CONFORMAL FLATNESS AND CONFORMAL VECTOR FIELDS ON UMBILICALLY SYNCHRONIZED SPACE-TIMES

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We study umbilically synchronized space-times M. First, we show that M with vanishing electric part of the Weyl tensor is conformally flat if either dim $M = 4$ or spatial slices Σ are conformally flat. Next, for the vacuum case, we show that the scalar curvature of spatial slices Σ is a non-positive function of time t (this includes the case when M is Schwarzchild exterior space-time), and if, in addition, M is geodesic (acceleration-free) and electric part of the Weyl tensor vanishes, then M is a Lorentzian cone over a hyperbolic space which is, in dimension 4, an expanding hyperbolic cosmological model. Finally, we provide some characterizations of conformal (including inheriting conformal) vector fields of an umbilically synchronized space-time.

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1. Introduction

The standard Friedmann–Lemaitre–Robertson–Walker (FLRW) cosmological model is described by the space-time M with the line-element

$$\mathrm{d}s^2 = g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = -\mathrm{d}t^2 + a^2(t)\gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,, \tag{1}$$

where α, β denote the space-time indices running over $0,1,2,3$, the spatial indices i,j run over $1,2,3$, time coordinate $t = x^0$, the warping function $a(t)$ is the scale function, and γ_{ij} is the fixed spatial metric of constant curvature. We know that *(i)* M is conformally flat, *(ii)* the spatial slices Σ ($t =$ constant) have constant curvature and are homothetic to one another, and *(iii)* Σ are umbilical in M and have constant mean curvature. A general $(n+1)$-dimensional space-time is described in the ADM (Misner, Thorne, and Wheeler [1]) formalism by the metric $g_{\alpha\beta}$ with the line-element

$$\mathrm{d}s^2 = g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = -N^2\,\mathrm{d}t^2 + g_{ij}\left(\mathrm{d}x^i + S^i\,\mathrm{d}t\right)\left(\mathrm{d}x^j + S^j\,\mathrm{d}t\right)\,, \tag{2}$$

where the Greek indices α, β run over $0, 1, \ldots, n$ and Latin indices i, j over $1, \ldots, n$; N is the Lapse function that depends on t and x^i, and represents the clock rates for an observer relative to a reference system of clocks, and S^i is a vector field on the n-dimensional slice Σ ($t =$ constant) which represents two observers in relative motion with velocity S^i. In this paper, we assume that the shift vector S^i is zero, *i.e.* the evolution vector field $\frac{\partial}{\partial t}$ is orthogonal to the spatial slices Σ and also that Σs are totally umbilical in M (which are true for the FLRW space-time). However, the mean curvature need not be constant on any slice. Space-times foliated by such Σs are called umbilically synchronized space-times (see Ferrando, Morales, and Portilla [2]) and are shear-free and vorticity-free with respect to an observer whose congruence is given by the unit vector $\boldsymbol{n} = \frac{1}{N}\frac{\partial}{\partial t}$ normal to Σ. The acceleration vector field $\boldsymbol{A} = \bar{\nabla}_{\boldsymbol{n}}\boldsymbol{n}$ need not vanish. Treciokas and Ellis [3] have shown that the shear-free and vorticity-free time-like congruences constitute a large class among the observers measuring an isotropic distribution function obeying the Boltzmann equation. Conformally flat umbilical synchronizations exist in any space-time admitting natural symmetric frames (Coll and Morales [4]).

