DARBOUX TRANSFORMATION OF THE COUPLED MASSIVE THIRRING MODELS AND EXACT SOLUTIONS

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Based on the gauge transformation between the corresponding Lax pair, we derive a Darboux transformation of the coupled massive Thirring system. As an application, using the Darboux transformation and the reduction technique, various exact solutions for the coupled massive Thirring system and the classical massive Thirring model are obtained, including one-soliton solution, two-soliton solution, periodic solution, and others.

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1. Introduction

Soliton equations are important models describing nonlinear phenomena that occur in nature and have many applications in various fields of physical science such as nonlinear waves, nonlinear optics, plasma physics, and magnetic fluids [1–3]. In all sorts of soliton models, the massive Thirring model was proposed firstly by Thirring to study some features of relativistic field theories [4]. In [5, 6], authors gave the connection between the classical massive Thirring model and the sine-Gordon model. The connection has made numerous research of the classical massive Thirring model been conducted smoothly, like the quasi-periodic solutions by employing the theory of algebraic curves and the periodic problem [7], soliton solutions by the inverse scattering transform [8], and multi-soliton solutions to the Thirring model through the reduction method [9]. Recently, Darboux polynomial matrices of the classical massive Thirring model has been obtained by introducing a novel algorithm [10].

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It is known that the Darboux transformation is a powerful tool for solving soliton equations. With the aid of all kinds of trivial solutions, the Darboux transformation can produce another set of new solutions. This process can be done continually and will usually yield a series of multi-soliton solutions [11–14], which is its advantage over the inverse scattering transformation [15], the direct method [16], the Bäcklund transformation [17], the algebro-geometric method [18–22], and others [23–26]. The aim of the present paper is to construct a Darboux transformation for the coupled massive Thirring system

$$
\begin{aligned}
&q_x + 2\mathrm{i}\left(\ell_0 uv + \beta_0\right) q + 2\mathrm{i}\alpha_0 u = 0\,, \quad r_x - 2\mathrm{i}\left(\ell_0 uv + \beta_0\right) r - 2\mathrm{i}\alpha_0 v = 0\,,\\
&u_t + 2\mathrm{i}q + \mathrm{i}\alpha_0^{-1}\left(1 - 2\ell_0\right) uqr = 0\,, \quad v_t - 2\mathrm{i}r - \mathrm{i}\alpha_0^{-1}\left(1 - 2\ell_0\right) vqr = 0\,,
\end{aligned}
\tag{1}
$$

with the aid of a gauge transformation between the corresponding 2×2 matrix spectral problems, by which some explicit solutions of the coupled massive Thirring system are given, where ℓ_0, α_0, and β_0 are three real constants. As a reduction, a Darboux transformation of the generalized classical massive Thirring model can be written as

$$
\begin{aligned}
&q_x + 2\mathrm{i}\left(\ell_0 |u|^2 + \beta_0\right) q + 2\mathrm{i}\alpha_0 u = 0\,,\\
&u_t + 2\mathrm{i}q + \mathrm{i}\alpha_0^{-1}\left(1 - 2\ell_0\right) u|q|^2 = 0\,,
\end{aligned}
\tag{2}
$$

and its explicit solutions are obtained. A systematic algebraic procedure is given in detail to solve equations (1) and (2).

The present paper is organized as follow. In Section 2, based on the introduced gauge transformation between two 2×2 spectral problems, we derive a Darboux transformation with multi-parameters for the coupled massive Thirring system (1), from which the solutions of the coupled massive Thirring system (1) are reduced to solving a linear algebraic system and two first-order ordinary differential equations. In Section 3, as an application of the Darboux transformation, we obtain various exact solutions of the coupled massive Thirring system (1), such as one-soliton solution, periodic solution, and plane wave solution. In Section 4, we arrive at a Darboux transformation of the generalized classical massive Thirring model (2) by means of the reduction technique. Furthermore, with the help of the Darboux transformation, we obtain one-soliton and two-soliton solutions of the generalized classical massive Thirring model (2).

2. Darboux transformation

In this section, we shall construct a Darboux transformation for the coupled massive Thirring system (1), which can be derive from the compatibility condition

$$U_t - V_x + [U, V] = 0\,,$$

between the 2×2 matrix spectral problem

$$\psi_x = U\psi\,, \qquad U = \mathrm{i}\begin{pmatrix} \lambda - (\ell_0 uv + \beta_0)\lambda u \\ v & -\lambda + (\ell_0 uv + \beta_0) \end{pmatrix}\,, \tag{3}$$

and the auxiliary problem

$$\psi_t = V\psi\,, \qquad V = \mathrm{i}\begin{pmatrix} \alpha_0\lambda^{-1} - \dfrac{1-2\ell_0}{2\alpha_0}qr & q \\ r\lambda^{-1} & -\alpha_0\lambda^{-1} + \dfrac{1-2\ell_0}{2\alpha_0}qr \end{pmatrix}\,, \tag{4}$$

where $\psi = \begin{pmatrix}\psi_1 & \psi_2\end{pmatrix}^T$, $u = u(x,t)$, $v = v(x,t)$, $q = q(x,t)$, $r = r(x,t)$ are four complex potentials with two real independent variables x and t, λ is a constant spectral parameter, and ℓ_0, α_0, and β_0 are three real constants that are unrelated to λ. Especially, Eq. (1) can be reduced to Eq. (2) as $r = q^*$ and $v = u^*$.

In order to derive a Darboux transformation of the coupled massive Thirring system (1), we introduce a gauge transformation of the spectral problems (3) and (4)

$$\hat{\psi} = T\psi\,, \qquad T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\,, \tag{5}$$

with

$$A = A_N\left(\lambda^N + \sum_{k=0}^{N-1} A_k\lambda^k\right)\,, \quad B = A_N\sum_{k=1}^{N} B_k\lambda^k,$$
$$C = \frac{1}{A_N}\sum_{k=0}^{N-1} C_k\lambda^k\,, \qquad D = \frac{1}{A_N}\left(\lambda^N + \sum_{k=0}^{N-1} D_k\lambda^k\right)\,,$$

where A_N will be determined later. Let us assume that $\phi = (\phi_1, \phi_2)^T$, $\varphi = (\varphi_1,\ \varphi_2)^T$ are two basic solutions of the spectral problems (3) and (4), and let us define a linear algebraic system

$$\sum_{k=0}^{N-1} A_k\lambda_j^k + \alpha_j\sum_{k=1}^{N} B_k\lambda_j^k = -\lambda_j^N\,, \qquad \sum_{k=0}^{N-1}\left(C_k + \alpha_j D_k\right)\lambda_j^k = -\alpha_j\lambda_j^N\,, \tag{6}$$

