THE MASS DENSITY CONTRAST IN PERTURBED FRIEDMAN–LEMAÎTRE–ROBERTSON–WALKER COSMOLOGIES

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We analyze the evolution of the mass density contrast in spherical perturbations of flat Friedman–Lemaître–Robertson–Walker cosmologies. Both dark matter and dark energy are included. In the absence of dark energy, the evolution equation coincides with that obtained by Bonnor within the "Newtonian cosmology".

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1. Introduction

We shall analyze the evolution of perturbations of flat FLRW spacetimes using the $1+3$ splitting of the spacetime. The original aim of this paper was just to find the general relativistic version of the well-known result of Bonnor [1] assuming isothermal perturbations and using the comoving coordinates. The main conclusion concerning the temporal behaviour of the mass density contrast — in the absence of dark energy — coincides with that of Bonnor and also with a later analysis of [2] for perturbations comoving with the background matter. The case of the nonzero cosmological constant was not investigated by Bonnor. In such a case, the evolution equation for the mass density contrast differs from that found earlier by Martel [2].

2. Selfgravitating fluids within spherically symmetric spacetimes

We shall assume only spherical symmetry, without spatial homogeneity. Some of the resulting Einstein equations were found by Lemaître in the 1930s [3, 4], who studied stability of Einstein's static universes. Tolman and Bondi extended results of Lemaître for a selfgravitating dust [5, 6]. The resulting class of metrics is often referred to as the Lemaître–Tolman–Bondi spacetimes. In the 1960s, Misner and Sharp [8], and Podurets [9] again analyzed these equations, but in the case of perfect gas; they, in particular, extended the Lemaître–Tolman–Bondi concept of the quasilocal material mass. Its expression will be given below.

We assume the Einstein equations $R_{\mu\nu} - g_{\mu\nu}\frac{R}{2} = 8\pi T_{\mu\nu} - \Lambda g_{\mu\nu}$, where the stress-energy tensor is defined as $T_{\mu\nu} = (\varrho + p)\, U_\mu U\nu + p g_{\mu\nu}$ and Λ is the cosmological constant. The coordinate 4-velocity is normalized, $U_\mu U^\mu = -1$. Here, ϱ and p denote the mass density and pressure, respectively.

We shall assume that we are given a $1+3$ foliation, with foliation leaves characterized by constant time, $t = \mathrm{const}$. The line element is taken in the form

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \hat{a}\mathrm{d}r^2 + R^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right), \tag{1}$$

where the radius $0 \le r < \infty$ and the angular variables satisfy $0 \le \phi < 2\pi$, $-\pi/2 \le \theta \le \pi/2$. The lapse N and the areal radius R depend on time t and the coordinate radius r. We adopt the standard condition that the speed of light c and the gravitational constant G are equal to unity.

This metric is diagonal, so we shall calculate extrinsic curvatures from the formula $K_{ij} = \frac{1}{2N}\partial_t g_{ij}$ [7]. The condition of isotropy implies that two of them are equal, $K^\phi_\phi = K^\theta_\theta$. The nonzero components of K_{ij} read

$$\begin{aligned} \mathrm{tr}K &= \frac{\partial_t\left(\sqrt{\hat{a}}R^2\right)}{N\sqrt{\hat{a}}R^2}, \qquad K^r_r = \frac{1}{2N\hat{a}}\partial_t\hat{a}, \\ K^\phi_\phi &= K^\theta_\theta = \frac{\partial_t R}{NR} = \frac{1}{2}\left(\mathrm{tr}K - K^r_r\right). \end{aligned} \tag{2}$$

Usually, one assumes that coordinates are comoving. We shall impose a foliation condition as in the standard $1+3$ formulations of the Einstein equations, by putting a condition onto extrinsic curvatures of leaves of a foliation. We shall assume the following:

$$\Delta(R(r,t),t) = \left(\frac{R\left(\mathrm{tr}K - K^r_r\right)}{2}\right)^2, \tag{3}$$

where Δ is defined as [10]

$$\begin{aligned} \Delta(R(r,t),t) &= \frac{-3}{4R}\int_0^R \tilde{R}^2 (K^r_r)^2 \mathrm{d}\tilde{R} + \frac{1}{4R}\int_0^R \tilde{R}^2(\mathrm{tr}K)^2\mathrm{d}\tilde{R} \\ &\quad + \frac{1}{2R}\int_0^R \mathrm{tr}K K^r_r \tilde{R}^2\mathrm{d}\tilde{R}. \end{aligned} \tag{4}$$

Differentiation of both sides of Eq. (4) with respect to the coordinate radius r yields, using the momentum constraint of the Einstein equations [7] and the definition of the mean curvature $\hat{p} = 2\partial_r \ln R/\sqrt{\hat{a}}$ [10],

$$R\left(\mathrm{tr}K - K_r^r\right)\frac{16\pi j_r R}{\hat{p}} = 0\,. \tag{5}$$

Herein, we define $j_r = NT^0r/\sqrt{\hat{a}}$.