In this paper, we study umbilically synchronized space-times (M, g). We note that these include the classical Schwarzschild exterior and FLWR space-times. First, we derive the components of its Weyl conformal tensor in terms of the geometric quantities of spatial slices Σ ($t =$ constant). As FLRW space-times are conformally flat, we obtain a condition for an umbilically synchronized space-time (M, g) to be conformally flat, in terms of vanishing of the electric components of the Weyl tensor. An example of a conformally flat non-FLRW umbilically synchronized space-time is the spherically symmetric Stephani model with a non-uniform pressure fluid as an exact solution of Einstein's field equations. This example is a special case discussed in Theorem 1. Another example of a non-conformally flat (not FLRW) umbilically synchronized space-time with vanishing electric components of the Weyl tensor can be constructed from Theorem 1, as the warped product of the time-line with the product: $S^2 \times S^2$ of two unit spheres, with metric $-\mathrm{d}t^2 + a^2(t)(\mathrm{d}\theta_1^2 + \sin^2\theta_1 \mathrm{d}\phi_1^2 + \mathrm{d}\theta_2^2 + \sin^2\theta_2 \mathrm{d}\phi_2^2)$, because $S^2 \times S^2$ is Einstein but does not have constant curvature. Next, for the vacuum case, we show that the scalar curvature of spatial slices Σ is a non-positive function of time t (this includes the case when M is the Schwarzchild exterior space-time), and if, in addition, M is geodesic (acceleration-free) and electric part of the Weyl tensor vanishes, then M is a Lorentzian cone over a hyperbolic space which is, in dimension 4, an expanding hyperbolic cosmological model. Finally, motivated by the fact that FLRW space-times admit a maximal conformal group, we provide some characterizations of conformal (including inheriting conformal) vector fields of an umbilically synchronized space-time.

2. Basic equations

We denote the Levi-Cita connection, the Riemann curvature tensor, the Ricci tensor, scalar curvature, and the Weyl tensor of the metric g_{ij} by ∇, R_{ijkl}, R_{ij}, R, and C_{ijkl}. Corresponding quantities of the space-time metric $g_{\alpha\beta}$ are denoted by bars over the corresponding symbols with Greek indices $\alpha, \beta, \gamma, \delta$ in lieu of the Latin indices i, j, k, l. As indicated earlier, the unit vector field $\boldsymbol{n} = \frac{1}{N}\frac{\partial}{\partial t}$ is normal to Σ, and the acceleration vector field $\boldsymbol{A} = \bar{\nabla}_{\boldsymbol{n}}\boldsymbol{n}$ is tangential to the space-like slices Σ, and can be shown by direct computation, to be equal to the spatial gradient of $\ln N$. Denoting the co-ordinate basis of the tangent space of Σ by ∂_i, we have the second fundamental form K_{ij} of Σ defined by $g(\bar{\nabla}_{\partial_i}\boldsymbol{n}, \partial_j)$, where g is the space-time metric.

Definition 1 *A space-like hypersurface Σ of space-time (M, g) is said to be umbilical if the second fundamental form K_{ij} of Σ is pointwise proportional to the induced metric tensor g_{ij} on Σ.*

We are assuming that space-time is umbilically synchronized, *i.e.* space-like slices Σ are umbilical in M, and hence we have

$$K_{ij} = \tau g_{ij} , \tag{3}$$

where τ stands for the mean curvature of Σ. It can be verified by an easy computation (see Fischer and Marsden [5]) that

$$K_{ij} = \frac{1}{2N}\partial_t g_{ij} . \tag{4}$$

Equations (3) and (4) imply the linear differential equation $\partial_t g_{ij} = 2N\tau g_{ij}$ whose solution g_{ij} splits off a time-independent metric γ_{ij} such that

$$g_{ij} = a^2\left(t, x^k\right)\gamma_{ij} \tag{5}$$

for a positive function a that depends on t and x^k. Thus, the line-element of the umbilically synchronized space-time assumes the following form:

$$-N^2\,\mathrm{d}t^2 + a^2\left(t, x^k\right)\gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j .$$

In particular, for $N = 1$, a a function of only t, and γ any fixed time-independent Riemannian metric, we get generalized Robertson–Walker space-time (Alias, Romero, and Sanchez [6]) which becomes FLRW space-time when γ is of constant curvature. Comparing equation (3) with (4) and using (5) yields the relation

$$\tau = \frac{\dot{a}}{aN} , \tag{6}$$

where the over-dot denotes partial differentiation with respect to t. The classical Gauss and Codazzi equations for Σ are

$$\begin{aligned} \bar{R}_{ijkl} &= R_{ijkl} + K_{il}K_{jk} - K_{ik}K_{jl}\,, && (7)\\ \bar{R}_{ijk0} &= \nabla_j K_{ki} - \nabla_i K_{kj}\,. && (8) \end{aligned}$$

We also have the following mixed components as given in [5]:

$$\bar{R}_{0i0j} = N^2\left(\frac{1}{N}\partial_t K_{ij} - K_i^k K_{kj} - \frac{1}{N}\nabla_i\nabla_j N\right). \tag{9}$$

