ON THE CALCULATION OF INVARIANT TENSORS IN GAUGE THEORIES*

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We present an efficient method for finding the independent invariant tensors of a gauge theory. Our method uses a theorem relating invariant tensors and D-flat directions in field space. We apply our method to several examples — SO(3) with symmetric tensors, SU(2) with a dimension-4 representation, and SU(3) with matter in the sextet — and find the set of independent invariant tensors in these theories.

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1. Introduction

A gauge theory is specified by the gauge group and the representation of the matter fields under the gauge group, but all observables, including the physical spectrum, are gauge-invariant combinations of fields. The structure of these objects is found by contracting the gauge-covariant fields with *invariant tensors* to form invariant objects. While there are an infinite number of invariants, there is expected to be a basis set of invariants such that all other invariants can be generated from these basis invariants; that is to say, any invariant should be expressible as a linear combination of products of members of this basis set. These motivate us to understand the basis set of invariant tensors for a general gauge theory.

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Invariant tensors are known for the fundamental representations of the classical groups[1]. However, the tensors for many other representations are not classified. For many groups such as E_6, even the full set of tensors for the fundamental representation has not been found [13].

A brief example will suffice to show the kinds of difficulties that may occur. It is known that in the group SO(3), the invariant tensors are δ^{ij}, ϵ^{ijk}. These tell us that in a theory where all fields V_i^I (we are using lowercase letters for the gauge representation, and uppercase to label the fields — a flavor index) are in the fundamental representation, the complete set of invariant polynomials is generated by $V_i^I V_j^J \delta^{ij}$ and $V_i^I V_j^J V_k^K \epsilon^{ijk}$. But in a theory with SO(3) gauge symmetry and with fields V_{ij}^I in the symmetric tensor representation, one can produce an infinite set of invariants by contracting an arbitrarily long sequence $V_{ij}^{I_1} V_{jk}^{I_2} \ldots V_{li}^{I_M}$ (and there exist further invariants involving epsilon tensors). Only a small set of these is independent, but finding these is nontrivial. For a small number of fields, the method of Hilbert series (*e.g.* [14–26]) can help, but this becomes difficult to use when there are a large number of fields.

In this paper, we present a new approach to finding a minimal set of invariants for more complicated gauge theories. Our approach will be to use the connection between invariant tensors and D-flat directions in field space, which was originally described for supersymmetric field theories (specifically in the context of dualities in these theories [27–30].) This theorem asserts that the independent gauge-invariant polynomial invariants of the theory are in 1–1 correspondence with the orbits of constant field configurations where configurations differing by a complex gauge transformation (*i.e.* gauge transformations where the parameters are complexified) are identified [32–37]. It is clear that at any point on configuration space, we can calculate the value of any gauge-invariant combination of the fields. The theorem states that this can be reversed; a knowledge of the values of all the independent gauge-invariant polynomials is sufficient to reconstruct the orbits of constant field configurations quotiented by complex gauge transformations. That is, if the gauge-fixed configuration is described by parameters $(a_1, a_2 \ldots a_n)$, then the statement of the theorem is that not only every invariant is a polynomial in these parameters, but also, every parameter can be expressed as a polynomial in the invariants.

This theorem is often used in simple theories to characterize the field space in terms of the known operators. Here, we will reverse the implication, and use the field space to find a complete set of gauge-invariant objects in various theories.

[1] For some of the mathematical literature relevant to invariant theory see, [1–12].

The procedure for finding the tensors is then as follows. We take a set of fields in the relevant representation, and set each component to an arbitrary constant value. We then use a complex gauge transformation to set some field components to zero. If the gauge transformations are completely fixed by this procedure, then the nonzero components parametrize the orbits, and we must find a set of invariants such that each of these parameters can be written as a combination of invariants. Such a set of invariants would then be a basis set of invariants for the theory.

In practice, we find that the full gauge symmetry is not easy to fix with a single field. In each case, we find a remnant discrete symmetry, and occasionally, a larger continuous symmetry. One possibility is to use further fields to completely gauge-fix the symmetry, but in each case we analyze below, we find that the residual symmetry is simple enough that we can find the complete set of combinations which are invariant under the residual symmetry. We will refer to these below as "gauge-fixed" invariants. These gauge-fixed invariants are also in 1–1 correspondence with the parameters of the configurations, and hence they are in 1–1 correspondence with the full set of basis invariants. We can therefore search for a set of invariants under the full group such that each of the gauge-fixed invariants can be written as a combination of invariants. Such a set of invariants would then be a basis set of invariants for the theory.

