ON FRACTIONAL AND DISTRIBUTED ORDERS NONLINEAR CHAOTIC JERK MODELS AND THEIR CONTROL

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In this paper, we have introduced two versions of the jerk model. These versions are commensurate fractional and distributed orders. They appear in several applications of physics and engineering, *e.g.*, laser physics, damped harmonic oscillators, and secure communications. The sufficient condition for the existence and uniqueness of the solution of commensurate fractional-order (CFO) jerk model is studied. We state and prove a theorem to test the dependence of the solutions of the CFO jerk model on initial conditions. The dynamics of the three versions of the jerk model are investigated. Using the largest Lyapunov exponent (LLE), we determine the values of the parameters at which these proposed versions have chaotic solutions. The linear feedback control is used to stabilize the chaotic solutions of these versions. Numerical simulations are used to show the chaotic solutions after control.

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1. Introduction

In the last few decades, fractional calculus has been used in many applications such as chaotic models [1], biological population models [2], fluid mechanics [3], neural networks [4], and signal processing [5]. Compared to integer-order derivatives, fractional-order (FO) ones offer a great tool for describing memory and the inherited characteristics of distinct materials and processes [6]. Therefore, using FO derivatives instead of integer-order ones may yield more accurate results. The FO hyperchaotic complex systems have been studied [7, 8]. On the other hand, Caputo put forward the concept

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behind the distributed-order (DO) calculus [9]. The DO dynamical models have numerous uses in physics and engineering [10, 11]. Mahmoud *et al.* [10] introduced the DO hyperchaotic forced and unforced complex van der Pol models. The hyperchaotic masking for a text is investigated using these models. The DO neural networks which are the generalization of integer and FOs neural networks were investigated [11]. Many FO and DO hyperchaotic models were presented [12].

Gottlieb jerk models [13] can be given by $\frac{\mathrm{d}^3x}{\mathrm{d}t^3} = J(x, \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}^2x}{\mathrm{d}t^2})$, where the jerk function is the time derivative of the acceleration [14]. The jerk model can be used to practically represent real-world phenomena such as a special case of the Nosé–Hoover thermostated dynamic model [15]. There exist many forms of jerk models in terms of type of its nonlinearity. A jerk circuit using a sine function is introduced [16]. Louodop *et al.* [17] investigated a jerk model with the cubic function. The authors of Ref. [18] proposed a jerk model with an exponential function. Quadratic jerk models and their complex behaviour analysis are given by Innocenti *et al.* [19]. Ramadoss *et al.* [20] used a six-order Tchebytchev polynomial to propose a novel chaotic jerk model as follows:

$$\begin{aligned} \dot{u}_1 &= u_2\,, \\ \dot{u}_2 &= au_3\,, \\ \dot{u}_3 &= -cu_2 - du_3 - \tfrac{1}{2}\eta(u_1)\,, \\ \eta(u_1) &= 32u_1^6 - 48u_1^4 + 18u_1^2 - 1 + b\,, \end{aligned} \tag{1}$$

where a, b, c, d are real constant parameters and $u_1, u_2, u_3 \in \mathbf{R}$. Model (1) can be written in a simple form as

$$\dot{u} = \digamma(u)\,, \tag{2}$$

where $u = (u_1, u_2, u_3)^T$ and $\digamma(u) = (\digamma_1, \digamma_2, \digamma_3)^T = (u_2, au_3, -cu_2 - du_3 - \frac{1}{2}(32u_1^6 - 48u_1^4 + 18u_1^2 - 1 + b))^T$.

In this work, we propose two versions of model (2):

1. The chaotic commensurate fractional-order (CFO) version:

$$^{\mathrm{C}}D^{\alpha}u = \digamma(u)\,, \tag{3}$$

where $^{\mathrm{C}}D^{\alpha}$ is the Caputo FO derivative of the order of $0 < \alpha \leq 1$ [9].

2. The chaotic distributed-order (CDO) version:

$$D^{w(\alpha)}u = \digamma(u)\,, \tag{4}$$

where $D^{w(\alpha)}$ is the DO derivative [21].

