THE ROLE OF SCALAR AND MASS INTERACTIONS IN A RELATIVISTIC MODEL OF THE CHARMONIUM SPECTRUM*

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Lorentz scalar and mass interactions are studied in more detail in the framework of a reduced Dirac equation for heavy quark–antiquark mesons. A microscopic model for these interactions is proposed and analyzed. The charmonium mass spectrum is reproduced by means of two free parameters; a third parameter is fixed by means of a phenomenological hypothesis in accordance with the model.

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1. Introduction

The study of hadronic spectroscopy still represents a challenge in theoretical physics — considering that Quantum Chromo-Dynamics (QCD) cannot be applied directly to this study and numerical lattice simulations require huge computational efforts.

On the other hand, when developing phenomenological models, many different aspects must be taken into account at the same time. In the framework of a constituent model with a fixed number of valence quarks, one has to face primarily the problem of implementing a relativistic wave equation for the hadronic system; within the relativistic equation model, it is also necessary to select a suitable interaction for the quarks.

We stress once again that a completely consistent relativistic two-body (or many-body) wave equation does not exist but, on the other hand, it necessarily represents the starting point for the study of hadronic spectroscopy.

A series of works has been previously developed by the author with the aim of constructing a consistent model for that phenomenology. In particular, in work [1], a reduced Dirac-like equation (RDLE) was introduced for studying the spectroscopy of quark composed systems. This equation has a

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local form in the coordinate space. Furthermore, our RDLE is particularly suitable when a vector plus scalar interaction is considered. The same procedure of reduction was also applied to other relativistic equations obtaining very similar numerical results. An accurate calculation of the charmonium spectrum was performed using a small number of free parameters in Ref. [2]. Furthermore, in a subsequent work [3], the Lorentz structure of the interaction terms was studied in more detail, considering a covariant form of the relativistic equation of the model.

In all those works a specific form of regularized *vector interaction* was used. That interaction had been introduced and studied previously in Ref. [4]. We highlight here that a vector interaction alone is not sufficient to give an accurate reproduction of the charmonium spectrum. To this aim, the contribution of a *scalar interaction* was included.

Starting from this observation, in the present study, we analyze in more detail the scalar interaction. We also consider the possibility of replacing the scalar interaction with a *mass interaction*.

Incidentally, we note that the use of a fully relativistic model with an accurate interaction may also help to study the main properties of higher excitation states in order to understand if these states can be described as quark–antiquark states or further (exotic) physical effects must be taken into account [5].

Going back to the present model, we also recall that, while the vector interaction can be related to an effective reduction of the one-gluon exchange QCD interaction, in the scalar and mass case, one should take into account, by means of specific techniques, many-gluon exchange processes.

Furthermore, also in the framework of a rigorous derivation of a nonrelativistic potential, obtained for example by matching the short-distance perturbative part to long-distance lattice QCD results [6], there is no clear way to distinguish between the contributions given by the time component of a vector interaction and those given by a hypothetical effective scalar interaction.

In any case, within our model, the scalar and the mass interactions can be considered phenomenological interactions given by an underlying field that has the same quantum numbers of the vacuum. This field is necessary not only to reproduce the charmonium spectrum but also (for the consistency of the model) to compensate the vector repulsive quark self-energy with an attractive counterterm, as determined by the *energy balance*, given in Eq. (23) of Ref. [2]. In the present work, the balance equation will be generalized to include the case of the mass interaction, as it will be shown in Eqs. (25) and (26) of Sec. 4.

The problem of a scalar quark interaction has been faced in many different ways. We quote here only a few works that may be partially related to the present study. Historically, a scalar interaction, quartic with respect to the fermionic field, was introduced in the Nambu–Jona-Lasinio (NJL) model [7, 8]. This model was extensively applied to the study of quark matter interactions, also highlighting possible unusual bound states in quark matter [9, 10]. The NJL model has been studied and modified in Ref. [11] including a running coupling obtained by a fractal approach to QCD.

A nonlinear Klein–Gordon equation for the study of condensates in hadronic particles has been proposed in Ref. [12]. However, considering the difficulty of applying the NJL approach to our relativistic equation for a two-body bound state, we shall not use directly this method in the present work, preferring a standard relativistic interaction with a spatial phenomenological potential.

We recall that the role of a scalar interaction, in the case of two interacting scalar particles, was deeply studied by means of a model quantum field theory [13, 14].

The *linear sigma model* was introduced for the interaction of hadronic particles considering the exchange of the (scalar) sigma meson. This model, considered as an effective field theory [15], was also applied to the quark scalar interaction. For a relatively recent application, see for example Ref. [16]. In the present work, when studying the underlying structure of the scalar and mass interaction, we establish, in a different way, a tentative connection with the first scalar hadronic resonances, now denoted as $f_0(500)$ (formerly sigma meson) and $f_0(980)$.