This implies that fluids are comoving in chosen coordinates,

$$j_r = 0\,, \tag{6}$$

provided that there are no minimal surfaces, $\hat{p} \neq 0$ and $\mathrm{tr}K \neq K_r^r$. On the other hand, it appears that in comoving coordinates $\mathrm{tr}K = \partial_R(R^3(\mathrm{tr}K - K_r^r))/(2R^2)$ (see Section 4.1). The areal velocity $R(\mathrm{tr}K - K_r^r)/2$ constitutes a part of the initial data of the Einstein equations — see forthcoming equation (15). Thus, under the conditions of $\hat{p} \neq 0$ and $\mathrm{tr}K \neq K_r^r$, our foliation equation (3) is equivalent to the standard assumption of comoving coordinates.

Notice that now the material energy-momentum tensor reads $T_0^0 = -\varrho$, $j_r = 0$ and $T_r^r = p = T_\theta^\theta$; we deal with perfect fluids. The cosmological constant is responsible for the dark energy ϱ_Λ and pressure p_Λ contributions

$$\varrho_\Lambda = \frac{\Lambda}{8\pi}\,, \qquad p_\Lambda = -\frac{\Lambda}{8\pi}\,. \tag{7}$$

In such a case, the quasilocal mass of Misner and Sharp [8], and Podurets [9], contained in a coordinate sphere of a radius r, is given by the formula

$$m(R(r)) = 2\pi \int\limits_0^r \tilde{R}^3\hat{p}\sqrt{a}\,(\rho + \varrho_\Lambda)\,\mathrm{d}r\,. \tag{8}$$

For the sake of concise notation, we shall define

$$U(r) = \frac{R(r)}{2}\left(\mathrm{tr}K(r) - K(r)_r^r\right)\,; \tag{9}$$

this quantity represents the areal velocity of a comoving particle of gas, $U = \partial_0 R/N$. The mean curvature $\hat{p}$ of centered spheres can be calculated to be [10]

$$\hat{p} = \frac{2}{R(r)}\sqrt{1 - \frac{2m(R(r))}{R(r)} + U^2(r)}\,. \tag{10}$$

One can show that the mass defined in (8) changes as follows [8]:

$$\partial_t m(R(r)) = -4\pi \left[N R^2 U \left(p + p_\Lambda\right)\right](r) \,. \tag{11}$$

Moreover, by direct calculation, one gets from (8)

$$\frac{\partial_r m(R)}{\sqrt{\hat{a}}} = 2\pi R^3 \hat{p} \varrho \,. \tag{12}$$

These equations should be supplemented by two conservation equations

$$N\partial_r p + \partial_r N(p + \varrho) = 0 \tag{13}$$

and

$$\partial_t \varrho = -N \mathrm{tr} K(p + \rho) \,. \tag{14}$$

The Einstein evolution equations reduce to the single equation

$$\partial_t U = -\frac{m(r)}{R^2} - 4\pi \left(p + p_\Lambda\right) RN + \frac{\hat{p}R}{2\sqrt{\hat{a}}} \partial_r N \,. \tag{15}$$

3. The Friedman-type solution

Assuming that matter consists of dust and imposing in addition homogeneity on slices of constant time t, one gets from equations (8)–(15) the Friedman metric $\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2 \left(\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2\right)$. Thus, the lapse $N = 1$. The conformal factor $a(t)$ satisfies Friedman equations

$$\begin{aligned} \varrho_0 + \varrho_\Lambda &= \frac{3}{8\pi} H^2 \,, \\ -\frac{\mathrm{d}H}{\mathrm{d}t} &= 4\pi \varrho_0 \,, \\ \frac{\mathrm{d}\varrho_0}{\mathrm{d}t} &= -3H\varrho_0 \,. \end{aligned} \tag{16}$$

(Only two of the three equations are independent.)