with

$$\alpha_j = \frac{\phi_2\left(\lambda_j\right) - \gamma_j\varphi_2\left(\lambda_j\right)}{\phi_1\left(\lambda_j\right) - \gamma_j\varphi_1\left(\lambda_j\right)}\,, \qquad 1 \le j \le 2N\,, \tag{7}$$

where parameters λ_j ($\lambda_j \neq \lambda_k$ as $k \neq j$) and γ_j are suitably chosen such as the determinant of the coefficients for (6) are non-zero. Then A_k, B_k, C_k, D_k $(1 \leq k \leq N-1)$, A_0, B_N, C_0, D_0 are uniquely determined by (6), and A_N will be given latter (see theorem 1 and theorem 2). It is easy to see from (6) that

$$\left(\lambda_j^N + \sum_{k=0}^{N-1} A_k \lambda_j^k\right)\left(\lambda_j^N + \sum_{k=0}^{N-1} D_k \lambda_j^k\right) = \sum_{k=1}^{N}\sum_{l=0}^{N-1} B_k C_l \lambda_j^{k+l}\,, \tag{8}$$

which implies that λ_j ($1 \leq j \leq 2N$) are $2N$ roots of $2N^{\text{th}}$-order polynomial $\det T$. Hence, $\det T$ can be written as

$$\begin{aligned}\det T &= \left(\lambda^N + \sum_{k=0}^{N-1} A_k \lambda^k\right)\left(\lambda^N + \sum_{k=0}^{N-1} D_k \lambda^k\right) - \sum_{k=1}^{N}\sum_{l=0}^{N-1} B_k C_l \lambda^{k+l} \\ &= \prod_{j=1}^{2N} (\lambda - \lambda_j)\,. \end{aligned} \tag{9}$$

With the help of the gauge transformation (5), the spectral problems (3) and (4) are transformed into the following spectral problems of $\hat{\psi}$:

$$\hat{\psi}_x = \hat{U}\hat{\psi}\,, \qquad \hat{\psi}_t = \hat{V}\hat{\psi}\,, \tag{10}$$

when $\lambda \neq \lambda_j$ ($1 \leq j \leq 2N$), where

$$\hat{U} = (T_x + TU)T^{-1}\,, \qquad \hat{V} = (T_t + TV)T^{-1}\,. \tag{11}$$

It is not difficult to verify that $\lambda = \lambda_j (1 \leq j \leq 2N)$ are removable isolated singularities of $\hat{U}$ and $\hat{V}$. Therefore, we can define $\hat{U}$ and $\hat{V}$ for all λ by analytic continuation. As shown in Ref. [11], a gauge transformation of a spectral problem is called a Darboux transformation of the spectral problem if it transforms the spectral problem into another spectral problem of the same type. In what follows, we are going to prove that the gauge transformation (5) is a Darboux transformation of (3) and (4).

Theorem 1. *Suppose that* $A_k, B_k, C_k, D_k (1 \leq k \leq N-1), A_0, B_N, C_0, D_0$ *are uniquely given by (6), and* A_N *is determined by the following first-order ordinary differential equation*

$$\partial_x \ln A_N = \mathrm{i}\,(1 - 2\ell_0)\,(uC_{N-1} - vB_N - 2B_N C_{N-1})\,. \tag{12}$$

Then the matrix $\hat{U}$ *determined by the first equation of (11) has the same form as* U*, i.e.*

$$\hat{U} = \mathrm{i}\begin{pmatrix} \lambda - (\ell_0 \hat{u}\hat{v} + \beta_0) & \lambda\hat{u} \\ v & -\lambda + (\ell_0 \hat{u}\hat{v} + \beta_0) \end{pmatrix}, \tag{13}$$

where the transformation formulations from the old potentials u, v into new ones are given by

$$\hat{u} = uA_N^2 - 2A_N^2 B_N\,, \qquad \hat{v} = \frac{v}{A_N^2} + \frac{2C_{N-1}}{A_N^2}\,. \tag{14}$$

Transformation (5) and (14): $(\psi, u, v) \to (\hat{\psi}, \hat{u}, \hat{v})$ is a Darboux transformation of the spectral problem (3).

Proof. Let $T^{-1} = T^*/\det T$ and

$$(T_x + TU)T^* = \begin{pmatrix} f_{11}(\lambda) & f_{12}(\lambda) \\ f_{21}(\lambda) & f_{22}(\lambda) \end{pmatrix}, \tag{15}$$

It is obvious that $f_{11}(\lambda)$, $f_{12}(\lambda)$, $\lambda f_{21}(\lambda)$, $f_{22}(\lambda)$ are $(2N+1)$th-order polynomials in λ and the lowest order of $f_{11}(\lambda)$, $f_{21}(\lambda)$, $f_{22}(\lambda)$ is 0, the lowest order of $f_{12}(\lambda)$ is 1. By using (3), (6), and (7), we arrive at

$$\begin{aligned} \alpha_{j,x} &= -\mathrm{i}\lambda_j u\alpha_j^2 - 2\mathrm{i}\left[\lambda_j - (\ell_0 uv + \beta_0)\right]\alpha_j + \mathrm{i}v\,, \\ A_x(\lambda_j) &= -B_x(\lambda_j)\alpha_j - B(\lambda_j)\alpha_{j,x}\,, \\ C_x(\lambda_j) &= -D_x(\lambda_j)\alpha_j - D(\lambda_j)\alpha_{j,x}\,, \qquad (1 \le j \le 2N)\,. \end{aligned} \tag{16}$$

It is easy to see that λ_j ($1 \le j \le 2N$) are roots of $f_{st}(\lambda)$ $(s,\ t = 1,2)$ through (15) and (16). We have

$$(T_x + TU)T^* = (\det T)P(\lambda)\,, \qquad P(\lambda) = \begin{pmatrix} p_{11}^{(1)}\lambda + p_{11}^{(0)} & \lambda p_{12} \\ p_{21} & p_{22}^{(1)}\lambda + p_{22}^{(0)} \end{pmatrix}, \tag{17}$$

where $p_{st}^{(l)}$ and p_{st} $(s,t = 1,2,\ l = 0,1)$ are independent of λ. Therefore, Eq. (15) can be written as

$$T_x + TU = P(\lambda)T. \tag{18}$$

Equating the coefficients of λ^{N+1} and λ^N in (18), we find

$$p_{11}^{(1)} = -p_{22}^{(1)} = \mathrm{i}\,,\ p_{12} = \mathrm{i}A_N^2\,(u - 2B_N) = \mathrm{i}\hat{u}\,,\ p_{21} = \frac{\mathrm{i}\,(v + 2C_{N-1})}{A_N^2} = \mathrm{i}\hat{v}\,, \tag{19}$$

$$p_{11}^{(0)} = \partial_x \ln A_N + \left(\mathrm{i} - p_{11}^{(1)}\right) A_{N-1} + \mathrm{i}vB_N - \frac{p_{12}C_{N-1}}{A_N^2} - \mathrm{i}\,(\ell_0 uv + \beta_0)\,, \tag{20}$$

$$p_{22}^{(0)} = -\partial_x \ln A_N - p_{21}A_N^2 B_N + \mathrm{i}uC_{N-1} - \left(p_{22}^{(1)} + \mathrm{i}\right) D_{N-1} + \mathrm{i}\,(\ell_0 uv + \beta_0)\,. \tag{21}$$

