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THIRD-0RDER CORRECTIONS TO THE GROUND-STATE ENERGY OF THE POLARIZED DILUTED GAS OF SPIN 1/2 FERMIONS

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We present the results of the computation of the third-order corrections to the ground-state energy of the diluted gas of nonrelativistic spin 1/2 fermions interacting through a spin-independent repulsive two-body potential. The corrections are computed within the effective field theory approach which does not require specifying the interaction potential explicitly but to characterize it by only a few parameters — the scattering lengths a_0, $a_1, \dots$ and effective radii $r_0, \dots$ — measurable in low-energy fermion–fermion elastic scattering. The corrections are computed semi-analytically, that is, are expressed in terms of two functions of the system's polarization. The functions are given by the integrals which can be easily evaluated using the Mathematica built-in routine for numerical integration.

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1. Introduction

The classic model of the many-body quantum mechanics, the diluted gas of nonrelativistic fermions interacting through a spin independent repulsive two-body potential [1, 2], has attracted in recent time renewed attention due to the advent of a new generation of experiments with cold atomic gases in which the interaction strength can be tuned in a wide range by exploiting the existence and properties of the so-called Feshbach resonance [3]. The experiments have stimulated intensive numerical studies of the system [4–6] aiming at computing its properties mainly relating to the possible application of the model to the problem of the emergence of the so-called itinerant ferromagnetism in systems of interacting fermions.

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On the other hand, the application to the model of the effective field theory method in the pioneering work [7] (see also [8–11]) has opened new possibilities to investigate properties of the system of interacting fermions analytically. The proposed approach has in particular greatly simplified perturbative computations of the ground-state energy of the system, automatically yielding its expansion in powers of $k_{\rm F}R$, where R is the length scale characterizing the interaction potential and $\hbar k_{\rm F} = (3\pi^2N/V)^{1/3}\hbar$ is the Fermi momentum of the system of N fermions enclosed in the volume V.

The simplifications offered by the effective field theory approach allowed to complete recently [12] the computation of the fourth-order, $(k_{\rm F}R)^4$, contribution to the ground-state energy of the system of spin s fermions with equal densities of fermions of different spin projections (unpolarized system) and to reproduce [13] semi-analytically but in the universal setting, that is, without specifying the underlying interaction potential, the order of $(k_{\rm F}R)^2$ correction to the ground-state energy of the spin $1/2$ fermions with the arbitrary ratio of the densities of spin-up and spin-down fermions (arbitrarily polarized system) which in the past has been computed by Kanno [14] using the hard spheres model interaction. This result has recently been generalized to the system of spin s fermions with arbitrary proportions of densities of the $g_s = 2s + 1$ possible spin projections [15].

Computations of the ground-state energy as a function of the system's polarization P directly relate to the possibility of the phase transition to the ordered state ($P \neq 0$) at zero temperature with increasing the strength of the interaction potential (reflected in the effective field theory approach by the increasing magnitude of the scattering lengths a_0, $a_1, \ldots$ and the effective radii $r_0, \ldots$) and/or of the system's overall density characterized by its Fermi momentum $\hbar k_{\rm F}$. The first order of the perturbative expansion (equivalent to the mean-field approximation) predicts that such a transition occurs for $k_{\rm F}a_0 = \pi/2$ (the Stoner's criterion [1, 16]). Numerical investigations [4] which necessarily use a concrete form of the interaction potential indicate that the transition occurs at $k_{\rm F}a_0 \approx 0.85$. Inclusion of the second-order contribution in the perturbative expansion of the ground-state energy yields $k_{\rm F}a_0 \approx 1.054$ as the critical value of the expansion parameter.

In this letter we compute the third-order correction to the ground-state energy of the system of interacting spin $1/2$ fermions for an arbitrary value of the system's polarization P. As in [13], we apply the effective field theory approach proposed first in [7] and regularize the divergent integrals over wave vectors with the help of the cutoff Λ. We explicitly demonstrate the cancellation of the terms diverging in the limit $\Lambda \to \infty$ after the couplings of the effective theory Lagrangian are expressed in terms of the scattering lengths computed up to the appropriate order using the same cutoff Λ. The final result is expressed in terms of two functions of the polarization P which are given by the integrals which can be computed with sufficient accuracy by the Mathematica package built-in routine for numerical evaluation of the multidimensional integrals over prescribed domains.

2. Computation

Assuming that the underlying "fundamental" spin-independent two-body interaction of nonrelativistic spin s fermions of mass m_f is consistent with the Galileo, parity and time-reversal symmetries, the most general interaction term of the effective theory Hamiltonian which captures properties of low-density system of N such fermions as well as characteristics of their low-energy ("in vacuum") scattering reads [7]

$$V_{\rm int} = \frac{C_0}{2}\int \mathrm{d}^3\boldsymbol{x} \sum_{\alpha\beta} \psi^\dagger_\alpha \psi^\dagger_\beta \psi_\beta \psi_\alpha - \frac{C_2}{16}\int \mathrm{d}^3\boldsymbol{x} \sum_{\alpha,\beta} \left[\psi^\dagger_\alpha \psi^\dagger_\beta \left(\psi_\beta \overleftrightarrow{\boldsymbol{\nabla}}^2 \psi_\alpha\right) + \text{H.c.}\right]$$
$$- \frac{C_2'}{8}\int \mathrm{d}^3\boldsymbol{x} \sum_{\alpha,\beta} \left(\psi^\dagger_\alpha \overleftrightarrow{\boldsymbol{\nabla}} \psi^\dagger_\beta\right)\cdot\left(\psi_\beta \overleftrightarrow{\boldsymbol{\nabla}} \psi_\alpha\right) + \frac{D_0}{2}\int \mathrm{d}^3\boldsymbol{x} \sum_{\alpha,\beta,\gamma} \psi^\dagger_\alpha \psi^\dagger_\beta \psi^\dagger_\gamma \psi_\gamma \psi_\beta \psi_\alpha + \dots, \tag{1}$$

where ψ_α and $\psi^\dagger_\alpha$ are the usual field operators of the second quantization formalism [2] (in the case of spin 1/2 fermions, all terms with $(\psi^\dagger_\uparrow\psi_\uparrow + \psi^\dagger_\downarrow\psi_\downarrow)^n$ with $n \geq 3$, such as the one proportional to D_0 are absent due to the Pauli exclusion encoded in the anticommutativity of the field operators). The coupling constants C_0, $C_2, \dots$ multiplying the local operator structures of decreasing length dimensions can be determined by computing using this interaction the amplitude the elastic scattering of two fermions parametrized in the low-energy limit in terms of the scattering lengths. The result of such a procedure is [7, 13, 17]

$$C_0 = \frac{4\pi\hbar^2}{m_f}\, a_0 \left(1 + \frac{2}{\pi}\, a_0 \Lambda + \frac{4}{\pi^2}\, a_0^2 \Lambda^2 + \dots\right),$$
$$C_2 = \frac{4\pi\hbar^2}{m_f}\, \frac{1}{2}\, a_0^2 r_0 + \dots, \qquad C_2' = \frac{4\pi\hbar^2}{m_f}\, a_1^3 + \dots\,, \tag{2}$$

where Λ is the UV cutoff imposed on the wave-vectors of the loop integrals. Divergences, absent in the underlying "fundamental" theory, appear as a result of the local (*i.e.* singular) nature of the interaction terms of the effective interaction Hamiltonian (1).

