NEW TRANSFORMATION METHOD FOR GENERATION OF EXACTLY SOLVABLE CENTRALLY SYMMETRIC POWER-LAW POTENTIALS IN THE QUANTUM MECHANICS OF KLEIN–GORDON EQUATION

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Received 6 September 2023, accepted 27 December 2023, published online 9 January 2024

We present a new transformation method in the framework of higher-dimensional relativistic Quantum Mechanics of the Klein–Gordon equation for the generation of exactly solvable quantum mechanical potentials from the already known exactly solvable centrally symmetric power-law potentials. The method is based on a coordinate transformation supplemented by a functional transformation along with a set of indispensable ansatzes. The efficacy of our method is investigated by (re)generating two of the most fundamental potentials — harmonic oscillator and Coulomb potentials in D-dimensional Euclidean space. The pertinent issue of normalisability for the generated wavefunctions can be elegantly examined in our formalism. The present work reveals a relative parent–daughter family relationship between the Coulomb and harmonic oscillator potentials in the relativistic regime of higher-dimensional quantum mechanics.

DOI:10.5506/APhysPolB.54.12-A3

1. Introduction

The quest for exact solutions of differential wave equations in non-relativistic and relativistic regimes has still remained an essential research area since the birth of Quantum Mechanics (QM). Exact solutions always play a decisive role in the development of QM, as they always set benchmarks to test the accuracy and reliability of approximate/perturbative methods. However, the number of Exactly Solvable Potentials (ESPs) is unfortunately very small, and there lies the importance of the generation/construction of

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ESPs in QM. In the relativistic regime of QM, the Klein–Gordon (KG) equation for spin-0 particles is the most widely used wave equation for solving various practical problems in nuclear physics [1], astrophysics [2, 3], laser physics [4], particle and quarkonium physics [5–7]. The KG equation is found to be solved exactly and very often quasi-exactly with both centrally symmetric potentials (CSPs) and non-central potentials (NCPs) by employing various methods/techniques/approaches, *e.g.* Standard Method [8–14], Nikiforov–Uvarov (NU) Method [15–22], Supersymmetric Quantum Mechanics (SUSYQM) Approach [16, 21, 23–25], Asymptotic Iteration Method (AIM) [26, 27], Path Integral Approach [28], Factorization Method [29], Darboux Transformation Method [30], Laplace Transformation Method [31], Algebraic Method of Matrix Recurrences [32], SU(1,1) Lie Algebraic Dynamical Symmetry Group Approach [33]. It is also noted that certain researchers have intelligently utilized a finite-difference calculus approach to conduct theoretical studies of QSs with both CSPs and NCPs in the relativistic QM [34–37]. Inspired by the work of Ahmed *et al.* [38–42] in the non-relativistic QM, we present here a transformation method comprising a coordinate transformation (CT) followed by a functional transformation (FT) along with a set of ansatzes for the generation of centrally symmetric Quantum Mechanical Exactly Solvable Potentials (QMESPs) from an already known genuine centrally symmetric QMESP in the QM of KG equation. The three striking points in our transformation method are as follows: *(i)* this method appears as a labour-saving technique, because it maps the wavefunction of the parent quantum system (QS) to that of the generated (daughter) QS in a straightforward manner; *(ii)* here, the transformation carries the normalisability property from the parent genuine QS to the generated QS, because of which, the generated wavefunction is almost always normalisable; and *(iii)* with this method, dimensional extension or dimensional reduction of the generated QS is possible. Though the KG equation is solved analytically for centrally symmetric Coulomb and harmonic oscillator potentials by various researchers using a number of techniques, *e.g.* Large-N Expansion Approximate Method [43], Operator Analysis [44], Levi-Civita Transformation [45], Snyder–de Sitter Algebra [46], Modified Commutation Relation Approach [47], we here apply our transformation method to (re)generate these two fundamental potentials along with their wavefunctions and energy spectrums, where solving of differential equations is not required at all, and consequently, it is found that a cyclic mapping between these two fundamental potentials is possible via our method in higher-dimensional relativistic QM of KG equation.