The extrinsic curvatures of this solution are equal to the Hubble parameter $H \equiv \frac{\mathrm{d}a}{a\mathrm{d}t}$,

$$K_r^r = K_\theta^\theta = K_\phi^\phi = H \,, \tag{17}$$

while its trace is $\mathrm{tr}K = 3H$. The velocity U reads now $U = HR$. The mean curvature of centered 2-spheres within the $t = \mathrm{const}$ slice is now the same as in the flat space: $\hat{p} = 2/R$.

This solution describes a flat, homogeneous, and isotropic universe filled with comoving dust of the density ϱ_0 that is expanding with the Hubble recession velocity $H = \dot{a}/a$. The product $\varrho_0 a^3$ is constant in time.

4. Evolution of small spherical inhomogeneities in a FLRW universe

We assume that the background (Friedman-type) universe is dotted by isolated, locally isotropic mass density perturbations $\delta\varrho$, so that the mass density is split into the background part ϱ_0 and the perturbation $\delta\varrho$: $\varrho = \varrho_0 + \delta\varrho$. The mass perturbations are isothermal — they exert pressure $p = c_{\rm s}^2\delta\varrho$. The metric of the perturbed spacetime reads $\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \hat{a}\mathrm{d}r^2 + R^2\mathrm{d}\Omega^2$; we use comoving coordinates. Far from these perturbations the lapse N tends to 1 and the spatial part of the metric is approaching the background metric $a^2\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right)$. We assume — similarly as Bonnor in his analysis of [1] — that this perturbing isothermal gas is comoving with the background dust. (Let us remark that perturbations do not have to comove with the background dust — see a different scenario discussed in [11].) For the matter of convenience, we shall locate our coordinate system in the symmetry center of a perturbation.

The areal velocity $U = \partial_0 R/N = R(\mathrm{tr}K - K_r^r)/2$ is split into the background and perturbed parts as follows:

$$U = H(t)R + \delta_U\,, \tag{18}$$

where $H(t)$ is the Hubble constant at the time t.

We need initial data — for the areal velocity $U = \partial_0 R/N$ and the mass density ϱ — for the two evolution equations (14) and (15). They are defined as follows at an initial hypersurface labelled by the world time t_0. The initial value of the perturbing component δ_U is small but otherwise, it is a free datum. The initial mass density ϱ is given as the sum of the background mass density ϱ_0 at the time t_0 and the small initial perturbation $\delta\varrho$, with the condition that far from the center ϱ approaches $\varrho_0(t_0)$.

The main aim of the forthcoming calculation is the derivation of the wave equation that rules the evolution of the mass density contrast $\delta\varrho/\varrho_0$. We shall get also an evolution equations for the velocity perturbation δ_U.

4.1. The extrinsic curvature

The first part of the calculation is actually exact — we do not need the assumption of small perturbations in order to get the trace of the extrinsic curvature

$$\mathrm{tr}K = \frac{\partial_R\left(R^2U\right)}{R^2} \tag{19}$$

of hypersurfaces of constant world time t.

Formula (19) is valid in all slicings of spherically symmetric spacetimes cosmological models that asymptotically coincide with flat slicings of cosmological flat FLRW models. We allow for dark energy (cosmological constant) and various forms of *comoving* matter — dust and fluids. This formula is known (see, for instance, [10, 12]), but we derive it here for the sake of completeness.

We have from the definition of extrinsic curvatures

$$\mathrm{tr}K = \frac{\partial_0\left(R^2\sqrt{\hat{a}}\right)}{NR^2\sqrt{\hat{a}}}\,. \tag{20}$$

The quantity $\sqrt{\hat{a}}$ in the nominator of (20) can be replaced by

$$\sqrt{\hat{a}} = \frac{2\partial_r R}{\hat{p}R}\,; \tag{21}$$

here, $\hat{p}$ is the mean curvature of the coordinate sphere $r=$ const. Thus, (20) yields

$$\mathrm{trK} = 2\frac{\partial_0 R}{NR} + \frac{2\partial_0\partial_r R}{N\hat{p}R\sqrt{\hat{a}}} + \frac{\hat{p}R}{N\sqrt{\hat{a}}}\partial_0\frac{1}{\hat{p}R}\,. \tag{22}$$

The first term is just $2U/R$. Changing the order of differentiation, we can write the second term as

$$\frac{2\partial_r\left(\frac{\partial_0 R}{N}N\right)}{N\hat{p}R\sqrt{\hat{a}}} = 2\frac{\partial_r U}{\hat{p}R\sqrt{\hat{a}}} + 2U\frac{\partial_r N}{N\hat{p}R\sqrt{\hat{a}}}\,.$$