The ground-state energy density of the system of N noninteracting nonrelativistic spin s fermions (enclosed in the volume V) is

$$\frac{E_{\Omega_0}}{V} = \frac{1}{6\pi^2} \sum_{\alpha=1}^{2s+1} \frac{\hbar^2}{2m_f}\, \frac{3}{5}\, p^5_{\mathrm{F}\alpha}\,. \tag{3}$$

$p_{\mathrm{F}\alpha} = ((1/6\pi^2))N_\alpha/V)^{1/3}$ are the respective Fermi wave vectors of N_α fermions with the spin projection α in the system; $N = \sum_{\alpha=1}^{2s+1} N_\alpha$. Since the energy of the system of spin 1/2 fermions is (in the absence of an external magnetic field) invariant with respect to the interchange $N_\uparrow \leftrightarrow N_\downarrow$, we will in the following, as in [13], denote N_+ (and, correspondingly, $p_{\mathrm{F}+}$) the number of spin-up fermions if $N_\uparrow \geq N_\downarrow$, and will write the polarization P, $0 \leq P \leq 1$, of the system of spin 1/2 fermions as

$$P = \frac{N_+ - N_-}{N_+ + N_-} \equiv \frac{1-r^3}{1+r^3}\,, \qquad \text{where} \quad r \equiv \frac{p_{\mathrm{F}-}}{p_{\mathrm{F}+}} = \frac{N_-^{1/3}}{N_+^{1/3}} = \left(\frac{1-P}{1+P}\right)^{1/3}. \tag{4}$$

It will be also convenient to express the results in terms of the average Fermi wave number $k_{\mathrm{F}} = ((6\pi^2/g_s)(N/V))^{1/3}$ which does not change when the numbers $N_\pm$ of fermions of different spin projections are varied (keeping constant $N = N_+ + N_-$); the wave numbers of the spin up and spin-down fermions are then directly expressed in terms of the polarization P: $p_{\mathrm{F}\pm} = k_{\mathrm{F}}(1 \pm P)^{1/3}$.

The first nontrivial correction to the ground-state energy was computed long time ago by Lenz [18]. Further corrections to E_Ω are most easily computed using the general formula[1]

$$\lim_{T\to\infty} \exp(-iT(E_\Omega - E_{\Omega_0})/\hbar) = \lim_{T\to\infty} \langle \Omega_0 | \mathrm{T} \exp\left(-\frac{i}{\hbar}\int\limits_{-T/2}^{T/2} \mathrm{d}t\, V^I_{\mathrm{int}}(t)\right) |\Omega_0\rangle\,, \tag{5}$$

in which $V^I_{\mathrm{int}}(t)$ is the interaction part of the theory Hamiltonian taken in the interaction picture. In application to the considered system, this formula, which can be evaluated using the standard Dyson expansion, gives the corrections $(E_\Omega - E_{\Omega_0})/V$ to the ground-state energy density as a sum of the momentum space connected vacuum Feynman diagrams (called in this context also the Hugenholtz diagrams) multiplied by $i\hbar$.

As the effective theory interaction (1) consists of an in principle infinite number of operator structures, diagrams which should be taken into account to obtain the order of $(k_{\mathrm{F}}R)^\nu$ contribution to $(E_\Omega - E_{\Omega_0})/V$ are selected by the power counting rules [7, 19]

$$\nu = 5 - \sum_i V_i \Delta_i = 2 + 3L + \sum_i V_i(d_i - 2)\,, \tag{6}$$

in which V_i is the number of the vertices of type i with d_i derivatives and n_i lines attached to the vertex, L is the number of closed loops and $\Delta_i =$

[1] The symbol T of the chronological ordering should not be confused with T denoting time.

$5-d_i-\frac{3}{2}n_i$ characterizes the dimension of the interaction vertices; $\Delta_{C_0} = -1$, $\Delta_{C_2,C_2'} = -3$, *etc.* The dimensional analysis shows that the magnitude of the coupling C_i multiplying the vertex of type i is $(4\pi\hbar^2/m_f)R^{-\Delta_i}$, where R is the characteristic length scale of the underlying interaction potential (which, if in the assumed absence of any resonant or anomalous behaviour, implies that all $a_\ell \sim r_\ell \sim R$).

The power counting rules (6) tell that the to the order $k_{\rm F}^5(k_{\rm F}R)^2$ correction to $(E_\Omega - E_{\Omega_0})/V$ contribute only diagrams with two C_0 vertices and three loops. There is only one nonvanishing such diagram, which has been first evaluated in [7] for the case of unpolarized system of spin s fermions and shown to straightforwardly reproduce the well-known classic result obtained first in [20] with the help of rather cumbersome methods (and since then reproduced using a variety of different approaches). In [13] the corresponding three loop diagram has been evaluated semi-analytically for the case of the polarized system of spin 1/2 fermions and the result found to numerically coincide with the analytic formula obtained by Kanno [14] within the hard-spheres model of the two-body interaction (extension of the result of [13] to the arbitrarily polarized system of spin s fermions has been presented very recently in [15]).

The order of $k_{\rm F}^5(k_{\rm F}R)^3$ correction is given by the Hugenholtz diagrams with either three C_0 vertices and four loops or by two-loop diagrams with a single C_2 or C_2' vertex. There are only two nonvanishing diagrams of the first kind [7] shown in figure 1. Both these diagrams come with the combinatoric factor of 2 (when the interaction term of spin 1/2 fermions is written as $C_0\int\psi_+^\dagger\psi_+\psi_-^\dagger\psi_-$) and both are given by an integral of a product of three identical blocks which consist of two terms each; of the arising $2^3 = 8$ terms two vanish as a result of the integration, while the remaining six give rise to only two different terms; the resulting factor $2\cdot 3$ cancels the factor $1/3!$ arising from the expansion of the exponent.

