Acta Physica Polonica B **54**, 11-A4 (2023)

PERTURBATIVE COMPUTATION OF THERMAL CHARACTERISTICS OF THE STONER PHASE TRANSITION

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Received 30 November 2023, accepted 7 December 2023,
published online 20 December 2023

We apply the thermal (imaginary time) perturbative expansion to the relevant effective field theory to compute characteristics of the phase transition to the ordered state which can occur at low temperatures in the gas of (nonrelativistic) spin-1/2 fermions interacting through a short-range spin-independent repulsive binary interaction potential. We show how to obtain a systematic expansion of the system's free energy depending on the densities n_+ and n_- of spin-up and spin-down fermions. In this paper, we truncate this expansion at the second order and determine, by numerically minimizing the free energy, the equilibrium proportions of n_+ and n_- (that is, the system's polarization) as functions of the temperature, the system's overall density $n = n_+ + n_-$, and the strength of the interaction.

DOI:10.5506/APhysPolB.54.11-A4

1. Introduction

There is a qualitative argument that in the gas of spin-1/2 fermions interacting through a short-range repulsive spin-independent binary potential, a phase transition to the ordered state should occur if the interaction strength and/or the system's density is sufficiently large. Indeed, at zero temperature, when the entropy factor does not intervene, the configuration of the system in which there are more fermions in one spin state than in the other one may be energetically favoured. This is because, due to the Pauli exclusion principle, the s-wave interaction of fermions in the same spin state is impossible and the resulting decrease in the interaction energy may be greater than the associated increase of the kinetic energy (increase of the Fermi energy of the more populated spin state).

Theoretical investigation of this phenomenon, called the Stoner transition, taking into account its temperature dependence, requires the full machinery of statistical mechanics. The standard textbook treatment of the problem [1, 2], equivalent to the so-called mean field approach or the Hartree–Fock approximation, employs the pseudo-potential method which allows to determine in the first-order approximation the Hamiltonian spectrum and to compute the Canonical Ensemble partition function of the system. In this approximation, the phase transition is continuous (with divergent magnetic susceptibility characterized by the critical exponent $\gamma = 1$ and a finite discontinuity of the heat capacity) and at low temperatures (where the Sommerfeld expansion can be used to obtain an analytical expression for the relevant chemical potentials), it occurs when [1–3]

$$k_{\rm F}a_0 \geq \frac{\pi}{2}\left[1+\frac{\pi^2}{12}\left(\frac{k_{\rm B}T}{\varepsilon_{\rm F}}\right)^2+\dots\right],$$

where the (overall) Fermi wave vector and energy

$$k_{\rm F} = \left(3\pi^2\,\frac{N}{V}\right)^{1/3}, \qquad \varepsilon_{\rm F} = \frac{\hbar^2 k_{\rm F}^2}{2m_f}, \tag{1}$$

characterize the density of the system and $a_0 > 0$ is the s-wave scattering length characterizing the strength of the (repulsive) interaction. The continuous character of the Stoner transition obtained in this approximation is, however, accidental — it is due to a numerical coincidence specific for a (three-dimensional) system of spin $s = 1/2$ fermions only (in the same approximation, the transition is of the first order if $s > 1/2$ and/or $D \neq 3$). In fact, computing the system's free energy beyond the mean field approximation, using the ordinary second-order perturbative expansion, it was found [4] that at low temperatures it is of the first order, just as had been suggested in [5] on the basis of the generic presence of nonanalytic terms (resulting from the coupling of the order parameter to the gap-less modes) in the free energy which cause the transition to have the first-order character.

The character of the considered transition (its dependence on the parameter $k_{\rm F}a_0$) can be most easily investigated at zero temperature because then the problem reduces to the computation of the ground-state energy density E_Ω/V of the system of fermions interacting through a binary spin-independent repulsive potential as a function of the system's density $n = N/V$ and its polarization

$$P = (N_+ - N_-)/N\,. \tag{2}$$

Such a computation is most easily performed using the modern effective field theory approach, the application of which to this problem has been pioneered in [6]. In this approach, the underlying spatially nonlocal field-theory interaction (see *e.g.* [7] for the exposition of the relevant formalism of the second quantization), resulting from the ordinary potential two-body interaction, is replaced by an infinite set of local (contact) effective interactions

$$\hat{V}_{\text{int}} = C_0 \int \mathrm{d}^3\boldsymbol{x}\ \psi_+^\dagger \psi_+ \psi_-^\dagger \psi_- + \hat{V}_{\text{int}}^{(C_2)} + V_{\text{int}}^{(C_2')} + \dots \tag{3}$$

$\psi_\pm(\boldsymbol{x})$ are here the usual field operators of spin-up and spin-down fermions; the terms $V_{\text{int}}^{(C_2)}$, $V_{\text{int}}^{(C_2')}$ (which will be not needed in this work) represent local operators of lower length dimension with four fermionic fields and two spatial derivatives, and the ellipsis stands for other local operators (with more derivatives and/or field operators) of yet lower length dimension (see [6]). The amount of work needed to obtain the systematic expansion of the ground-state energy in powers (which can be modified by logarithms) of $k_\text{F}R$, where R is the characteristic length scale of the underlying two-body spin-independent interaction potential, is in this way greatly reduced. This is because in this approach the coupling constants, like C_0 in (3), of the effective local interactions are directly determined in terms of the scattering lengths a_ℓ and the effective radii r_ℓ, $\ell = 0, 1, \dots$ (which are assumed to be of the order of $\sim R$), parametrizing the low-energy expansion in powers of the relative momentum $\hbar|\boldsymbol{k}|$ of the elastic scattering amplitude of two fermions. The simplifications brought in by the effective field theory method allowed to easily reproduce [8] and generalize to arbitrary repulsive potentials and arbitrary spins s [9] the old result of Kanno [10] who computed the order $(k_\text{F}R)^2$ correction to the energy density using the specific hard-sphere interaction of spin $s = 1/2$ fermions. The first-order character of the phase transition at $T = 0$ is then clearly seen in the form of the energy density obtained in this approximation plotted as a function of the order parameter P: starting from some value of $k_\text{F}a_0$, the energy density develops the second minimum well away from the one at $P = 0$ and at $k_\text{F}a_0 = 1.054$ (for $s = 1/2$), this second minimum becomes deeper than that at $P = 0$.

However, the analysis of the dependence on the order parameter of the system's energy density which includes the complete order $(k_\text{F}R)^3$ corrections obtained recently in [11, 12] using the same effective field theory approach shows that, independently of the value s of the fermion spin, they have the effect of erasing the first-order character of the Stoner transition, making it almost indistinguishable from the continuous one. This is reflected in the fact that the height of the hill separating the minimum at $P \neq 0$ from the one at $P = 0$ is greatly reduced (for higher spins also the position of the nontrivial minimum of E_{Ω}/V as a function of the relevant order parameter is strongly shifted towards its small values) compared to the situation without these corrections. Moreover, there are claims [13] based on a resummation of an infinite subclass of Feynman diagrams contributing to the ground-state energy density that the transition (at $T = 0$) is indeed continuous. Although it is not obvious that the contributions taken into account in this resummation are really the dominant ones [12], the results it leads to seem to agree well, as far as the critical value of $k_{\rm F}a_0$ is concerned, with the numerical quantum Monte Carlo simulations [14].

In view of this situation, it is desirable to investigate how the higher-order corrections influence the character of the Stoner phase transition at nonzero temperatures. With this goal in mind, in this paper, we formulate a systematic perturbative expansion of the thermodynamic potentials of the system in question applying the standard imaginary time formalism [7] within the effective field theory. We show that the expansion of the free energy is in this approach particularly simple being given by the same connected vacuum Feynman diagrams which give nonzero contributions to the energy density expressed in terms of the chemical potentials of the noninteracting system. In the numerical analysis, we restrict ourselves in this paper only to the second-order contributions reproducing the results obtained in [4], but with more labour the computations can be extended to higher orders as well.

2. Perturbative expansion of the thermodynamic potential $\boldsymbol{\Omega(T, V, \mu_+, \mu_-)}$

The natural equilibrium statistical physics formalism in which to treat the problem of the gas of fermions the interactions of which preserve their spins, and therefore the numbers N_σ of particles with the spin projection σ, is the Grand Canonical Ensemble with separate chemical potentials μ_σ associated with the individual spin projections. One is, therefore, interested in the statistical operator (as usually, $\beta \equiv 1/k_{\rm B}T$)

$$\hat{\rho} = \frac{1}{\varXi_{\rm stat}}\,{\rm e}^{-\beta\hat{K}}\,, \qquad \text{in which} \qquad \hat{K} = \hat{H}_0 - \sum_\sigma \mu_\sigma \hat{N}_\sigma + \hat{V}_{\rm int} \equiv \hat{K}_0 + \hat{V}_{\rm int}\,, \tag{4}$$

and in computing the statistical sum (we specify the notation to the case of spin-1/2 fermions, so that $\sigma = +, -$)

$$\varXi_{\rm stat}(T, V, \mu_+, \mu_-) = {\rm Tr}\left({\rm e}^{-\beta\hat{K}}\right)\,, \tag{5}$$

from which all the necessary thermodynamic potentials can, in principle, be obtained by performing the standard steps. The free part $\hat{K}_0$ of the operator

$\hat{K} = \hat{K}_0 + \hat{V}_{\rm int}$, in which $\hat{V}_{\rm int}$ will be taken in the form (3), reads

$$\hat{K}_0 = \sum_{\boldsymbol{p},\sigma} (\varepsilon_{\boldsymbol{p}} - \mu_\sigma)\, a^\dagger_{\boldsymbol{p},\sigma} a_{\boldsymbol{p},\sigma} = \sum_\sigma \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} (\varepsilon_{\boldsymbol{p}} - \mu_\sigma)\, a^\dagger_\sigma(\boldsymbol{p})\, a_\sigma(\boldsymbol{p})\,, \tag{6}$$

with $\varepsilon_{\boldsymbol{p}} \equiv \hbar^2 \boldsymbol{p}^2/2m_f$, in the normalizations in the finite volume V and in an infinite space, respectively.