The organization of the paper is as follows. Section 2 is dedicated to the formalism of our transformation method for generation of CSPs (power law) in any desired dimensional Euclidean space from a genuine already known exactly solved CSP (power law) in the framework of relativistic QM of KG equation. Then the normalisability of the generated wavefunctions is discussed. Section 3 and Section 4 are kept reserved for the investigation of the efficacy of our method by applying it in two known QSs — Coulomb and harmonic oscillator. The renormalisability of the generated wavefunctions is also checked as per our formalism. Finally, a few discussions have been made along with the concluding remarks in Section 5.

2. Formalism

The KG equation for spin-0 particle of rest mass M_0 (in atomic unit $\hbar = c = 1$)

$$\left(\Box_{D+1} + M_0^2\right) \phi(t,\vec{r}) = 0 \tag{1}$$

in $(D+1)$ dimensional spacetime gets transmutated to the following equation [10] with the consideration of interaction described by the vector potential $V(t,\vec{r})$ and scalar potential $S(t,\vec{r})$ as

$$\left[-\vec{\nabla}_D^2 + \{M_0 + S(t,\vec{r})\}^2\right] \phi(t,\vec{r}) = \left[i\frac{\partial}{\partial t} - V(t,\vec{r})\right]^2 \phi(t,\vec{r})\,. \tag{2}$$

For the time-independent potentials, the time sector in the above KG equation can be separated out by writing $\Phi(t,\vec{r}) = \mathrm{e}^{-iEt}\Psi(\vec{r})$ [11], where E is the relativistic energy, to have a time-independent KG equation for an interacting spin-0 particle in arbitrary D dimensions as

$$\left[-\vec{\nabla}_D^2 + \{M_0 + S(\vec{r})\}^2\right] \Psi(\vec{r}) = \left[E - V(\vec{r})\right]^2 \Psi(\vec{r})\,. \tag{3}$$

Here, $\vec{r} = \vec{r}(r,\Omega)$, Ω denotes a set of $D-1$ angular variables. If the potentials are centrally symmetric, the separation of variable method can be implemented successfully by considering

$$\Psi(\vec{r}) = \Psi(r,\Omega) = \psi(r) Y(\Omega)\,,$$

where $\psi(r)$ is the radial wavefunction and $Y(\Omega) = Y^{(l_{D-1}=l)}_{l_1,l_2,\ldots l_{D-2}}(\theta_1,\theta_2,\ldots\theta_{D-1})$ is the normalised hyperspherical harmonics [10]. Applying

$$\vec{\nabla}_D^2 = \frac{\partial^2}{\partial r^2} + \frac{D-1}{r}\frac{\partial}{\partial r} - \frac{L^2}{r^2}\,, \tag{4}$$

where L^2 is the square of angular momentum operator such that $L^2 Y = l(l+D-2)Y$ [43], in equation (3) and then simplifying, we get the following time-independent radial KG equation for a spin-0 particle in D_A-dimensional Euclidean space ($D_A \geq 3$) for a QS, hereafter called parent QS-A, as

$$\psi''_A(r) + \frac{D_A - 1}{r}\psi'_A(r) + \left[\{E_A - V_A(r)\}^2 - \{M_0 + S_A(r)\}^2 - \frac{l_A(l_A + D_A - 2)}{r^2}\right]\psi_A(r) = 0\,, \quad (5)$$

where the normalised radial eigenfunction $\psi_A(r)$ and relativistic eigenenergy E_A are supposed to be known exactly for the QS-A.