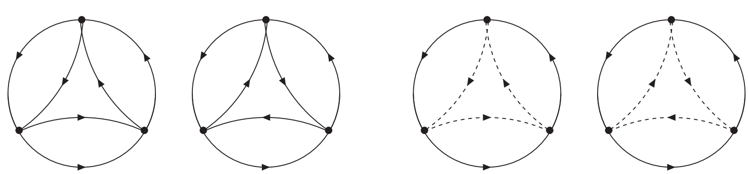

Fig. 1. The two nonvanishing order C_0^3 effective theory connected vacuum diagrams (the "particle-particle" diagram and the "particle-hole" diagram) contributing to the correction $E_\Omega^{(3)}$ to the ground-state energy of the system of spin s fermions with equal densities of different spin projections and their counterparts in the case of the system of spin 1/2 fermions and $N_\uparrow \neq N_\downarrow$. In this second case, the solid and dashed lines represent propagators of fermions with opposite spin projections.

After the standard steps (for more details, see [13]), the contribution of the "particle–particle" diagram to the energy density can be written in the form of

$$\frac{E_{\Omega}^{(3)p-p}}{V} = \frac{128 m_f^2 C_0^3}{(2\pi)^8 \hbar^4} \left[G^{(1)}(p_{\mathrm{F}-}, p_{\mathrm{F}+}) + G^{(2)}(p_{\mathrm{F}-}, p_{\mathrm{F}+})\right], \tag{7}$$

with

$$G^{(1)}(p_{\mathrm{F}-}, p_{\mathrm{F}+}) = \int_0^{s_{\max}} \mathrm{d}s\, s^2 \frac{1}{4\pi} \int \mathrm{d}^3\boldsymbol{t}\, \theta(p_{\mathrm{F}-} - |\boldsymbol{t}+\boldsymbol{s}|)\, \theta(p_{\mathrm{F}+} - |\boldsymbol{t}-\boldsymbol{s}|)\, (g(t,s))^2, \tag{8}$$

$$G^{(2)}(p_{\mathrm{F}-}, p_{\mathrm{F}+}) = \int_0^{s_{\max}} \mathrm{d}s\, s^2 \frac{1}{4\pi} \int \mathrm{d}^3\boldsymbol{t}\, \theta(|\boldsymbol{t}+\boldsymbol{s}| - p_{\mathrm{F}-})\, \theta(|\boldsymbol{t}-\boldsymbol{s}| - p_{\mathrm{F}+})\, (h(t,s))^2, \tag{9}$$

where the functions $g(t,s) \equiv g(|\boldsymbol{t}|, |\boldsymbol{s}|)$ and $h(t,s) \equiv h(|\boldsymbol{t}|, |\boldsymbol{s}|)$ are given below and $s_{\max} = \frac{1}{2}(p_{\mathrm{F}+} + p_{\mathrm{F}-})$. The analogous contribution of the "particle–particle" diagram to the energy density of the unpolarized system of spin s fermions is obtained by multiplying (7) by the spin factor $\frac{1}{2}g_s(g_s-1)$ and setting $p_{\mathrm{F}-} = p_{\mathrm{F}+} = k_{\mathrm{F}}$.

The function $g(t,s)$ is the one which appeared in [13] in evaluating the order C_0^2 contribution to the energy density; it can be written in the form

$$g(t,s) = -\Lambda + g_{\mathrm{fin}}(t,s) + \frac{t^2}{\Lambda} + \mathcal{O}\frac{1}{\Lambda^2}, \tag{10}$$

in which Λ is the UV cutoff imposed on the divergent integral over the wave vectors. The finite part $g_{\mathrm{fin}}(t,s)$ is for $0 < s \le s_0 \equiv \frac{1}{2}(p_{\mathrm{F}+} - p_{\mathrm{F}-})$ given by

$$g(t,s) = \frac{1}{2}\, p_{\mathrm{F}+} + \frac{t}{4} \ln\frac{(p_{\mathrm{F}+} - t)^2 - s^2}{(p_{\mathrm{F}+} + t)^2 - s^2} + \frac{p_{\mathrm{F}+}^2 - s^2 - t^2}{8s} \ln\frac{(p_{\mathrm{F}+} + s)^2 - t^2}{(p_{\mathrm{F}+} - s)^2 - t^2}, \tag{11}$$

and for $\frac{1}{2}(p_{\mathrm{F}} - p_{\mathrm{F}-}) < s \le s_{\max}$ by

$$\begin{aligned} g(t,s) &= \frac{1}{4}(p_{\mathrm{F}+} + p_{\mathrm{F}-} + 2s) + \frac{t}{4}\ln\frac{p_{\mathrm{F}+} + s - t}{p_{\mathrm{F}+} + s + t} + \frac{t}{4}\ln\frac{p_{\mathrm{F}-} + s - t}{p_{\mathrm{F}-} + s + t} \\ &+ \frac{p_{\mathrm{F}+}^2 - t^2 - s^2}{8s} \ln\frac{(p_{\mathrm{F}+} + s)^2 - t^2}{u_0^2 - t^2} + \frac{p_{\mathrm{F}-}^2 - t^2 - s^2}{8s} \ln\frac{(p_{\mathrm{F}-} + s)^2 - t^2}{u_0^2 - t^2}, \end{aligned} \tag{12}$$

where

$$u_0^2 = \frac{1}{2}\left(p_{\mathrm{F}+}^2 + p_{\mathrm{F}-}^2\right) - s^2\,. \tag{13}$$

Compared to the form of $g(t,s)$ given in [13], we have retained in (10) the term proportional to $1/\Lambda$ to show explicitly that the finite contributions of such terms (which are absent in the dimensional regularization used in [7]) cancel out. $h(t,s)$ is a new function given by the finite integral

$$h(t,s) = \frac{1}{4\pi}\int \mathrm{d}^3\boldsymbol{u}\,\frac{\theta(p_{\mathrm{F}-} - |\boldsymbol{u}+\boldsymbol{s}|)\,\theta(p_{\mathrm{F}+} - |\boldsymbol{u}-\boldsymbol{s}|)}{\boldsymbol{t}^2 - \boldsymbol{u}^2 - i0}\,. \tag{14}$$