To compute perturbatively the statistical sum $\varXi_{\rm stat}(T, V, \mu_+, \mu_-)$, one introduces [7] the (imaginary time) interaction picture evolution operator

$$\mathcal{U}_{\rm I}(\tau_2, \tau_1) = e^{\tau_2 \hat{K}_0}\, e^{-(\tau_2-\tau_1)\hat{K}}\, e^{-\tau_1 \hat{K}_0}\,, \tag{7}$$

which satisfies the differential equation

$$\frac{d}{d\tau_2}\, \mathcal{U}_{\rm I}(\tau_2, \tau_1) = -V^{\rm I}_{\rm int}(\tau_2)\, \mathcal{U}_{\rm I}(\tau_2, \tau_1)\,,$$

$(V^{\rm I}_{\rm int}(\tau_2) = e^{\tau_2 \hat{K}_0} V_{\rm int} e^{-\tau_2 \hat{K}_0})$ with the "initial" condition $\mathcal{U}_{\rm I}(\tau, \tau) = \hat{1}$ and which formally can be written in the form

$$\mathcal{U}_{\rm I}(\tau_2, \tau_1) = {\rm T}_\tau \exp\left\{-\int_{\tau_1}^{\tau_2} d\tau\; V^{\rm I}_{\rm int}(\tau)\right\}\,,$$

in which ${\rm T}_\tau$ is the symbol of the "chronological" ordering. Since $e^{-\beta\hat{K}} = e^{-\beta\hat{K}_0}\,\mathcal{U}_{\rm I}(\beta, 0)$, the statistical sum can be represented as

$$\varXi_{\rm stat} = {\rm Tr}\left(e^{-\beta\hat{K}_0}\, \mathcal{U}_{\rm I}(\beta, 0)\right) \equiv \varXi^{(0)}_{\rm stat}\, {\rm Tr}\left(\hat{\rho}^{(0)}\, \mathcal{U}_{\rm I}(\beta, 0)\right)\,, \tag{8}$$

with $\hat{\rho}^{(0)}$ and $\varXi^{(0)}_{\rm stat}$ the statistical operator and the statistical sum of the noninteracting system, respectively. The perturbative expansion of $\varXi_{\rm stat}$ is then given by the series

$$\varXi_{\rm stat} = \varXi^{(0)}_{\rm stat} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta d\tau_n \ldots \int_0^\beta d\tau_1 {\rm Tr}\left(\hat{\rho}^{(0)}\; {\rm T}_\tau\left[V^{\rm I}_{\rm int}(\tau_n) \ldots V^{\rm I}_{\rm int}(\tau_1)\right]\right)\,. \tag{9}$$

The corresponding expansion of the potential

$$\varOmega(T, V, \mu_+, \mu_-) = -\frac{1}{\beta} \ln \varXi_{\rm stat}(T, V, \mu_+, \mu_-)$$

is

$$\Omega = \Omega^{(0)} - \frac{1}{\beta}\ln\left\{\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^\beta d\tau_n \dots \int_0^\beta d\tau_1 \,\mathrm{Tr}\Big(\hat{\rho}^{(0)}\mathrm{T}_\tau\left[V^{\mathrm{I}}_{\mathrm{int}}(\tau_n)\dots V^{\mathrm{I}}_{\mathrm{int}}(\tau_1)\right]\Big)\right\}, \tag{10}$$

its first term $\Omega^{(0)}$ being the textbook expression [1] ($\varepsilon_{\boldsymbol{k}} = \hbar^2\boldsymbol{k}^2/2m_f$)

$$\Omega^{(0)}(T, V, \mu_\sigma) = -\frac{1}{\beta}\sum_\sigma V\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\ln\Big(1 + e^{-\beta(\varepsilon_{\boldsymbol{k}}-\mu_\sigma)}\Big). \tag{11}$$

Or, since the logarithm picks up connected contributions only,

$$\Omega = \Omega^{(0)} - \frac{1}{\beta}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^\beta d\tau_n \dots \int_0^\beta d\tau_1 \,\mathrm{Tr}\Big(\hat{\rho}^{(0)}\mathrm{T}_\tau\left[V^{\mathrm{I}}_{\mathrm{int}}(\tau_n)\dots V^{\mathrm{I}}_{\mathrm{int}}(\tau_1)\right]\Big)^{\mathrm{con}}. \tag{12}$$

In this form, the expression for Ω is just the thermal analog of the expansion of the formula[1]

$$E_\Omega = E_{\Omega_0} - \lim_{T\to\infty}\frac{\hbar}{iT}\langle\Omega_0|\mathrm{T}_t \exp\left(-\frac{i}{\hbar}\int_{-T/2}^{T/2} dt\, V^{\mathrm{I}}_{\mathrm{int}}(t)\right)|\Omega_0\rangle^{\mathrm{con}} \tag{13}$$

used in [6, 8, 11, 12] for computing the ground state energy E_Ω of the system. It is clear that the correspondence between the two formalisms is $\beta \leftrightarrow iT/\hbar$ (it transforms the K-picture operators into the Heisenberg picture ones and *vice versa*). The formula (13) for the ground-state energy is thus obtained from the thermal expansion (12) by taking the limit $\beta \to \infty$ and simultaneously adjusting the chemical potential μ_σ so that there are N_σ particles with the spin projection σ (see below).

Evaluation of the successive terms of the expansion (12) reduces, owing to the thermal analog of the Wick formula (see [7]), to drawing all possible connected Feynman diagrams with a given number of different interaction vertices arising from $\hat{V}_{\mathrm{int}}$ joined by the oriented lines and integrating over the positions $\boldsymbol{x}$ and "times" τ ascribed to these vertices the corresponding products of free thermal propagators

$$\begin{aligned}-\mathcal{G}^{(0)}_{\sigma_2,\sigma_1}(\tau_2-\tau_1;\boldsymbol{x}_2-\boldsymbol{x}_1) &= \frac{1}{\beta}\sum_n\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\, e^{-i\omega^F_n(\tau_2-\tau_1)}\, e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_2-\boldsymbol{x}_1)}\\ &\quad\times\Big(-\tilde{\mathcal{G}}^{(0)}_{\sigma_2,\sigma_1}(\omega^F_n,\boldsymbol{k})\Big),\end{aligned}$$

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[1] Here T denotes time and not the temperature.

associated with the (oriented) lines connecting the vertices of the diagram. The Fourier transforms $-\tilde{\mathcal{G}}^{(0)}_{\sigma_2,\sigma_1}$ have the form [7] (the definition of ω^F_n as well as of ω^B_n are given in (A.1) and (A.2))

$$-\tilde{\mathcal{G}}^{(0)}_{\sigma_2,\sigma_1}\left(\omega^F_n,\boldsymbol{k}\right)=\frac{-\delta_{\sigma_2,\sigma_1}}{i\omega^F_n-(\varepsilon_{\boldsymbol{k}}-\mu_{\sigma_1})}\,.$$

The resulting Feynman rules in the "momentum" space are almost identical to the ordinary ones except that the integrations over frequencies are replaced by summations over the (fermionic) Matsubara frequencies $\omega^F_n=(\pi/\beta)(2n+1)$, $n\in\mathbb{Z}$. In this way, one obtains the expansion of the potential $\Omega(T,V,\mu_+,\mu_-)$ the successive terms of which depend on the chemical potentials μ_+ and μ_- which must be adjusted in successive orders of the expansion to yield through the relations

$$N_\pm=-\left(\partial\Omega/\partial\mu_\pm\right)_{T,V}\,, \tag{14}$$

the prescribed densities $n_+=N_+/V$ and $n_-=N_-/V$ of particles with the spin projections up and down.

It will be instructive to recover first, using this formalism, the textbook results [1, 2] of the mean field approximation. The first correction $\Omega^{(1)}$ to the grand potential is given by the single diagram shown in figure 1. The corresponding expression reads

$$\begin{aligned}\Omega^{(1)}&=\frac{1}{\beta}C_0\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^3\boldsymbol{x}\,\mathrm{Tr}\left(\hat{\rho}^{(0)}\mathrm{T}_\tau\left[\hat{\psi}^{\dagger\mathrm{I}}_+\hat{\psi}^{\mathrm{I}}_+\hat{\psi}^{\dagger\mathrm{I}}_-\hat{\psi}^{\mathrm{I}}_-\right]\right)\\&=C_0V\mathcal{G}^{(0)}_{++}(0,\mathbf{0})\mathcal{G}^{(0)}_{--}(0,\mathbf{0})\,.\end{aligned}\tag{15}$$

Using the summation formula (A.1), one obtains

$$\mathcal{G}^{(0)}_{\pm\pm}(0,\mathbf{0})=\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\left[1+\mathrm{e}^{\beta(\varepsilon_{\boldsymbol{k}}-\mu_\pm)}\right]^{-1}\,. \tag{16}$$

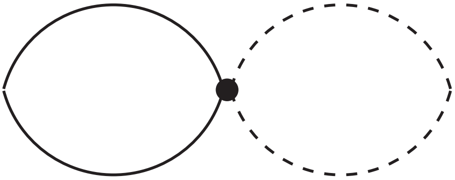

Fig. 1. The first-order correction $\Omega^{(1)}$ to the thermodynamic potential $\Omega(T,V,\mu_+,\mu_-)$. Solid and dashed lines represent fermions with opposite spin projections.

As will be shown in the next section, to the first order in the coupling C_0 the free energy $F(T,V,N_+,N_-)$ is given by

$$\begin{aligned} F(T,V,N_+,N_-) \;&=\; \Omega^{(0)}\left(T,V,\mu_+^{(0)},\mu_-^{(0)}\right) + N_+\mu_+^{(0)} + N_-\mu_-^{(0)} \\ &\quad + \Omega^{(1)}\left(T,V,\mu_+^{(0)},\mu_-^{(0)}\right) + \dots\,, \end{aligned} \tag{17}$$

where $\mu_\pm^{(0)}$ are the zeroth order chemical potentials determined by the conditions analogous to (14) but with Ω replaced by $\Omega^{(0)}$ given by (11). It is convenient to define the function

$$f(\nu) \equiv \frac{3}{2}\int_0^\infty \mathrm{d}\xi\, \frac{\xi^{1/2}}{1+\mathrm{e}^{\xi-\nu}} \equiv \frac{3\sqrt{\pi}}{4}\, f_{3/2}(\nu)\,, \tag{18}$$

and to rewrite these conditions in the form

$$f(\nu_\pm) = \left(\frac{\varepsilon_{\mathrm{F}}^{(0)}(n_\pm)}{k_{\mathrm{B}}T}\right)^{3/2}, \tag{19}$$

in which $\nu_\pm \equiv \mu_\pm/k_{\mathrm{B}}T$ and $\varepsilon_{\mathrm{F}}^{(0)}(n) = (6\pi^2 n)^{2/3}\hbar^2/2m_f$ is the Fermi energy of the system of $N = nV$ spin-0 noninteracting fermions enclosed in the volume V. The function $f(\nu)$, which is a decent monotonically growing function of ν mapping $\mathbb{R}$ onto $\mathbb{R}_+$, has the inverse, so after writing $n_\pm$ as $(n/2)(1 \pm P)$, the solutions take the form[2]

$$\frac{\mu_\pm^{(0)}}{k_{\mathrm{B}}T} = f^{-1}\left((1\pm P)\left(\frac{\varepsilon_{\mathrm{F}}(n)}{k_{\mathrm{B}}T}\right)^{3/2}\right), \tag{20}$$

in which $\varepsilon_{\mathrm{F}}(n)$ is the system's overall Fermi energy (1).

Expressed in terms of the zeroth order chemical potentials $\mu_\pm^{(0)}$, the first-order correction (15) can be simply written as $\Omega^{(1)}(T,V,\mu_+^{(0)},\mu_-^{(0)}) = C_0 V(N_+/V)(N_-/V)$, *i.e.* it is independent (when expressed in terms of the particle densities) of the temperature. Minimization with respect to N_+

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[2] Inverting the appropriate expansions of the integral in (18) given *e.g.* in [1], it is straightforward to find that asymptotically

$$f^{-1}(x) = \begin{cases} \ln\left(\sqrt{2} - \sqrt{2 - \left(\frac{4x}{3}\right)\sqrt{\frac{8}{\pi}}}\right) + \dots\,, & x \ll 1 \\ x^{2/3}\left[1 - \left(\frac{\pi^2}{12}\right)x^{-4/3} - \left(\frac{\pi^4}{80}\right)x^{-8/3} - \left(\frac{1511\pi^6}{207360}\right)x^{-4} + \dots\right], & x \gg 1 \end{cases}.$$

of $F(T,V,N_+,N-N_+)$ truncated to the first order in the coupling C_0 (at fixed N) then leads to the equilibrium condition

$$\mu_+^{(0)}(N_+)-\mu_-^{(0)}(N-N_+)+\frac{C_0}{V}(N-2N_+)=0\,,$$

which, because $N-2N_+=-NP$ and (to this order) $C_0=(4\pi\hbar^2/m_f)a_0$, can be rewritten [6, 8] in the familiar form [1]

$$\mu_+^{(0)}(N_+)-\mu_-^{(0)}(N_-)=\frac{8}{3\pi}\varepsilon_{\rm F}(k_{\rm F}a_0)\,P\,. \tag{21}$$

This leads to the continuous phase transition.