It can be shown that there exist mathematical similarities between the non-relativistic radial Schrödinger equation and relativistic radial KG equation under the condition of equal vector and scalar potentials, while the potential term is scaled from $2V_A(r)$ to $V_A(r)$. With the assumption $S_A(r) = +V_A(r)$, equation (5) can be shaped into the following Schrödinger-like equation:

$$\psi''_A(r) + \frac{D_A - 1}{r}\psi'_A(r) + \left[\tilde{E}_A - \tilde{V}_A(r) - \frac{l_A(l_A + D_A - 2)}{r^2}\right]\psi_A(r) = 0\,. \quad (6)$$

Here, in the line of Schrödinger QM, $\tilde{V}_A(r) = (E_A + M_0)V_A(r)$ is defined as the modulated potential, and $\tilde{E}_A = (E_A^2 - M_0^2)$ is the redefined energy corresponding to the modulated potential $\tilde{V}_A(r)$ for the parent QS-A.

Now applying the following CT, which is the primary part of the whole transformation method,

$$r \to g_B(r)$$

into equation (6), the following intermediate auxiliary differential equation has been structured

$$\psi''_A(g_B) + \frac{D_A - 1}{g_B}\psi'_A(g_B) + \left[\tilde{E}_A - \tilde{V}_A(g_B) - \frac{l_A(l_A + D_A - 2)}{g_B^2}\right]\psi_A(g_B) = 0\,, \quad (7)$$

where the prime denotes differentiation with respect to the argument. Here, the transformation function $g_B(r)$ is a differentiable function of at least class C^2 and its importance in the transformation method lies in the fact that it helps us to generate a new potential function $V_B(r)$ for the daughter QS-B through one of the ansatzes in equation (13).

In the second part of the transformation, the following FT is applied in equation (7):

$$\psi_B(r) = f_B^{-1}(r)\psi_A(g_B(r))\,, \tag{8}$$

which is the radial wavefunction for the newly generated QS-B. The modulating function $f_B^{-1}(r)$ plays a vital role in choosing dimensionality for Euclidean space where the QS-B is going to be generated.

After applying the FT in equation (7), we have

$$\begin{aligned}&\psi''_B(r) + \left(\frac{\mathrm{d}}{\mathrm{d}r}\ln\frac{f_B^2 g_B^{D_A-1}}{g'_B}\right)\psi'_B(r) + \left[\left(\frac{\mathrm{d}}{\mathrm{d}r}\ln f_B\right)\left(\frac{\mathrm{d}}{\mathrm{d}r}\ln\frac{f'_B g_B^{D_A-1}}{g'_B}\right)\right.\\ &\left.+g'^2_B\left\{\tilde{E}_A - \tilde{V}_A(g_B) - \frac{l_A(l_A+D_A-2)}{g_B^2}\right\}\right]\psi_B(r) = 0\,.\end{aligned} \tag{9}$$

To cast equation (9) in the form of a Schrödinger-like equation similar to equation (7) for the generated QS-B in an arbitrarily chosen D_B-dimensional Euclidean space, we, at first, require the coefficient of the first-order derivative of ψ_B equal to $\frac{D_B-1}{r}$, *i.e.*

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\ln\frac{f_B^2 g_B^{D_A-1}}{g'_B}\right) = \frac{D_B-1}{r}\,,$$

which fixes $f_B(r)$ and its derivative as functions of $g_B(r)$. As it is found from the above equation that

$$f_B(r) = g_B(r)'^{\frac{1}{2}} g_B(r)^{-\frac{D_A-1}{2}} r^{\frac{D_B-1}{2}}\,, \tag{10}$$

using $f_B(r)$ in equation (9), we move forward to

$$\begin{aligned}&\psi''_B(r) + \frac{D_B-1}{r}\psi'_B(r)\\ &+\left[\frac{1}{2}\{g_B,r\} - \frac{D_A-1}{2}\frac{D_A-3}{2}\left(\frac{g'_B}{g_B}\right)^2 + \frac{D_B-1}{2}\frac{D_B-3}{2}\left(\frac{1}{r^2}\right)\right.\\ &\left.+g'^2_B\left\{\tilde{E}_A - \tilde{V}_A(g_B) - \frac{l_A(l_A+D_A-2)}{g_B^2}\right\}\right]\psi_B(r) = 0\,,\end{aligned} \tag{11}$$

where $\{g_B,r\} = \frac{g'''_B}{g'_B} - \frac{3}{2}\left(\frac{g''_B}{g'_B}\right)^2$ is the Schwartzian derivative [38].