We now show the practicality of this approach by explicitly finding the basis set of invariants for three gauge theories — SO(3) with symmetric tensor matter, SU(2) with matter in the dimension-4 representation, and SU(3) with matter in the sextet. To our knowledge, the last two are completely new analyses (the first case has been analyzed previously in [31]).

2. SU(2) with fields in the dimension-4 representation

2.1. Overview

We will take as our first example a theory with a gauge group SU(2) and a field content where there are N fields in a representation of dimension 4; this is the simplest case for which the independent set of invariants has (to our knowledge) not been worked out.

The fields can be represented as three-index completely symmetric tensors V^I_{abc} where $a, b, c = 1, 2$ are acted on by the gauge symmetry, and $I = 1 \ldots N$ labels the different fields (we shall consistently use lowercase indices for gauge indices and uppercase indices to label the different fields, similar to a flavor index). The fields can also be represented as a column vector with four elements

$$V^I = \begin{pmatrix} V^I_{111} \\ V^I_{112} \\ V^I_{122} \\ V^I_{222} \end{pmatrix} . \tag{1}$$

The invariant tensor is ϵ^{ab}, but one can write an infinite set of invariants, and it is hard to find relations between them. We, therefore, find the gauge-fixed configuration space, and attempt to characterize this space by invariants.

2.2. Counting candidate invariants

We first perform a preliminary counting to establish some candidate choices for the invariants. The invariants can be classified by the symmetry of the fields (more specifically by the flavor symmetry). For each candidate flavor symmetry, we use the program LiE [38] to find the number of invariants with that symmetry. We then subtract any invariants that can be written as a product of smaller invariants.

For instance, if we consider two fields, they can be the symmetric combination $\ydiagram{2}$ or the antisymmetric combination $\ydiagram{1,1}$. From LiE, we find that there is one invariant with the symmetry $\ydiagram{1,1}$ and none with $\ydiagram{2}$.

Proceeding further, we find from LiE that there is one invariant with four antisymmetrized fields. However, we also have the product of two of the invariants $\ydiagram{1,1}$, which could have the same flavor symmetry. Subtracting this, we find there are actually no invariants with four antisymmetrized fields.

We proceed with this process. For each candidate flavor symmetry, in the first column (shown as a list of the number of boxes per row), we use the program LiE to find the number of invariants (shown in the second column) with that symmetry. We then subtract any invariants that can be written as a product of smaller invariants; these are shown in the following column of the table. The net number of independent invariants after subtraction is shown in the last column.

Tableau	# Invts	Subtract	Net
(2)	0	0	0
(1,1)	1	0	1
(3)	0	0	0
(2,1)	0	0	0
(1,1,1)	0	0	0
(4)	1	0	1
(3,1)	0	0	0
(2,2)	1	$(1,1) \otimes (1,1)$	0
(2,1,1)	0	0	0
(1,1,1,1)	1	$(1,1) \otimes (1,1)$	0
(5)	0	0	0
(4,1)	0	0	0
(3,2)	0	0	0
(3,1,1)	0	0	0
(2,2,1)	0	0	0
(2,1,1,1)	0	0	0
(1,1,1,1,1)	0	0	0
(6)	0	0	0
(5,1)	1	$(4) \otimes (1,1)$	0
(4,2)	0	0	0
(3,3)	2	$(1,1) \otimes (1,1) \otimes (1,1)$	1
(4,1,1)	1	$(4) \otimes (1,1)$	0
(3,2,1)	0	0	0
(2,2,2)	0	0	0
(3,1,1,1)	0	0	0
(2,2,1,1)	1	$(1,1) \otimes (1,1) \otimes (1,1)$	0
(2,1,1,1,1)	0	0	0
(1,1,1,1,1,1)	0	0	0

We have also verified that no new invariants of degree 8 exist.

We, therefore, find that to six fields, we have three candidate invariants, of symmetry (1,1), (4), (3,3).

2.3. Construction of invariants

Even though all invariants of eight fields canceled above, this cannot prove that we have achieved a complete count. To have confidence in our result, we apply the theorem from the introduction, by gauge fixing some fields.