The effect of the external magnetic field $\mathcal{H}$ can be also taken into account by simply including the interaction with it in the free part of the Hamiltonian, *i.e.* by replacing $\mu_\pm$ in (6) by $\tilde{\mu}_\pm=\mu_\pm\pm\mathcal{H}$ (the magnetic moment has been here included in $\mathcal{H}$ which has therefore the dimension of energy). Since ultimately the free energy will be cast in the form in which its dependence on $N_\pm$ and $\mathcal{H}$ enters only through $\tilde{\mu}_\pm^{(0)}$ which should be determined from the conditions

$$\frac{\tilde{\mu}_\pm^{(0)}}{k_{\rm B}T}=\frac{\mu_\pm^{(0)}\pm\mathcal{H}}{k_{\rm B}T}=f^{-1}\left((1\pm P)\left(\frac{\varepsilon_{\rm F}(n)}{k_{\rm B}T}\right)^{3/2}\right)\,, \tag{22}$$

this prescription remains valid to all orders of the expansion. In particular, in the first-order approximation the equilibrium condition, written in the convenient dimensionless variables

$$t\equiv\frac{T}{T_{\rm F}}\equiv\frac{k_{\rm B}T}{\varepsilon_{\rm F}}\,,\qquad h\equiv\frac{\mathcal{H}}{\varepsilon_{\rm F}}\,,\qquad \delta_\pm\equiv\frac{\mu_\pm^{(0)}}{\varepsilon_{\rm F}}\,, \tag{23}$$

takes the form

$$\frac{8}{3\pi}\,(k_{\rm F}a_0)\,P+2h=t\left[f^{-1}\left(\frac{1+P}{t^{3/2}}\right)-f^{-1}\left(\frac{1-P}{t^{3/2}}\right)\right]\,. \tag{24}$$

If the asymptotic expansion of $f^{-1}(x)$ for $x\gg1$ is used, this reproduces the equilibrium condition derived in [1].

For further applications, it will be convenient to write down explicitly the formula (17) (including the external magnetic field $\mathcal{H}$) expressing it through the introduced dimensionless variables (23) and the polarization (2):

$$\frac{6\pi^2}{k_{\rm F}^3}\,\frac{F}{\varepsilon_{\rm F}V}=-\frac{3\sqrt{\pi}}{4}\,t^{5/2}\left[f_{5/2}(\tilde{\nu}_+)+f_{5/2}(\tilde{\nu}_-)\right]$$
$$+(1+P)\left(\tilde{\delta}_+-h\right)+(1-P)\left(\tilde{\delta}_-+h\right)+(k_{\rm F}a_0)\,\frac{4}{3\pi}\,\left(1-P^2\right)+\dots \tag{25}$$

Here[3],

$$f_p(\nu) = \frac{1}{\Gamma(p)} \int\limits_0^\infty \frac{\mathrm{d}\xi\, \xi^{p-1}}{1+\mathrm{e}^{\xi-\nu}}\,, \tag{26}$$

and $\tilde{\nu}_\pm$ (and $\tilde{\delta}_\pm \equiv t\tilde{\nu}_\pm$) are given by (22). In the limit of $T \to 0$ ($t \to 0$) in which $\tilde{\nu}_\pm \gg 1$, $f_{5/2}(\nu) = (4/3\sqrt{\pi}\,)(2/5)\nu^{5/2} + \ldots$, while (*cf.* the expansion of the function $f^{-1}(x)$ given in footnote 2) $\tilde{\nu}_\pm = ((1 \pm P)^{2/3})/t + \ldots$ and the right-hand side of (25) tends to

$$\begin{aligned}&-\frac{2}{5}\left[(1+P)^{5/3} + (1-P)^{5/3}\right] + (1+P)\left[(1+P)^{2/3} - h\right]\\ &+(1-P)\left[(1-P)^{2/3} + h\right] + (k_\mathrm{F}a_0)\,\frac{4}{3\pi}\,\left(1-P^2\right)\,,\end{aligned}$$

reproducing, of course, the well-known formula for the ground-state energy given (for $\mathcal{H} = 0$) *e.g.* in [12].

3. Expansion of the free energy

From the thermodynamic point of view, much more convenient to work with than the potential Ω is the free energy $F = \Omega + \mu_+ N_+ + \mu_- N_-$ which canonically depends on T, V, and the particle numbers $N_\pm$. It turns out that the expansion of this potential is also simpler. We will derive it here up to the third order following the method outlined in [15]. To make the notation more transparent, we will denote the chemical potentials as

$$\mu_+ \equiv x = x_0 + x_1 + x_2 + \ldots\,, \qquad \mu_- \equiv y = y_0 + y_1 + y_2 + \ldots\,, \tag{27}$$

where the successive terms x_n, y_n correspond to the successive terms $\Omega^{(n)}$ of the expansion of the potential Ω. Introducing the notation $\Omega^{(n)}_x$, $\Omega^{(n)}_y$, $\Omega^{(n)}_{xx}$, *etc.* for the first, second, *etc.* derivatives of $\Omega^{(n)}$ with respect to their chemical potential arguments and expanding the right-hand side of the relation ($N_x \equiv N_+$, $N_y \equiv N_-$)

$$\begin{aligned} F \;=\; & \Omega^{(0)}(x_0 + x_1 + x_2 + x_3 + \ldots,\, y_0 + y_1 + y_2 + y_3 + \ldots)\\ &+\Omega^{(1)}(x_0 + x_1 + x_2 + \ldots,\, y_0 + y_1 + y_2 + \ldots)\\ &+\Omega^{(2)}(x_0 + x_1 + \ldots,\, y_0 + y_1 + \ldots) + \Omega^{(3)}(x_0 + \ldots, y_0 + \ldots) + \ldots\\ &+(x_0 + x_1 + x_2 + x_3 + \ldots)N_x + (y_0 + y_1 + y_2 + y_3 + \ldots)N_y\,,\end{aligned}$$

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[3] By the appropriate change of variables and the integration by parts, $\Omega^{(0)}$ given by (11) is written in terms of the standard integral (26) with $p = 5/2$ [1].

one obtains, using the zeroth order relations $\Omega_x^{(0)} = -N_x$ and $\Omega_y^{(0)} = -N_y$, and the fact that $\Omega^{(0)}(x_0, y_0) = \Omega^{\text{free}}(x_0) + \Omega^{\text{free}}(y_0)$ (*cf.* formula (11)), *i.e.* that $\Omega_{xy}^{(0)} = 0$,

$$\begin{aligned}
F = {} & \left(\Omega^{(0)} + x_0 N_x + y_0 N_y\right) + \left(\Omega^{(1)}\right) \\
& + \left(\Omega^{(2)} + x_1 \Omega_x^{(1)} + y_1 \Omega_y^{(1)} + \frac{1}{2} x_1^2 \Omega_{xx}^{(0)} + \frac{1}{2} y_1^2 \Omega_{yy}^{(0)}\right) \\
& + \Bigg(\Omega^{(3)} + x_1 \Omega_x^{(2)} + y_1 \Omega_y^{(2)} + \frac{1}{2} x_1^2 \Omega_{xx}^{(1)} + \frac{1}{2} y_1^2 \Omega_{yy}^{(1)} + x_1 y_1 \Omega_{xy}^{(1)} \\
& + x_2 \Omega_x^{(1)} + y_2 \Omega_y^{(1)} + x_1 x_2 \Omega_{xx}^{(0)} + y_1 y_2 \Omega_{yy}^{(0)} + \frac{1}{6} x_1^3 \Omega_{xxx}^{(0)} + \frac{1}{6} y_1^3 \Omega_{yyy}^{(0)}\Bigg) + \dots ,
\end{aligned} \tag{28}$$

all functions being now evaluated at x_0 and y_0 (at $\tilde{x}_0 = \mu_+^{(0)} + \mathcal{H}$ and $\tilde{y}_0 = \mu_-^{(0)} - \mathcal{H}$ if there is an external magnetic field). The terms in the successive brackets are the successive terms of the expansion of the free energy. The first-order correction $F^{(1)}$ used in the preceding section is indeed given by $\Omega^{(1)}(x_0, y_0)$ (by $\Omega^{(1)}(\tilde{x}_0, \tilde{y}_0)$). Furthermore, expanding around x_0 and y_0 (or $\tilde{x}_0$ and $\tilde{y}_0$) the right-hand side of the relation (14) which determines the chemical potential x

$$\begin{aligned}
-N_x = {} & \Omega_x^{(0)} + (x_1 + x_2)\Omega_{xx}^{(0)} + \frac{1}{2} x_1^2 \Omega_{xxx}^{(0)} + \Omega_x^{(1)} + x_1 \Omega_{xx}^{(1)} \\
& + y_1 \Omega_{xy}^{(1)} + \Omega_x^{(2)} + \dots ,
\end{aligned}$$

and the other similar relation for y, and taking into account that x_0 and y_0 are such that $-N_x = \Omega_x^{(0)}$, $-N_y = \Omega_y^{(0)}$, one obtains

$$\begin{aligned}
x_1 &= -\frac{\Omega_x^{(1)}}{\Omega_{xx}^{(0)}} , \\
x_2 &= -\frac{\Omega_x^{(2)}}{\Omega_{xx}^{(0)}} + \frac{\Omega_{xx}^{(1)} \Omega_x^{(1)}}{\left[\Omega_{xx}^{(0)}\right]^2} + \frac{\Omega_{xy}^{(1)} \Omega_y^{(1)}}{\Omega_{xx}^{(0)} \Omega_{yy}^{(0)}} - \frac{\Omega_{xxx}^{(0)} \left[\Omega_x^{(1)}\right]^2}{2\left[\Omega_{xx}^{(0)}\right]^3} .
\end{aligned} \tag{29}$$

y_1 and y_2 are given by the analogous formulae. Inserting the corrections to the chemical potentials determined in this way into the formulae for $F^{(2)}$ and $F^{(3)}$, one finds that (again, all functions are evaluated at x_0 and y_0 or at $\tilde{x}_0$ and $\tilde{y}_0$)

$$F^{(2)} = \Omega^{(2)} - \frac{\left[\Omega_x^{(1)}\right]^2}{2\Omega_{xx}^{(0)}} - \frac{\left[\Omega_y^{(1)}\right]^2}{2\Omega_{yy}^{(0)}} , \tag{30}$$

and (the formulae for x_1 and y_1 immediately imply that the first four terms in the last line of the formula (28) sum up to zero) that

$$\begin{aligned} F^{(3)} &= \Omega^{(3)} - \frac{\Omega^{(2)}_x \Omega^{(1)}_x}{\Omega^{(0)}_{xx}} - \frac{\Omega^{(2)}_y \Omega^{(1)}_y}{\Omega^{(0)}_{yy}} + \frac{\Omega^{(1)}_{xx}\left[\Omega^{(1)}_x\right]^2}{2\left[\Omega^{(0)}_{xx}\right]^2} + \frac{\Omega^{(1)}_{yy}\left[\Omega^{(1)}_y\right]^2}{2\left[\Omega^{(0)}_{yy}\right]^2} \\ &\quad + \frac{\Omega^{(1)}_{xy}\Omega^{(1)}_x\Omega^{(1)}_y}{\Omega^{(0)}_{xx}\Omega^{(0)}_{yy}} - \frac{\Omega^{(0)}_{xxx}\left[\Omega^{(1)}_x\right]^3}{6\left[\Omega^{(0)}_{xx}\right]^3} - \frac{\Omega^{(0)}_{yyy}\left[\Omega^{(1)}_y\right]^3}{6\left[\Omega^{(0)}_{yy}\right]^3} . \end{aligned} \tag{31}$$

It will be seen that the extra terms in (30) precisely cancel the contributions to $\Omega^{(2)}$ of those diagrams which do not contribute to the expansion of the formula (13) for the ground-state energy density. The analogous cancellation of the extra terms in (31) and in $\Omega^{(3)}$ is demonstrated in Appendix B.