For the successful implementation of the combined transformation of CT and FT, it is mandatory to invoke the following set of two ansatzes in equation (11) so as to retrieve the Schrödinger-like equation similar to equation (7) which will be conceived with a potential and its corresponding energy for the QS-B:

$$g_B'^2 \tilde{V}_A(g_B) = -\tilde{E}_B \tag{12}$$

and

$$g_B'^2 \tilde{E}_A = -\tilde{V}_B(r)\,, \tag{13}$$

where $\tilde{V}_B(r) = (E_B + M)V_B(r)$ is the modulated potential and $\tilde{E}_B = (E_B^2 - M^2)$ is the redefined energy corresponding to the modulated potential $\tilde{V}_B(r)$ for the generated QS-B. By putting the transformation function $g_B(r)$, which is obtained from the ansatz in equation (12), in the ansatz in equation (13), the modulated potential function $\tilde{V}_B(r)$, and hence $V_B(r)$ are also generated for the QS-B in our formalism.

After using the ansatzes from equations (12) and (13) in equation (11), we get the following differential equation:

$$\begin{aligned} &\psi_B''(r) + \frac{D_B-1}{r}\psi_B'(r) \\ &+\Bigg[\tilde{E}_B - \tilde{V}_B(r) - \Bigg\{ -\frac{1}{2}\{g_B, r\} + \frac{D_A-1}{2}\frac{D_A-3}{2}\left(\frac{g_B'}{g_B}\right)^2 \\ &-\frac{D_B-1}{2}\frac{D_B-3}{2}\left(\frac{1}{r^2}\right) + g_B'^2\frac{l_A(l_A+D_A-2)}{g_B^2}\Bigg\}\Bigg]\psi_B(r) = 0\,. \end{aligned} \tag{14}$$

The quantities inside the pair of big curly brackets can be moulded to give the correct form of the 'centrifugal barrier' term in the D_B-dimensional space, whenever the parent potential $V_A(r)$ is a power-law type [38], *i.e.*

$$\begin{aligned} &-\frac{1}{2}\{g_B, r\} + \frac{D_A-1}{2}\frac{D_A-3}{2}\left(\frac{g_B'}{g_B}\right)^2 - \frac{D_B-1}{2}\frac{D_B-3}{2}\left(\frac{1}{r^2}\right) \\ &+g_B'^2\frac{l_A(l_A+D_A-2)}{g_B^2} = \frac{l_B(l_B+D_B-2)}{r^2}\,. \end{aligned} \tag{15}$$

Hence equation (14) yields the following exact form of a Schrödinger-like equation for the generated QS-B:

$$\psi_B''(r) + \frac{D_B-1}{r}\psi_B'(r) + \left[\tilde{E}_B - \tilde{V}_B(r) - \frac{l_B(l_B+D_B-2)}{r^2}\right]\psi_B(r) = 0\,. \tag{16}$$

Normalisability of the generated wavefunction $\psi_B(r)$ can be checked elegantly by using the general expressions in equations (8), (10), and (12) as follows. If the radial wavefunction $\psi_B(r)$ for D_B-dimensional QS-B is normalisable, then [10]

$$I(\infty,0) = \int_0^\infty \psi_B^*(r)\psi_B(r) r^{D_B-1}\mathrm{d}r = \text{finite}\,. \tag{17}$$