We begin by considering a single field V^1. By a complex gauge transformation, one can set the second and third components to zero, and set the first component to 1. The field then has the form

$$V^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ d \end{pmatrix} . \tag{2}$$

This breaks the continuous gauge symmetry but preserves a discrete symmetry, which can be understood as follows: if we interchange every gauge 1 index with a 2 index, this is equivalent to taking $\epsilon^{ab} \to -\epsilon^{ab}$. Then any invariant with $4n+2$ fields will pick up a minus sign, while any invariant with $4n$ fields is unchanged. This then indicates that the combined transformation of interchanging every 1 index with a 2 index, and multiplying every field by an overall factor of i should be a symmetry (this is, in fact, the gauge symmetry corresponding to a rotation by π around the x-axis).

Under this symmetry, we have

$$V^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ d \end{pmatrix} \to \begin{pmatrix} id \\ 0 \\ 0 \\ i \end{pmatrix} . \tag{3}$$

We can further use a gauge transformation by e^{iL_3} (*i.e.* a gauge transform by the L_3 subgroup of SU(2)) to transform

$$V^I = \begin{pmatrix} V^I_{111} \\ V^I_{112} \\ V^I_{122} \\ V^I_{222} \end{pmatrix} \to \begin{pmatrix} \mathrm{e}^{3i\alpha} V^I_{111} \\ \mathrm{e}^{i\alpha} V^I_{112} \\ \mathrm{e}^{-i\alpha} V^I_{122} \\ \mathrm{e}^{-3i\alpha} V^I_{222} \end{pmatrix} . \tag{4}$$

A suitable choice of complex α allows us to bring

$$V^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ d \end{pmatrix} \to \begin{pmatrix} id \\ 0 \\ 0 \\ i \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \\ 0 \\ -d \end{pmatrix} . \tag{5}$$

We have therefore produced a configuration of the form (2) but with a change in the sign of d. This implies that the sign of d can be changed by a gauge transformation. The gauge-invariant combination is $(d)^2$.

As explained in the introduction, this parameter $(d)^2$ should be expressible as a polynomial in the invariants. We, therefore, look for an SU(2) invariant tensor such that gauge fixing the field to be of the form (2) allows us to deduce the value of $(d)^2$.

There is no nonzero bilinear invariant involving V^1 alone. We can however find an invariant of degree 4 in V^1 as follows: we first construct a symmetric combination of two fields

$$W^{IJ}_{ab} = \left(V^I\right)_{acd}\left(V^J\right)_b{}^{cd} + \left(V^I\right)_{bcd}\left(V^J\right)_a{}^{cd} \tag{6}$$

(as always in SU(2), indices are raised and lowered by the epsilon tensor). We can then construct the invariant

$$I_4^{IJKL} = W^{IJ}_{ab} W^{KLab} \,. \tag{7}$$

We now evaluate

$$I_4^{1111} = -8(d)^2 \,. \tag{8}$$

The knowledge of the invariant I_4^{1111} is therefore sufficient to deduce the value of d^2, and therefore is sufficient to completely parametrize the gauge-inequivalent configurations of a single field. By the theorem cited in the introduction, I_4^{IJKL} is a basis set of invariants for a single field in the dimension-4 representation of SU(2).

Note that this agrees with the counting above, as there is only one completely symmetric invariant.

We now consider multiple fields. These can be brought by a complex gauge transformation to the form

$$V^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ d^1 \end{pmatrix}, \qquad V^I = \begin{pmatrix} a^I \\ b^I \\ c^I \\ d^I \end{pmatrix} \quad \text{for} \quad I > 1 \,. \tag{9}$$

Exactly as above, this parametrization breaks the gauge symmetry but preserves a discrete Z_2 gauge symmetry. Under this symmetry, we have

$$V^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ d^1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ -d^1 \end{pmatrix} . \tag{10}$$

This breaks the Z_2, but preserves a subgroup under which each 1 index gets a factor ω, and each 2 index receives a factor ω^2, where ω is a cube root of 1. This corresponds to a map $a^I \to a^I$, $d^I \to d^I$, $b^I \to \omega b^I$, $c^I \to \omega^2 c^I$. The invariants are then a^I, d^I, $b^I c^J$, $c^I c^J c^K$, $b^I b^J b^K$.