Replace now the coordinate radius r with the areal radius R and notice that $\frac{2\partial_r}{\hat{p}R} = \partial_R$. We obtain the following form of the second term of (22):

$$\frac{2\partial_0\partial_r R}{N\hat{p}R} = \partial_R U + U\frac{\partial_R N}{N}.$$

The calculation of the third term in (22) is a little bit longer. Recall (see formula (10)) that the mean curvature $\hat{p}R = 2\sqrt{1-\frac{2m(R(r,t),t)}{R(r,t)} + U^2(r,t)}$. Its differentiation with respect to time yields, after using the mass conservation equation (11) and the Einstein equation describing the evolution of $U = R(\mathrm{tr} - K_r^r)/2$ (see equation (15)),

$$\partial_t\frac{2}{\hat{p}R} = -\frac{2U}{\hat{p}R}\partial_R N\,. \tag{23}$$

Combining the three terms of (22), we arrive at formula (19).

In the case of small spherically symmetric perturbations, we can use the splitting (18) of the radial velocity. We immediately arrive at the following corollary.

Conclusion. Assume a perturbed FLRW flat universe. The trace of the extrinsic curvature of constant time hypersurfaces, in the foliation defined by the assumption of comoving particles, is given by

$$\mathrm{tr}K = 3H + \frac{\partial_R \left(R^2 \delta_U\right)}{R^2}\,. \tag{24}$$

Remark. *The alternative way to derive formula (19) is to write down the momentum constraint (i.e., the Einstein equation* $R_{0i} - \frac{R}{2} g_{0i} = 8\pi T_{0i}$*), using metric (1). The* r*-component of the constraint can be expressed as (19) in comoving coordinates.*

4.2. The lapse

In what follows, we need the lapse function N; it can be obtained from (13). We assumed that the pressure is isothermal in perturbed FLRW universes, $p = c_{\mathrm{s}}^2 \delta\varrho = c_{\mathrm{s}}^2 \varrho_0 \delta$, where we introduced the mass density contrast

$$\delta \equiv \frac{\delta\varrho}{\varrho_0}\,. \tag{25}$$

If the mass density contrast is small, $\delta \ll 1$, then (13) yields $\partial_r N \approx -c_{\mathrm{s}}^2 \partial_r \delta$. Far from the center, $N \to 1$; thus

$$N \approx 1 - c_{\mathrm{s}}^2 \delta\,. \tag{26}$$

This implies that the time derivative of the areal radius evolves as

$$\partial_0 R = UN \approx (HR + \delta_U)\left(1 - c_{\mathrm{s}}^2 \delta\right) \approx HR + \delta_U - c_{\mathrm{s}}^2 \delta HR\,. \tag{27}$$

4.3. Evolution of the mass density contrast

We investigate perturbations of (flat) FLRW universes with dust (including dark matter) and dark energy. Let us summarise the relevant information. The material pressure $p_0 = 0$ and the sum of the background energy density satisfies $\varrho_0 + \varrho_\Lambda = \frac{3H^2}{8\pi}$. The metric scale factor $a(t)$ of the background metric can be obtained from equations (16). The lapse up to the first perturbation is given by (26) and Eq. (27) reads now $\partial_0 R \approx HR + \delta_U - c_{\mathrm{s}}^2 HR$.

Equation (15) can be written as

$$\partial_0 U = -\frac{m(R)}{R^2} N - 4\pi \left(c_{\mathrm{s}}^2 R \varrho_0 \delta + p_\Lambda\right) RN + \frac{\hat{p}^2 R^2}{4} \partial_R N\,. \tag{28}$$

Zeroth order terms (see Section 3) drop out. Thus the linear perturbations satisfy the equation

$$\frac{1}{a} \partial_0 \left(a \delta_U\right) = -\frac{\delta m(R)}{R^2} - c_{\mathrm{s}}^2 \partial_R \delta\,. \tag{29}$$

We employed (18) and (26)–(28) in the process of deriving (29).