Since the integral

$$\begin{aligned}
I(\infty,0) &= \int_0^\infty \left(g_B(r)'^{\frac{1}{2}} g_B(r)^{-\frac{D_A-1}{2}} r^{\frac{D_B-1}{2}}\right)^{-2} \psi_A^*(g_B)\psi_A(g_B) r^{D_B-1}\frac{\mathrm{d}r}{\mathrm{d}g_B}\mathrm{d}g_B \\
&= \int_{g_B(0)}^{g_B(\infty)} g_B'^{-2}\psi_A^*(g_B)\psi_A(g_B) g_B^{D_A-1}\mathrm{d}g_B \\
&= \int_{g_B(0)}^{g_B(\infty)} \psi_A^*(g_B)\left[-\frac{\tilde{V}_A(g_B)}{\tilde{E}_B}\right]\psi_A(g_B) g_B^{D_A-1}\mathrm{d}g_B \\
&= \text{constant}\int_0^\infty \psi_A^*(r)V_A(r)\psi_A(r) r^{D_A-1}\mathrm{d}r \\
&= \text{constant}\,\langle V_A(r)\rangle\,,
\end{aligned}$$

and $\langle V_A(r)\rangle$ always exists for the genuine parent QS-A, $I(\infty,0)$ takes a finite value, and hence $\psi_B(r)$ is normalisable.

If the transformation function $g_B(r)$ is not badly behaved, so far as its local and asymptotic properties, *i.e.* $g(0) = 0$ and $g(\infty) = \infty$, are concerned, the transformation always carries forward the normalisability property from the genuine parent QS-A to its daughter (generated) QS-B [42].

3. Generation of harmonic oscillator potential in D_B-dimensional Euclidean space

We start with considering the Coulomb potential as the known genuine QS-A in three-dimensional Euclidean space ($D_A = 3$) [11] for which $V_A(r) = \frac{z_A}{r}$, where z_A is the charge coupling parameter proportional to the product of the charge number and the fine structure constant; the wavefunctions are

$$\psi_A(r) = C_A r^{l_A}\,\mathrm{e}^{\frac{-1}{2}\rho_A r} F\left(-n_\mathrm{r}, 2(l_A+1), \rho_A r\right)\,, \tag{18}$$

where C_A is the normalisation constant, $n_{\rm r}(=0,1,2\ldots)$ is the radial quantum number, $l_A(=0,1,2\ldots)$ is the orbital quantum number, $F(-n_{\rm r},2(l_A+1),\rho_A r)$ is the confluent hypergeometric function [11], and

$$\rho_A = -z_A\left(\frac{E_A+M_0}{n_A}\right); \tag{19}$$

the energy eigenvalues are

$$E_A = M_0\left(\frac{n_A^2-\frac{z_A^2}{4}}{n_A^2+\frac{z_A^2}{4}}\right) \tag{20}$$

with $n_A = n_{\rm r}+l_A+1$ as the principal quantum number for the QS-A.

A simple integration of equation (12) taking $V_A(g_B) = \frac{z_A}{g_B}$ along with the local property of the transformation function $g_B(0)=0$ yields

$$g_B(r) = \sigma_B r^2\,, \tag{21}$$

where

$$\sigma_B = \frac{\left(E_B^2-M_0^2\right)}{4z_A\left(E_A+M_0\right)}\,. \tag{22}$$

After putting the transformation function $g_B(r)$ from equation (21) into equation (13), and using the following judicious assumption:

$$\frac{M_0\omega^2}{2} = -\frac{4\sigma_B^2\left(E_A^2-M_0^2\right)}{\left(E_B+M_0\right)}\,, \tag{23}$$

the generated potential for the QS-B is fixed as the well-known harmonic oscillator, *i.e.*

$$V_B(r) = \frac{M_0\omega^2 r^2}{2}\,. \tag{24}$$

Now using equations (10), (18), and (21) in equation (8), and $l_B = \frac{1}{2}(4l_A+4-D_B)$ by equation (15), and defining the principal quantum number $n_B = 2n_{\rm r}+l_B$, the radial wavefunction of the QS-B (harmonic oscillator) is generated as

$$\psi_B(r) = C_B r^{l_B}\,{\rm e}^{\frac{-1}{2}\rho_B r^2} F\left(-n_{\rm r},\frac{1}{2}(2l_B+D_B),\rho_B r^2\right)\,, \tag{25}$$

where C_B is the normalisation constant and $\rho_B = -\frac{E_B^2-M_0^2}{2(n_B+\frac{D_B}{2})}$.