Next, using the umbilicity condition (3) and the above curvature components in the definition $\bar{R}_{\alpha\beta} = g^{\gamma\delta}\bar{R}_{\gamma\alpha\beta\delta}$, we obtain

$$\begin{aligned} \bar{R}_{ij} &= R_{ij} + \left(n\tau^2 + \frac{\dot{\tau}}{N}\right) g_{ij} - \frac{1}{N}\nabla_i\nabla_j N\,, && (10)\\ \bar{R}_{i0} &= (1-n)\nabla_i\tau\,, && (11)\\ \bar{R}_{00} &= N\left(\Delta N - Nn\tau^2 - n\dot{\tau}\right)\,, && (12)\\ \bar{R} &= R + 2\frac{n}{N}\dot{\tau} + n\tau^2 + n^2\tau^2 - \frac{2}{N}\Delta N\,. && (13) \end{aligned}$$

At this point, we recall that the Weyl conformal tensor $\bar{C}$ of the $(n+1)$-dimensional space-time is given by the components

$$\begin{aligned} \bar{C}_{\alpha\beta\gamma\delta} &= \bar{R}_{\alpha\beta\gamma\delta} - \frac{1}{n-1}\left(\bar{R}_{\beta\gamma}g_{\alpha\delta} - \bar{R}_{\alpha\gamma}g_{\beta\delta} + g_{\beta\gamma}\bar{R}_{\alpha\delta} - g_{\alpha\gamma}\bar{R}_{\beta\delta}\right)\\ &\quad + \frac{\bar{R}}{n(n-1)}\left(g_{\beta\gamma}g_{\alpha\delta} - g_{\alpha\gamma}g_{\beta\delta}\right). \end{aligned}$$

Using this definition along with equations (7)–(9) and (10)–(13), and after a lengthy computations and arrangements, we obtain

$$\begin{aligned} \bar{C}_{ijkl} &= C_{ijkl} + \frac{1}{(n-1)(n-2)}\left[g_{il}\left(R_{jk} - \frac{R}{n}g_{jk}\right) - g_{jl}\left(R_{ik} - \frac{R}{n}g_{ik}\right)\right.\\ &\quad \left. + g_{jk}\left(R_{il} - \frac{R}{n}g_{il}\right) - g_{ik}\left(R_{jl} - \frac{R}{n}g_{jl}\right)\right]\\ &\quad + \frac{1}{N(n-1)}\left[g_{il}\left(\nabla_j\nabla_k N - \frac{\Delta N}{n}g_{jk}\right) - g_{jl}\left(\nabla_k\nabla_i N - \frac{\Delta N}{n}g_{ki}\right)\right.\\ &\quad \left. + g_{jk}\left(\nabla_i\nabla_l N - \frac{\Delta N}{n}g_{il}\right) - g_{ik}\left(\nabla_j\nabla_l N - \frac{\Delta N}{n}g_{jl}\right)\right], && (14)\\ \bar{C}_{ijk0} &= 0\,, && (15)\\ \bar{C}_{0i0j} &= -\frac{N^2}{n-1}\left[R_{ij} - \frac{R}{n}g_{ij} + \frac{n-2}{N}\left(\nabla_i\nabla_j N - \frac{\Delta N}{n}g_{ij}\right)\right]. && (16) \end{aligned}$$

3. A conformal flatness criterion

Let us recall (Stephani *et al.* [7]) that the electric part of the Weyl tensor of the space-time with respect to $\boldsymbol{n}$ is $E_{ij} = \bar{C}(\partial_i, \boldsymbol{n}, \partial_j, \boldsymbol{n}) = \frac{1}{N^2}\bar{C}_{i0j0}$. So, if $E_{ij} = 0$, then equation (16) provides

$$R_{ij} - \frac{R}{n} g_{ij} + \frac{n-2}{N}\left(\nabla_i \nabla_j N - \frac{\Delta N}{n} g_{ij}\right) = 0\,. \tag{17}$$

If M is 4-dimensional, then Σ is 3-dimensional, and hence $C_{ijkl} = 0$. Using this in (14) shows that $\bar{C}_{ijkl} = 0$ and hence M is conformally flat. For dimension $M > 4$, equations (14) and (17) imply $\bar{C}_{ijkl} = C_{ijkl}$, and so if $C_{ijkl} = 0$, *i.e.* Σs are conformally flat, then $\bar{C}_{ijkl} = 0$, and hence M is conformally flat. Converse is evident. We state these findings as the following result.