The present phenomenological study has been developed taking into account the complexity of the quark interaction, with no intent to draw definitive conclusions. To avoid repetitions, the reader is frequently referred to the previous works [1–4] that can also help to gain a better understanding of the whole subject. The contents of the present paper are organized as follows. In Sec. 1.1, the symbols and notation of the work are introduced. In Sec. 2, the Lorentz structure of the scalar interaction is revised and the mass interaction is introduced. In Sec. 3, a general discussion about the reproduction of the charmonium spectrum is given. In Sec. 4, we construct the scalar and mass interaction of the model, starting from an elementary interaction of point-like particles, then introducing finite density distributions for the interacting quarks. The corresponding self-energies are also determined and the balance equation is generalized to the case of the mass interaction. In Sec. 5, we try to construct the scalar and mass interactions introducing an underlying scalar field in order to improve the consistency of the model and to interpret the physical meaning of its parameters. Finally, the results are summarized and discussed. In the Appendix A, we give, for the mass interaction, the reduced expressions to be inserted in the RDLE of the model.

1.1. Symbols and notation

The following notation is used in the paper:

The invariant product between four vectors is standardly written as $V^\mu U_\mu = V^0 U^0 - \vec{V} \cdot \vec{U}$.

The lower index $i(j) = 1, 2$ is the *particle index*, referred to the quark and to the antiquark.

The Dirac wave functions will be represented by the letter Ψ.

The subindex X will be used to indicate, for different quantities, the scalar ($X = S$) or mass ($X = M$) character of the corresponding interaction. In the text, we shall also write, in general, "x-interaction", "x-charge density", *etc.*

Furthermore, the subindex E (always associated to X) will be used to indicate, for a given quantity, the " elementary" or "point-like" character of the corresponding x-interaction.

For the *general* case of two different x-charge densities, in Sec. 4, the subindex G will be used.

Throughout the work, we use the standard natural units, that is $\hbar = c = 1$.

2. The Lorentz structure of the scalar and mass interaction

We analyze here the Lorentz structure of the *scalar* and *mass* interactions. In the first case we have, for a two-body scalar interaction, the following standard expression:

$$\mathcal{V}_S = \bar{\Psi} V_S(r) \Psi = \Psi^\dagger \gamma_1^0 \gamma_2^0 V_S(r) \Psi \,. \tag{1}$$

The scalar character of this term is obvious. This interaction, given in Eq. (11) of Ref. [2] was used in that work to study in detail the charmonium spectrum. For the potential, we used there the notation $V^S_{(2)}(r)$ to indicate its two-body character. Here, for the same potential, we simply write $V_S(r)$.

We recall that the distance r between the quark and the antiquark is defined in the center of mass reference frame (CMRF), that will be always used in this work.

In more detail, as shown in Eq. (30) of Ref. [3], that interaction can be written in the momentum space, with the CMRF momentum transfer of Eq. (31); then, the covariant integration of Eq. (25) is performed, leading to the covariant integral equation shown in Eq. (34) of the same work.

On the other hand, the *mass interaction* operator can be formally introduced by means of the following substitution in the two-body Dirac equation

$$m_i \to m_i + U_i^M(r)\,. \tag{2}$$

For the charmonium, we have $m_i = m_q$. Symmetry with respect to interchange of c and $\bar{c}$ requires

$$U_i^M(r) = \frac{V_M(r)}{2}\,. \tag{3}$$

By using Eqs. (34), (35) and (36) of Ref. [3], one finds that the corresponding mass interaction term, in the covariant form of the equation, takes the form of

$$\mathcal{V}_M = \bar{\Psi}\frac{V_M(r)}{2}\frac{P_\mu\left(\gamma_1^\mu + \gamma_2^\mu\right)}{M}\Psi\,. \tag{4}$$

For the calculation of the charmonium spectrum, in the CMRF, where $P^\mu = (M, \vec{0})$, Eq. (4) can be written as

$$\mathcal{V}_M^{\mathrm{CMRF}} = \bar{\Psi}\frac{V_M(r)}{2}\left(\gamma_1^0 + \gamma_2^0\right)\Psi = \Psi^\dagger\frac{V_M(r)}{2}\left(\gamma_1^0 + \gamma_2^0\right)\Psi\,. \tag{5}$$

From the last equation, we derive the reduced expression of the mass interaction, shown in Eqs. (A.1)–(A.4) of Appendix A. That expression is obtained by using the vinculated wave functions of our relativistic model, applying the reduction operators as shown in Eq. (8) of Ref. [2].

We recall that the reduced expression of the scalar interaction was given in Eqs. (C.1)–(C.3) of Ref. [1].