After simplifying equation (23) with the help of σ_B from equation (22), we have obtained the relativistic energy spectrum in the form of an irrational equation for the QS-B (harmonic oscillator) in D_B-dimensional space as

$$(E_B - M_0)\sqrt{\frac{(E_B + M_0)}{2M_0}} = \omega\left(n_B + \frac{D_B}{2}\right), \tag{26}$$

which agrees with the already published result [11] for $D_B = 3$.

Since the normalisation integral in equation (17) is found to be proportional to the expectation value of the genuine Coulomb potential $V_A(r)$, the generated radial wavefunction $\psi_B(r)$ is normalisable.

4. Generation of Coulomb potential in D_C-dimensional Euclidean space

Repeating the transformation on equation (16) for harmonic oscillator potential as mentioned in equation (24) considering QS-B as a parent QS in D_B-dimensional space with wavefunction given by equation (25) and the energy spectrum given by equation (26), Coulomb potential in D_C-dimensional space is generated as follows.

After applying the CT

$$r \to g_C(r)$$

and the FT

$$\psi_C(r) = f_C^{-1}(r)\psi_B(g_C(r)), \tag{27}$$

into equation (16), we have the following equation similar to equation (9):

$$\psi''_C(r) + \psi'_C(r)\left(\frac{\mathrm{d}}{\mathrm{d}r}\ln\frac{f_C^2 g_C^{D_B-1}}{g'_C}\right) + \left[\left(\frac{\mathrm{d}}{\mathrm{d}r}\ln f_C\right)\left(\frac{\mathrm{d}}{\mathrm{d}r}\ln\frac{f'_C g_C^{D_B-1}}{g'_C}\right)\right]$$
$$+g'^2_C\left[\tilde{E}_B - \tilde{V}_B(g_C) - \frac{l_B(l_B + D_B - 2)}{g_C^2}\right]\psi_C(r) = 0. \tag{28}$$

To cast equation (28) in the form of a Schrödinger-like equation similar to equation (16) for the generated QS-C in an arbitrarily chosen D_C-dimensional Euclidean space, we need to call equation (10)

$$f_C(r) = g_C(r)'^{\frac{1}{2}} g_C(r)^{-\frac{D_B-1}{2}} r^{\frac{D_C-1}{2}}, \tag{29}$$

with which equation (28) converts to

$$\psi_C''(r) + \frac{D_C - 1}{r}\psi_C'(r) + \left[\frac{1}{2}\{g_C, r\} - \frac{D_B - 1}{2}\frac{D_B - 3}{2}\left(\frac{g_C'}{g_C}\right)^2 + \frac{D_C - 1}{2}\frac{D_C - 3}{2}\left(\frac{1}{r^2}\right) + g_C'^2\{\tilde{E}_B - \tilde{V}_B(g_C) - \frac{l_B(l_B + D_B - 2)}{g_C^2}\right]\psi_C(r) = 0\,. \tag{30}$$

After applying the following ansatzes as stated in equations (12) and (13), *i.e.*

$$g_C'^2\tilde{V}_B(g_C) = -\tilde{E}_C\,, \tag{31}$$

and

$$g_C'^2\tilde{E}_B = -\tilde{V}_C(r)\,, \tag{32}$$

we get the following differential equation:

$$\psi_C''(r) + \frac{D_C - 1}{r}\psi_C'(r) + \left[\tilde{E}_C - \tilde{V}_C(r) - \left\{-\frac{1}{2}\{g_C, r\} + \frac{D_B - 1}{2}\frac{D_B - 3}{2}\left(\frac{g_C'}{g_C}\right)^2 - \frac{D_C - 1}{2}\frac{D_C - 3}{2}\left(\frac{1}{r^2}\right) + g_C'^2\frac{l_B(l_B + D_B - 2)}{g_C^2}\right\}\right]\psi_C(r) = 0\,, \tag{33}$$