Substituting (19) into (20) and (21) yields

$$p_{11}^{(0)} = -p_{22}^{(0)}\,.$$

Finally, inserting (12) and (19) into (20) and through tedious calculations, one can achieve

$$p_{11}^{(0)} = -p_{22}^{(0)} = -\mathrm{i}\left[\ell_0\left(u-2B_N\right)\left(v+2C_{N-1}\right)+\beta_0\right] = -\mathrm{i}\left(\ell_0\hat{u}\hat{v}+\beta_0\right). \quad (22)$$

□

Theorem 2. *Assume the hypotheses of Theorem 1. Suppose that the time dependence of A_N obey the following first-order ordinary differential equation with respect to the variable t*

$$\partial_t \ln A_N = \frac{\mathrm{i}\left(1-2\ell_0\right)\left(-qA_0C_0+rB_1D_0+2\alpha_0B_1C_0\right)}{A_0D_0}. \quad (23)$$

Then the matrix $\hat{V}$ defined by the second equation of (11) has the same form as V, in which the old potentials q and r are mapped into the new ones $\hat{q}$ and $\hat{r}$ where the transformation formulations are given by

$$\hat{q} = \frac{A_N^2\left(qA_0-2\alpha_0B_1\right)}{D_0}, \qquad \hat{r} = \frac{\left(rD_0+2\alpha_0C_0\right)}{A_0A_N^2}. \quad (24)$$

Transformation (5), (14), and (24): $(\psi,u,v,q,r)\to(\hat{\psi},\hat{u},\hat{v},\hat{q},\hat{r})$ is a Darboux transformation of the two spectral problems (3) and (4).

Proof. Let $T^{-1}=T^*/\det T$ and

$$\left(T_t+TV\right)T^* = \begin{pmatrix} g_{11}(\lambda) & g_{12}(\lambda) \\ g_{21}(\lambda) & g_{22}(\lambda) \end{pmatrix}. \quad (25)$$

It is easy to note that $g_{11}(\lambda),\ g_{12}(\lambda),\ \lambda g_{21}(\lambda),\ g_{22}(\lambda)$ are $2N$th-order polynomials in λ and the lowest order of $g_{11}(\lambda),\ g_{21}(\lambda),\ g_{22}(\lambda)$ is -1, the lowest order of $g_{12}(\lambda)$ is zero. From (4), (6), and (7), we see that

$$\begin{aligned} \alpha_{j,t} &= -\mathrm{i}q\alpha_j^2-\mathrm{i}\left[2\alpha_0^2\lambda_j^{-1}-\left(1-2\ell_0\right)qr\right]\frac{\alpha_j}{\alpha_0}+\mathrm{i}r\lambda_j^{-1}, \quad 1\le j\le 2N, \\ A_t\left(\lambda_j\right) &= -B_t\left(\lambda_j\right)\alpha_j-B(\lambda_j)\alpha_{j,t}, \\ C_t\left(\lambda_j\right) &= -D_t\left(\lambda_j\right)\alpha_j-D\left(\lambda_j\right)\alpha_{j,t}. \end{aligned} \quad (26)$$

It can be verified that λ_j ($1\le j\le 2N$) are roots of $g_{st}(\lambda)$ ($s,\ t=1,2$) by using (25) and (26). Therefore, we get

$$(T_t+TV)T^* = (\det T)Q(\lambda), Q(\lambda) = \begin{pmatrix} q_{11}^{(1)}\lambda^{-1}+q_{11}^{(0)} & q_{12} \\ q_{21}\lambda^{-1} & q_{22}^{(1)}\lambda^{-1}+q_{22}^{(0)} \end{pmatrix},$$

where $q_{st}^{(l)}$ and q_{st} $(s,t=1,2,\ l=0,1)$ are independent of λ. Therefore, Eq. (25) can be written as

$$T_t+TV=Q(\lambda)T\,. \tag{27}$$

By comparing the coefficients of λ^{-1}, λ^0 and λ^N in (27), we find

$$\begin{aligned} q_{11}^{(1)} &= -q_{22}^{(1)}=\mathrm{i}\alpha_0\,, \qquad q_{12}=\frac{\mathrm{i}A_N^2\left(qA_0-2\alpha_0B_1\right)}{D_0}=\mathrm{i}\hat{q}\,,\\ q_{21} &= \frac{\mathrm{i}\left(rD_0+2\alpha_0C_0\right)}{A_0A_N^2}=\mathrm{i}\hat{r}\,, \end{aligned}$$

and

$$q_{11}^{(0)}=-q_{22}^{(0)}=\partial_t\ln A_N-\frac{\mathrm{i}qr\left(1-2\ell_0\right)}{2\alpha_0}\,. \tag{28}$$

By using (23) and (28), we can arrive at

$$q_{11}^{(0)}=-q_{22}^{(0)}=\frac{\mathrm{i}\left(2\ell_0-1\right)\left(qA_0-2\alpha_0B_1\right)\left(rD_0+2\alpha_0C_0\right)}{2\alpha_0A_0D_0}=\frac{2\ell_0-1}{2\alpha_0}\mathrm{i}\hat{q}\hat{r}.$$

□

Based on Theorems 1 and 2, transformations (5), (14) and (24) transform the Lax pair (3) and (4) into another Lax pair of the same type in view of (11). Naturally, both of the Lax pairs lead to the same (1). Therefore, transformation (14) and (24) is also called a Darboux transformation of the coupled massive Thirring system (1). We get immediately the following fact.