Considering, for the reduced expressions of the two interactions, a "non-relativistic" expansion in powers of p/m (being n the power of each term) one can easily find that the leading term, with $n = 0$, is the same for both interactions; the following terms, with $n = 4$, have an opposite sign for the two cases.

We recall that for the scalar potential $V_S(r)$, a Gaussian function was used to fit the charmonium spectrum. Different functions with the same number of parameters were tested but the fit to the experimental data strongly favored the Gaussian spatial dependence. A constant function significantly worsened the reproduction of the data. The author has also tested that a large distance linear scalar potential (similar to that of the Cornell model) is unable to reproduce the data with the same accuracy.

Furthermore, in work [2], a model was studied with a *two region potential*, with the potential of the outer spatial region of Yukawa form in order to investigate if the scalar interaction can be related to the standard mechanism of one scalar meson exchange. However, the comparison with

the experimental data did not show a significant improvement with respect to the Gaussian potential function, suggesting that the scalar interaction is not originated in this way. This point will be examined more deeply also in Sec. 5 of this work.

3. Study of the charmonium spectrum

In the present work, we study again the scalar interaction and also consider the mass interaction of Eq. (5), inserting the reduced operator $\hat{W}^M$ of Eq. (A.4) in the RDLE. On the other hand, the regularized vector interaction, used in Ref. [2], is left unchanged. We recall that this interaction (that includes the self-energy term) is zero at $r = 0$ and approaches the value $\bar{V}_V$ as $r \to \infty$.

The technique for solving the RDLE and the fit procedure are exactly the same as in Ref. [2]. For the charmonium spectrum we use here the new experimental data [17] that present some small differences with respect to the old data [18] used in our previous work [2].

For the quality of the fit, we define

$$\Theta = \sqrt{\frac{\sum_k \left(E_k^{\rm th} - M_k^{\rm exp}\right)^2}{N_d}}\,, \tag{6}$$

where $E_k^{\rm th}$ and $M_k^{\rm exp}$ respectively represent the result of the theoretical calculation and the experimental value of the mass, for the k-th resonance and $N_d = 16$ is the number of the fitted resonances.

Taking into account the results obtained in the previous work for the scalar interaction and those obtained in the present work, with many different trials, for both interactions, we make the following general comments.

(i) *Results of the same quality* are obtained with the scalar and mass interaction. As discussed before, a difference between the scalar and mass reduced interactions would appear only at the order $n = 4$ of a nonrelativistic expansion. In consequence, one can argue that the motion of the charm quark and antiquark (due to their relatively high mass) is not sufficiently relativistic to distinguish between the two interactions. We conclude that, at least for the charmonium case, both the scalar and the mass interaction are able to reproduce, with high accuracy, the spectrum.

(ii) To obtain a good fit, $V_X(r)$ must have a Gaussian form, both for the scalar and for the mass interaction case. We can write the x-potential in the form of

$$V_X(r) = -\bar{V}_X \exp\left(-\frac{r^2}{r_X^2}\right)\,. \tag{7}$$

(iii) A fit of the same quality as that of Ref. [2] is now obtained also for the mass interaction, enforcing the same *balance equation*, shown in Eq. (23) of Ref. [2]. In this work, the balance equation will be rewritten, for a general x-interaction, in Eqs. (25) and (26) of Sec. 4.

(iv) Finally, the values obtained in the fit for r_M and $\bar{V}_M$ are numerically very similar to r_S and $\bar{V}_S$.

The results of the spectrum are shown in Table 1. The values of the parameters for the interaction are given in Table 2. For the mass of the quark we have taken the same value of the previous work [2], that is $m_q = 1.27$ GeV. This value represents the "running" charm quark mass in the $\overline{MS}$ scheme [17].

For reasons that will be explained in Sec. 5, when choosing the *free parameters* of the fit, we set here a slightly different strategy with respect to the previous work [2]. In that work the free parameters were α_V, d and r_S, representing respectively the adimensional coupling constant of the vector interaction, the regularization distance of the vector interaction' and the distance parameter of the Gaussian scalar potential. The dependent parameters were $\bar{V}_V$ and $\bar{V}_S$; $\bar{V}_V$ is the two quark vector self-energy that depends on α_V and d, as shown in Eq. (16) of that work; $\bar{V}_S$ is the two quark scalar self-energy, determined by the balance equation Eq. (23) of the same work.