Its analytic form

$$\begin{aligned} h(t,s) \;=\; & -\frac{1}{2}\,p_{\mathrm{F}-} - \frac{t}{4}\ln\frac{t-(p_{\mathrm{F}-}-s)}{t+(p_{\mathrm{F}-}-s)} - \frac{t}{4}\ln\frac{t-(p_{\mathrm{F}-}+s)}{t+(p_{\mathrm{F}-}+s)} \\ & + \frac{t^2 - \left(p_{\mathrm{F}-}^2 - s^2\right)}{8s}\ln\frac{t^2-(p_{\mathrm{F}-}+s)^2}{t^2-(p_{\mathrm{F}-}-s)^2}\,, \end{aligned} \tag{15}$$

for $s < s_0 \equiv \frac{1}{2}(p_{\mathrm{F}+} - p_{\mathrm{F}-})$ and

$$\begin{aligned} h(t,s) = & \frac{1}{2}\left(2s - p_{\mathrm{F}-} - p_{\mathrm{F}+}\right) - \frac{t}{4}\ln\frac{t-(p_{\mathrm{F}-}-s)}{t+(p_{\mathrm{F}-}-s)} - \frac{t}{4}\ln\frac{t-(p_{\mathrm{F}+}-s)}{t+(p_{\mathrm{F}+}-s)} \\ & - \frac{1}{8s}\left[(p_{\mathrm{F}+}-s)^2 + (p_{\mathrm{F}-}-s)^2 - 2u_0^2\right] \\ & - \frac{t^2 - p_{\mathrm{F}+}^2 + s^2}{8s}\ln\frac{t^2-(p_{\mathrm{F}+}-s)^2}{t^2-u_0^2} - \frac{t^2 - p_{\mathrm{F}-}^2 + s^2}{8s}\ln\frac{t^2-(p_{\mathrm{F}-}-s)^2}{t^2-u_0^2} \end{aligned} \tag{16}$$

for $s_0 \le s \le s_{\max} = \frac{1}{2}(p_{\mathrm{F}+} + p_{\mathrm{F}-})$, can be obtained by the same technique, introduced in [17], which served to obtain the function $g(t,s)$ (the analysis of the ranges of the integrations over $|\boldsymbol{u}|$, $|\boldsymbol{t}|$, and s shows that the pole at $\boldsymbol{u}^2 = \boldsymbol{t}^2$ of the integrand in (14) is never reached and, therefore, $-i0$ can be dropped). Both these functions vanish for $s > s_{\max}$, therefore, the integrals over $s = |\boldsymbol{s}|$ in (8) and (9) are finite. Similarly manifestly finite is the integral over $t = |\boldsymbol{t}|$ in (8), while the analogous integral in (9) is finite owing to the fact that $h(t,s) \sim 1/t^2$ as $t \to \infty$.

The contribution to the energy density of the “particle–hole” diagram of figure 1 can be written in the form of

$$\frac{E_{\Omega}^{(3)\mathrm{p-h}}}{V} = -\frac{32 m_f^2 C_0^3}{(2\pi)^8 \hbar^4}\left[K^{(1)}(p_{\mathrm{F}-}, p_{\mathrm{F}+}) + K^{(2)}(p_{\mathrm{F}-}, p_{\mathrm{F}+})\right], \tag{17}$$

(the corresponding contribution of the left "particle–hole" diagram of figure 1 to the energy density of the unpolarized system of spin s fermions is obtained by multiplying (17) by the spin factor $\frac{1}{2}g_s(g_s-1)(3-g_s)$ and setting $p_{\rm F-} = p_{\rm F+} = k_{\rm F}$). The functions $K^{(1)}(p_{\rm F-}, p_{\rm F+})$ and $K^{(2)}(p_{\rm F-}, p_{\rm F+})$ are given by

$$K^{(1)}(p_{\rm F-}, p_{\rm F+}) = \int\limits_0^\infty {\rm d}s\, s^2\, \frac{1}{4\pi}\int {\rm d}^3\boldsymbol{t}\,\theta(|\boldsymbol{t}+\boldsymbol{s}|-p_{\rm F-})\,\theta(p_{\rm F+}-|\boldsymbol{t}-\boldsymbol{s}|)\,(f_1(\boldsymbol{t}\cdot\boldsymbol{s},\, s))^2\,, \quad (18)$$

$$K^{(2)}(p_{\rm F-}, p_{\rm F+}) = \int\limits_0^\infty {\rm d}s\, s^2\, \frac{1}{4\pi}\int {\rm d}^3\boldsymbol{t}\,\theta(p_{\rm F-}-|\boldsymbol{t}+\boldsymbol{s}|)\,\theta(|\boldsymbol{t}-\boldsymbol{s}|-p_{\rm F+})\,(f_2(\boldsymbol{t}\cdot\boldsymbol{s},\, s))^2\,, \quad (19)$$

and the functions $f_1(\boldsymbol{t}\cdot\boldsymbol{s},\, s)$ and $f_2(\boldsymbol{t}\cdot\boldsymbol{s},\, s)$ are given by the integrals

$$f_1(\boldsymbol{t}\cdot\boldsymbol{s},\, s) = \frac{1}{4\pi}\int {\rm d}^3\boldsymbol{u}\, \frac{\theta(p_{\rm F-}-|\boldsymbol{u}+\boldsymbol{s}|)\,\theta(|\boldsymbol{u}-\boldsymbol{s}|+p_{\rm F+})}{(\boldsymbol{u}-\boldsymbol{t})\cdot\boldsymbol{s}+i0}\,, \quad (20)$$

and

$$f_2(\boldsymbol{t}\cdot\boldsymbol{s},\, s) = \frac{1}{4\pi}\int {\rm d}^3\boldsymbol{u}\, \frac{\theta(|\boldsymbol{u}+\boldsymbol{s}|-p_{\rm F-})\,\theta(p_{\rm F+}+|\boldsymbol{u}-\boldsymbol{s}|)}{(\boldsymbol{u}-\boldsymbol{t})\cdot\boldsymbol{s}-i0}\,. \quad (21)$$

Both integrals defining the functions f_1 and f_2 are over manifestly finite domains: the one defining f_1 is over the interior of the ball of radius $p_{\rm F-}$ and the exterior of the sphere of radius $p_{\rm F+}$ and the one defining f_2 — the other way around. In $K_1(p_{\rm F-}, p_{\rm F+})$ (18), the function f_1 is then integrated (over ${\rm d}^3\boldsymbol{t}$) again over a manifestly finite domain — namely over the interior of the ball of radius $p_{\rm F+}$ and the exterior of the sphere of radius $p_{\rm F-}$, while the function f_2 is in $K_2(p_{\rm F-}, p_{\rm F+})$ (19) integrated over the interior of the ball of $p_{\rm F-}$ and the exterior of the sphere of radius $p_{\rm F+}$. The straightforward analysis shows that the poles at $\boldsymbol{u}\cdot\boldsymbol{s} = \boldsymbol{t}\cdot\boldsymbol{s}$ are never within the integration domains. Hence, the factors $\pm i0$ are irrelevant. It is also clear that $K_2(k_{\rm F}, k_{\rm F}) = K_1(k_{\rm F}, k_{\rm F})$.