We study the CFO version (3) of the jerk model (1). The other version (4) of this model is also investigated. The existence and uniqueness of the solutions of the CFO jerk model (3) are studied. To verify whether the CFO jerk model solution (3) is dependent on initial conditions, we establish and prove a theorem. The dynamics of these versions such as symmetry, dissipative, fixed points, and stability are studied. We also determine the intervals of parameters at which these versions have chaotic solutions and solutions that go to fixed points. The new versions of jerk model (3) can be used in many applications such as mechanical oscillators, laser physics, and electrical circuits. The linear feedback control is applied to show that the solutions of our versions approach to fixed points. The solutions of the CDO version of jerk model (3) after control take less time to converge to fixed points than the fractional version. We used the Predictor–Corrector method [22] in numerical treatments for fractional version (3) and spectral numerical method [23] for the solution of CDO one (4). Numerical calculations are presented to verify the analytical results of these theorems.

The rest of this paper is organized as follows. In Section 2, we present and prove two theorems for the existence and uniqueness of the CFO jerk model solution (3) and the dependence of its solution on initial conditions. The dynamics of our equations (3) and (4) are investigated. The values of the parameters of these equations at which the proposed versions have chaotic solutions are calculated based on the sign of their largest Lyapunov exponent (LLE) via the modified technique of the Wolf algorithm [24]. The control of the chaotic versions (3) and (4) is illustrated in Section 3. Finally, Section 4 concludes our investigations.

2. Dynamics of chaotic versions (3) and (4) of jerk models

In this section, we study the dynamics of our new versions (3) and (4) of the jerk model (1). The dynamics of these equations contain symmetry, dissipation, fixed points, stability, and chaotic solutions.

2.1. The existence and uniqueness of CFO jerk version (3)

We study the existence and uniqueness of the solution in the $\Lambda \times [0,T]$ region, where $\Lambda = \{(u_1,u_2,u_3) : \max(|u_1|,|u_2|,|u_3|) \leq \zeta\}$.

For the class of continuous functions $C[0,T]$, the supremum norm used in the analysis that follows is defined as $\|\psi\| = \sup\limits_{t\in(0,T]} |\psi(t)|$, $\psi(t) \in C[0,T]$. Then the norm of the matrix $M = [m_{ij}[t]]$ is $\|M\| = \sum\limits_{i,j} \sup\limits_{t\in(0,T]} |m_{ij}[t]|$.

Theorem 1. *In the $\Lambda \times [0,T]$ region with initial conditions $U(0) = U_0$ and $t \in [0,T]$, a sufficient condition for the existence and uniqueness of the solution of FO jerk version* (3) *is*

$$B = \frac{T^\alpha}{\Gamma(\alpha+1)} \max\{a+b, 1+c, A\} < 1\,. \tag{5}$$

Proof. The FO jerk version (3) can be written as

$$D^\alpha U(t) = \varphi(U(t))\,, \qquad t \in (0,T]\,, \qquad U(0) = U_0\,, \tag{6}$$

where

$$U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \qquad U_0 = \begin{pmatrix} u_{1_0} \\ u_{2_0} \\ u_{3_0} \end{pmatrix},$$

$$\varphi(U) = \begin{pmatrix} u_2 \\ au_3 \\ -cu_2 - du_3 - 16u_1^6 + 24u_1^4 - 9u_1^2 + \frac{1}{2} - \frac{1}{2}b \end{pmatrix}. \tag{7}$$

Thus, the solution of CFO jerk version (3) is obtained as

$$U = U_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} \varphi(U(\theta))\,\mathrm{d}\theta = H(U)\,, \tag{8}$$

so

$$H(U_1) - H(U_2) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} (\varphi(U_1(\theta)) - \varphi(U_2(\theta)))\,\mathrm{d}\theta\,. \tag{9}$$