Under the broken Z_2, these combinations are acted on as

$$\begin{aligned} d^1 &\to -d^1\,, & a^I\left(d^1\right) &\leftrightarrow -d^I\,, \\ b^I c^J &\to -b^J c^I\,, & b^I b^J b^K d^1 &\leftrightarrow c^I c^J c^K\,. \end{aligned}$$

The gauge-invariant even combinations are

$$\begin{aligned} i_1^I &= a^I\left(d^1\right) - d^I\,, \\ i_2^{IJ} &= b^I c^J - b^J c^I\,, \\ i_3^{IJK} &= b^I b^J b^K d^1 + c^I c^J c^K\,. \end{aligned} \tag{11}$$

A product of two combinations odd under the Z_2 is even under the Z_2. We can therefore generate a new set of gauge-invariant even combinations

$$\begin{aligned} i_4^I &= d^1\left(a^I d^1 + d^I\right)\,, \\ i_5^{IJ} &= d^1\left(b^I c^J + b^J c^I\right)\,, \\ i_6^{IJK} &= d^1\left(b^I b^J b^K d^1 - c^I c^J c^K\right)\,. \end{aligned} \tag{12}$$

The combinations $i_{1\ldots6}$ are the complete set of invariants under the residual U(1) group; by virtue of the theorem cited in the introduction, they are in 1–1 correspondence with the space of gauge-inequivalent configurations.

Now the set of SU(2) invariants is also in 1–1 correspondence with the gauge-inequivalent configurations. This implies that the SU(2) invariants are in 1–1 with the $i_{1\ldots6}$ found above. In particular, we should be able to find the value of each of the $i_{1\ldots6}$ as functions of the complete set of SU(2) invariants. We, therefore, look for SU(2) invariants that can reproduce $i_{1\ldots6}$; that is, we look for SU(2) invariant polynomials in the fields, such that when these are evaluated on the gauge-fixed configuration (9), their values are sufficient to reconstruct combinations (11), (12).

The flavor symmetry is a guide. For instance, the combinations with one free index *i.e.* i_1^I, i_4^I should be reproduced from operators with one free index. One such operator is provided by the operator I_4^{IJKL}, where three indices are replaced by 1. Another operator that we can consider is the antisymmetric bilinear

$$I_2^{IJ} = V^I_{abc} V^{Jabc}\,. \tag{13}$$

Indeed, we find that on configuration (9),

$$I_2^{I1} = i_1^I\,, \qquad I_4^{111I} = 4 i_4^I\,. \tag{14}$$

Hence, a knowledge of invariants (7), (13) indeed allows us to reproduce the combinations with one free flavor index i_1^I, i_4^I.

The combinations with two free flavor indices are i_2^{IJ}, i_5^{IJ}. We find that on configuration (9),

$$I_2^{IJ} = -3i_2^{IJ} + \dots\,, \qquad I_4^{11IJ} = -4i_5^{IJ} + \dots\,, \tag{15}$$

where the ellipses represent products of invariants with fewer indices.

We have two combinations with three flavor indices *i.e.* i_3^{IJK}, i_6^{IJK}. One can be reproduced from I_4^{1IJK}, but we need another invariant. We, therefore, consider

$$I_6^{IJKLMN} = W^{IJ}_{ab} W^{KL}_{cd} V^{Mab}{}_{e} V^{Necd}\,. \tag{16}$$

We find on configuration (9)

$$I_4^{1IJK} = -8i_3^{IJK} + \dots\,, \tag{17}$$

$$I_6^{111IJK} = -8i_6^{IJK} + \dots\,, \tag{18}$$

where we have omitted terms which are composed of products of invariants of lower degree.

We find then that the invariants

$$I_2^{IJ}\,, \qquad I_4^{11IJ}\,, \qquad I_6^{IJKLMN} \tag{19}$$

are sufficient to reconstruct the gauge-invariant parameter space of this theory. The theorem from the introduction then tells us that these are a complete set of independent polynomial invariants for the dimension-4 representation of SU(2) (that is, none of the invariants in this set can be expressed as a polynomial in the others, but any other invariant is a polynomial in these invariants). This also agrees with the counting of the previous subsection.

3. SO(3) with fields in the dimension-5 representation

We will take as our next example a theory with a gauge group SO(3) and a field content where there are N fields that are in a representation of dimension 5; such a field is a traceless symmetric tensor V_{ij}^I of SO(3), where we take $i, j = 1 \dots 3$, and $I = 1 \dots N$ is a flavor index labeling the different fields. The field can therefore be written as

$$V^I = \begin{pmatrix} V_{11}^I & V_{12}^I & V_{13}^I \\ V_{12}^I & V_{22}^I & V_{23}^I \\ V_{13}^I & V_{23}^I & V_{33}^I \end{pmatrix} \tag{20}$$

with $V_{11}^I + V_{22}^I + V_{33}^I = 0$.