On the other hand, in the present work, we take as *free parameters* $\bar{V}_X$, d, and r_X. $\bar{V}_V$ is determined by the balance equation, then α_V is obtained by means of Eq. (16) of Ref. [2], as function of $\bar{V}_V$ and d, that is

$$\alpha_V = \sqrt{\pi}\frac{3}{4}\bar{V}_V d\,. \tag{8}$$

We now make some comments on the parameter $\bar{V}_X$. In our previous work [2], the fit procedure with the old data [18] gave $\bar{V}_S = 0.7268$ GeV (see Table II of Ref. [2]). With the new data [17], we now obtain $\bar{V}_S = 0.7050$ GeV. For the case of the mass interaction, we obtain $\bar{V}_M = 0.7237$ GeV. *Instead of these values*, according to the phenomenological model that will be discussed in Sec. 5, we give, in Table 1 and Table 2, only the results obtained by fixing $\bar{V}_X$ (for both $X = S$ and $X = M$) at the value $\bar{V}_X = 0.7350$ GeV, as it will be discussed in Sec. 5; see, in particular, Eq. (34). This value is not very different with respect to the results obtained by the fit; in consequence, this choice of $\bar{V}_X$ (instead of taking the fit results) does not alter significantly the reproduction of the charmonium spectrum.

Table 1. Comparison between the experimental average values [17] of the charmonium spectrum (last column) and the theoretical results of the model. All the masses are in MeV. The quantum numbers n, L, S and J, introduced in Ref. [2], respectively represent the principal quantum number, the orbital angular momentum, the spin, and the total angular momentum. The results of the columns "Scalar" and "Mass" respectively refer to the scalar (S) and mass (M) interaction. A line divides the resonances below and above the open Charm threshold. At the bottom, the quantity Θ, formally in MeV, defined in Eq. (6), gives an indication of the quality of the fit.

Name	$n^{2S+1}L_J$	Scalar	Mass	Experiment
$\eta_c(1S)$	1^1S_0	2998	2981	2983.9 ± 0.4
$J/\psi(1S)$	1^3S_1	3090	3102	3096.9 ± 0.006
$\chi_{c0}(1P)$	1^3P_0	3420	3405	3414.71 ± 0.30
$\chi_{c1}(1P)$	1^3P_1	3498	3497	3510.67 ± 0.05
$h_c(1P)$	1^1P_1	3510	3514	3525.38 ± 0.11
$\chi_{c2}(1P)$	1^3P_2	3564	3577	3556.17 ± 0.07
$\eta_c(2S)$	2^1S_0	3648	3641	3637.5 ± 1.1
$\psi(2S)$	2^3S_1	3679	3680	3686.10 ± 0.06
$\psi(3770)$	1^3D_1	3796	3795	3773.7 ± 0.4
$\psi_2(3823)$	1^3D_2	3831	3833	3823.7 ± 0.5
$\chi_{c1}(3872)$	2^3P_1	3893	3887	3871.65 ± 0.06
$\chi_{c2}(3930)$	2^3P_2	3928	3932	3922.5 ± 1.0
$\psi(4040)$	3^3S_1	4014	4014	4039 ± 1
$\chi c1(4140)$	3^3P_1	4144	4143	4146.5 ± 3.0
$\psi(4230)$	4^3S_1	4211	4216	4222.7 ± 2.6
$\chi c1(4274)$	4^3P_1	4267	4273	4286 ± 9
Θ		13.4	12.8	

Concerning the form of the x-potential, we consider that the Gaussian form is strongly favored by the fit to the data. For this reason, in all this work, we focus our attention on the Gaussian x-potential function: in the next Section 4, we try to develop a microscopic model for this interaction and for the corresponding contribution to the quark self-energies; as anticipated, in Sec. 5, we shall discuss a possible phenomenological model for the origin of this interaction.

Finally, we note that the model is unable to reproduce the resonance $\chi_{c0}(3915)$. The new experimental data [17] give, for this resonance, a mass of 3921.7 ± 1.8 MeV. Our model, taking the quantum numbers 2^3P_0, gives the

Table 2. Numerical values of the free and dependent parameters of the model; m_q is fixed at the value of Ref. [17], as explained in the text; according to the discussion of Sec. 5, $\bar{V}_X$ has the value given by Eq. (34) for both scalar and mass case; in consequence, $\bar{V}_V$ has the value determined by Eq. (26); d and r_X are the truly free parameters of the model; finally, α_V is a dependent parameter, determined by Eq. (8).

			Units
m_q	1.27		GeV
$\bar{V}_X$	0.735		GeV
$\bar{V}_V$	1.805		GeV
	Scalar	Mass	
d	0.1511	0.14045	fm
r_X	1.849	1.846	fm
α_V	1.838	1.708	

mass values of 3862 MeV and 3850 MeV, for the S and the M interactions, respectively. Our model and other quark models give a wrong order for the masses of this resonance and its partner $\chi_{c1}(3872)$. On the other hand, the analysis of the decay processes leaves open the possibility of different quantum number assignments and of a description in terms of multiquark states, as discussed, for example, in the work [19]. For all these reasons, this resonance has not been included in the fit of Table 1.