Theorem 1 *Let the electric part of the Weyl tensor of an umbilically synchronized space-time M of dimension ≥ 4 be zero. If dimension $M = 4$, then M is conformally flat. For dimension $M > 4$, M is conformally flat if and only if Σs are conformally flat.*

This result is a generalization of the classical result: "A generalized Robertson–Walker space-time M with metric: $-\mathrm{d}t^2 + a^2(t)\gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$ (where γ is a time-independent Riemannian metric) is conformally flat if and only if γ_{ij} has constant curvature (and hence conformally flat). More generally, the electric part of the Weyl tensor of M is zero if and only if γ_{ij} is Einstein." (Sharma and Duggal [8]).

4. Vacuum case

For umbilically synchronized space-times that are vacuum, *i.e.* $\bar{R}_{\alpha\beta} = 0$, equations (10), (11), (12), and (13) provide

$$R_{ij} - \frac{R}{n} g_{ij} = \frac{1}{N}\left(\nabla_i \nabla_j N - \frac{\Delta N}{n} g_{ij}\right), \tag{18}$$

$$(1-n)\nabla_i \tau = 0\,, \tag{19}$$

$$\Delta N = n\left(\dot{\tau} + N\tau^2\right), \tag{20}$$

$$R = -2\frac{n}{N}\dot{\tau} - n\tau^2 - n^2\tau^2 + \frac{2}{N}\Delta N\,. \tag{21}$$

Equation (19) immediately shows that the mean curvature $\tau = \frac{\mathrm{Tr}\,K_{ij}}{n}$ of Σ is a function of only t, *i.e.* the mean curvature of each spatial slice is constant on that slice.

Further, equations (20) and (21) imply that

$$R = -n(n-1)\tau^2 \tag{22}$$

showing that the scalar curvature of spatial slices Σ is a non-positive function of t.

Remark 1 *For Schwarzschild exterior space-time metric:* $-(1-\frac{2m}{r})\mathrm{d}t^2 + (1-\frac{2m}{r})^{-1}\mathrm{d}r^2 + (r^2)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)$*, we know that the spatial slices* Σ *are totally geodesic (*$\tau = 0$*) in the space-time,* $R = 0$*, and* $N = (1-\frac{2m}{r})^{1/2}$ *depends only on spatial coordinate* r *and hence* $\Delta N = 0$ *(i.e.* N *is a harmonic function), from equation (20). Consequently, (18) assumes the form* $R_{ij} = \frac{1}{N}(\nabla_i\nabla_j N)$*. From this, it follows by a straightforward computation that the only non-zero components of the Ricci tensor of* Σ *are* $R^1_1 = \frac{-2m}{r^3}$*,* $R^2_2 = \frac{m}{r^3}$*,* $R^3_3 = \frac{m}{r^3}$*, and hence* Σ *is asymptotically flat, which is well known.*

At this point, we assume that (M, g) is also geodesic, *i.e.* the acceleration vector $\boldsymbol{A}$ vanishes. Thus, N depends only on t and hence can be taken equal to 1 by time-rescaling. Thus, equation (20) reduces to $\dot{\tau} = -\tau^2$ and hence integrates as $\tau = 1/t$. The use of (6) in the foregoing equation and time integration gives $a = tX$, where X is an arbitrary function of x^k and can be absorbed in γ_{ij}. Consequently, the space-time metric on M becomes the Lorentzian cone: $-\mathrm{d}t^2 + t^2\gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$ and $R_{ij} = -\frac{n-1}{t^2}g_{ij}$. Hence, g_{ij} is Einstein, and from (16) the electric components of the Weyl tensor of M vanish. For dim $M = 4$, Σ are 3-dimensional and hence of constant negative curvature, consequently, M is Minkowski. Thus, it represents the expanding hyperbolic model in the Minkowski space-time (Misner, Thorne, and Wheeler [1]). This leads to the following result.