As a result, we get

$$|H(U_1) - H(U_2)| \le \int_0^t \left| \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} (\varphi(U_1(\theta)) - \varphi(U_2(\theta))) \right| \mathrm{d}\theta\,, \tag{10}$$

which is condensed to

$$\|H(U_1) - H(U_2)\| \le \frac{T^\alpha}{\Gamma(\alpha+1)} \max\{a+b, 1+c, A\} \|U_1 - U_2\| \le B\|U_1 - U_2\|\,,$$

where

$$A = 96\zeta^5 + 96\zeta^3 + 36\zeta\,, \qquad B = \frac{T^\alpha}{\Gamma(\alpha+1)} \max\{a+b, 1+c, A\}\,.$$

Then the mapping $U = H(U)$ is a contraction mapping, therefore the proof of this theorem is completed. □

2.2. Continuous dependence on initial conditions

The consistent reliance on starting conditions is a characteristic of system solutions that contradicts the sensitive reliance on initial conditions, which is a defining feature of chaotic behaviour. The aim of this subsection is to investigate the period of time and parameter range in which the solution of the system exhibits a consistent reliance on initial conditions. Therefore, being aware of this parameter range allows researchers to identify the time period T and parameter range where the system does not display chaotic dynamics.

Theorem 2. *Suppose that the FO jerk version* (3) *is satisfying the condition of Theorem 1 where B is given by* (5)*. Then,$\forall\ \epsilon > 0\ \exists\ \delta(\epsilon) = (1-B)\epsilon > 0$ such that $\|U_{0_1} - U_{0_2}\| \leq \delta$ implies that $\|U_1 - U_2\| \leq \epsilon$, i.e. the solution exhibits continuous dependence on initial conditions.*

Proof. Assume that there are two sets of initial conditions to system (6), U_{0_1} and U_{0_2}, which satisfy

$$\|U_{0_1} - U_{0_2}\| \leq \delta\,.$$

Firstly, we suppose that the condition of the last theorem holds. Thus,

$$\begin{aligned} U_1(t) &= U_{0_1} + \int_0^t \varphi(U_1(\theta))\,\mathrm{d}\theta\,, \\ U_2(t) &= U_{0_2} + \int_0^t \varphi(U_2(\theta))\,\mathrm{d}\theta\,, \end{aligned}$$

and also we obtain

$$\|U_1 - U_2\| \leq \|U_{0_1} - U_{0_2}\| + B\,\|U_1 - U_2\|\,,$$

hence

$$(1-B)\|U_1 - U_2\| \leq \|U_{0_1} - U_{0_2}\|\,,$$

where $0 < B < 1$ is defined by (5). Finally, we get $\|U_1 - U_2\| \leq \epsilon$, where $\epsilon = \frac{\delta}{1-B}$. □

Remark 1. *The existence and uniqueness of the solutions and their dependence on the initial conditions for model (4) can be similarly studied.*

2.3. *The FO jerk version* (3)

Model (3) is not symmetric. The divergence of (3) is

$$\nabla \cdot F = \sum_{i,j=1}^{3} \frac{\partial \dot{u_{ij}}}{\partial u_{ij}} = -d\,.$$

Therefore model (3) is dissipative for $d > 0$. Model (3) has six fixed points (FPs): $\bar{E} = (\bar{u}_1, 0, 0)$, where $\bar{u}_1$ is the solution of the equation $32u_1^6 - 48u_1^4 + 18u_1^2 - 1 + b = 0$. If $b = 0.45$, the values of the six FPs are: $E_{1,2} = (\pm 0.183, 0, 0)^T$, $E_{3,4} = (\pm 0.78, 0, 0)^T$, and $E_{5,6} = (\pm 0.943, 0, 0)^T$.