The most difficult part of the computation is obtaining analytical expressions for the functions f_1 and f_2. The formulae for f_1 (for f_2) have been obtained by shifting the center of the $\boldsymbol{u}$-space in the regime of small s to the center of the $p_{\rm F+}$-sphere (of the $p_{\rm F-}$-sphere) and to the center of the $p_{\rm F-}$-sphere (of the $p_{\rm F+}$-sphere) in the regime of large s, introducing then the polar coordinated and taking the resulting integrals analytically with the help of the Mathematica routines; the results of the symbolic integrations have been then simplified manually by exploiting the relations which follow from the definitions of the integration domains (details will be published elsewhere [21]). In this way, we have arrived at

$$f_1(\boldsymbol{t}\cdot\boldsymbol{s},\, s) = \frac{1}{2s}\times\begin{cases} f_1^{(a)}(t\,\eta - s,\, s) & \text{for} \quad s < \frac{1}{2}p_{\mathrm{F}+} \\ f_1^{(b)}(t\,\eta + s,\, s) & \text{for} \quad \frac{1}{2}p_{\mathrm{F}+} < s < s_{\max} \\ f_1^{(c)}(t\,\eta + s,\, s) & \text{for} \quad s_{\max} < s \end{cases},$$

$$f_2(\boldsymbol{t}\cdot\boldsymbol{s},\, s) = \frac{1}{2s}\times\begin{cases} f_2^{(a)}(t\,\eta + s,\, s) & \text{for} \quad s < \frac{1}{2}p_{\mathrm{F}+} \\ f_2^{(b)}(t\,\eta - s,\, s) & \text{for} \quad \frac{1}{2}p_{\mathrm{F}+} < s < s_{\max} \\ f_2^{(c)}(t\,\eta - s,\, s) & \text{for} \quad s_{\max} < s \end{cases}, \tag{22}$$

where

$$\begin{aligned} f_1^{(a)}(t,s) &= -2s^2 + \frac{t}{2}\,(p_{\mathrm{F}+} - p_{\mathrm{F}-} - 2s) - s\,p_{\mathrm{F}-} - s\,p_{\mathrm{F}+}\,\xi_0 \\ &- \frac{p_{\mathrm{F}+}^2}{2}\ln\!\left(\frac{t - p_{\mathrm{F}+}\xi_0}{t + p_{\mathrm{F}+}}\right) + \frac{t^2}{2}\ln\!\left(\frac{t + 2s + p_{\mathrm{F}-}}{t + p_{\mathrm{F}+}}\right) \\ &+ \frac{1}{4}\,(p_{\mathrm{F}-}^2 - 4s^2 - 4st)\left\{-2\ln\!\left(\frac{p_{\mathrm{F}+} + 4s\xi_0}{p_{\mathrm{F}-} - 2s}\right)\right. \\ &\left. + \ln\frac{[t^2 - (p_{\mathrm{F}-}^2 - 4s^2 - 4st)\xi_0^2][tp_{\mathrm{F}+} - (p_{\mathrm{F}-}^2 - 4s^2 - 4st)\xi_0][tp_{\mathrm{F}-} - p_{\mathrm{F}-}^2 + 4s^2 + 2st]}{[t^2 - p_{\mathrm{F}-}^2 + 4s^2 + 4st][tp_{\mathrm{F}+} + (p_{\mathrm{F}-}^2 - 4s^2)\xi_0][tp_{\mathrm{F}-} + p_{\mathrm{F}-}^2 - 4s^2 - 2st]}\right\}, \end{aligned}$$

$$\begin{aligned} f_1^{(b)}(t,s) &= -\frac{t}{2}\,p_{\mathrm{F}-}\left(1 + \frac{1}{\xi_0'}\right) + 2s^2 + \frac{1}{2}(t - 2s)p_{\mathrm{F}+} - st + \frac{t}{2\xi_0'}p_{\mathrm{F}-} - sp_{\mathrm{F}-}\xi_0' \\ &+ \frac{t^2}{2}\ln\!\left(1 + \frac{p_{\mathrm{F}-}}{t}\right) + \frac{p_{\mathrm{F}-}^2}{2}\ln\!\left(\frac{t - \xi_0' p_{\mathrm{F}-}}{t + p_{\mathrm{F}-}}\right) - \frac{t^2}{2}\ln\!\left(1 + \frac{p_{\mathrm{F}+} - 2s}{t}\right) \\ &+ \frac{1}{4}(p_{\mathrm{F}+}^2 - 4s^2 + 4st)\left\{-2\ln\!\left(\frac{2s + p_{\mathrm{F}+}}{4s\xi_0' - p_{\mathrm{F}-}}\right)\right. \\ &\left. + \ln\frac{[p_{\mathrm{F}+}^2 - 4s^2 + 4st - t^2][tp_{\mathrm{F}+} + p_{\mathrm{F}+}^2 - 4s^2 + 2st][tp_{\mathrm{F}-} + (p_{\mathrm{F}+}^2 - 4s^2)\xi_0']}{[t^2 - (p_{\mathrm{F}+}^2 - 4s^2 + 4st)\xi_0'^2][tp_{\mathrm{F}+} - p_{\mathrm{F}+}^2 + 4s^2 - 2st][(p_{\mathrm{F}+}^2 - 4s^2 + 4st)\xi_0' - tp_{\mathrm{F}-}]}\right\}, \end{aligned}$$

$$f_1^{(c)}(t,s) = -tp_{\mathrm{F}-} + \frac{1}{2}\,(p_{\mathrm{F}-}^2 - t^2)\ln\!\left(\frac{t - p_{\mathrm{F}-}}{t + p_{\mathrm{F}-}}\right)\Bigg\}\,,$$

and

$$\begin{aligned} f_2^{(a)}(t,s) &= \frac{t}{2}p_{\mathrm{F}-} + 2s^2 + \frac{1}{2}(2s - t)p_{\mathrm{F}+} - st - \xi_0' sp_{\mathrm{F}-} \\ &+ \frac{1}{2}p_{\mathrm{F}-}^2\ln\!\left(\frac{t - \xi_0' p_{\mathrm{F}-}}{t - p_{\mathrm{F}-}}\right) - \frac{1}{2}t^2\ln\!\left(\frac{t - 2s - p_{\mathrm{F}+}}{t - p_{\mathrm{F}-}}\right) \end{aligned}$$