4. Computation of $F^{(2)}$

Diagrams contributing to $\Omega^{(2)}$ are shown in figures 2 and 3 (the left one). It is straightforward to check that the contributions $\Omega^{(2)b}$ and $\Omega^{(2)c}$ of the ones of figure 2 cancel against the last two terms in the formula (30). Indeed, with the help of the summation rules collected in Appendix A and taking into account that these contributions are evaluated at x_0 and y_0, one easily obtains ($\Omega^{(2)c}$ is given by an analogous formula)

$$\begin{aligned} \Omega^{(2)b} &= \frac{C_0^2 V}{2}\left(\frac{N_-}{V}\right)^2 \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\left[\frac{\mathrm{d}}{\mathrm{d}a}\frac{1}{1+\mathrm{e}^{\beta a}}\right]_{a=\varepsilon_{\boldsymbol{p}}-x_0} \\ &= -\frac{1}{2}C_0^2 V\beta(n_x - n_{xx})n_y^2 , \end{aligned} \tag{32}$$

where the second form of $\Omega^{(2)b}$ is given in the notation introduced in Appendix B. With the help of the formulae (B.1) and (B.2), it is immediately seen that it is canceled by the second term of (30). Thus,

$$\Omega^{(2)b} + \Omega^{(2)c} - \frac{\left[\Omega^{(1)}_x\right]^2}{2\Omega^{(0)}_{xx}} - \frac{\left[\Omega^{(1)}_y\right]^2}{2\Omega^{(0)}_{yy}} = 0 .$$

Hence, $F^{(2)} = \Omega^{(2)a}$ evaluated at x_0 and y_0 (or at $\tilde{x}_0$ and $\tilde{y}_0$).

The integrals and sums corresponding to the left diagram of figure 3 giving $\Omega^{(2)a}$ can be written in three different forms (corresponding to three

Fig. 2. The order C_0^2 contributions $\Omega^{(2)b}$ and $\Omega^{(2)c}$.

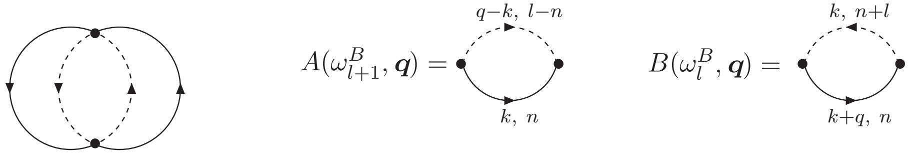

Fig. 3. The order C_0^2 diagram contributing to the correction $\Omega^{(2)}$ and two "elementary" one-loop diagrams out of which the second-order and third-order corrections with the C_0 couplings can be constructed. Solid and dashed lines denote propagators of fermions with the spin projections + and −, respectively.

different routings of the internal momenta and frequencies) of which two can be composed out of two "elementary" blocks A and B shown in figure 3, right

$$\begin{aligned}\Omega^{(2)a} &= -\frac{1}{2}\,C_0^2 V\,\frac{1}{\beta}\sum_{l\in\mathbb{Z}}\int\frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\left[A\left(\omega^B_l,\boldsymbol{q}\right)\right]^2\\ &= -\frac{1}{2}\,C_0^2 V\,\frac{1}{\beta}\sum_{l\in\mathbb{Z}}\int\frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\left[B\left(\omega^B_l,\boldsymbol{q}\right)\right]^2\,,\end{aligned}$$

where (here $n_\pm(\boldsymbol{p})\equiv[1+\exp\{\beta(\varepsilon_{\boldsymbol{p}}-\mu^{(0)}_\pm)\}]^{-1}$, $\mu^{(0)}_+\equiv x_0$, $\mu^{(0)}_-\equiv y_0$)

$$\begin{aligned}A\left(\omega^B_{l+1},\boldsymbol{q}\right) &= \frac{1}{\beta}\sum_{n\in\mathbb{Z}}\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\,\frac{1}{i\omega^F_n-(\varepsilon_{\boldsymbol{k}}-x_0)}\,\frac{1}{i\omega^F_{l-n}-(\varepsilon_{\boldsymbol{q}-\boldsymbol{k}}-y_0)}\\ &= \int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\,\frac{n_+(\boldsymbol{k})+n_-(\boldsymbol{q}-\boldsymbol{k})-1}{i\omega^B_{l+1}-(\varepsilon_{\boldsymbol{k}}-x_0+\varepsilon_{\boldsymbol{q}-\boldsymbol{k}}-y_0)}\,,\end{aligned}\tag{33}$$

and

$$\begin{aligned}B\left(\omega^B_{l},\boldsymbol{q}\right) &= \frac{1}{\beta}\sum_{n\in\mathbb{Z}}\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\,\frac{1}{i\omega^F_n-(\varepsilon_{\boldsymbol{k}+\boldsymbol{q}}-x_0)}\,\frac{1}{i\omega^F_{n+l}-(\varepsilon_{\boldsymbol{k}}-y_0)}\\ &= \int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\,\frac{n_+(\boldsymbol{k}+\boldsymbol{q})-n_-(\boldsymbol{k})}{i\omega^B_{l}-(\varepsilon_{\boldsymbol{k}}-y_0-\varepsilon_{\boldsymbol{k}+\boldsymbol{q}}+x_0)}\,.\end{aligned}\tag{34}$$

(The contributions $\Omega^{(3)a}$ and $\Omega^{(3)b}$ of the left and right diagrams shown in figure 7 can be written analogously with $[A(\omega^B_l,\boldsymbol{q})]^3$ and $[B(\omega^B_l,\boldsymbol{q})]^3$, respectively [11, 12]).

With the help of the sum rule (A.5), the sum over l of two A-blocks can be done and gives (the symbol $\int_{\boldsymbol{k}}$ stands for the integral over the measure $\mathrm{d}^3\boldsymbol{k}/(2\pi)^3$)

$$\int_{\boldsymbol{k}}\int_{\boldsymbol{p}} \frac{[n_+(\boldsymbol{k})+n_-(\boldsymbol{q}-\boldsymbol{k})-1]\,[n_+(\boldsymbol{p})+n_-(\boldsymbol{q}-\boldsymbol{p})-1]}{\varepsilon_{\boldsymbol{k}}+\varepsilon_{\boldsymbol{q}-\boldsymbol{k}}-\varepsilon_{\boldsymbol{p}}-\varepsilon_{\boldsymbol{q}-\boldsymbol{p}}}$$
$$\times\left(\frac{1}{1-\mathrm{e}^{\beta(\varepsilon_{\boldsymbol{k}}-x_0)}\mathrm{e}^{\beta\left(\varepsilon_{\boldsymbol{q}-\boldsymbol{k}}-y_0\right)}}-\frac{1}{1-\mathrm{e}^{\beta(\varepsilon_{\boldsymbol{p}}-x_0)}\mathrm{e}^{\beta(\varepsilon_{\boldsymbol{q}-\boldsymbol{p}}-y_0)}}\right).$$

The identity

$$n_+(\boldsymbol{k})+n_-(\boldsymbol{q}-\boldsymbol{k})-1=n_+(\boldsymbol{k})\,n_-(\boldsymbol{q}-\boldsymbol{k})\left[1-\mathrm{e}^{\beta(\varepsilon_{\boldsymbol{k}}-x_0)}\mathrm{e}^{\beta\left(\varepsilon_{\boldsymbol{q}-\boldsymbol{k}}-y_0\right)}\right] \tag{35}$$

and the fact that the two terms in the bracket above give equal contributions allow then to write

$$\Omega^{(2)a}=C_0^2V\int_{\boldsymbol{q}}\int_{\boldsymbol{p}}\int_{\boldsymbol{k}}\frac{n_+(\boldsymbol{k})\,n_-(\boldsymbol{q}-\boldsymbol{k})[1-n_+(\boldsymbol{p})-n_-(\boldsymbol{q}-\boldsymbol{p})]}{\varepsilon_{\boldsymbol{k}}+\varepsilon_{\boldsymbol{q}-\boldsymbol{k}}-\varepsilon_{\boldsymbol{p}}-\varepsilon_{\boldsymbol{q}-\boldsymbol{p}}}\,. \tag{36}$$

It is interesting to notice that because the integral of the quartic product $n_+(\boldsymbol{k})n_-(\boldsymbol{q}-\boldsymbol{k})n_+(\boldsymbol{p})n_-(\boldsymbol{q}-\boldsymbol{p})$ vanishes (the numerator is even with respect to the interchange $\boldsymbol{k}\leftrightarrow\boldsymbol{p}$, while the denominator is odd), the expression for $\Omega^{(2)a}$ can be written (after the change $\boldsymbol{p}=-\boldsymbol{u}+\boldsymbol{s}$, $\boldsymbol{k}=-\boldsymbol{t}+\boldsymbol{s}$, $\boldsymbol{q}=2\boldsymbol{s}$ of the integration variables) in the form completely analogous to the expression giving E_Ω/V (see [8]), the only modification being the change in the prefactor and the replacement of $\theta(k-|\boldsymbol{v}|)$ and $\theta(|\boldsymbol{v}|-k)$ by $n(|\boldsymbol{v}|)$ and $1-n(|\boldsymbol{v}|)$, respectively. (Curiously enough, we have found that this simple analogy does not work for the diagrams of figure 7).

It is straightforward to see that the expression (36) is divergent, the divergence arising from the unity in the square bracket in the numerator. In the variables $\boldsymbol{s}$, $\boldsymbol{t}$, and $\boldsymbol{u}$, the integral over $\boldsymbol{u}$ is the one evaluated with the cutoff Λ in [8] and using this result[4] and changing once more the variables to $\boldsymbol{k}=\boldsymbol{t}-\boldsymbol{s}$, $\boldsymbol{p}=\boldsymbol{t}+\boldsymbol{s}$, after adding the contribution $\Omega^{(1)}$ and expressing C_0 in terms of the scattering lengths a_0

$$C_0(\Lambda)=\frac{4\pi\hbar^2}{m_f}\,a_0\left(1+\frac{2}{\pi}\,a_0\Lambda+\dots\right), \tag{37}$$

---

[4] In [8], this integral has been computed with the $+i0$ prescription in dealing with the pole on the integration axis; its principal value form needed here is obtained by simply discarding the imaginary part.