Theorem 3. *Every solution (u,v,q,r) of the coupled massive Thirring system (1) is mapped into a new solution $(\hat{u},\hat{v},\hat{q},\hat{r})$ of the coupled massive Thirring system (1) under the Darboux transformation (14) and (24), where A_0,B_1,B_N,C_0,C_{N-1}, and D_0 are given by the linear algebraic system (6), and A_N is given by the two first-order ordinary differential equations (12) and (23) uniquely.*

3. Explicit solutions of the coupled massive Thirring system

In this section, we show the explicit solutions of the coupled massive Thirring system (1) by using the Darboux transformations (14) and (24). Substituting the trivial solution $u=v=q=r=0$ of the coupled massive Thirring system (1) into (3) and (4), two basic solutions $\phi(\lambda)$ and $\varphi(\lambda)$ are chosen as

$$\phi(\lambda)=\begin{pmatrix}\mathrm{e}^{\mathrm{i}x(\lambda-\beta_0)+\frac{\mathrm{i}\alpha_0t}{\lambda}}\\ 0\end{pmatrix},\qquad \varphi(\lambda)=\begin{pmatrix}0\\ -\mathrm{e}^{-\mathrm{i}x(\lambda-\beta_0)-\frac{\mathrm{i}\alpha_0t}{\lambda}}\end{pmatrix}. \tag{29}$$

For $N=1$, according to (6), we can get

$$A_0 = \frac{\lambda_1\lambda_2\left(\gamma_2 e^{\frac{2i\alpha_0 t}{\lambda_1}+2i\lambda_1 x} - \gamma_1 e^{\frac{2i\alpha_0 t}{\lambda_2}+2i\lambda_2 x}\right)}{\left(\lambda_1\gamma_1 e^{\frac{2i\alpha_0 t}{\lambda_2}+2i\lambda_2 x} - \lambda_2\gamma_2 e^{\frac{2i\alpha_0 t}{\lambda_1}+2i\lambda_1 x}\right)},$$

$$B_1 = \frac{(\lambda_2-\lambda_1)\, e^{2i\left[\frac{\alpha_0 t(\lambda_1+\lambda_2)}{\lambda_1\lambda_2}+x(-\beta_0+\lambda_1+\lambda_2)\right]}}{\left(\lambda_1\gamma_1 e^{\frac{2i\alpha_0 t}{\lambda_2}+2i\lambda_2 x} - \lambda_2\gamma_2 e^{\frac{2i\alpha_0 t}{\lambda_1}+2i\lambda_1 x}\right)},$$

$$C_0 = \frac{(\lambda_1-\lambda_2)\,\gamma_1\gamma_2 e^{2i\beta_0 x}}{\left(\gamma_1 e^{\frac{2i\alpha_0 t}{\lambda_2}+2i\lambda_2 x} - \gamma_2 e^{\frac{2i\alpha_0 t}{\lambda_1}+2i\lambda_1 x}\right)},$$

$$D_0 = \frac{\left(\lambda_1\gamma_1 e^{\frac{2i\alpha_0 t}{\lambda_2}+2i\lambda_2 x} - \lambda_2\gamma_2 e^{\frac{2i\alpha_0 t}{\lambda_1}+2i\lambda_1 x}\right)}{\left(\gamma_2 e^{\frac{2i\alpha_0 t}{\lambda_1}+2i\lambda_1 x} - \gamma_1 e^{\frac{2i\alpha_0 t}{\lambda_2}+2i\lambda_2 x}\right)}.$$

With the help of Eqs. (12), (23) and through complex calculations, we can obtain

$$A_1 = \rho_1 \Delta_1^{2\ell_0-1}, \qquad \Delta_1 = \frac{\gamma_1 - \gamma_2 e^{\frac{2i(\lambda_1-\lambda_2)(\lambda_1\lambda_2 x-\alpha_0 t)}{\lambda_1\lambda_2}}}{\gamma_1\lambda_1 - \gamma_2\lambda_2 e^{\frac{2i(\lambda_1-\lambda_2)(\lambda_1\lambda_2 x-\alpha_0 t)}{\lambda_1\lambda_2}}}, \tag{30}$$

where ρ_1 is a complex constant of integration. From Eqs. (14), (24), and (30), one can arrive at explicit solution of the coupled massive Thirring system (1) (see Figs. 1 and 2)

$$\hat{u} = \frac{2(\lambda_1-\lambda_2)\,\rho_1^2 e^{-2i\beta_0 x}\Delta_1^{4\ell_0-2}}{\left(\gamma_1\lambda_1 e^{-\frac{2i\alpha_0 t}{\lambda_1}-2i\lambda_1 x} - \gamma_2\lambda_2 e^{-\frac{2i\alpha_0 t}{\lambda_2}-2i\lambda_2 x}\right)},$$

$$\hat{v} = \frac{2\gamma_1\gamma_2(\lambda_1-\lambda_2)\, e^{2i\beta_0 x}\Delta_1^{2-4\ell_0}}{\left[\rho_1^2\left(\gamma_1 e^{\frac{2i\alpha_0 t}{\lambda_2}+2i\lambda_2 x} - \gamma_2 e^{\frac{2i\alpha_0 t}{\lambda_1}+2i\lambda_1 x}\right)\right]},$$

$$\hat{q} = \frac{2\alpha_0(\lambda_1-\lambda_2)\,\rho_1^2 e^{-2i\beta_0 x}\Delta_1^{4\ell_0-1}}{\left(\lambda_2\gamma_2 e^{-\frac{2i\alpha_0 t}{\lambda_2}-2i\lambda_2 x} - \lambda_1\gamma_1 e^{-\frac{2i\alpha_0 t}{\lambda_1}-2i\lambda_1 x}\right)},$$

$$\hat{r} = \frac{2\alpha_0\gamma_1\gamma_2(\lambda_2-\lambda_1)\, e^{2i\beta_0 x}\Delta_1^{1-4\ell_0}}{\left[\lambda_1\lambda_2\rho_1^2\left(\gamma_1 e^{\frac{2i\alpha_0 t}{\lambda_2}+2i\lambda_2 x} - \gamma_2 e^{\frac{2i\alpha_0 t}{\lambda_1}+2i\lambda_1 x}\right)\right]}. \tag{31}$$

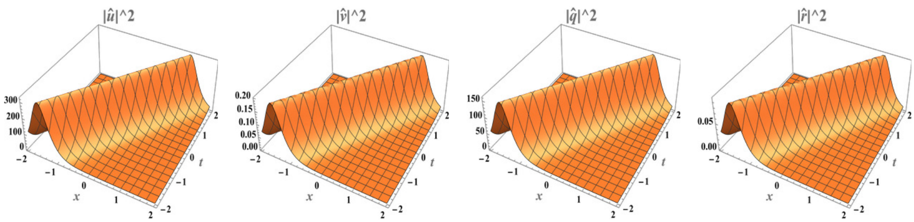

Fig. 1. Single-hump soliton solution (31): $\alpha_0 = 0.7, \beta_0 = 2, \ell_0 = 0.5, \rho_1 = 3, \lambda_1 = \mathrm{i}, \lambda_2 = -\mathrm{i}, \gamma_1 = 2, \gamma_2 = \mathrm{i}$.