It will prove convenient to define a product

$$(A \cdot B)_{ij} = A_{ik} B^{k}_{\ j} \tag{21}$$

as well as a trace

$$\mathrm{Tr}(A) = A_{ij} \delta^{ij} \,. \tag{22}$$

We may then write a sequence of invariants

$$\begin{aligned} I_2^{IJ} &= \mathrm{Tr}\left(V^I \cdot V^J\right) \,, \\ I_3^{IJK} &= \mathrm{Tr}\left(V^I \cdot V^J \cdot V^K\right) \,, \\ I_4^{IJKL} &= \mathrm{Tr}\left(V^I \cdot V^J \cdot V^K \cdot V^L\right) \,, \\ I_5^{IJKLM} &= \mathrm{Tr}\left(V^I \cdot V^J \cdot V^K \cdot V^L \cdot V^M\right) \,, \end{aligned} \tag{23}$$

and so on.

To find the independent set of invariants, we now find the gauge-fixed configuration space and attempt to characterize this space by invariants.

3.1. Counting candidate invariants

We follow the counting procedure described in the previous section. For each candidate tableau in the first column (shown as a list of the number of boxes per row), we use the program LiE to find the number of invariants (shown in the second column) with that symmetry. We then subtract any invariants that can be written as a product of smaller invariants; these are shown in the following column of the table. The net number of independent invariants after subtraction is shown in the last column.

Tableau	# Invts	Subtract	Net
(2)	1	0	1
(1,1)	0	0	0
(3)	1	0	1
(2,1)	0	0	0
(1,1,1)	0	0	0
(4)	1	$(2) \otimes (2)$	0
(3,1)	0	0	0
(2,2)	2	$(2) \otimes (2)$	1
(2,1,1)	0	0	0
(1,1,1,1)	0	0	0
(5)	1	$(2) \otimes (3)$	0
(4,1)	1	$(2) \otimes (3)$	0
(3,2)	1	$(2) \otimes (3)$	0
(3,1,1)	0	0	0
(2,2,1)	1	0	1
(2,1,1,1)	0	0	0
(1,1,1,1,1)	1	0	1
(6)	2	$(3) \otimes (3),\ (2) \otimes (2) \otimes (2)$	0
(5,1)	0	0	0
(4,2)	3	$(2,2) \otimes (2),\ (2) \otimes (2) \otimes (2),\ (3) \otimes (3)$	0
(3,3)	0	0	0
(4,1,1)	0	0	0
(3,2,1)	1	$(2,2) \otimes (2)$	0
(2,2,2)	2	$(2,2) \otimes (2),\ (2) \otimes (2) \otimes (2)$	0
(3,1,1,1)	1	0	1
(2,2,1,1)	0	0	0
(2,1,1,1,1)	0	0	0
(1,1,1,1,1,1)	0	0	0

We have also verified that no new invariants of degree 8 exist.

We construct candidate structures for each of the tableaux with a net nonzero number of invariants

$$
\begin{aligned}
(2) : I_2^{IJ} &= \mathrm{Tr}\left(V^I \cdot V^J\right) , && (24)\\
(3) : I_3^{IJK} &= \mathrm{Tr}\left(V^I \cdot V^J \cdot V^K\right) , &&\\
(2,2) : \tilde{I}_4^{IJKL} &= \mathrm{Tr}\left(V^{[I} \cdot V^{J]} \cdot V^{[K} \cdot V^{L]}\right) , && (25)\\
(2,2,1) : I_5^{[IJKLM]} &= \mathrm{Tr}\left(V^{[I} \cdot V^{J]} \cdot V^{[K} \cdot V^L \cdot V^{M]}\right) , && (26)\\
(1,1,1,1,1) : \tilde{I}_5^{IJLKM} &= \mathrm{Tr}\left(V^{[I} \cdot V^J \cdot V^K \cdot V^L \cdot V^{M]}\right) , && (27)\\
(3,1,1,1) : I_6^{IJLKMN} &= \mathrm{Tr}\left(V^{[I} \cdot V^J \cdot V^K \cdot V^{L]} \cdot V^M \cdot V^N\right) . && (28)
\end{aligned}
$$

3.2. Construction of invariants

Even though all invariants of eight fields canceled above, this cannot prove that we have achieved a complete count. To have confidence in our result, we apply the theorem from the introduction by gauge-fixing some fields.