[11, 12, 16], one arrives at the finite (to the second order) result

$$\begin{aligned}\Omega^{(1)} + \Omega^{(2)a} &= \frac{4\pi\hbar^2}{m_f}\, a_0 \left(1 + \frac{2}{\pi}\,\Lambda a_0 + \dots\right) V\,\frac{N_-}{V}\,\frac{N_+}{V} \\ &\quad - \frac{\Lambda}{2\pi^2}\left(\frac{4\pi\hbar^2}{m_f}\,a_0\right)^2 \frac{m_f}{\hbar^2}\, V\,\frac{N_-}{V}\,\frac{N_+}{V} + \Omega^{(2)a}_{\rm finite} \\ &= \frac{4\pi\hbar^2}{m_f}\, a_0\, V\,\frac{N_-}{V}\,\frac{N_+}{V} + \Omega^{(2)a}_{\rm finite}\,.\end{aligned}$$

The finite part of $\Omega^{(2)a}$,

$$\Omega^{(2)a}_{\rm finite} = -C_0^2 V \int_{\boldsymbol{q}}\int_{\boldsymbol{k}} n_+(\boldsymbol{k})\; n_-(\boldsymbol{q}-\boldsymbol{k}) \int_{\boldsymbol{p}} \frac{n_+(\boldsymbol{p}) + n_-(\boldsymbol{q}-\boldsymbol{p})}{\varepsilon_{\boldsymbol{k}} + \varepsilon_{\boldsymbol{q}-\boldsymbol{k}} - \varepsilon_{\boldsymbol{p}} - \varepsilon_{\boldsymbol{q}-\boldsymbol{p}}}\,,$$

upon setting first $\boldsymbol{k} = \boldsymbol{k}_1$, $\boldsymbol{q} - \boldsymbol{k} = \boldsymbol{k}_2$ and then replacing in the term with $n_-(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{p})$, the variable $\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{p}$ by $\boldsymbol{p}'$ (upon which $\varepsilon_{\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{p}} \to \varepsilon_{\boldsymbol{p}'}$ but at the same time $\varepsilon_{\boldsymbol{p}} \to \varepsilon_{\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{p}'}$) can be cast in the convenient symmetric form

$$\Omega^{(2)a}_{\rm finite} = F^{(2)} = -C_0^2 V \int_{\boldsymbol{k}_1}\int_{\boldsymbol{k}_2} n_+(\boldsymbol{k}_1) n_-(\boldsymbol{k}_2) \int_{\boldsymbol{p}} \frac{n_+(\boldsymbol{p}) + n_-(\boldsymbol{p})}{\varepsilon_{\boldsymbol{k}_1} + \varepsilon_{\boldsymbol{k}_2} - \varepsilon_{\boldsymbol{p}} - \varepsilon_{\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{p}}}\,. \tag{38}$$

The expression (38) is very similar[5] to the formula (5) used in [4] as the second-order contribution to the system's internal energy density u, except that the latter has an extra factor of 2. The foundation of the formula for $f = u - Ts$ (which, apart from this factor of 2, is equivalent to our one) used in [4] is, however, somewhat unclear: to obtain their second-order correction to the energy density u, these authors took the expression (15) given in Section 11.4 of [2] which is obtained by simply using the finite temperature distributions in place of the zero-temperature ones in the ordinary second-order correction to the ground-state energy of the system and have taken the entropy density s as given by the zeroth order textbook formula. In contrast, our expression (38) results from a systematic, well-founded expansion, and the coefficient in (38) is unambiguously fixed by the cancellation of the divergence.

After integrating over the cosine of the angle between $\boldsymbol{p}$ and $\boldsymbol{k}_1 + \boldsymbol{k}_2$, one can write the resulting expression in the form

---

[5] Recall the standard rule $\sum_{\boldsymbol{k}} \to V \int \mathrm{d}^3\boldsymbol{k}/(2\pi)^3 \equiv V \int_{\boldsymbol{k}}$ for passing from the box normalization used in [2, 4] to the continuum one used here.

$$F^{(2)} = -V\,\frac{C_0^2 m_f}{(2\pi)^2\hbar^2}\int\limits_{\boldsymbol{k}_1}\int\limits_{\boldsymbol{k}_2}\frac{n_+(\boldsymbol{k}_1)\,n_-(\boldsymbol{k}_2)}{|\boldsymbol{k}_1+\boldsymbol{k}_2|}\int\limits_0^\infty \mathrm{d}p\,p\,[n_+(\boldsymbol{p})+n_-(\boldsymbol{p})] \times \ln\left|\frac{(p-\Delta_+)(p-\Delta_-)}{(p+\Delta_+)(p+\Delta_-)}\right|, \tag{39}$$

in which

$$\Delta_\pm \equiv \frac{1}{2}\,|\boldsymbol{k}_1+\boldsymbol{k}_2| \pm \frac{1}{2}\,|\boldsymbol{k}_1-\boldsymbol{k}_2|\,.$$

It is clear that the singularity at $|\boldsymbol{k}_1+\boldsymbol{k}_2| = 0$ in (39) is spurious: if $\boldsymbol{k}_1+\boldsymbol{k}_2=\boldsymbol{0}$, then $\Delta_- = -\Delta_+$ and the innermost integral vanishes.

5. Numerical evaluation

The most difficult part of the computation is the accurate and efficient numerical evaluation of the multiple integrals in the expression (39). Rescaling the momentum integration variables $\boldsymbol{k}_1 = k_\mathrm{F}\boldsymbol{v}_1$, *etc.* and inserting $C_0 = (4\pi\hbar/m_f)a_0$, one can write the second-order contribution to the right-hand side of (25) as

$$\frac{6\pi^2}{k_\mathrm{F}^3}\,\frac{F^{(2)}}{\varepsilon_\mathrm{F}V} = -(k_\mathrm{F}a_0)^2\,\frac{6}{\pi^2}\int\limits_0^\infty \mathrm{d}v_1\;v_1^2\;n(v_1,\nu_+,t)\int\limits_0^\infty \mathrm{d}v_2\;v_2^2\;n(v_2,\nu_-,t) \times\sum_{\sigma=\pm}\sum_{\sigma'=\pm}\int\limits_{-1}^{1}\mathrm{d}\xi\,\frac{I(\Delta_\sigma,\nu_{\sigma'},t)}{\sqrt{v_1^2+v_2^2+2\xi v_1v_2}}\,, \tag{40}$$

where as previously $\nu_\pm = \mu_\pm^{(0)}/k_\mathrm{B}T \equiv \delta_\pm/t$ ($\delta_\pm = \mu_\pm^{(0)}/\varepsilon_\mathrm{F}$),

$$n(v,\nu,t) = \left[1+\exp\left(\frac{v^2}{t}-\nu\right)\right]^{-1},$$

$$I(\Delta,\nu,t) = \int\limits_0^\infty \mathrm{d}u\;u\;n(u,\nu,t)\;\ln\left|\frac{u-\Delta}{u+\Delta}\right|,$$

and

$$\Delta_\pm(v_1,v_2,\xi) = \frac{1}{2}\sqrt{v_1^2+v_2^2+2\xi v_1v_2} \pm \frac{1}{2}\sqrt{v_1^2+v_2^2-2\xi v_1v_2}\,.$$

The trick allowing to realize the numerical computation is to make first, for fixed values of t (temperature) and P (the system's polarization) which together, through (20) determine ν_+ and ν_-, an interpolation of the functions $I(|\Delta|, \nu_+, t)$ and $I(|\Delta|, \nu_-, t)$ (because, obviously, $I(-|\Delta|, \nu_\pm, t) = -I(|\Delta|, \nu_\pm, t)$) in the variable $w = 1/(1+|\Delta|)$ (to interpolate on the compact interval $[0, 1]$), and then performing numerically the integrations over v_1, v_2, and ξ using these interpolations. In the actual code written in the Python programming language, the functions $I(|\Delta|, \nu_\pm, t)$ are evaluated with the help of the adaptive integration routine (scipy.integrate.quad; the integration domain is splitted into three subdomains to accurately handle the logarithmic singularity — in the relevant regions near $w = 1/(1 + \Delta) \equiv w_0$, we substitute $r^3 = |w - w_0|$ so that the integrand behaves like $r^2 \ln(r)$ and can be treated using the quadrature methods — and its sharp falloff, especially for small temperatures t, near $u^2 = t\nu$ of the distribution $n(u, \nu, t)$), and then interpolated using the cubic spline interpolation routines of Python. The remaining triple integral over v_1, v_2, and ξ are performed with the help of the Clenshaw–Curtis quadrature in the variables $w_{1,2} = 1/(1+v_{1,2})$ (again to have a compact integration domain and again splitting it into subdomains to better handle the regions $v_1^2 \approx t\nu_+$ and $v_2^2 \approx t\nu_-$); the spurious singularity at $|\boldsymbol{v}_1 + \boldsymbol{v}_2| = 0$ is taken care of by simply taking somewhat different numbers for the v_1 and v_2 grids.

To check the correctness of the code, we have first compared its results for $t \to 0$ (replacing the distributions $n(u, \nu, t)$ by the Heaviside theta functions) with the second-order correction $E_\Omega^{(2)}$ to the system's ground-state energy which as a function of P is known analytically [9, 10] (the function $J_K(x, y)$ is given *e.g.* by formula (4) in [12])

$$\frac{6\pi^2}{k_{\rm F}^3} \frac{E_\Omega^{(2)}}{\varepsilon_{\rm F} V} = (k_{\rm F} a_0)^2 \, \frac{6}{5\pi^2} \, J_K\left((1+P)^{1/3}, (1-P)^{1/3}\right) .$$

At $P = 0$ (equal densities of spin-up and spin-down fermions), $J_K = 4(11 - \ln 4)/21$ and the right-hand side of the above formula (setting in all these comparisons $k_{\rm F} a_0 = 1$) equals 0.222644842, while the Python code for the right-hand side of (40) gives the value 0.22264522. For $P = 0.5$, the code gives 0.17184256 to be compared with 0.17184207, while at $P = 0.9$, the numbers to be compared are 0.046470057 and 0.046470077 (at $P = 1$ both are zero reflecting the impossibility of the s-wave interactions of two fermions in the same spin state). For nonzero temperatures, the results obtained using the Clenshaw–Curtis quadrature have been compared with the ones obtained using the more accurate (but more time-consuming) adaptive integration routine. The comparison shows that the relative uncertainty Δ_F (the difference between the results of the two methods divided by their mean) is typically of the order of 10^{-5}, varying rather irregularity with P and increasing somewhat with t; in our further estimates, we set $\Delta_F = 10^{-5}$

for $t \lesssim 0.1$, $\varDelta_F = 1.5 \times 10^{-5}$ for $0.1 < t \leq 0.2$, and $\varDelta_F = 2 \times 10^{-5}$ for $0.2 < t$. While this accuracy superficially looks quite satisfactory, it is, nevertheless, barely sufficient: for values of the parameters (t and/or $k_{\mathrm{F}}a_0$) at which spontaneous ordering appears, there is a very delicate cancellation between different contributions to F and the (relative) error of the order of 10^{-5} in $F^{(2)}$ can, and in some cases indeed does, lead to the appearances of very shallow fake minimum near $P = 0$.

6. Results

For a fixed value of the temperature, the system's free energy F as a function of the polarization (and of the parameter $k_{\mathrm{F}}a_0$ in which, in the approximation to which our analysis is restricted, it is a polynomial of the second order), can be efficiently obtained by evaluating numerically the integrals in (40) for several values of P and constructing the cubic spline interpolation. The resulting free energy differences, $F(P) - F(0)$, are plotted in figure 4 as functions of the polarization P for two temperatures: $t = 0.1$ and 0.15 and several values of $k_{\mathrm{F}}a_0$ (obtained by constructing the interpolation based on 11 points in P only).