4. A model for the scalar and mass interaction

In this section, we try to develop a model for the scalar and mass interaction, starting from an elementary x-interaction. This model, analogously to the vector interaction case, takes into account a finite x-charge distribution of the quarks. The same form, for the scalar and mass interactions, will be considered. We assume that the quarks, with an extended distribution of x-charge, represent the source of the attractive x-interaction.

The procedure that determines the interaction also allows to analyze in more detail the *self-energy* of those charge distributions. This topic was not examined with sufficient accuracy in our previous works where the scalar interaction was only introduced and used to fit the charmonium spectrum.

As a starting point, we now make the hypothesis that an *elementary* x-potential $V_{EX}(r)$ between two point-like charges at distance r has the following Gaussian form

$$V_{EX}(r) = -\bar{V}_{EX} \exp\left(-\frac{r^2}{r_{EX}^2}\right), \tag{9}$$

where r_{EX} represents the distance parameter of the elementary x-interaction. Furthermore, analogously to the vector interaction case, we also consider a Gaussian distribution for the quark x-charge density

$$\rho_{X}(x) = \frac{1}{(2\pi d_X^2)^{3/2}} \exp\left(-\frac{\vec{x}^{\,2}}{2d_X^2}\right), \tag{10}$$

where d_X is related the radius of x-charge distribution density.

In order to construct the global x-potential avoiding cumbersome calculations, we introduce the corresponding Fourier transformed quantities where $\vec{q}$ represents the vertex momentum transfer and also, $q = |\vec{q}|$. For the potential, transforming Eq. (9), we obtain

$$V_{EX}(q) = -\bar{V}_{EX} \frac{r_{EX}^3}{8\pi^{3/2}} \exp\left(-\frac{q^2 r_{EX}^2}{4}\right). \tag{11}$$

From the x-charge density of Eq. (10), by means of the Fourier transform, we obtain the standard vertex form factor

$$F_{X}(q) = \exp\left(-\frac{q^2 d_X^2}{2}\right) \tag{12}$$

with the normalization $F_X(q=0) = 1$. The global interaction in the $\vec{q}$ space is obtained inserting the form factors at the two vertices. For further developments, we consider the *general* case of two different x-charge distributions for the two sources, with spatial parameters d_{1X} and d_{2X}. We have

$$V_G(q) = F_{1X}(q) V_{EX}(q) F_{2X}(q) = -\bar{V}_{EX} \frac{r_{EX}^3}{8\pi^{3/2}} \exp\left(-\frac{q^2 r_G^2}{4}\right), \tag{13}$$

where r_G is defined as:

$$r_G = \sqrt{r_{EX}^2 + 2\left(d_{1X}^2 + d_{2X}^2\right)}. \tag{14}$$

Eq. (13) is easily transformed to the $\vec{r}$ space. The result is

$$V_G(r) = -\bar{V}_{EX} \left(\frac{r_{EX}}{r_G}\right)^3 \exp\left(-\frac{r^2}{r_G^2}\right). \tag{15}$$

Finally, to reproduce the x-potential of the present model, given by Eq. (7), we take for the two quarks $d_{1X} = d_{2X} = d_X$. By means of Eq. (14), we obtain

$$r_X = \sqrt{r_{EX}^2 + 4d_X^2} \tag{16}$$

and, identifying in Eq. (15)

$$\bar{V}_X = \bar{V}_{EX} \left(\frac{r_{EX}}{r_X} \right)^3 , \tag{17}$$

we have the same expression for the x-potential given by Eq. (7).

We now calculate the self-energy of a spherical Gaussian x-charge distribution density. To this aim, we consider (taking the *general* Eq. (15)) the potential of a spherical Gaussian distribution interacting with a point-like particle, setting $d_{1X} = d_X$ and $d_{2X} = 0$ in Eq. (14). In this case, we have

$$r_{1X} = \sqrt{r_{EX}^2 + 2d_X^2} . \tag{18}$$

Defining

$$\bar{V}_{1X} = \bar{V}_{EX} \left(\frac{r_{EX}}{r_{1X}} \right)^3 , \tag{19}$$

we can write this potential as

$$V_{1X}(r) = -\bar{V}_{1X} \exp\left(-\frac{r^2}{r_{1X}^2} \right) . \tag{20}$$

We now replace the point-like x-charge with the x-charge $\mathrm{d}Q$ contained in a volume element. For the x-charge density we use the expression of Eq. (10). We have

$$\mathrm{d}Q = r^2 \mathrm{d}r \, \mathrm{d}\Omega \rho_X(r) . \tag{21}$$

The integration for determining the self-energy is performed taking into account the spherical symmetry of the problem. Furthermore, we insert a factor $1/2$ to avoid double counting in the integral for the total self-energy. In this way, the total self-energy of the x-charge distribution of one quark is written in the form of

$$W_{1X}^{\mathrm{self}} = \frac{1}{2} 4\pi \int_0^\infty \mathrm{d}r \, r^2 V_{1X}(r) \rho_X(r) . \tag{22}$$

A standard calculation finally gives

$$W_{1X}^{\mathrm{self}} = \frac{1}{2} V_X(0) = -\frac{1}{2} \bar{V}_X . \tag{23}$$

Obviously, for charmonium (and $q\,\bar{q}$ systems) the *total* self-energy due to the x-interaction is

$$W_X^{\mathrm{self}} = 2W_{1X}^{\mathrm{self}} = V_X(0) = -\bar{V}_X . \tag{24}$$

We now comment critically the obtained results.