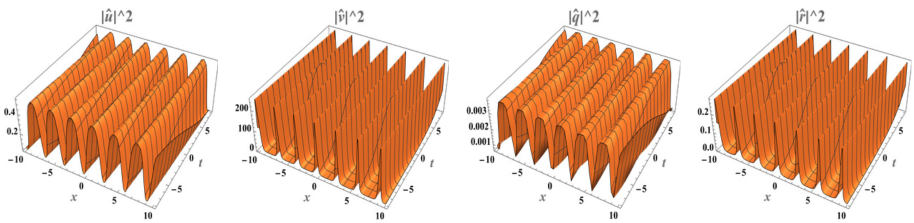

Fig. 2. Periodic soliton solution (31): $\alpha_0 = 0.1, \beta_0 = 1, \ell_0 = -0.6, \rho_1 = -1, \lambda_1 = 1, \lambda_2 = 2, \gamma_1 = 4, \gamma_2 = 1$.

For $N = 2$, let $\lambda_2 = -\lambda_1$, $\lambda_3 = 2\lambda_1$, $\lambda_4 = -2\lambda_1$, $\gamma_3 = -\gamma_1$, and $\gamma_4 = -\gamma_2$ for convenience. Similarly, we can get another explicit solution of the coupled massive Thirring system (1) (see Figs. 3 and 4)

$$
\begin{aligned}
\hat{u} &= -12\rho_2^2 \Delta_2^{4\ell_0-2} \Delta_3^{1-4\ell_0} \mathrm{e}^{\mathrm{i}\left[\frac{\alpha_0 t}{\lambda_1}+2x(\lambda_1-\beta_0)\right]} \left(\mathrm{e}^{\frac{i\alpha_0 t}{\lambda_1}} + \mathrm{e}^{2\mathrm{i}\lambda_1 x}\right) \\
&\quad \times \left(\gamma_1 + \gamma_2 \mathrm{e}^{\frac{3\mathrm{i}\alpha_0 t}{\lambda_1}+6\mathrm{i}\lambda_1 x}\right), \\
\hat{v} &= 12\gamma_1\gamma_2\lambda_1\rho_2^{-2} \Delta_2^{1-4\ell_0} \Delta_3^{4\ell_0-2} \mathrm{e}^{i\left[\frac{\alpha_0 t}{\lambda_1}+2x(\beta_0+\lambda_1)\right]} \\
&\quad \times \left(-\gamma_1 \mathrm{e}^{\frac{i\alpha_0 t}{\lambda_1}} + \gamma_2 \mathrm{e}^{\frac{3\mathrm{i}\alpha_0 t}{\lambda_1}+8\mathrm{i}\lambda_1 x} + 2\gamma_2 \mathrm{e}^{\frac{4\mathrm{i}\alpha_0 t}{\lambda_1}+6\mathrm{i}\lambda_1 x} - 2\gamma_1 \mathrm{e}^{2\mathrm{i}\lambda_1 x}\right), \\
\hat{q} &= -6\alpha_0\rho_2^2\lambda_1^{-1} \Delta_2^{4\ell_0-1} \Delta_3^{-4\ell_0} \mathrm{e}^{\mathrm{i}\left[\frac{\alpha_0 t}{\lambda_1}+2x(\lambda_1-\beta_0)\right]} \\
&\quad \times \left(-2\gamma_1 \mathrm{e}^{\frac{i\alpha_0 t}{\lambda_1}} + 2\gamma_2 \mathrm{e}^{\frac{3\mathrm{i}\alpha_0 t}{\lambda_1}+8\mathrm{i}\lambda_1 x} + \gamma_2 \mathrm{e}^{\frac{4\mathrm{i}\alpha_0 t}{\lambda_1}+6\mathrm{i}\lambda_1 x} - \gamma_1 \mathrm{e}^{2\mathrm{i}\lambda_1 x}\right), \\
\hat{r} &= -12\alpha_0\gamma_1\gamma_2\rho_2^{-2} \Delta_2^{-4\ell_0} \Delta_3^{4\ell_0-1} \mathrm{e}^{\mathrm{i}\left[\frac{\alpha_0 t}{\lambda_1}+2x(\beta_0+\lambda_1)\right]} \left(\mathrm{e}^{\frac{i\alpha_0 t}{\lambda_1}} + \mathrm{e}^{2\mathrm{i}\lambda_1 x}\right) \\
&\quad \times \left(\gamma_1 + \gamma_2 \mathrm{e}^{\frac{3\mathrm{i}\alpha_0 t}{\lambda_1}+6\mathrm{i}\lambda_1 x}\right),
\end{aligned}
\tag{32}
$$

where

$$\begin{aligned}\Delta_2 &= {\gamma_1}^2 - \gamma_1\gamma_2 e^{\frac{2i\alpha_0 t}{\lambda_1}+4i\lambda_1 x}\left(9e^{\frac{2i\alpha_0 t}{\lambda_1}} + 16e^{\frac{i\alpha_0 t}{\lambda_1}+2i\lambda_1 x} + 9e^{4i\lambda_1 x}\right)\\ &\quad + {\gamma_2}^2 e^{\frac{6i\alpha_0 t}{\lambda_1}+12i\lambda_1 x},\\ \Delta_3 &= {\gamma_1}^2 + \gamma_1\gamma_2 e^{\frac{2i\alpha_0 t}{\lambda_1}+4i\lambda_1 x}\left(9e^{\frac{2i\alpha_0 t}{\lambda_1}} + 20e^{\frac{i\alpha_0 t}{\lambda_1}+2i\lambda_1 x} + 9e^{4i\lambda_1 x}\right)\\ &\quad + {\gamma_2}^2 e^{\frac{6i\alpha_0 t}{\lambda_1}+12i\lambda_1 x},\end{aligned}$$

and ρ_2 is a complex constant of integration.

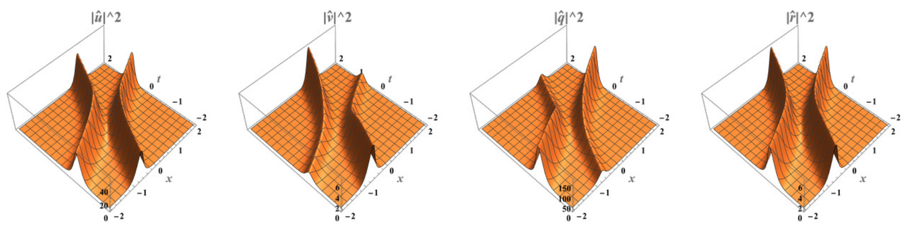

Fig. 3. Double-hump soliton solution (32): $\alpha_0 = 2$, $\beta_0 = 0.9$, $\ell_0 = 0$, $\rho_2 = 1.5$, $\lambda_1 = i$, $\gamma_1 = 1$, $\gamma_2 = i$.