$$+\frac{1}{4}\left(p_{\mathrm{F}+}^2-4s^2+4st\right)\left\{2\ln\left(\frac{2s+p_{\mathrm{F}+}}{p_{\mathrm{F}-}}\right)\right.$$
$$\left.+\ln\frac{[t^2-p_{\mathrm{F}+}^2+4s^2-4st][tp_{\mathrm{F}+}-p_{\mathrm{F}+}^2+4s^2-2st][tp_{\mathrm{F}-}+(p_{\mathrm{F}+}^2-4s^2)\xi_0']}{[t^2-(p_{\mathrm{F}+}^2-4s^2+4st)\xi_0'^2][tp_{\mathrm{F}+}+p_{\mathrm{F}+}^2-4s^2+2st][tp_{\mathrm{F}-}-(p_{\mathrm{F}+}^2-4s^2+4st)\xi_0']}\right\},$$

$$f_2^{(b)}(t,s)=-2s^2-ts+sp_{\mathrm{F}-}+\frac{t}{2}p_{\mathrm{F}-}-\frac{t}{2\xi_0}p_{\mathrm{F}+}-\xi_0 sp_{\mathrm{F}+}$$
$$-\frac{t^2}{2\xi_0^2}\ln\left(1-\frac{\xi_0 p_{\mathrm{F}+}}{t}\right)+\frac{t^2}{2}\ln\left(1+\frac{2s-p_{\mathrm{F}-}}{t}\right)$$
$$+\frac{1}{4}\left(p_{\mathrm{F}-}^2-4s^2-4st\right)\left\{-2\ln\left(\frac{2s-p_{\mathrm{F}-}}{p_{\mathrm{F}+}}\right)\right.$$
$$\left.+\ln\frac{[t^2-(p_{\mathrm{F}-}^2-4s^2-4st)\xi_0^2][tp_{\mathrm{F}+}-(p_{\mathrm{F}-}^2-4s^2-4st)\xi_0][tp_{\mathrm{F}-}+p_{\mathrm{F}-}^2-4s^2-2st]}{[t^2-p_{\mathrm{F}-}^2+4s^2+4st][tp_{\mathrm{F}+}+(p_{\mathrm{F}-}^2-4s^2)\xi_0][tp_{\mathrm{F}-}-p_{\mathrm{F}-}^2+4s^2+2st]}\right\}$$
$$-\frac{t}{2}p_{\mathrm{F}+}(1-1/\xi_0)-\frac{1}{2}p_{\mathrm{F}+}^2\ln\left(\frac{\xi_0 p_{\mathrm{F}+}-t}{p_{\mathrm{F}+}-t}\right)-\frac{t^2}{2}\ln\left(1-\frac{p_{\mathrm{F}+}}{t}\right)+\frac{t^2}{2\xi_0^2}\ln\left(1-\frac{\xi_0 p_{\mathrm{F}+}}{t}\right),$$

$$f_2^{(c)}(t,s)=-tp_{\mathrm{F}+}+\frac{1}{2}\left(p_{\mathrm{F}+}^2-t^2\right)\ln\left(\frac{t-p_{\mathrm{F}+}}{t+p_{\mathrm{F}+}}\right).$$

In these formulae,

$$\xi_0\equiv\frac{p_{\mathrm{F}-}^2-p_{\mathrm{F}+}^2-4s^2}{4sp_{\mathrm{F}+}}, \qquad \xi_0'\equiv\frac{p_{\mathrm{F}-}^2-p_{\mathrm{F}+}^2+4s^2}{4sp_{\mathrm{F}-}}. \tag{23}$$

Once the functions $f_1(t,s)$ and $f_2(t,s)$ are given in their analytic forms, the functions $K^{(1)}(p_{\mathrm{F}-},p_{\mathrm{F}+})$ and $K^{(2)}(p_{\mathrm{F}-},p_{\mathrm{F}+})$ can be evaluated using the Mathematica package built-in instruction for numerical integration over a specified domain.

Evaluation of the contribution to the energy density of the interactions proportional to the couplings C_2 and C_2' is straightforward (no complicated integrals are involved). The result is

$$\begin{aligned}\frac{E_\Omega^{(C_2)}}{V}&=\frac{C_2}{240\pi^4}\,p_{\mathrm{F}-}^3p_{\mathrm{F}+}^3\left(p_{\mathrm{F}-}^2+p_{\mathrm{F}+}^2\right),\\ \frac{E_\Omega^{(C_2')}}{V}&=\frac{C_2'}{120\pi^4}\left[p_{\mathrm{F}+}^8+p_{\mathrm{F}-}^8+\frac{1}{2}\,p_{\mathrm{F}+}^3p_{\mathrm{F}-}^3\left(p_{\mathrm{F}+}^2+p_{\mathrm{F}-}^2\right)\right].\end{aligned} \tag{24}$$

These formulae agree for $p_{\mathrm{F}-}=p_{\mathrm{F}+}=k_{\mathrm{F}}$ with the ones for $g_s=2$ obtained in [7].

Combining (7) with (17) and (24), adding the result (3) (for $\alpha = +, -$), the known order $k_{\rm F}^5(k_{\rm F}R)$ contribution $C_0 p_{\rm F-}^2 p_{\rm F+}^2/36\pi^4$, the contribution of order $k_{\rm F}^5(k_{\rm F}R)^2$

$$\frac{64 m_f C_0^2}{(2\pi)^6\hbar^2}\int\limits_0^{s_{\rm max}} {\rm d}s\, s^2\,\frac{1}{4\pi}\int {\rm d}^3\boldsymbol{t}\,\theta\,\theta\left(-\Lambda + g_{\rm fin}(t,s) + \frac{t^2}{\Lambda}\right), \tag{25}$$

obtained in [13], and finally expressing the couplings C_0, C_2 and C_2' in terms of the s- and p-wave scattering lengths a_0, a_1 and the s-wave effective radius r_0 using (2), one easily finds (using the results of [13]) that up to the order of $k_{\rm F}^5(k_{\rm F}R)^3$ all terms divergent in the limit $\Lambda \to \infty$ cancel out. One observes that the finite contribution arising in (25) from the term proportional to $1/\Lambda$ after it is multiplied by the term $\propto \Lambda$ present in C_0^2 cancels against the finite term $-2t^2$ arising from squaring function (10) in the contribution of the "particle–particle" diagram. Such terms must cancel because they would be absent had one used Dimensional Regularization instead of the cutoff Λ. Defining then,