In view of the mentioned uncertainty in the computation of $F^{(2)}$, the critical value of $k_{\mathrm{F}}a_0$ and the value of the polarization P at the transition must be determined by requiring that the value of F at a minimum developing away from $P = 0$ differs from the one at $P = 0$ at least by $\varDelta_F F^{(2)}(0)$. In this way, one can properly handle the mentioned fake minima close to $P = 0$ one of which can be observed in the right panel of figure 4 (for $t = 0$ that such a minimum is indeed produced by the inaccuracies of the numerical code can be substantiated by comparing with the analytically known dependence of the ground-state energy on P). The actual procedure which has been adopted to determine the polarization and its uncertainty is as follows. For a fixed value of the parameter $k_{\mathrm{F}}a_0$ (which is successively increased from 0 in steps $\Delta(k_{\mathrm{F}}a_0) = 0.001$), the values of F on a preliminary grid of P-values $P_n = n\Delta P$ with $n = 0, \dots, n_{\max} = 32$ are obtained. If the minimal value of F occurs for $n_{\min} = n_{\max}$, the polarization is taken as maximal ($P = 1$); if $n_{\min} = 0$ or $|F(P_{n_{\min}}) - F(0)| \leq \varDelta_F F^{(2)}(0)$, the polarization is taken as vanishing ($P = 0$). If $n_{\min} \neq 0, n_{\max}$ and $|F(P_{n_{\min}}) - F(0)| > \varDelta_F F^{(2)}(0)$ the polarization is taken as truly nonvanishing. If it is nonvanishing for the first time (as far as the increasing values of $k_{\mathrm{F}}a_0$ are concerned), one determines n_{down} and n_{up} such that $|F(P_{n_{\min}}) - F(P_{n_{\mathrm{down/up}}})| < 2\varDelta_F F^{(2)}(0)$ (of course, $n_{\mathrm{down}} = 0$ and/or $n_{\mathrm{up}} = n_{\max}$ if these criteria cannot be fulfilled for intermediate values of n) and finds the values of F on a finer grid of P-values with $|P_{j+1} - P_j| = 0.0001$ and $P_{n_{\mathrm{down}}} \leq P_j \leq P_{n_{\mathrm{up}}}$. If the minimum of F found on the finer grid occurs for $P_{j_{\min}} < 0.02$, it is assumed that

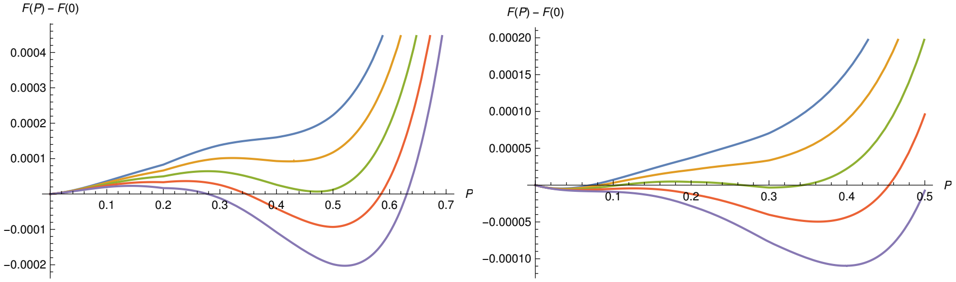

Fig. 4. Plots of the differences $F(P) - F(0)$ in units of $(k_{\rm F}^3/6\pi^2)(\hbar^2 k_{\rm F}^2/2m_f)$ of the system of spin-1/2 fermions as a function of the order parameter P for two representative values of the temperature $t \equiv T/T_{\rm F}$ as obtained in the second order of the perturbative expansion. Left: $t = 0.1$; the successive curves (from below) correspond to $k_{\rm F}a_0 = 1.0718$ (the lowest, blue line), $k_{\rm F}a_0 = 1.0723$ (yellow), $k_{\rm F}a_0 = 1.0728$ (green), 1.0733 (red) and 1.0738 (the highest, blue line). Right: $t = 0.15$; the successive curves (from below) correspond to $k_{\rm F}a_0 = 1.0978$ (the lowest, blue line), $k_{\rm F}a_0 = 1.983$ (yellow), $k_{\rm F}a_0 = 1.0988$ (green), 1.0993 (red) and 1.0998 (the highest, blue line).

it is a numerical artifact and the polarization is taken as vanishing. In the opposite case, the polarization is taken to be nonvanishing and that value of $k_{\rm F}a_0$ is recorded as the critical one (for the considered temperature). In this case on the finer grid, one seeks a range $(P_{j_{\rm down}}, P_{j_{\rm up}})$ of P around $P_{j_{\rm min}}$ in which $|F(P_{j_{\rm min}}) - F(P_j)| > \mathit{\Delta}_F F^{(2)}(0)$ for $P_{j_{\rm down}} \leq P_j \leq P_{j_{\rm up}}$; if such a range cannot be found, the transition is classified as continuous (the polarization at the considered temperature is assumed to increase continuously from zero as $k_{\rm F}a_0$ is increased), while if a nontrivial range is obtained, the transition is classified as first order and $P_{j_{\rm min}} - P_{j_{\rm down}}$ and $P_{j_{\rm up}} - P_{j_{\rm min}}$ are taken as the uncertainties of the determination of the polarization right at the transition. For values of $k_{\rm F}a_0$ higher than the critical one (determined as described above for the considered temperature), F is evaluated on a finer grid of points P_j with $P_{n_{\rm min}-1} \leq P_j \leq P_{n_{\rm min}+1}$ and $P_{j+1} - P_j = 0.001$ and the corresponding polarization is determined as the position of the minimum of F on this finer grid. In this way, one finds that $(k_{\rm F}a_0)_{\rm cr} = 1.05409$ at $t = 0$ (which perfectly agrees with the known value obtained by computing the system's ground-state energy [11] and with [4]), $(k_{\rm F}a_0)_{\rm cr} = 1.05858$ at $t = 0.05$, $(k_{\rm F}a_0)_{\rm cr} = 1.07282$ at $t = 0.1$, $(k_{\rm F}a_0)_{\rm cr} = 1.09881$ at $t = 0.15$, and $(k_{\rm F}a_0)_{\rm cr} = 1.13845$ at $t = 0.2$. The corresponding values of the polarization right at the transition point are $P_{\rm cr} = 0.575^{+0.017}_{-0.019}$ (again in agreement with the value found in [11]), $0.558^{+0.017}_{-0.017}$, $0.477^{+0.019}_{-0.021}$, $0.325^{+0.035}_{-0.048}$, and $0.197^{+0.045}_{-0.096}$.

The dependence of the polarization as a function of the "gas parameter" $k_{\mathrm{F}}a_0$ is shown, for a few values of the temperature t, in the left panel of figure 5. This is essentially the same plot as the one presented in [4] (the agreement with the critical values of the gas parameters at successive temperatures that can be read off from the plot there seems to be quite good) except that in figure 5 marked are also the uncertainties in the determination (following from the procedure just described) of the polarization right at the transition.

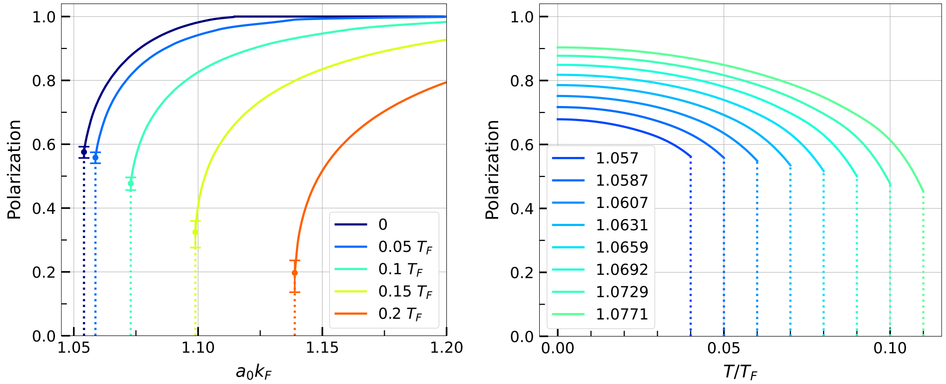

Fig. 5. Polarization $P = (N_+ - N_-)/N$ of the system of spin-1/2 fermions with a short-range repulsive interaction obtained from the free energy F computed up to the second order of the perturbative expansion. In the left panel as a function of the "gas parameter" $k_{\mathrm{F}}a_0$ for several values of the temperature (counting from the left): $t \equiv T/T_{\mathrm{F}} = 0$, 0.1, 0.15 and 0.2 ($T_{\mathrm{F}} \equiv \varepsilon_{\mathrm{F}}/k_{\mathrm{B}}$); marked are also uncertainties of the value of P right at the transition points. In the right panel as a function of the temperature for several fixed values of $k_{\mathrm{F}}a_0$.

Owing to the efficiency of our numerical code (stemming basically from the trick with the interpolations), the procedure of finding the polarization of the system described above can be applied also at fixed values of $k_{\mathrm{F}}a_0$ (replacing the grid in $k_{\mathrm{F}}a_0$ by a one in t). The resulting polarization of the system as a function of the temperature for several fixed values of the gas parameter is shown in the right panel of figure 5. Knowing the polarization as a function of the other parameters, it is possible to construct the free energy $F(T,V,N) \equiv F(T,V,N,P(T,N/V))$ for several values of $k_{\mathrm{F}}a_0$ and to determine also other thermodynamic characteristics of the system. For example, using the grid in t, the second derivative of the free energy $F(T,V,N)$ with respect to the temperature can in principle be obtained yielding the system's heat capacity. The result of such an exercise is shown in figure 6 for two values of the "gas parameter". It shows that the discontinuity of the heat capacity at the transition point grows with the value of $k_{\mathrm{F}}a_0$ (*i.e.* also with the increasing temperature, if $k_{\mathrm{F}}a_0$ is varied). However, for higher values of $k_{\mathrm{F}}a_0$, the numerical inaccuracies do not allow for a reliable computation. Indeed, as the transition at higher temperatures becomes continuous, a divergence of the heat capacity probably starts to build up making the numerical computation of the second derivative of the free energy unstable for $t \gtrsim 0.12$. Similarly, it is in principle possible to determine the system's polarization taking into account an infinitesimally weak external magnetic field (this as explained influences only the determination of the zeroth order chemical potentials $\tilde{\mu}_{\pm}^{(0)}$ from the conditions of (22)) and to compute the system's magnetic susceptibility χ_T by constructing the derivative of the polarization with respect to $\mathcal{H}$. While such a computation seems to indicate that at least at low temperatures, at which the transition is (in the approximation to which our computation is restricted) first order, the susceptibility also has a finite discontinuity at the transition point, it is not sufficiently stable numerically to yield reliable values of χ_T.