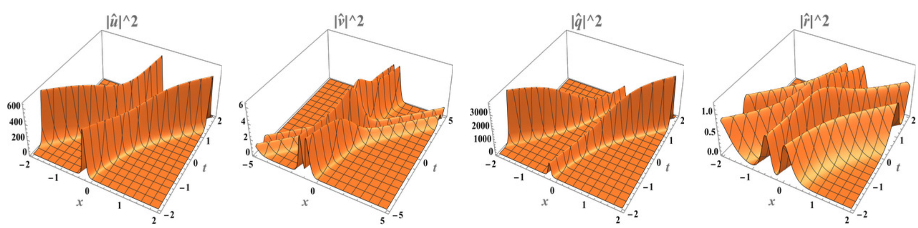

Fig. 4. Solution (32): $\alpha_0 = 1$, $\beta_0 = 0$, $\ell_0 = 1$, $\rho_2 = 1$, $\lambda_1 = i$, $\gamma_1 = 1 + i$, $\gamma_2 = -i$. $\hat{u}$, $\hat{q}$ are two double-hump solitons, $\hat{v}$ is the collision of two double-hump solitons and $\hat{r}$ is four-hump soliton.

4. The reduction of Darboux transformation and explicit solutions

In this section, we will derive a Darboux transformation of the generalized classical massive Thirring model (2) through the reduction technique and its explicit solutions. We choose the two solutions of (3) and (4)

$$\phi(\lambda) = (\phi_1(\lambda),\ \phi_2(\lambda))^T\,, \qquad \varphi(\lambda) = (\lambda\phi_2^*(\lambda^*)\,, -\phi_1^*(\lambda^*))^T\,, \tag{33}$$

when $v = u^*$, $r = q^*$. Furthermore, suppose that

$$\lambda_{2j} = \lambda^*_{2j-1}\,, \qquad \gamma_{2j} = -\lambda^{*-1}_{2j-1}\gamma^{*-1}_{2j-1}\,, \qquad 1 \le j \le N\,. \tag{34}$$

It is easy to prove that $\alpha_{2j} = -\lambda^{*-1}_{2j-1}\alpha^{*-1}_{2j-1}$, $C^*_k = -B_{k+1}$, $D^*_k = A_k$, $(1 \le j \le N,\ 0 \le k \le N-1)$. Here, α_{2j-1} can be expressed as

$$\alpha_{2j-1} = \frac{\phi_2\left(\lambda_{2j-1}\right) + \gamma_{2j-1}\phi^*_1\left(\lambda^*_{2j-1}\right)}{\phi_1\left(\lambda_{2j-1}\right) - \gamma_{2j-1}\lambda_{2j-1}\phi^*_2\left(\lambda^*_{2j-1}\right)}\,. \tag{35}$$

Functions A_k and B_{k+1} $(0 \le K \le N-1)$ are determined by the following linear algebraic system:

$$\begin{aligned} &\sum_{k=0}^{N-1} A_k\lambda^k_{2j-1} + \alpha_{2j-1}\sum_{k=1}^{N} B_k\lambda^k_{2j-1} = -\lambda^N_{2j-1}\,, \\ &\sum_{k=0}^{N-1}\left(\alpha^*_{2j-1}A_k - B_{k+1}\right)\lambda^{*k}_{2j-1} = -\alpha^*_{2j-1}\lambda^{*N}_{2j-1}\,, \qquad (1 \le j \le N)\,. \end{aligned} \tag{36}$$

Therefore, we obtain the following assertion.

Theorem 4. *Assume that (u, q) is a solution of the generalized classical massive Thirring model (2). Suppose that A_N be a solution of the two first-order ordinary differential equations*

$$\begin{aligned} \partial_x \ln A_N &= 2\mathrm{i}\left(1 - 2\ell_0\right)\left[-\mathrm{Re}\left(u^*B_N\right) + |B_N|^2\right]\,, \\ \partial_t \ln A_N &= \frac{2\mathrm{i}\left(1 - 2\ell_0\right)\left[\mathrm{Re}\left(qA_0B^*_1\right) - \alpha_0|B_1|^2\right]}{|A_0|^2}\,, \end{aligned} \tag{37}$$

and $|A_N| = 1$, where Re *stands for real part. Then $(\hat{u}, \hat{q})$ determined by the Darboux transformation*

$$\hat{u} = uA^2_N - 2A^2_N B_N\,, \qquad \hat{q} = \frac{A^2_N\left(qA_0 - 2\alpha_0B_1\right)}{A^*_0}\,, \tag{38}$$

is a new solution of the generalized classical massive Thirring model (2).

In fact, $|A_N| = 1$ implies that the constant of integration for $\ln A_N$ is selected to be purely imaginary and $A^*_N = A^{-1}_N$. Hence, we can arrive at

$$\begin{aligned} \hat{v} &= \frac{v}{A^2_N} + \frac{2C_{N-1}}{A^2_N} = u^*A^{*2}_N - 2A^{*2}_N B^*_N = \left(uA^2_N - 2A^2_N B_N\right)^* = \hat{u}^*\,, \\ \hat{r} &= \frac{\left(rD_0 + 2\alpha_0C_0\right)}{A_0A^2_N} = \frac{A^{*2}_N\left(q^*A^*_0 - 2\alpha_0B^*_1\right)}{D^*_0} = \left[\frac{A^2_N\left(qA_0 - 2\alpha_0B_1\right)}{D_0}\right]^* = \hat{q}^*\,. \end{aligned}$$

This means that the result of Theorem 4 holds.