Theorem 2 *If a vacuum umbilically synchronized space-time* (M, g) *is geodesic (acceleration-free), then it is a Lorentzian cone over a negatively Einstein manifold. If the dimension of* M *is 4, then it represents the expanding hyperbolic cosmological model.*

5. Conformal vector fields

Let us recall the fact that FLRW-space-time admits a maximal conformal group (Maartens and Maharaj [9]). Intrigued by this, we would like to examine a 1-parameter group of conformal motions generated by a conformal vector field V defined by

$$\pounds_V g = 2\psi g\,, \tag{23}$$

where $\pounds$ denotes the Lie-derivative operator and ψ a smooth function called the conformal-scale function. We divide our analysis into the following three modules.

5.1. Module I

Let us first consider a conformal vector field V in the direction of the unit vector field $\boldsymbol{n}$ orthogonal to spatial slices Σ of an umbilically synchronized space-time (M, g), *i.e.* $V = f\boldsymbol{n}$ for a function f on M. Substituting it into (23), we have

$$g\left(\bar{\nabla}_{\bar{X}} f\boldsymbol{n}, \bar{Y}\right) + g\left(\bar{\nabla}_{\bar{Y}} f\boldsymbol{n}, \bar{X}\right) = 2\psi g\left(\bar{X}, \bar{Y}\right) , \tag{24}$$

where $\bar{X}$ and $\bar{Y}$ are arbitrary vector fields on M. Now, denoting arbitrary tangent vector fields on Σ by X, Y, taking $(\boldsymbol{n}, \boldsymbol{n})$, $(X, \boldsymbol{n})$, and (X, Y) projections of (24), we obtain the following relations:

$$\boldsymbol{n}f = \psi\,, \tag{25}$$

$$Xf = fX \ln N\,, \tag{26}$$

and

$$f\tau = \psi\,, \tag{27}$$

where $\boldsymbol{n}f$ is understood as the action of the vector field $\boldsymbol{n}$ as a differential operator on the function f. It follows from equations (6), (25), and (27) that $\partial_t \ln f = \frac{\dot{a}}{a}$ which easily integrates as $f = aX$ for an arbitrary positive function of spatial coordinates. However, X can be absorbed by the fixed time-independent metric γ_{ij} defined by (5). Thus, we have

$$f = a\,. \tag{28}$$

Next, integrating equation (26) gives

$$f = N/T\,, \tag{29}$$

where T is a function only of t. Consequently, using (29), we find that

$$V = (1/T)\partial_t\,. \tag{30}$$

Thus, the conformal vector field along ∂_t is completely time-dependent. Also, equation (27) provides the conformal scale function

$$\psi = \frac{\dot{a}}{N}\,. \tag{31}$$

In particular, for the FLRW-space-time, $N = 1$, and hence (28) and (29) show that $a = 1/T$, and hence (30) yields the well-known time-like conformal vector field $V = a\partial_t$ with conformal scale function $\psi = \dot{a}$.

Let us consider the special case when V is homothetic, *i.e.* ψ is constant. The use of equations (27), (28), and (29) provides $\frac{\dot{a}}{a} = \psi T$ which integrates to $a = Y\mathrm{e}^{\psi\int(T)\mathrm{d}t}$, where Y is a function of only the spatial coordinates and hence can be absorbed by a. Thus, $a = \mathrm{e}^{\psi\int(T)\mathrm{d}t}$, and hence depends only on t. Now, equations (28) and (29) show that $a = N/T$. As a and T depend only on t, therefore so does N. By time-rescaling, we can therefore take $N = 1$. Consequently, it follows from (25) that $\dot{a} = \psi$ which integrates to $a = \psi t + c$ for a constant c. By rescaling and translating t, we can have $a = t$. Also, from (28), we have $f = t$. As a result, the line-element becomes $-\mathrm{d}t^2 + t^2\gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$, *i.e.* the Lorentzian cone, and the homothetic vector field along ∂_t becomes $t\partial_t$.

5.2. Module II

Next, let us consider a spatial conformal vector field V such that $V \perp \boldsymbol{n}$. Taking the $(\boldsymbol{n},\boldsymbol{n})$-component of the conformal equation (23), we have

$$g\left(\bar{\nabla}_{\boldsymbol{n}}V,\boldsymbol{n}\right) = -\psi\,.$$

But $g(V,\boldsymbol{n}) = 0$. Therefore, the above equation assumes the form $g(\bar{\nabla}_{\boldsymbol{n}}\boldsymbol{n},V) = \psi$. As we pointed out in Section 2 that $\bar{\nabla}_{\boldsymbol{n}}\boldsymbol{n}$ is the spatial gradient of $\ln N$, we find that $V\ln N = \psi$. Thus, we obtain the following result.