$$\begin{aligned} G(p_{\rm F-},\, p_{\rm F+}) &= G_{\rm fin}^{(1)}(p_{\rm F-},\, p_{\rm F+}) + G^{(2)}(p_{\rm F-},\, p_{\rm F+})\,,\\ K(p_{\rm F-},\, p_{\rm F+}) &= K^{(1)}(p_{\rm F-},\, p_{\rm F+}) + K^{(2)}(p_{\rm F-},\, p_{\rm F+})\,, \end{aligned} \tag{26}$$

where $G_{\rm fin}^{(1)}(p_{\rm F-},\, p_{\rm F+})$ is given by the formula (8) with $g(t,s)$ replaced by $g_{\rm fin}(t,s)$, one arrives at the final formula

$$\begin{aligned} \frac{E_\Omega}{V} &= \frac{1}{6\pi^2}\frac{\hbar^2}{2m_f}\left\{\frac{3}{5}\left(p_{\rm F-}^5 + p_{\rm F+}^5\right) + \frac{4}{3\pi}\,p_{\rm F-}^3 p_{\rm F+}^3\, a_0 + \frac{192}{\pi^2}\,a_0^2\,J(p_{\rm F-},\, p_{\rm F+})\right.\\ &\quad + \frac{384}{\pi^3}\,a_0^3\,G(p_{\rm F-},\, p_{\rm F+}) - \frac{96}{\pi^3}\,a_0^3\,K(p_{\rm F-},\, p_{\rm F+})\\ &\quad + \frac{1}{10\pi}\,a_0^2 r_0\,p_{\rm F-}^3 p_{\rm F+}^3\left(p_{\rm F-}^2 + p_{\rm F+}^2\right)\\ &\quad \left. + \frac{1}{5\pi}\,a_1^3\left[2p_{\rm F-}^8 + 2p_{\rm F+}^8 + p_{\rm F-}^3 p_{\rm F+}^3\left(p_{\rm F-}^2 + p_{\rm F+}^2\right)\right]\right\}. \end{aligned} \tag{27}$$

The function $J(p_{\rm F-}, p_{\rm F+})$ is defined in [13]. It is also easy to see that $J(p_{\rm F-},\, p_{\rm F+}) = p_{\rm F+}^7 J(r,1)$, $G(p_{\rm F-},\, p_{\rm F+}) = p_{\rm F+}^8 G(r,1)$, and $K(p_{\rm F-},\, p_{\rm F+}) = p_{\rm F+}^8 K(r,1)$, where $r = p_{\rm F-}/p_{\rm F+}$. The plot of the function $J(r,1)$ has been given in [13]. The functions $(192/\pi^3)G(r,1)$ and $(48/\pi^3)K(r,1)$ are shown here in figures 2 and 3, respectively.

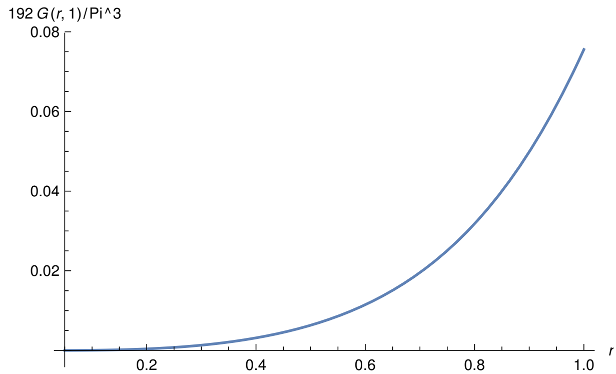

Fig. 2. Plot of the function $(192/\pi^3)G(r,1)$.

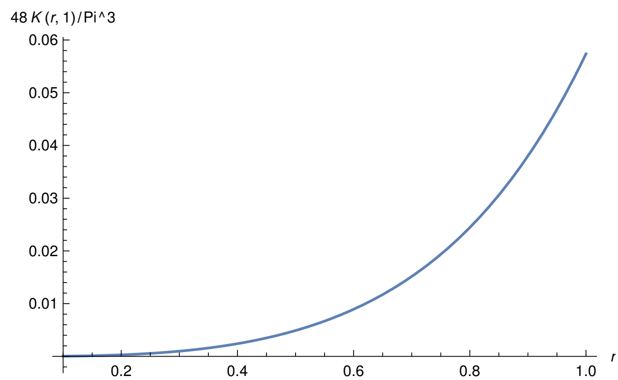

Fig. 3. Plot of the function $(48/\pi^3)K(r,1)$.

In the limit of $p_{\mathrm{F}-} = p_{\mathrm{F}+} = k_{\mathrm{F}}$ the result (27) should coincide with

$$\begin{aligned}\frac{E_\Omega}{V} &= \frac{1}{6\pi^2}\,\frac{\hbar^2}{2m_f}\left\{g_s\,\frac{3}{5}\,k_{\mathrm{F}}^5 + g_s(g_s-1)\,\frac{2}{3\pi}\,k_{\mathrm{F}}^6 a_0\right.\\ &\quad + g_s(g_s-1)\,\frac{4(11-2\ln 2)}{35\pi^2}\,k_{\mathrm{F}}^7 a_0^2\\ &\quad + [g_s(g_s-1)\,N_1 + g_s(g_s-1)(g_s-3)\,N_2]\,k_{\mathrm{F}}^8\,a_0^3\\ &\quad \left. + g_s(g_s-1)\,\frac{1}{10\pi}\,k_{\mathrm{F}}^8\,a_0^2 r_0 + g_s(g_s+1)\,\frac{1}{5\pi}\,k_{\mathrm{F}}^8\,a_1^3\right\}, \qquad (28)\end{aligned}$$

for $g_s = 2$ given in [7] and [12] with $N_1 = 0.07550 \pm 0.00003$ and $N_2 = 0.05741 \pm 0.00002$ in [7], and $N_1 = 0.0755732$ and $N_2 = 0.0573879$ in [12]. Numerical evaluation of the functions $(192/\pi^3)G(1,1)$ and $(48/\pi^3)K(1,1)$

— the endpoints in figures 2 and 3, respectively — gives $N_1 = 0.0755617$ and $N_2 = 0.057387$ in good agreement with the numbers obtained in [7] and [12]. This accuracy of the simple Mathematica routine is certainly sufficient for E_Ω computed up to the order of $(k_\mathrm{F}R)^3$; after inclusion of yet higher order corrections, more sophisticated numerical procedures will probably have to be employed to reach the accuracy achieved in [12]. (In [13], it has been found, by evaluating the relevant integral numerically, that $J(1,1) = 0.0114449$ which with sufficiently high accuracy equals $(11 - 2\ln 2)/840$).