In the following, we shall apply the Darboux transformation to give explicit solutions of the generalized classical massive Thirring model (2). Substituting the trivial solution $u = q = 0$ of (2) into (3) and (4) when $v = u^*$ and $r = q^*$, we still select the basic solutions (29), which also satisfies (33). Therefore, (35) and (37) can be written as

$$\alpha_{2j-1} = \gamma_{2j-1}\mathrm{e}^{-2\mathrm{i}x(\lambda_{2j-1}-\beta_0)-\frac{2\mathrm{i}\alpha_0 t}{\lambda_{2j-1}}}, \tag{39}$$

$$\partial_x \ln A_N = 2i\left(1-2\ell_0\right)|B_N|^2, \qquad \partial_t \ln A_N = -\frac{2i\alpha_0\left(1-2\ell_0\right)|B_1|^2}{|A_0|^2}. \tag{40}$$

For $N = 1$, $\gamma_1 = 1$ and $\lambda_1 = \mathrm{i}$, we obtain from the linear algebraic system (36) that

$$A_0 = \frac{\mathrm{i}\mathrm{e}^{4\alpha_0 t+2\mathrm{i}\beta_0 x} + e^{2(i\beta_0+2)x}}{-\mathrm{e}^{4\alpha_0 t+2\mathrm{i}\beta_0 x} - \mathrm{i}\mathrm{e}^{2(i\beta_0+2)x}}, \qquad B_1 = \frac{2\mathrm{e}^{2\alpha_0 t+2x}}{\mathrm{i}\mathrm{e}^{4\alpha_0 t+2\mathrm{i}\beta_0 x} - \mathrm{e}^{2(\mathrm{i}\beta_0+2)x}}, \tag{41}$$

which together with system (40) implies that

$$A_1 = \mathrm{e}^{\mathrm{i}\rho_3+2\mathrm{i}(1-2\ell_0)\arctan\left(\mathrm{e}^{4x-4\alpha_0 t}\right)}, \tag{42}$$

where ρ_3 is a real constant of integration. One-soliton solution (Fig. 5) of the generalized classical massive Thirring model (2) is obtained with the help of the Darboux transformation (38)

$$\hat{u} = \frac{4\sigma_1}{\mathrm{e}^{4x} - \mathrm{i}\mathrm{e}^{4\alpha_0 t}}, \qquad \hat{q} = \frac{4\alpha_0\left[\mathrm{e}^{4\alpha_0 t+2\mathrm{i}(\beta_0+\mathrm{i})x} - \mathrm{i}\mathrm{e}^{2x+2\mathrm{i}\beta_0 x}\right]\sigma_1}{\left(\mathrm{e}^{4\alpha_0 t} + \mathrm{i}\mathrm{e}^{4x}\right)\left[\mathrm{e}^{4\alpha_0 t+2\mathrm{i}(\beta_0+i)x} + \mathrm{i}\mathrm{e}^{2x+2\mathrm{i}\beta_0 x}\right]}, \tag{43}$$

where $\sigma_1 = \mathrm{e}^{2\left[\mathrm{i}\rho_3+\alpha_0 t+2\mathrm{i}(1-2\ell_0)\arctan\left(\mathrm{e}^{4x-4\alpha_0 t}\right)-\mathrm{i}\beta_0 x+x\right]}$.

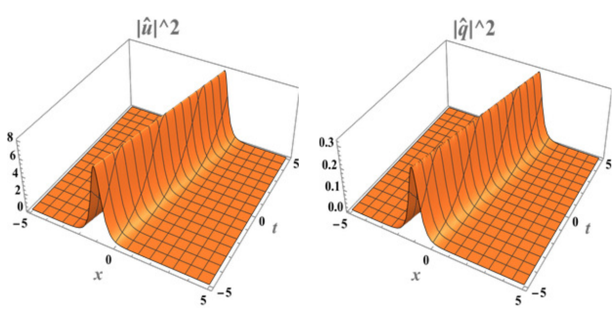

Fig. 5. One-soliton solution (43): $\alpha_0 = 0.2$, $\beta_0 = 0.4$, $\rho_3 = 0.3$, $\ell_0 = 5$.

For $N = 2$, $\gamma_1 = \gamma_3 = 1$, $\lambda_1 = \mathrm{i}$, and $\lambda_3 = 2i$, one can arrive at a two-soliton solution (Fig. 6) of the generalized classical massive Thirring model (2) with the aid of the Darboux transformation (38)

$$\begin{aligned} \hat{u} &= 12\left(-\mathrm{e}^{3\alpha_0 t} - 2\mathrm{i}\mathrm{e}^{\alpha_0 t+8x} + 2\mathrm{e}^{4\alpha_0 t+2x} + 2\mathrm{i}\mathrm{e}^{6x}\right)\sigma_2/\sigma_3\,, \\ \hat{q} &= 12\mathrm{i}\alpha_0\left(2\mathrm{i}\mathrm{e}^{\alpha_0 t+8x} - \mathrm{e}^{3\alpha_0 t} + \mathrm{e}^{4\alpha_0 t+2x} - \mathrm{i}\mathrm{e}^{6x}\right)\sigma_2\sigma_3^*/\sigma_3^2\,, \end{aligned} \tag{44}$$

where

$$\begin{aligned} \sigma_2 &= \mathrm{e}^{2\mathrm{i}\rho_4+\alpha_0 t+4\mathrm{i}(1-2\ell_0)\arctan\left[\frac{3\mathrm{e}^{2\alpha_0 t+4x}\left(-8\mathrm{e}^{\alpha_0 t+2x}+6\mathrm{e}^{2\alpha_0 t+4x}+3\right)}{\mathrm{e}^{6\alpha_0 t}-2\mathrm{e}^{12x}}\right]-2\mathrm{i}\beta_0 x+2x}\,, \\ \sigma_3 &= -\mathrm{i}\mathrm{e}^{6\alpha_0 t} + 9\mathrm{e}^{2\alpha_0 t+4x} - 24\mathrm{e}^{3\alpha_0 t+6x} + 18\mathrm{e}^{4\alpha_0 t+8x} + 2\mathrm{i}\mathrm{e}^{12x}\,, \end{aligned}$$

and ρ_4 is a real constant of integration.

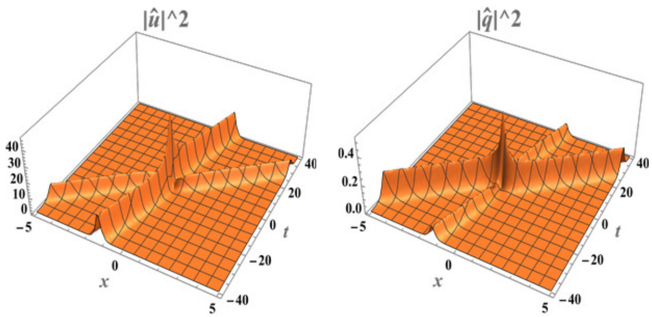

Fig. 6. Two-soliton solution (44): $\alpha_0 = 0.15$, $\beta_0 = 0.3$, $\rho_4 = 0.6$, $\ell_0 = 3$.