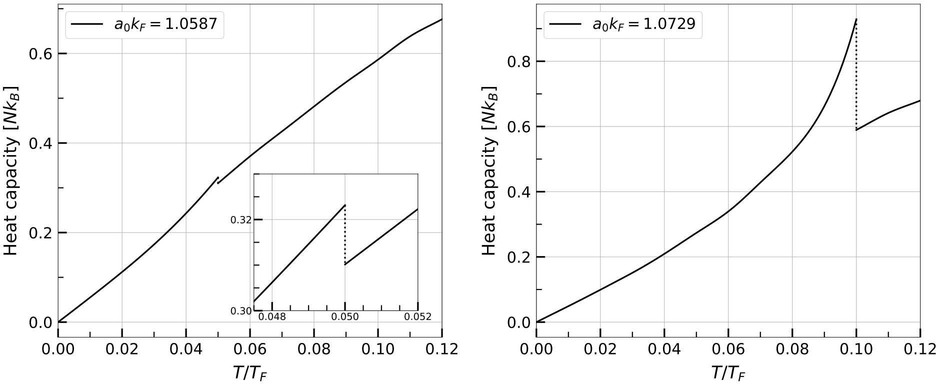

Fig. 6. Heat capacity (in units of Nk_{B}) of the system of spin-1/2 fermions with a short-range repulsive interaction as a function of the temperature for two different fixed values of the parameter $k_{\mathrm{F}}a_0$ obtained from the free energy F computed up to the second order.

7. Conclusions

We have developed a systematic perturbative expansion of the grand thermodynamic potential Ω and of the free energy F of the system of (non-relativistic) interacting spin-1/2 fermions. We have applied this expansion within the effective field theory in which the underlying repulsive spin-independent binary interaction of fermions is replaced by an infinite number of contact interaction terms and which allows to directly express computed quantities in terms of the scattering lengths and the effective radii which characterize the underlying interaction potential. We have shown (up to the third order but the result seems to be valid in general) that to the expansion of the free energy effectively contribute only those Feynman diagrams which give nonvanishing contributions to the ground-state energy of the system evaluated at zeroth order chemical potentials (associated with spin-up and spin-down fermions).

Our numerical analysis has been restricted here to the first nontrivial order of the perturbative expansion (*i.e.* the first one going beyond the textbook mean field approximation) in which the results are still universal, *i.e.* on the form of the underlying interaction depend only through the s-wave scattering length a_0 (in the next order, the results start to depend also on the p-wave scattering length a_1 and the effective radius r_0). We have devised a method for efficient numerical evaluation of the requisite nested integrals and used it to compute the system's polarization and its value right at the transition point paying attention to the uncertainty of the determination of the latter quantity which is crucial in assessing the character of the transition. For low temperatures, $T \lesssim 0.1\,T_{\rm F}$, we have also managed to determine the system's heat capacity encountering, however, some problems with the accuracy of the numerical evaluation of the derivatives of the free energy which seem to prevent obtaining (at least without substantial improvements in the method) reliable values of the heat capacity for higher temperatures as well as determining the system's magnetic susceptibility.

Of course, since the perturbative computation of the system's ground-state energy agrees with the results obtained (for specific forms of the underlying interaction) using the Quantum Monte Carlo approach only for $k_{\rm F}a_0 \lesssim 0.5$, the results presented here cannot be taken very seriously. Moreover, it is now known that already the inclusion of the third-order corrections to the system's ground-state energy (free energy at zero temperature) significantly weakens the first order character of the transition (at zero temperature) to the ordered state. For these reasons, our effort summarized here should be treated rather as a preliminary step taken towards extending the computation to a higher order and towards a possible implementation of a resummation of some class of the contributions to the free energy in the spirit of the approach of [13]. Such a resummation can probably also allow to overcome the limitation, inherent in the effective field theory approach, to sufficiently small temperatures only: as this approach relies on the clean separation of the scales ($R \ll k_{\rm F}^{-1}$, where R is the characteristic length of the underlying interaction), it cannot be applied, at least if restricted to a finite order of the perturbative expansion, when $k_{\rm B}T$ becomes comparable with the energy scale set by $\varepsilon_{\rm F}$. We plan to return to these issues in the forthcoming paper.

Appendix A

The following summation formulae hold [7] (the limit $\eta \to 0^+$ is implicit):

$$\frac{1}{\beta}\sum_{n\in\mathbb{Z}} \frac{\mathrm{e}^{i\eta\omega_n^F}}{i\omega_n^F - x} = \frac{1}{1+\mathrm{e}^{\beta x}}\,, \qquad \omega_n^F \equiv \frac{\pi}{\beta}(2n+1)\,, \tag{A.1}$$

$$\frac{1}{\beta}\sum_{n\in\mathbb{Z}} \frac{\mathrm{e}^{i\eta\omega_n^B}}{i\omega_n^B - x} = \frac{1}{1-\mathrm{e}^{\beta x}}\,, \qquad \omega_n^B \equiv \frac{2\pi}{\beta}\,n\,, \tag{A.2}$$

and, by decomposing into simple fractions,

$$\begin{aligned}
\frac{1}{\beta}\sum_{n\in\mathbb{Z}} \frac{1}{i\omega_n^F - x}\,\frac{1}{i\omega_{n+l}^F - y} &= \frac{1}{\beta}\sum_{n\in\mathbb{Z}} \frac{1}{i\omega_n^F - x}\,\frac{1}{i\omega_n^F - (y - i\omega_l^B)} \\
&= \frac{1}{i\omega_l^B - (y-x)}\left(\frac{1}{1+\mathrm{e}^{\beta x}} - \frac{1}{1+\mathrm{e}^{\beta y}}\right), \qquad \text{(A.3)} \\
\frac{1}{\beta}\sum_{n\in\mathbb{Z}} \frac{1}{i\omega_n^F - x}\,\frac{1}{i\omega_{l-n}^F - y} &= -\frac{1}{\beta}\sum_{n\in\mathbb{Z}} \frac{1}{i\omega_n^F - x}\,\frac{1}{i\omega_n^F - (i\omega_{l+1}^B - y)} \\
&= \frac{1}{i\omega_{l+1}^B - (y+x)}\left(\frac{1}{1+\mathrm{e}^{\beta x}} - \frac{1}{1+\mathrm{e}^{-\beta y}}\right). \qquad \text{(A.4)}
\end{aligned}$$

Similarly,

$$\frac{1}{\beta}\sum_{l\in\mathbb{Z}} \frac{1}{i\omega_l^B - x}\,\frac{1}{i\omega_l^B - y} = \frac{1}{x-y}\left(\frac{1}{1-\mathrm{e}^{\beta x}} - \frac{1}{1-\mathrm{e}^{\beta y}}\right). \tag{A.5}$$

Useful can be also the formula

$$\frac{1}{x-a_1}\,\frac{1}{x-a_2}\,\cdots\,\frac{1}{x-a_n} = \sum_{l=1}^{n}\left(\prod_{k\neq l}^{n}\frac{1}{a_l - a_k}\right)\frac{1}{x-a_l}\,. \tag{A.6}$$

Appendix B

Here, we demonstrate the cancellation of the additional terms in the formula (31) for $F^{(3)}$ against the contributions of diagrams which vanish at zero temperature. Analogous cancellation of the contribution of the diagrams shown in figure 2 against the additional terms in (30) has been checked in the main text. It will be convenient to introduce first the following notation:

$$n_x \equiv \int_{\boldsymbol{k}} \frac{1}{1+\mathrm{e}^{\beta(\varepsilon_{\boldsymbol{k}} - x_0)}}\,,$$

$$\begin{aligned} n_{xx} &\equiv \int\limits_{\boldsymbol{k}} \frac{1}{\left[1+\mathrm{e}^{\beta\left(\varepsilon_{\boldsymbol{k}}-x_0\right)}\right]^2}\,, \\ n_{xxx} &\equiv \int\limits_{\boldsymbol{k}} \frac{1}{\left[1+\mathrm{e}^{\beta\left(\varepsilon_{\boldsymbol{k}}-x_0\right)}\right]^3}\,, \end{aligned}$$

etc. From (11), one immediately obtains (all functions are evaluated at x_0 and y_0)

$$\begin{aligned} \Omega^{(0)}_{x} &= -V\,n_x\,, \\ \Omega^{(0)}_{xx} &= -V\beta\left(n_x-n_{xx}\right)\,, \\ \Omega^{(0)}_{xxx} &= -V\beta^2\left(n_x-3n_{xx}+2n_{xxx}\right)\,. \end{aligned} \tag{B.1}$$

Analogously can be written the derivatives of $\Omega^{(0)}$ with respect to y. The necessary derivatives of $\Omega^{(1)}=C_0V\,n_x\,n_y$ take the form

$$\begin{aligned} \Omega^{(1)}_{x} &= C_0V\beta\left(n_x-n_{xx}\right)n_y\,, \\ \Omega^{(1)}_{y} &= C_0V\beta\,n_x\left(n_y-n_{yy}\right)\,, \\ \Omega^{(1)}_{xx} &= C_0V\beta^2\left(n_x-3n_{xx}+2n_{xxx}\right)n_y\,, \\ \Omega^{(1)}_{yy} &= C_0V\beta^2\,n_x\left(n_y-3n_{yy}+2n_{yyy}\right)\,, \\ \Omega^{(1)}_{xy} &= C_0V\beta^2\left(n_x-n_{xx}\right)\left(n_y-n_{yy}\right)\,. \end{aligned} \tag{B.2}$$

To $\Omega^{(3)}$, in addition to the two "mercedes-type" diagrams shown in figure 7 (the contributions $\Omega^{(3)a}$ and $\Omega^{(3)b}$), contribute also the "mitsubishi-type" diagrams of figure 8 (the contributions $\Omega^{(3)c}$ and $\Omega^{(3)d}$), the two diagrams of figure 9 (the contributions $\Omega^{(3)e}$ and $\Omega^{(3)f}$), and the single "audi-type" diagram of figure 10 ($\Omega^{(3)g}$).

The computation of $\Omega^{(3)c}$ and $\Omega^{(3)d}$ is straightforward (it is analogous to that of $\Omega^{(2)b}$ and $\Omega^{(2)c}$) and yields

$$\Omega^{(3)c}+\Omega^{(3)d}=\frac{1}{6}C_0^3V\beta^2\left[\left(n_x-3n_{xx}+2n_{xxx}\right)n_y^3+n_x^3\left(n_y-3n_{yy}+2n_{yy}\right)\right]\,.$$

One then readily sees that in (31) this is cancelled by the last two terms

$$\Omega^{(3)a}+\Omega^{(3)b}-\frac{\Omega^{(0)}_{xxx}\left[\Omega^{(1)}_{x}\right]^3}{6\left[\Omega^{(0)}_{xx}\right]^3}-\frac{\Omega^{(0)}_{yyy}\left[\Omega^{(1)}_{y}\right]^3}{6\left[\Omega^{(0)}_{yy}\right]^3}=0\,.$$

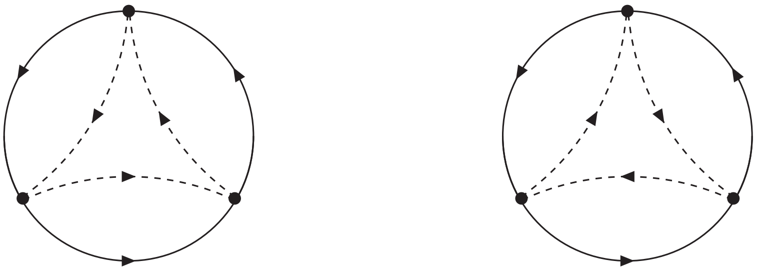

Fig. 7. The particle–particle and the particle–hole diagrams (the "mercedes-like" diagrams) contributing in the order C_0^3 to $\Omega^{(3)}$.