As successful achievements, in the first place, we note that the Gaussian interaction used in the fit can be obtained by means of an *elementary* Gaussian x-interaction and a Gaussian x-charge distribution for the quarks. Moreover, this model determines the self-energy $W_X^{\rm self}$ that was used in the balance equation, given in Eq. (23) of previous work [2]. For clarity, we generalize here this equation for a x-interaction:

$$W_V^{\rm self} = 2m_q + W_X^{\rm self} \tag{25}$$

or, more explicitly

$$\bar{V}_V = 2m_q - \bar{V}_X \,. \tag{26}$$

On the other hand, the following two inconveniencies are found.

(i) The negative x-self-energy $W_X^{\rm self}$ of Eq. (24) correctly appears in the balance equation but gives no contribution to the energies of the charmonium spectrum. Different trials have been performed to include directly this quantity in the potential functions but the quality of the fit is always greatly worsened. Note that, on the contrary, for the vector interaction, the self-energy $\bar{V}_V$ makes part of the regularized potential function and gives $V_V(0) = 0$, as shown in Eqs. (13)–(17) of Ref. [2]. Some mechanism should be found to cancel the (unwanted) negative x-energy $W_X^{\rm self}$.

(ii) As shown in Eq. (16) for r_X, this model as such is unable to determine independently the parameters r_{EX} and d_X, introduced respectively in Eqs. (9) and (10). The potential used for the x-interaction only depends on r_X, that is, in any case, a free fit parameter, determined by the charmonium spectroscopy. From Eq. (16) one can only obtain the following inequalities:

$$d_X \leq \frac{r_X}{2} \,, \qquad r_E \leq r_X \,. \tag{27}$$

In the limiting (extreme) case of a point-like x-interaction ($r_E = 0$), we have $d_X = \frac{r_X}{2}$. On the other hand, for a point-like x-charge distribution ($d_X = 0$) we have $r_E = r_X$.

To solve these two difficulties and to investigate the physical origin of the Gaussian form of the elementary x-interaction, we developed a phenomenological model that will be discussed in the following section.

5. A model for the origin of the scalar and mass interaction

As a starting point, we introduce an underlying Gaussian field for the $q\,\bar{q}$ system. This field depends only on the interquark distance $r = |\vec{r}|$; in this way, it carries a vanishing orbital angular momentum and can represent a *completely scalar* field with the same quantum numbers of the vacuum. We shall try to relate this field to the *elementary* potential $V_{EX}(r)$, introduced in Eq. (9) of the previous section. We can write

$$\Phi(r) = A_{\text{coupl}} \exp\left(-\frac{r^2}{\bar{r}^2}\right) , \tag{28}$$

where A_{coupl} represents a dimensional coupling constant that will be fixed in the following.

In order to understand the dynamical origin of that field, we apply to $\Phi(r)$ the operator $\vec{p}^{\,2} = -\vec{\nabla}^2$, obtaining the following equation:

$$\left[\vec{p}^{\,2} + \sigma^2 r^2\right] \Phi(r) = \mu^2 \Phi(r) \tag{29}$$

with

$$\sigma = \frac{2}{\bar{r}^2} \tag{30}$$

and

$$\mu = \frac{\sqrt{6}}{\bar{r}} . \tag{31}$$

Examining Eq. (29), we note that the second term in the brackets of the *l.h.s.* represents a harmonic "potential" that determines $\Phi(r)$ as a "confined" Gaussian field; in the *r.h.s.* we have the squared energy μ^2 that can be related to the mass (μ) of the *quantum* associated to the Gaussian field. We recall that in the case of a vector Coulombic interaction, the situation is completely different and no term of this kind is present.

At this point, in order to avoid the inconveniency (i) found at the end of the previous section, we make the hypothesis that the (positive) value of the quantum μ cancels the negative x-self-energy of the quarks. To this aim, with the help of Eq. (24), we fix:

$$\mu = -W_X^{\text{self}} = \bar{V}_X . \tag{32}$$

In consequence, from Eq. (31), the parameter $\bar{r}$ of the Gaussian field can be expressed in the form of

$$\bar{r} = \frac{\sqrt{6}}{\bar{V}_X} . \tag{33}$$

Taking into account the phenomenology of the hadronic interactions, we note, from Eq. (32), that the values of μ obtained by fitting the charmonium

spectrum lie between the masses of the first two scalar meson resonances that have the vacuum quantum numbers. More precisely, we recall that for the $f_0(500)$ and the $f_0(980)$, the peak of the mass is roughly estimated at 0.475 GeV and at 0.995 GeV, respectively [17]. In consequence, the mean value of these two peaks is, indicatively, at $\langle m_0 \rangle = 0.7350$ GeV.