One can show that in the leading order of $O(\delta)$ the following rule holds:

$$\partial_0 \partial_R \left(R^2 \delta_U\right) = \partial_R \left(\frac{R^2}{a} \partial_0 \left(a \delta_U\right)\right) . \tag{30}$$

The mass density conservation equation is given by (14). Using the derived earlier expressions for the lapse N and the trace of the extrinsic curvature trK, we get

$$\partial_0 \delta\varrho + 3H\delta\varrho + \frac{\varrho_0}{R^2} \partial_R \left(R^2 \delta_U\right) = 0 . \tag{31}$$

Dividing both sides by ϱ_0, we obtain

$$\partial_0 \delta + \frac{1}{R^2} \partial_R \left(R^2 \delta_U\right) = 0 . \tag{32}$$

Differentiate now both sides of (32) with respect to time, using formula (30) and equation (29). After straightforward calculation, we arrive at

$$\partial_0^2 \delta - \frac{c_\mathrm{s}^2}{R^2} \partial_R \left(R^2 \partial_R \delta\right) - \frac{3}{2} H^2 \delta + 2H\partial_0 \delta = 0 . \tag{33}$$

Notice also that Eq. (33) is a wave equation — thus, it possesses a kind of travelling wave pulses that move within the coordinate sphere that encloses the perturbed initial data.

Equation (33) is equivalent to the corresponding Bonnor equation describing the evolution of the mass density contrast [1, 2] when the cosmological constant is absent. In order to see this, perform the Fourier transformation of (33) and insert $H^2 = 8\pi\varrho_0/3$. Then one arrives exactly at the result of Bonnor.

Our equation (33) differs from the corresponding equation of Martel (see Eq. (8) in [2]) in the case of the nonzero cosmological constant.

The two descriptions differ in the part concerning the evolution of velocity perturbations. In the model of Bonnor, the perpendicular velocity components behave like $\vec{V}_T \propto 1/a(t)$ [1]; thus their length has to decrease. In the general relativistic analysis, we have only a partly coincident behaviour of velocity perturbations — $\partial_0(a\delta_U) \leq 0$, assuming that $\partial_R \delta \geq 0$. In the case of dust-like perturbations — with the vanishing speed of sound, $c_\mathrm{s}^2 = 0$ — the velocity perturbation δ_U is strictly decreasing. Positive velocity perturbations might decrease at least like the inverse of the scale factor, $1/a(t)$, but there is no a bound onto the absolute value of negative velocity disturbances δ_U.

5. The influence of dark energy

We shall investigate how dark energy would influence the evolution of the mass density contrast δ after the end of the recombination epoch, that is for times $t \geq t_{\rm re}$. We neglect — as in the whole paper — the contribution of the radiation energy. The speed of sound c_s is negligible in this period and the evolution equation becomes

$$\partial_0^2\delta - \frac{3}{2}H^2\delta + 2H\partial_0\delta = 0\,. \tag{34}$$

5.1. Absence of dark energy

In this case the conformal factor $a(t) \propto t^{2/3}$ and $H = 2/(3t)$. The increasing solution of (34) reads $\delta(t) \propto a(t) \propto t^{2/3}$. According to astronomical observations $a(t_0)/a(t_{\rm re}) \approx 1100$ [13]; here t_0 is the present age of the Universe. Thus, the mass density contrast δ of dust-like perturbations of dust Friedman universes would increase 1100 times since the end of the recombination era.

5.2. Including dark energy

In this case, the coefficients $H(t)$ and $H^2(t)$ are given as related solutions of the Friedman equations (see Section 3); the latter can be solved numerically, assuming dust and the cosmological constant. The evolution equation reads

$$\partial_0^2\delta - \frac{3}{2}H^2\delta + 2H\partial_0\delta = 0\,. \tag{35}$$

At the recombination era, the material density ϱ exceeds the dark energy density ϱ_Λ by a factor of the order of 10^8. Thus as initial data, we can choose

$$\delta(t_{\rm re}) = t_{\rm re}^{2/3}\,, \qquad \frac{\mathrm{d}\delta}{\mathrm{d}t}\bigg|_{t_{\rm re}} = \frac{2}{3t_{\rm re}^{1/3}} \tag{36}$$

— these are data dictated by the solution $\delta(t) \propto a(t)$, valid in the case of no-dark energy.

The solution of Eq. (35) with initial data (36) is very close to $\delta(t) = t^{2/3}$; the difference becomes clear at relatively late times $t \geq t_0/10$ [14].

Assuming a flat universe with present data $\Omega(t_0)_{\rm d} = 0.3$ and $\Omega_\Lambda(t_0) = 0.7$, one gets $\delta(t_0)/\delta(t_{\rm re}) \approx 975$ [14]. The cosmological constant slows the process of formation of bound structures; its influence is comparable to that obtained from the equation of Martel — see [13].

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