Proposition 1 *Let V be a conformal vector field on an umbilically synchronized space-time such that V is tangential to the spatial slices Σ. If the lapse function N is constant along V, then V is Killing.*

The above result is a generalization of the well-known result that conformal vector fields tangential to the space-like slices of an FLRW space-time are Killing [9], because the lapse function $N = 1$ for an FLRW space-time.

5.3. Module III

Finally, we find inheriting conformal vector fields on an umbilically synchronized space-time. Following Coley and Tupper [10], a conformal vector field is said to be an inheriting conformal vector field, if it preserves the flow lines along the unit time-like vector field up to a function multiple. In our context, let the conformal Killing vector field be decomposed as $V = \alpha\boldsymbol{n}+U$, where U is the component of V tangential to the spatial slices Σ. The $(\boldsymbol{n},\boldsymbol{n})$ projection of the conformal Killing equation (23) provides the relation

$$g\left(\bar{\nabla}_{\boldsymbol{n}}U,\boldsymbol{n}\right) = \boldsymbol{n}\alpha - \psi\,. \tag{32}$$

For an arbitrary vector field X tangent to Σ, we take the $(X, \boldsymbol{n})$-projection of (23) and use the umbilicity condition $\bar{\nabla}_X \boldsymbol{n} = \tau X$ in order to get

$$\bar{\nabla}_{\boldsymbol{n}} U = \tau U + \beta \boldsymbol{n} + N D\left(\frac{\alpha}{N}\right) , \tag{33}$$

where D is the spatial gradient operator. Using (32) in the above readily gives $\beta = \psi - \boldsymbol{n}\alpha$. Also, noting that $\bar{\nabla}_U \boldsymbol{n} = \tau U$ (as U is tangent to Σ) and using it in (33) gives $[U, \boldsymbol{n}] = (\boldsymbol{n}\alpha - \psi)\boldsymbol{n} - ND(\frac{\alpha}{N})$. Hence, we compute $\pounds_V \boldsymbol{n} = [\alpha\boldsymbol{n} + U, \boldsymbol{n}] = -(\boldsymbol{n}\alpha)\boldsymbol{n} + [U, \boldsymbol{n}] = -\psi\boldsymbol{n} - ND(\frac{\alpha}{N})$. From this, it follows that $\pounds_V \boldsymbol{n}$ is a multiple of $\boldsymbol{n}$ if and only if $D(\frac{\alpha}{N}) = 0$, *i.e.* $\alpha = NT$, where T is a function of only t. In this case, the inheriting conformal vector field assumes the form of $V = NT\boldsymbol{n} + U = T\partial_t + U$ and $\pounds_V \boldsymbol{n} = -\psi\boldsymbol{n}$. Also, for X, Y tangent to Σ, the (X, Y)-projection of the conformal Killing equation (23) gives $(\pounds_U g)(X, Y) = 2(\psi - \alpha\tau)g(X, Y)$. Summing up these findings, we obtain the following characterization of an inheriting conformal vector field on an umbilically synchronized space-time.

Theorem 3 *A conformal vector field V on an umbilically synchronized space-time is an inheriting conformal vector field with respect to the flow lines determined by the unit time-like vector field $\boldsymbol{n}$ if and only if $V = NT(t)\boldsymbol{n} + U$, where N is the lapse function, $T(t)$ is a function of only t, and U is orthogonal to $\boldsymbol{n}$.*

Remark 2 *Considering a time-like conformal vector field $V = \alpha\boldsymbol{n}$ for a scalar function α, we obviously see that V is an inheriting conformal vector field. These vector fields arise (see Israel [11]) as the inverse temperature function $(1/T)u^a$ (here $u = \boldsymbol{n}$ and T is the temperature). Their existence has been further supported by Stephani [12] in terms of complete exact reversible thermodynamics and argued in favor by Tauber and Weinberg [13] in terms of the isotropy of the cosmic microwave background.*

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