One has now to consider the $-\Omega_x^{(2)}\Omega_x^{(1)}/\Omega_{xx}^{(0)}$ and $-\Omega_y^{(2)}\Omega_y^{(1)}/\Omega_{yy}^{(0)}$ terms in (31). $\Omega^{(2)}$ is given by three diagrams shown in figures 3, left and 2. It is convenient to write the contribution the first one, $\Omega^{(2)a}$, in the form

$$\begin{aligned}\Omega^{(2)a} &= -\frac{C_0^2 V}{2}\frac{1}{\beta}\sum_{l\in\mathbb{Z}}\int_{\boldsymbol{q}}\int_{\boldsymbol{k}}\int_{\boldsymbol{p}}\frac{1}{\beta}\sum_{n\in\mathbb{Z}}\frac{1}{i\omega_n^F-(\varepsilon_{\boldsymbol{k}}-x)}\,\frac{1}{i\omega_{n+l}^F-(\varepsilon_{\boldsymbol{k}+\boldsymbol{q}}-x)}\\ &\quad\times\frac{1}{\beta}\sum_{m\in\mathbb{Z}}\frac{1}{i\omega_m^F-(\varepsilon_{\boldsymbol{p}}-y)}\,\frac{1}{i\omega_{m-l}^F-(\varepsilon_{\boldsymbol{p}-\boldsymbol{q}}-y)}\,. \qquad \text{(B.3)}\end{aligned}$$

Differentiating it with respect to x, one obtains the expression which is a sum of the two terms

$$\begin{aligned}\Omega_x^{(2)a} &= \frac{C_0^2 V}{2}\frac{1}{\beta}\sum_{l\in\mathbb{Z}}\int_{\boldsymbol{q}}\int_{\boldsymbol{k}}\int_{\boldsymbol{p}}\frac{1}{\beta}\sum_{m\in\mathbb{Z}}\frac{1}{i\omega_m^F-(\varepsilon_{\boldsymbol{p}}-y)}\,\frac{1}{i\omega_{m-l}^F-(\varepsilon_{\boldsymbol{p}-\boldsymbol{q}}-y)}\\ &\quad\times\frac{1}{\beta}\sum_{n\in\mathbb{Z}}\left\{\frac{1}{[i\omega_n^F-(\varepsilon_{\boldsymbol{k}}-x)]^2}\,\frac{1}{i\omega_{n+l}^F-(\varepsilon_{\boldsymbol{k}+\boldsymbol{q}}-x)}\right.\\ &\quad\left.+\frac{1}{i\omega_n^F-(\varepsilon_{\boldsymbol{k}}-x)}\,\frac{1}{[i\omega_{n+l}^F-(\varepsilon_{\boldsymbol{k}+\boldsymbol{q}}-x)]^2}\right\}.\end{aligned}$$

These two terms are equal — to see this, it suffices to set in the second term $\boldsymbol{q}=-\boldsymbol{q}'$, $\boldsymbol{k}=\boldsymbol{k}'+\boldsymbol{q}'$, $\boldsymbol{p}=\boldsymbol{p}'-\boldsymbol{q}'$, and $l=-l'$, $n=n'+l'$, $m=m'-l'$. Thus, multiplying the expression for $\Omega_x^{(2)a}$ by $-\Omega_x^{(1)}/\Omega_{xx}^{(0)}$ which simply equals $C_0\,n_y$, one readily sees (taking into account that $\mathcal{G}_{--}^{(0)}(0,\mathbf{0})=n_y$) that the resulting expression precisely cancels $\Omega^{(3)e}$. Thus, in $F^{(3)}$

$$\Omega^{(3)e}+\Omega^{(3)f}-\frac{\Omega_x^{(1)}}{\Omega_{xx}^{(0)}}\,\Omega_x^{(2)a}-\frac{\Omega_y^{(1)}}{\Omega_{yy}^{(0)}}\,\Omega_y^{(2)a}=0\,.$$

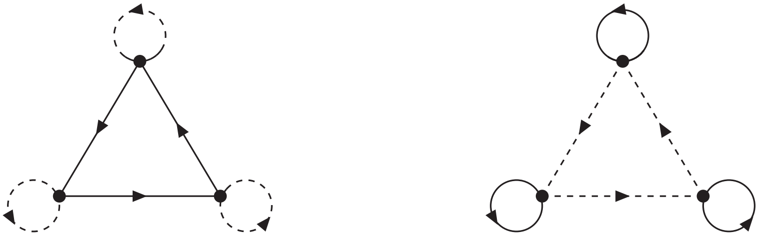

Fig. 8. Two order C_0^3, "mitsubishi-type" diagrams contributing to $\varOmega^{(3)}$.

After these cancellations, one is left with

$$\begin{aligned} F^{(3)} &= \varOmega^{(3)a} + \varOmega^{(3)b} + \varOmega^{(3)g} + C_0 n_y \left(\varOmega_x^{(2)b} + \varOmega_x^{(2)c}\right) + C_0 n_x \left(\varOmega_y^{(2)b} + \varOmega_y^{(2)c}\right) \\ &\quad + \frac{\varOmega_{xx}^{(1)} \left[\varOmega_x^{(1)}\right]^2}{2\left[\varOmega_{xx}^{(0)}\right]^2} + \frac{\varOmega_{yy}^{(1)} \left[\varOmega_y^{(1)}\right]^2}{2\left[\varOmega_{yy}^{(0)}\right]^2} + \frac{\varOmega_{xy}^{(1)} \varOmega_x^{(1)} \varOmega_y^{(1)}}{\varOmega_{xx}^{(0)} \varOmega_{yy}^{(0)}} \,. \end{aligned}$$

Explicitly, the last line of $F^{(3)}$ reads

$$\begin{aligned} &\frac{1}{2} C_0^3 V \beta^2 \left\{ (n_x - 3n_{xx} + 2n_{xxx})\, n_y^3 \right. \\ &\left. + n_x^3\, (n_y - 3n_{yy} + 2n_{yyy}) + 2\, n_x\, (n_x - n_{xx})(n_y - n_{yy})\, n_y \right\} \,, \end{aligned}$$

while the contribution $\varOmega^{(3)g}$ of the "audi-type" diagram of figure 10 can be written as

$$\varOmega^{(3)g} = C_0^3 V \beta^2\, n_x\, (n_x - n_{xx})\, (n_y - n_{yy})\, n_y \,.$$

Finally,

$$\begin{aligned} &\varOmega_x^{(2)b} + \varOmega_x^{(2)c} \\ &= -\frac{1}{2} C_0^2 V \beta^2 \left[(n_x - 3n_{xx} + 2n_{xxx})\, n_y^2 + 2\, n_x\, (n_x - n_{xx})(n_y - n_{yy}) \right] \,, \end{aligned}$$

(the sum $\varOmega_y^{(2)b} + \varOmega_y^{(2)c}$ is given by an analogous expression) and after a straightforward algebra all the extra terms cancel out, so that eventually, $F^{(3)} = \varOmega^{(3)a} + \varOmega^{(3)b}$, that is, it is given solely by the "mercedes-like" diagrams evaluated at the zeroth order chemical potentials x_0, y_0 (or $\tilde{x}_0$, $\tilde{y}_0$, if there is an external magnetic field). The diagrams canceled by the extra terms in the formulae (30) and (31) are precisely those (see *e.g.* [6]) which vanish at zero temperature, that is, do not contribute to the expansion of the formula (13) for the ground-state energy.

Fig. 9. The contributions to $\Omega^{(3)e}$ and $\Omega^{(3)f}$.

One can also simplify the formula (29) for the second-order correction x_2 to the μ_+ chemical potential (and the analogous formula for y_2). After a straightforward algebra, one obtains

$$x_2 = -\frac{\Omega_x^{(2)a}}{\Omega_{xx}^{(0)}}\,,$$

all other terms neatly cancelling. Of course, $x_1 = C_0\, n_-$, $y_1 = C_0\, n_+$ but it is perhaps more instructive to write[6]

$$x = x_0 - \frac{1}{\Omega_{xx}^{(0)}}\left(\Omega_x^{(1)} + \Omega_x^{(2)a} + \ldots\right).$$

The cancellation found here is obviously necessary for the equivalence of two ways of determining the system's polarization. Indeed, only if this cancellation holds, is the minimization of the free energy written in the form

$$F = \Omega^{(0)}(x_0, y_0) + (x_0\, n_x + y_0\, n_y)\, V + \Omega^{(1)}(x_0, y_0) + \Omega^{(2)a}(x_0, y_0) + \ldots\,,$$

with respect to n_x (keeping $n_y = n - n_x$), which (taking into account that $\Omega_x^{(0)} + n_x V = 0$, $\Omega_y^{(0)} + n_y V = 0$) amounts to

$$(x_0 - y_0)\, V = -\left[\Omega_x^{(1)} + \Omega_x^{(2)a} + \ldots\right]\frac{\partial x_0}{\partial n_x} + \left[\Omega_y^{(1)} + \Omega_y^{(2)a} + \ldots\right]\frac{\partial y_0}{\partial n_y}\,,$$

equivalent to the condition $\mu_+ = \mu_-$ written in the form $x_0 + x_1 + x_2 + \ldots = y_0 + y_1 + y_2 + \ldots$, that is

$$x_0 - y_0 = \frac{\Omega_x^{(1)} + \Omega_x^{(2)a} + \ldots}{\Omega_{xx}^{(0)}} - \frac{\Omega_y^{(1)} + \Omega_y^{(2)a} + \ldots}{\Omega_{yy}^{(0)}}\,.$$

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[6] This form clearly shows, since the cancellation of the divergences in the sum $\Omega^{(1)} + \Omega^{(2)a}$ has already been demonstrated, that the computed perturbatively chemical potentials x and y are to this order finite, after the cutoff dependence of the coupling C_0 is taken into account. The argument obviously generalizes to all orders: if the free energy F is made finite by the renormalization of the couplings, so must be the chemical potentials.

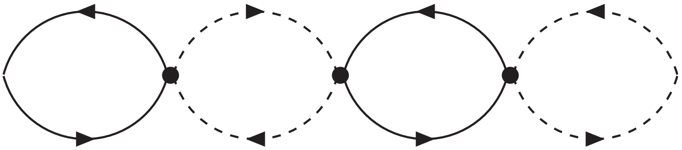

Fig. 10. The "audi"-type, order C_0^3 contribution to $\Omega^{(3)}$.

The equivalence follows from noticing that since $n_x = -\Omega_x^{(0)}/V$, $n_y = -\Omega_y^{(0)}/V$, the derivatives of x_0 and y_0 are precisely equal to

$$\frac{\partial x_0}{\partial n_x} = -\frac{V}{\Omega_{xx}^{(0)}}\,, \qquad \frac{\partial y_0}{\partial n_y} = -\frac{V}{\Omega_{yy}^{(0)}}\,.$$

From this argument it immediately follows that $x_3 = -(\Omega_x^{(3)a} + \Omega_x^{(3)b})/\Omega_{xx}^{(0)}$ and $y_3 = -(\Omega_y^{(3)a} + \Omega_y^{(3)b})/\Omega_{yy}^{(0)}$.

Restricted to the first order, the left-hand side of the equality $x_0 + x_1 = y_0 + y_1$ reads

$$x_0 + x_1 + \ldots = k_{\mathrm{B}}T\, f^{-1}\left(\left(\frac{\varepsilon_{\mathrm{F}}^{(0)}(n_+)}{k_{\mathrm{B}}T}\right)^{3/2}\right) + C_0\, n_- + \ldots$$

The right-hand side is given by the analogous formula. If one sets here $n_\pm = (N/2V)(1 \pm P)$, this reproduces the condition (21).

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