where $\tilde{V}_C(r) = (E_C + M)V_C(r)$ is the modulated potential and $\tilde{E}_C = (E_C^2 - M^2)$ is the redefined energy for generated QS-C. To introduce the 'centrifugal barrier term' for D_C-dimensional Euclidean space into the above equation, we adopt [38]

$$-\frac{1}{2}\{g_C, r\} + \frac{D_B - 1}{2}\frac{D_B - 3}{2}\left(\frac{g_C'}{g_C}\right)^2 - \frac{D_C - 1}{2}\frac{D_C - 3}{2}\left(\frac{1}{r^2}\right) + g_C'^2\frac{l_B(l_B + D_B - 2)}{g_C^2} = \frac{l_C(l_C + D_C - 2)}{r^2}\,. \tag{34}$$

Thus, the final Schrödinger-like equation for the QS-C becomes

$$\psi_C''(r) + \frac{D_C - 1}{r}\psi_C'(r) + \left[\tilde{E}_C - \tilde{V}_C(r) - \frac{l_C(l_C + D_C - 2)}{r^2}\right]\psi_C(r) = 0\,. \tag{35}$$

Integrating equation (31) with $V_B(g_C) = \frac{M_0\omega^2 g_C^2}{2}$ along with the local property of the transformation function $g_C(0) = 0$, the transformation function for QS-C is obtained as

$$g_C(r) = \sigma_C \sqrt{r}\,, \tag{36}$$

where

$$\sigma_C = \left[\frac{-8\left(E_C^2 - M_0^2\right)}{M_0\omega^2(E_B + M_0)}\right]^{\frac{1}{4}}. \tag{37}$$

Using the above $g_C(r)$ along with the following physically relevant assumption:

$$z_C = -\frac{\sigma_C^2\left(E_B^2 - M_0^2\right)}{4(E_C + M_0)}\,, \tag{38}$$

equation (32) yields the generated potential for the QS-C as

$$V_C(r) = \frac{z_C}{r}\,,$$

which is the well-known Coulomb potential with its characteristic constant z_C. This Coulomb potential is (re)generated in a D_C-dimensional Euclidean space from a harmonic oscillator potential in D_B-dimensional Euclidean space via our transformation method as described in Section 2.

Using equations (25), (29), and (36) in equation (27), the wavefunction of the QS-C is generated as

$$\psi_C(r) = C_C r^{l_C}\, \mathrm{e}^{\frac{-1}{2}\rho_C r} F\left(-n_\mathrm{r}, (2l_C + D_C - 1), \rho_C r\right)\,, \tag{39}$$

where, C_C is the normalisation constant, $l_C = \frac{D_B - 2D_C + 2l_B + 2}{4}$ by equation (34) and $\rho_C = \rho_B\sigma_B^2 = 2\sqrt{E_C^2 - M_0^2}$.

Combining equations (26), (37), and (38), we get the relativistic energy spectrum for Coulomb potential in the generated QS-C in D_C-dimensional space as

$$E_C = M_0 \frac{\left\{n_C + \frac{D_C - 3}{2}\right\}^2 - \frac{z_C^2}{4}}{\left\{n_C + \frac{D_C - 3}{2}\right\}^2 + \frac{z_C^2}{4}}\,, \tag{40}$$

defining $n_C = n_\mathrm{r} + l_C + 1$ as the principal quantum number for the QS-C. It resembles the energy spectrum of the 3-D Coulomb system as in equation (20).

Normalisability of the generated radial $\psi_C(r)$ wavefunction here is also easily tested as earlier by using equation (17). Since the normalisation integral for QS-C is proportional to the expectation value of the harmonic oscillator potential, it is finite, indicating that $\psi_C(r)$ is normalisable.

