}{m_1 + m_2} - \frac{1}{2} \frac{m_1^2 - m_2^2}{M}. \quad (\text{A.4})$$

For the case of two equal mass particles, the standard result of Eq. (A.3), can be also obtained in a different way: one can use the symmetry properties of the system and consider E_1^c and E_2^c as *mean values* of the corresponding Hamiltonian operators.

Due to the relevance of this argument in the context of this work, we summarize its derivation.

We introduce the CMRF Hamiltonian, schematically written in the following form:

$$H^c = H_{1 \text{ free}}^c + H_{2 \text{ free}}^c + W, \quad (\text{A.5})$$

where the first two terms represent the free Hamiltonian of the two particles and W is the interaction term. The corresponding eigenvalue equation is

$$H^c |\Psi^c\rangle = M |\Psi^c\rangle. \quad (\text{A.6})$$

Given that the two particles have the same mass, the interaction is symmetric with respect to particle interchange

$$P_{12} W P_{12} = W, \quad (\text{A.7})$$

where P_{12} represents the particle interchange operator.

In consequence, the eigenstates have definite symmetry

$$P_{12} |\Psi^c\rangle = (-1)^\sigma |\Psi^c\rangle \quad (\text{A.8})$$

with $\sigma = +1$ for *symmetric* and $\sigma = -1$ for *antisymmetric* states, respectively.

We can define the interacting Hamiltonian operator for each particle in the form of

$$H_i^c = H_{i \text{ free}}^c + \frac{1}{2} W \quad (\text{A.9})$$

with $i = 1, 2$. In this way, one can immediately sum the Hamiltonians of the two particles to obtain the expression of Eq. (A.5)

$$H^c = H_1^c + H_2^c. \quad (\text{A.10})$$

Furthermore, the standard interchange property

$$P_{ij} H_i^c P_{ij} = H_j^c \quad (\text{A.11})$$

is automatically satisfied.

By using Eqs. (A.6), (A.8), (A.10), and (A.11), one finds

$$\langle \Psi^c | H_1^c | \Psi^c \rangle = \langle \Psi^c | H_2^c | \Psi^c \rangle = \frac{M}{2}. \quad (\text{A.12})$$

Further investigation is needed to study the case of two particles with different masses.

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