As anticipated in Sec. 3, we take *this value* for both the cases ($X = M$ or $X = S$) obtaining an accurate reproduction of the spectrum. We have

$$\bar{V}_X = \mu = \langle m_0 \rangle = 0.7350 \text{ GeV}\,. \tag{34}$$

From the numerical value of μ fixed in the previous equation, we also obtain, from Eq. (33), $\bar{r} = 0.6575$ fm and, from Eq. (30), $\sigma = 0.9126$ GeV/fm.

Phenomenologically, one could consider $\bar{V}_X = \langle m_0 \rangle$ as an input of the model and not as a free parameter. In this way, it is possible to say that the charmonium spectrum is fitted using *only two* truly free parameters: d and r_X, as shown in Table 2.

We have seen that our field $\Phi(r)$ can be phenomenologically related to the mesonic excitations that have the vacuum quantum numbers. In this sense, Eq. (29) can be tentatively generalized to give the whole spectrum of these excitations

$$\left[\vec{p}^{\,2} + \sigma^2 r^2\right] \Phi_n(r) = (\mathcal{E}_n)^2 \Phi_n(r) \tag{35}$$

with

$$\mathcal{E}_n = \frac{2}{\bar{r}} \sqrt{n + \frac{3}{2}}\,. \tag{36}$$

The scalar resonances ($L = 0$), that couple to the vacuum, have $n = 0, 2, 4, \ldots$ The case discussed above for Eq. (29) corresponds to ground state with $n = 0$; more explicitly, in that case we have $\mu = \mathcal{E}_0$.

We stress that Eqs. (35) and (36) are not able, as such, to describe the spectrum of the scalar resonances. To this aim one should take into account further terms in the "potential" (besides $\sigma^2 r^2$) and, in any case, consider the quark contributions. Our model could represent a phenomenological starting point for investigating this highly controversial item of hadronic physics with a different method. All this subject goes beyond the scope of the present work.

In order to solve the inconvenience (*ii*) of the previous section, we now study a possible relationship between the Gaussian field $\Phi(r)$ and the phenomenological potential $V_X(r)$.

We first consider the case of a *direct coupling* to the fermion fields. In this case, the Gaussian field $\Phi(r)$ of Eq. (28) represents the elementary x-potential of Eq. (9), that is

$$V_{EX}(r) = \Phi(r) \tag{37}$$

that implies

$$A_{\text{coupl}} = A_{\text{dir}} = -\bar{V}_{EX}\,, \qquad \bar{r} = r_{_{EX}}\,. \tag{38}$$

This direct coupling is of the same kind of the one studied in Refs. [13, 14]. However, in our model, for the field $\Phi(r)$ no self-interaction term is considered. On the other hand, as shown in Eq. (29), we have introduced the "attractive term" $\sigma^2 r^2$.

By using Eq. (16) and r_{EX} from Eq. (38) it is now possible to determine the distance parameter of x-charge distribution density:

$$d_X = \frac{1}{2}\sqrt{r_X^2 - \bar{r}^2}\,. \tag{39}$$

Furthermore, by using Eq. (17), the elementary coupling of the x-interaction can be written as

$$\bar{V}_{EX} = \bar{V}_X \left(\frac{r_X}{\bar{r}}\right)^3\,, \tag{40}$$

where, as discussed above, we have $\bar{V}_X = \mu$. For completeness, we give the corresponding numerical values. Using the results of Table 2, for the scalar interaction, we have: $d_S = 0.8640$ fm and $\bar{V}_{ES} = 16.35$ GeV; for the mass interaction, we have: $d_M = 0.8625$ fm and $\bar{V}_{EM} = 16.27$ GeV.

Another possibility, at purely phenomenological level, consists in taking a *quadratic coupling* of the form

$$V_{EX}(r) = -\Phi^\dagger(r)\Phi(r)\,. \tag{41}$$

In this case,

$$A_{\text{coupl}} = A_{\text{quad}} = \sqrt{\bar{V}_{EX}}\,, \qquad \bar{r} = \sqrt{2}\; r_{_{EX}}\,. \tag{42}$$

Taking into account the expression of $\bar{r}$, the distance parameter of the x-charge distribution density is obtained from Eq. (16), in the form of

$$d_X = \frac{1}{2}\sqrt{r_X^2 - \frac{\bar{r}^2}{2}} \tag{43}$$

and the elementary coupling of the x-interaction, from Eq. (17), can be written as

$$\bar{V}_{EX} = \bar{V}_X \left(\frac{r_X\sqrt{2}}{\bar{r}}\right)^3 \tag{44}$$

with, as before, $\bar{V}_X = \mu$. The numerical values, for the scalar interaction, are: $d_S = 0.8948$ fm, $\bar{V}_{ES} = 46.23$ GeV; for the mass interaction, they are: $d_M = 0.8932$ fm, $\bar{V}_{EM} = 46.01$ GeV.