5. Discussion and conclusion

This paper deals with a new transformation method for (re)generation of QMESP from an already known genuine exactly solved CSP (power law) in the relativistic framework of the KG equation. The radial KG equation in equation (5) is the basis of the treatment along with CT followed by FT plus an indispensable set of ansatzes as stated in equations (12) and (13). Starting with a centrally symmetric genuine Coulomb potential for the parent QS-A in three-dimensional ($D_A = 3$) Euclidean space, it is found that this transformation quite efficiently generates the centrally symmetric harmonic oscillator potential for daughter QS-B in any desired D_B-dimensional Euclidean space ($D_B \geq 3$). The generated wavefunctions and the derived expression for energy spectrum in equations (25) and (26), respectively, for the generated harmonic oscillator agree with the relevant expressions in Ref. [11] in three-dimensional Euclidean space, which ascertains the reliability of our transformation method for the generation of CSPs (power law).

It is observed in our work that the repetition of the method on QS-B with the centrally symmetric harmonic oscillator potential yields again a Coulomb potential (QS-C) in any desired D_C-dimensional Euclidean space. Thus, as a by-product, our transformation method reveals a seamless cyclic mapping establishing a relative parent–daughter family relationship between the two most fundamental potentials, Coulomb and harmonic oscillator, in the higher-dimensional QM of the KG equation in a more general way compared to the work in [45]. Our method does not require solving of any differential equations, it simply maps the wavefunction from a QS (parent) to its daughter QS. In that sense, our transformation is a labour-saving method compared to the other works reported in Refs. [43–48].

In our formalism, we find out a general procedure, as described in the last part of Section 2, to address the essential issue of normalisability for the generated wavefunctions in the QM of the KG equation irrespective of the dimensions of Euclidean space. The proposed normalisability checking procedure vividly shows that if the parent QS is a physical one, the wavefunctions for the generated QS are almost always normalisable. Thus, the transformation acts as a smart carrier for the 'DNA' responsible for the normalisability character from the parent to its daughter QS. It means indirectly that our transformation method in principle has the capability to map one physical QS-A with a CSP (power law) in D_A-dimensional Euclidean space to another physical one with a new CSP (power law) in D_B ($D_B = D_A, D_B \neq D_A$) dimensional Euclidean space with the possibility of dimensional extension and reduction as per the situation demands.

The generating function $g(r)$ which is obtained by the application of one of the ansatzes in equation (12) plays a key role in mapping one QS to another one, provided $g(r)$ would be an invertible and differentiable function of at least class C^2, and also not a badly behaved function so far as its local and asymptotic properties, *i.e.* $g(0) = 0$ and $g(\infty) = \infty$, are concerned. It is observed from our work that if someone starts with the Coulomb-like potential $V(r) \sim \frac{1}{r}$ (parent system), the ansatz in equation (12) always yields a transformation function $g(r) \sim r^2$ in equation (21) which will invariably lead us to a harmonic oscillator potential $V(r) \sim r^2$ (daughter system) through the ansatz in equation (13), and if someone repeats the procedure starting with harmonic oscillator as the parent system, then the transformation function $g(r) \sim \sqrt{r}$ in equation (36) generates none other than a Coulomb-like system, immaterial about what the dimensions of Euclidean space are for the parent and daughter QSs. The learning is that our transformation stringently maps a Coulomb to harmonic oscillator and *vice versa*, which therefore seems to be an inherent constriction of this method. To break this cyclic mapping, we will need to formulate a new ansatz *in lieu* of the one in equation (12) so that suitable transformation functions other than $g(r) \sim r^2$ (if we start with Coulomb) and other than $g(r) \sim \sqrt{r}$ (if we start with harmonic oscillator) can be obtained so as to arrive at exactly solvable/solved CSPs other than harmonic oscillator and Coulomb, respectively.

The major inbuilt strength of our transformation method is its ability to generate genuine/physical QSs from already exactly solved genuine QSs with CSPs in the higher-dimensional QM of KG equation with equal vector and scalar potentials. As such, we are currently investigating the possibility of generating new and genuine CSPs (power law and non-power law) from already known CSPs (non-power law) with our transformation method, and the results attained hereafter will be communicated in our forthcoming publications.

The authors would like to thank the referee for positive and precious suggestions which have improved the manuscript greatly.

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