Expressed in terms of $k_\mathrm{F} = (3\pi^2 N/V)^{1/3}$ and r the formula (27) takes the form of

$$\begin{aligned}\frac{E_\Omega}{V} = \frac{k_\mathrm{F}^3}{3\pi^2}\frac{\hbar^2 k_\mathrm{F}^2}{2m_f}\frac{3}{5}\Bigg\{&\frac{1}{2}\left(1+r^5\right)\left(\frac{2}{1+r^3}\right)^{5/3} + \frac{10}{9\pi}r^3\left(\frac{2}{1+r^3}\right)^2 (k_\mathrm{F}a_0)\\ &+\frac{160}{\pi^2}\left(\frac{2}{1+r^3}\right)^{7/3} J(r,1)\,(k_\mathrm{F}a_0)^2\\ &+\frac{80}{\pi^3}\left(\frac{2}{1+r^3}\right)^{8/3}\left[4\,G(r,1) - K(r,1)\right](k_\mathrm{F}a_0)^3\\ &+\frac{1}{12\pi}\left(\frac{2}{1+r^3}\right)^{8/3}\left[r^3\left(1+r^2\right)\left(k_\mathrm{F}^3 a_0^2 r_0\right) + 2\left(2+2r^8+r^3+r^5\right)(k_\mathrm{F}a_1)^3\right]\Bigg\}.\end{aligned} \tag{29}$$

(The ground-state energy per particle, E_Ω/N, can be readily obtained by removing the factor $k_\mathrm{F}^3/3\pi^2$). The third-order corrections computed in this work (the last two lines in the above formula) are rather small. For $r_0 = a_1 = 0$ (*i.e.* without the contribution on the dimension R^{-6} operators), the ratio of the order of $(k_\mathrm{F}a_0)^3$ contribution to the first term in the curly brackets increases from 0.00003 at $k_\mathrm{F}a_0 = 0.1$ to 0.03 at $k_\mathrm{F}a_0 = 1$. This can be compared to the analogous ratio of the order of $(k_\mathrm{F}a_0)^2$ term which at $r = 1$ increases from 0.00185 to 0.185. These ratios decrease further with decreasing r (increasing polarization) and become exactly zero at $r = 0$ (*i.e.* at $P = 1$) due to the Pauli exclusion which forbids any contribution to the ground-state energy to be generated by the interaction operator proportional to C_0.

The plot of the system's ground-state energy density as a function of the polarization P related to r by $r(P) = ((1-P)/(1+P))^{1/3}$ is shown in Fig. 4 for three different values of the expansion parameter $k_\mathrm{F}a_0$ (keeping $r_0 = a_1 = 0$). All curves merge at $P = 1$ as a result of the Pauli exclusion principle. The curve corresponding to $k_\mathrm{F}a_0 = 0.6$ can be directly compared to the lowest curve shown in Fig. 3 of Ref. [4] which shows a numerical estimate of the exact ground state energy obtained using the Quantum Monte Carlo method for a specific model repulsive potential. Consistently with the comparison of the ground-state energies of the unpolarized system ($P = 0$ or $r = 1$) made in Fig. 2 of Ref. [4], our green curve (for $k_{\rm F}a_0 = 0.6$) is systematically below its counterpart in Fig. 3 of Ref. [4] but the comparison with the red curve of figure 4 shows that the third order correction computed in this work has the tendency to reduce the difference between the perturbative and Monte Carlo estimates. In general, the comparison with the results of Ref. [4] show that the perturbative expansion is reliable up to $k_{\rm F}a_0 \lesssim 0.5$.

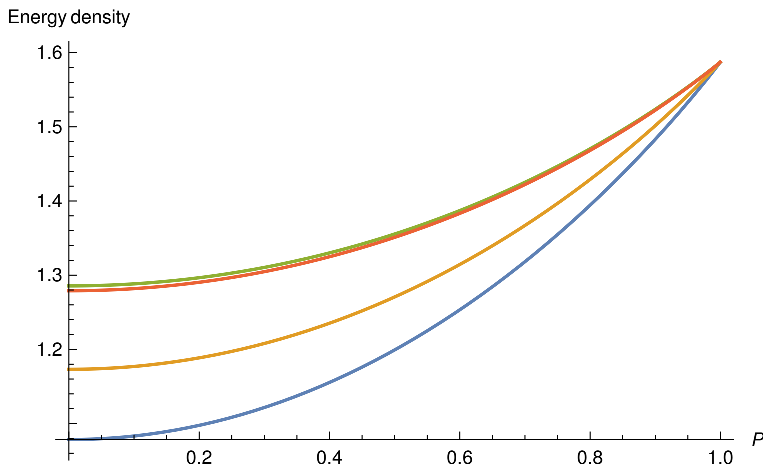

Fig. 4. Energy density E_Ω/V in units $(3/5)\hbar^2 k_{\rm F}^5/6\pi^2 m_f = (N/V)(\hbar^2 k_{\rm F}^2/2m_f)(3/5)$ of the gas of spin 1/2 fermions as a function of its polarization $P = (N_+ - N_-)/N$ for different values (from below) 0.2 (blue), 0.4 (yellow), 0.6 (green) of the expansion parameter $k_{\rm F}a_0$. The last curve (red) corresponding to $k_{\rm F}a_0 = 0.6$ shows the same quantity but without the order $(k_{\rm F}a_0)^3$ correction.

3. Summary

In this work, we have reproduced the third order formula for the ground-state energy of the unpolarized gas of spin s fermions and extended it to the case of the arbitrarily polarized gas of spin 1/2 fermions. We have checked the cancellation of all ultraviolet divergences occurring when the result is expressed in terms of the s-wave scattering length a_0 and worked out analytically the most important integrals occurring in the computation of the relevant Feynman (Hughenholtz) diagrams. This allowed to compute the remaining integrals numerically using the standard Mathematica built-in routine; the resulting final third order formula for the ground-state energy of the arbitrarily polarized gas of spin 1/2 fermions is given in terms of two new functions of the system's polarization for which the convenient interpolating formula can be easily obtained. The numerical results suggest that for $k_{\mathrm{F}}R \lesssim 0.5$, the perturbation series for the ground-state energy is well convergent but is not reliable for higher values of the expansion parameter for which the system is expected to exhibit the phase transition to the ordered phase. One may hope, however, that supplemented with a reliable extrapolation procedure, the perturbative series will be able to give valuable information about the nature of the phase transition. Further extension of our work to the case of an arbitrary mixture of different spin projections of spin s fermions (in the spirit of [15]) is straightforward.

Note added. After submission of this paper we have learned that the result obtained here can be, though not so easily, extracted from the works [22].

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