We point out that, in any case, the Gaussian character of the elementary interaction is essential to obtain the Gaussian spatial x-potential. In more detail, we have also tried to use a Yukawa function for the elementary interaction (as it would be given by the standard linear sigma model) with Gaussian x-densities for the quarks but the total x-interaction obtained in this way is not able to reproduce, with sufficient accuracy, the charmonium spectrum. Moreover, in Ref. [2], we tried to reproduce the experimental charmonium spectrum with a "two region" scalar potential. For the outer region, we used an exponential function with a spatial decay parameter $r_b = 0.7594$ fm, possibly corresponding to a mass $m_b = 1/r_b = 0.2598$ GeV. We observe that this last value cannot be associated to any relevant observable hadronic state. On the other hand, the Gaussian model studied in the present work allows for a possible physical interpretation in the framework of the hadronic phenomenology.

We can now try to summarize the results of the work and to draw some conclusions. Using a reduced relativistic, energy-dependent, equation, an accurate reproduction of the charmonium spectrum is obtained with a regularized vector interaction and a scalar or mass x-interaction. For the latter interactions, a Gaussian spatial potential is required to fit the data. A balance relationship among the quark mass and the vector and x-interaction self-energies is established. In this last section, we have shown that the elementary Gaussian x-interaction $V_{EX}(r)$ can be associated to a scalar field $\Phi(r)$ whose energy quantum μ cancels the negative self-energy of the x-charge distributions of the quarks. Phenomenologically, the mass μ is of the order of the first scalar hadronic resonances.

Furthermore, the scalar field $\Phi(r)$ can be related to $V_{EX}(r)$ by means of a direct or quadratic coupling allowing to determine, in the two cases, the distance parameter d_X of the x-charge distribution density of the quarks and the coupling $\bar{V}_{EX}$ of the elementary x-interaction.

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Appendix A

Reduction of the two-body mass interaction

For the mass interaction of Eq. (5), we apply the same procedure used in Ref. [2] for the vector and scalar interactions. We use the reduction operators K_1 and K_2, as in Eq. (8) of Ref. [2]. For more generality, we start by taking

$U_1(r) \neq U_2(r)$ and obtain

$$\begin{aligned}\hat{W}^M &= K_1^\dagger K_2^\dagger \left[\gamma_1^0 U_1^M(r) + \gamma_2^0 U_2^M(r)\right] K_2 K_1 \\ &= U_+^M(r) - \frac{1}{(m_1+E_1)^2}\vec{\sigma}_1\cdot\vec{p}_1 U_-^M(r)\vec{\sigma}_1\cdot\vec{p}_1 \\ &\quad + \frac{1}{(m_2+E_2)^2}\vec{\sigma}_2\cdot\vec{p}_2 U_-^M(r)\vec{\sigma}_2\cdot\vec{p}_2 \\ &\quad - \frac{1}{(m_1+E_1)^2(m_2+E_2)^2}(\vec{\sigma}_1\cdot\vec{p}_1)(\vec{\sigma}_2\cdot\vec{p}_2)U_+^M(r)(\vec{\sigma}_2\cdot\vec{p}_2)(\vec{\sigma}_1\cdot\vec{p}_1)\,,\end{aligned} \tag{A.1}$$

where we have defined

$$U_\pm^M(r) = U_1^M(r) \pm U_2^M(r)\,. \tag{A.2}$$

For equal mass quarks, we have $m_1 = m_2 = m_q\,,\ \ E_1 = E_2 = E/2$. Recalling Eq. (3), one also has

$$U_-^M(r) = 0,\ U_+^M(r) = V_M(r)\,. \tag{A.3}$$

Furthermore, using $\vec{p}_2 = -\vec{p}_1 = \vec{p}$ we obtain the following expression that is used for the calculation of the charmonium spectrum:

$$\hat{W}^M = V_M(r) - \frac{1}{\left(m_q + \frac{E}{2}\right)^4}\left(\vec{\sigma}_1\cdot\vec{p}\right)\left(\vec{\sigma}_2\cdot\vec{p}\right)V_M(r)\left(\vec{\sigma}_2\cdot\vec{p}\right)\left(\vec{\sigma}_1\cdot\vec{p}\right)\,. \tag{A.4}$$

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