To study the stability of our FPs, we calculate the Jacobian matrix for the FP $\bar{E} = (\bar{u}_1, 0, 0)$ as

$$J_{\bar{E}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & a \\ \eta'(\bar{u}_1) & -c & -d \end{pmatrix}, \tag{11}$$

where $\eta'(\bar{u}_1) = 96\bar{u}_1^5 - 96\bar{u}_1^3 + 18\bar{u}_1$. The corresponding characteristic equation of our model at the FP $\bar{E}$ is

$$\mu^3 + d\mu^2 + ac\mu - 96a\bar{u}_1^5 + 96a\bar{u}_1^3 - 18a\bar{u}_1 = 0\,. \tag{12}$$

If $\| \arg(\mu_i(J_{\bar{E}}))\| > \frac{\alpha\pi}{2}$, $i = 1, 2, 3$, then the FP $\bar{E}$ is stable, where $\mu_i(J_{\bar{E}})$, $i = 1, 2, 3$ are the solutions of Eq. (12).

2.3.1. The behaviour of solutions of the CFO jerk model (3)

We study the solution behaviours of the CFO jerk model (3) numerically and evaluate the largest Lyapunov exponent (LLE) by a modified technique of the Wolf algorithm [24]. We calculate the LLE for the initial values $(u_1, u_2, u_3)^T(0) = (0.1, 0.2, 0.14)^T$ for different values for the parameters as: **Fix $\alpha = 0.99$, $a = 27.5$, $b = 0.45$, $d = 1.9$ and vary c.**
The LLE of model (3) is calculated for the values of $c \in [0.65, 10]$ as shown in figure 1. It has chaotic solutions for $c \in [0.65, 0.84)$, $[1.02, 1.06)$, $[1.09, 1.13)$, and $[1.16, 1.18)$. For the values of $c \in [0.84, 1.02)$, $[1.06, 1.09)$, $[1.13, 1.16)$, and $[1.18, 10]$, the solutions of model (3) approach FPs. For the choice of $\alpha = 0.99$, $a = 27.5$, $b = 0.45$, $d = 1.9$, and $c = 0.65$, we calculated the LLE which is $\lambda_{\mathrm{L}} = 0.3354$. This means that model (3) has a chaotic solution as shown in figure 2.

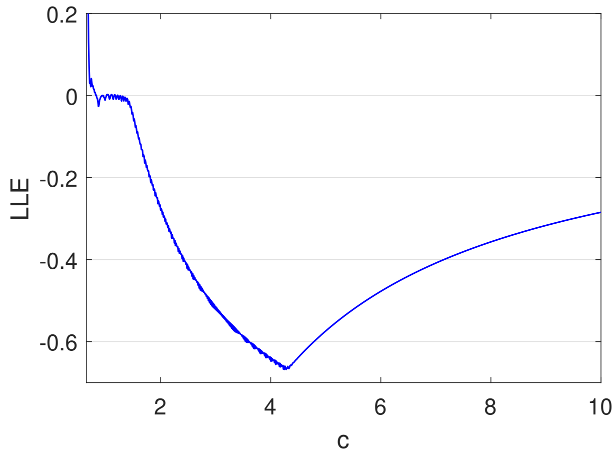

Fig. 1. The LLE for model (3) *versus* $c \in [0.65, 10]$.

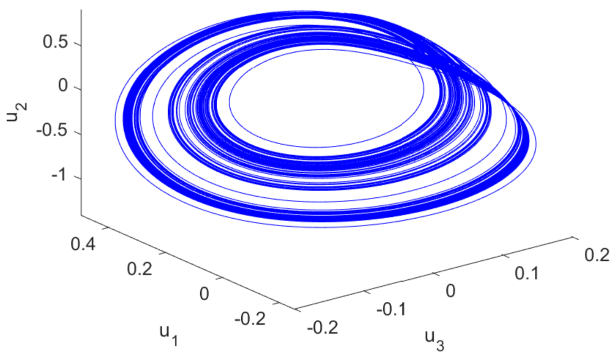

Fig. 2. A chaotic attractor for model (3) for $\alpha = 0.99$, $a = 27.5$, $b = 0.45$, $d = 1.9$, and $c = 0.65$ and the initial conditions $u_0 = (0.1, 0.2, 0.14)^T$.