For $N = 2$, $\gamma_1 = \mathrm{i}$, $\gamma_3 = 2$, $\lambda_1 = \mathrm{i}$, and $\lambda_3 = 2\mathrm{i}$, one can arrive at a double-hump soliton solution (Fig. 7) of the generalized classical massive Thirring model (2):

$$\begin{aligned} \hat{u} &= 12\mathrm{i}\left(\mathrm{i}\mathrm{e}^{3\alpha_0 t} - 8\mathrm{e}^{\alpha_0 t+8x} + 4\mathrm{e}^{4\alpha_0 t+2x} + 4\mathrm{i}\mathrm{e}^{6x}\right)\sigma_4/\sigma_5\,, \\ \hat{q} &= 12\alpha_0\left(\mathrm{i}\mathrm{e}^{3\alpha_0 t} + 8\mathrm{e}^{\alpha_0 t+8x} + 2\mathrm{e}^{4\alpha_0 t+2x} - 2\mathrm{i}\mathrm{e}^{6x}\right)\sigma_4\sigma_5^*/\sigma_5^2\,, \end{aligned} \tag{45}$$

where

$$\begin{aligned} \sigma_4 &= \mathrm{e}^{2\mathrm{i}\rho_5+\alpha_0 t+4\mathrm{i}(1-2\ell_0)\arctan\left[\frac{9\mathrm{e}^{2\alpha_0 t+4x}\left(8\mathrm{e}^{2\alpha_0 t+4x}+1\right)}{\mathrm{e}^{6\alpha_0 t}-16\mathrm{e}^{3\alpha_0 t+6x}-8\mathrm{e}^{12x}}\right]-2\mathrm{i}\beta_0 x+2x}\,, \\ \sigma_5 &= \mathrm{e}^{6\alpha_0 t} + 9\mathrm{i}\mathrm{e}^{2\alpha_0 t+4x} - 16\mathrm{e}^{3\alpha_0 t+6x} + 72\mathrm{i}\mathrm{e}^{4\alpha_0 t+8x} - 8\mathrm{e}^{12x}\,, \end{aligned}$$

and ρ_5 is a real constant of integration.

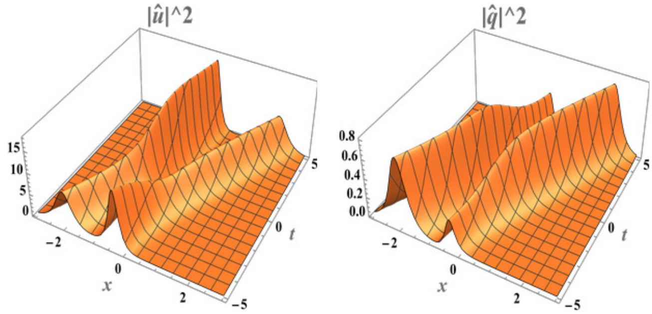

Fig. 7. Double-hump soliton solution (45): $\alpha_0 = 0.3,\ \beta_0 = 0,\ \rho_5 = 0,\ \ell_0 = 4.5$.

For $N = 2$, $\gamma_1 = \gamma_3 = 1$, $\lambda_1 = 1 + \mathrm{i}$, $\lambda_3 = -1 + \mathrm{i}$, and $\ell_0 = 0.5$, one can arrive at a breather solution (Fig. 8) of the generalized classical massive Thirring model (2)

$$\begin{aligned}\hat{u} &= (4+4\mathrm{i})\sigma_7\left[\mathrm{i}\mathrm{e}^{2\alpha_0 t}-(1+\mathrm{i})\mathrm{e}^{2\mathrm{i}\alpha_0 t+(4+4\mathrm{i})x}+\mathrm{e}^{(2+2\mathrm{i})\alpha_0 t+4\mathrm{i}x}+(1+\mathrm{i})\mathrm{e}^{4x}\right]\Big/\sigma_6\,,\\ \hat{q} &= 4\alpha_0\sigma_7\left[(1-\mathrm{i})\mathrm{e}^{4x}-\mathrm{i}\mathrm{e}^{2\alpha_0 t}-(1+\mathrm{i})\mathrm{e}^{2\mathrm{i}\alpha_0 t+(4+4\mathrm{i})x}+\mathrm{i}\mathrm{e}^{(2+2\mathrm{i})\alpha_0 t+4\mathrm{i}x}\right]\\ &\quad\times\left[\sigma_6+2\mathrm{e}^{2\mathrm{i}(\alpha_0 t+2x)}\left(2\mathrm{e}^{8x}-\mathrm{e}^{4\alpha_0 t}\right)\right]\Big/\sigma_6^2\,, \end{aligned} \tag{46}$$

where

$$\begin{aligned}\sigma_6 &= (1-\mathrm{i})\mathrm{e}^{(2+4\mathrm{i})(\alpha_0 t+2x)}-(1+\mathrm{i})\mathrm{e}^{2\alpha_0 t+4x}-2e^{2\mathrm{i}\alpha_0 t+(8+4\mathrm{i})x}+\mathrm{e}^{(4+2\mathrm{i})\alpha_0 t+4\mathrm{i}x}\\ &\quad+4\mathrm{i}\mathrm{e}^{(2+2\mathrm{i})(\alpha_0 t+2x)}\,,\\ \sigma_7 &= \mathrm{e}^{2\mathrm{i}\rho_6+(1+\mathrm{i})\alpha_0 t-2\mathrm{i}\beta_0 x+(2+2\mathrm{i})x}\,,\end{aligned}$$

and ρ_6 is a real constant of integration.

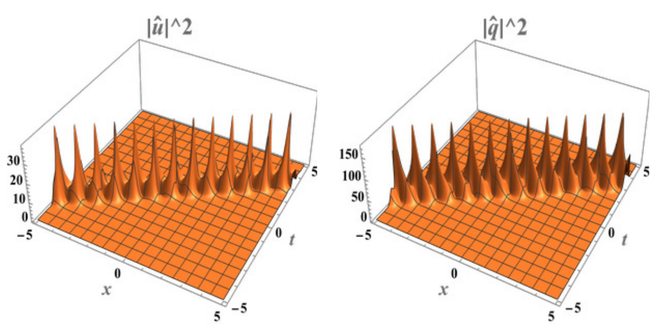

Fig. 8. Breather solution (46): $\alpha_0 = 3,\ \beta_0 = 2,\ \rho_6 = 1.5$.

5. Conclusion

In this study, we have constructed a Darboux transformation for the coupled massive Thirring system (1) by means of the zero-curvature equation and the polynomial expansion of the spectral parameter. On this basis, a DT for its reduction (2) have been obtained by resorting to the reduction technique. Furthermore, we have displayed their soliton, breather, and so on.

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