2.4. The DO jerk model (4)

Model (4) has the same dynamics as model (3) but the study of stability is different. Firstly, the first condition of Theorem 3.1 [25] is tested as

$$\begin{aligned}\lim_{\|u(t)\|\to 0} \frac{\|f(u(t))\|}{\|u(t)\|} &= \lim_{\|u(t)\|\to 0} \frac{|32u_1^6 - 48u_1^4 + 18u_1^2|}{\|u(t)\|} \\ &= \lim_{\|u(t)\|\to 0} \frac{u_1^2|32u_1^4 - 48u_1^2 + 18|}{\|u(t)\|} \\ &\le \lim_{\|u(t)\|\to 0} \frac{\|u\|^2(32\|u\|^4 + 48\|u\|^2 + 18)}{\|u(t)\|} = 0\,, \quad (13)\end{aligned}$$

where $u = (u_1, u_2, u_3)^T$.

By testing the second condition in Theorem 3.1 [25], we put $m = 40$, $\omega(\alpha) = \frac{\alpha^2}{1000(1-\alpha)^2}$, $i = 1$, and $\Delta\tau_1 = \frac{1}{m}$; then

$$-\frac{\theta_0}{\theta_1} = 7.8 \times 10^3 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & a \\ \eta'(\bar{u}_1) & -c & -d \end{pmatrix}, \tag{14}$$

where $\theta_0 = -A$, $\theta_1 = \frac{1}{40} I \omega(\frac{1}{80})$, A is the Jacobian of the linear part of model (4) and I is the (3×3) identity matrix.

The characteristic equation of matrix (14) is the same as in model (3). If the condition $|\arg(\mu_k(-\frac{\theta_0}{\theta_1}))| > \frac{\pi}{80}$, $(k = 1, 2, 3)$ holds, then the FP $\bar{E}$ is stable. The previous condition is verified for the other values of i as

$$\left|\arg\left(\mu_k\left(-\frac{\theta_{i-1}}{\theta_i}\right)\right)\right| = \pi > \frac{\mu_i \pi}{2} = \frac{\pi}{40}, \qquad i = 2, 3, ..., m; \qquad k = 1, 2, 3. \tag{15}$$

2.4.1. The solution behaviours of model (4)

For the choice of $a = 27.5, b = 0.45, c = 0.65, d = 2$, $\omega(\alpha) = \frac{\alpha^2}{1000(1-\alpha)^2}$, and the initial conditions $u_0 = (0.1, 0.2, 0.14)^T$, the LLE has the value $\lambda_L = 0.0073$. This means that model (4) has a chaotic solution as shown in figure 3.

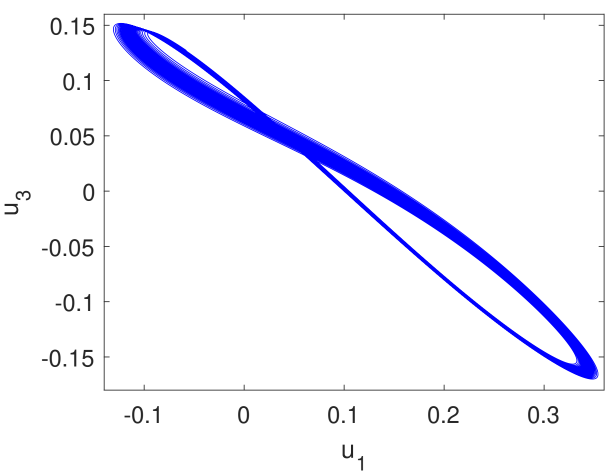

Fig. 3. A chaotic attractor for model (4), for $a = 27.5, b = 0.45, c = 0.65, d = 2$, $\omega(\alpha) = \frac{\alpha^2}{1000(1-\alpha)^2}$, and $u_0 = (0.1, 0.2, 0.14)^T$.

Remark 2. *The DO jerk model* (4) *has chaotic behaviour and solutions that reach FPs by varying parameter* a*, while model* (3) *has only chaotic behaviour.*

3. Control of chaotic new versions (3) and (4)

In this section, we use the linear feedback method to stabilize the solutions of the chaotic FO jerk model (3) and the chaotic DO jerk model (4). We discussed the difference between these results. Equations (3) and (4) can be stabilized after adding the control function V to the right-hand side of their second equations, where $V = -ku_2$ is the linear control function.

For the same choice of parameters as in figures 2 and 3, the solutions of models (3) and (4) after adding the linear control approach to the fixed point E_1 as shown in figures 4 and 5. The solutions of these models take different times to converge to E_1, which are, $t = 26.8$ and $t = 5$, respectively. This means that the DO version of jerk model (4) takes less time to reach to fixed point than the fractional version (3).

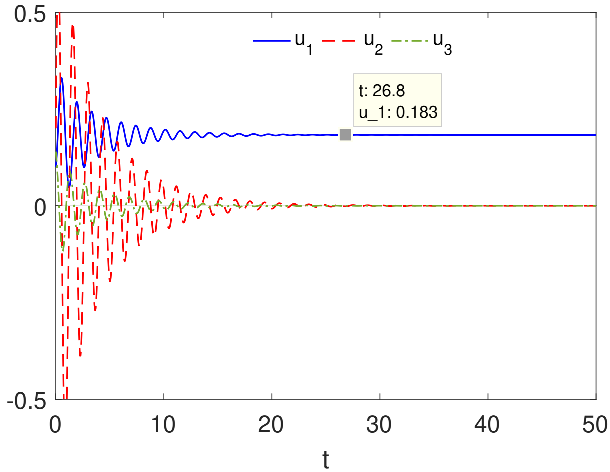

Fig. 4. The chaotic attractor of Fig. 2 after adding control for $k = 2$ and approaches to the $E_1 = (0.183, 0, 0)^T$.

Remark 3. *According to figures 4 and 5, it is noticed that the solutions of the DO version of the jerk model* (4) *reach FP faster than the fractional version (3) by using control.*

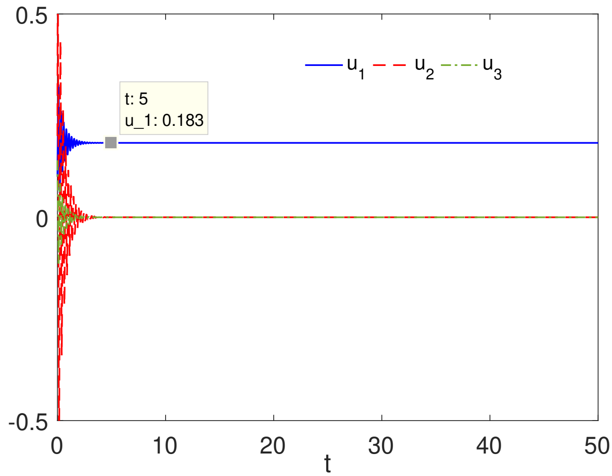

Fig. 5. The chaotic attractor of Fig. 3 after adding control for $k = 2$ and approaches to the $E_1 = (0.183, 0, 0)^T$.

4. Conclusion

In this work, we introduced the chaotic FO (3) and DO (4) versions of jerk model (1). We also stated and proved Theorems 1 and 2 for the existence and uniqueness of the FO jerk model solutions (3) and the dependence of their solutions on initial conditions. The dynamics of these models such as symmetry, dissipative, fixed points, and stability are investigated. The intervals of the parameters at which these models have chaotic solutions and solutions that go to FPs are calculated in Section 2. There are many differences between the two versions and they are mentioned in Remark 2. We controlled these models using the linear feedback control as shown in figures 4 and 5. These figures show the chaotic solutions after adding control and approach to fixed points. It can be noticed that the solutions of the DO version of jerk model (4) approach to fixed point faster than the fractional version by using control. Numerical simulations are used in these investigations as depicted in figures 1–5. Jerk models appeared in many applications in physics, biology, and engineering, *e.g.* laser physics and jerk circuit.

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