We first consider the case where the matter content is a single field V^1. We can use complex gauge transformations to bring this to the form of a diagonal matrix

$$
V^1 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \tag{29}
$$

with $a + b + c = 0$.

This generically fixes the continuous gauge symmetry, but the ordering of the three diagonal elements can be changed by a gauge transformation. There is therefore a residual discrete $Z_2 \times Z_2$ gauge symmetry. In the special case when two eigenvalues coincide, the continuous symmetry is partially unbroken and there is a residual U(1) symmetry.

The invariants are the symmetric combinations

$$
\begin{aligned}
i_1 &= a + b + c = 0 , && (30)\\
i_2 &= a^2 + b^2 + c^2 , && (31)\\
i_3 &= a^3 + b^3 + c^3 . && (32)
\end{aligned}
$$

We look for SO(3) invariants which can reproduce these combinations. We find

$$
I_2^{11} = i_2 , \qquad I_3^{111} = i_3 . \tag{33}
$$

The invariants I_2^{IJ}, I_3^{IJK} are hence sufficient to completely parametrize the gauge-inequivalent configurations of a single field, and hence form a complete set of invariants for one field.

We now consider a generic configuration of multiple fields V^I. We gauge fix V^1 as before. The configuration is now

$$\begin{aligned} V^1 &= \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix} , \\ V^I &= \begin{pmatrix} V^I_{11} & V^I_{12} & V^I_{13} \\ V^I_{21} & V^I_{22} & V^I_{23} \\ V^I_{31} & V^I_{32} & V^I_{33} \end{pmatrix} \quad \text{for} \quad I > 1 \, . \end{aligned} \tag{34}$$

We first consider the special case where two eigenvalues of V^1 coincide. Here we have that

$$\begin{aligned} V^1 &= a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} , \\ V^I &= \begin{pmatrix} V^I_{11} & V^I_{12} & V^I_{13} \\ V^I_{21} & V^I_{22} & V^I_{23} \\ V^I_{31} & V^I_{32} & V^I_{33} \end{pmatrix} \quad \text{for} \quad I > 1 \, . \end{aligned} \tag{35}$$

This particular configuration preserves a U(1) subgroup of the SU(2), so we could proceed by further fixing the gauge to find the completely gauge-fixed hypersurface in field space. But the gauge group is simple enough at this point that we can straightforwardly write down a set of independent polynomial combinations (invariant under the U(1) symmetry) which parametrize the configuration space.

The fields in V^I can be organized into combinations with specific charges 2, 1, 0, -1, -2 under the U(1); these are

$$\begin{aligned} T^I_2 &= 2V^I_{12} + i\left(V^I_{11} - V^I_{22}\right) , \\ T^I_1 &= V^I_{13} - iV^I_{23} \, , \\ T^I_0 &= V^I_{11} + V^I_{22} \, , \end{aligned} \tag{36}$$

where the subscripts denote the respective charges. The negatively charged fields are the complex conjugates of the positive charges.

In addition to the U(1), there is also a discrete Z_2 symmetry corresponding to charge conjugation

$$T^I_2 \leftrightarrow -T^I_{-2} \, , \qquad T^I_1 \leftrightarrow -T^I_{-1} \, , \qquad T^I_0 \leftrightarrow T^I_0 \, . \tag{37}$$

The combinations of fields invariant under the U(1) symmetry are

$$T^I_2 T^J_{-2} \, , \qquad T^I_1 T^J_{-1} \, , \qquad T^I_0 \, , \qquad T^I_2 T^J_{-1} T^K_{-1} \, , \qquad T^I_{-2} T^J_1 T^K_1 \, .$$

Under the Z_2 action, these combinations are acted on as

$$\begin{aligned} T_0^I &\leftrightarrow T_0^I\,, \\ T_2^I T_{-2}^J &\leftrightarrow T_2^J T_{-2}^I\,, \\ T_1^I T_{-1}^J &\leftrightarrow T_1^J T_{-1}^I\,, \\ T_2^I T_{-1}^J T_{-1}^K &\leftrightarrow -T_{-2}^I T_1^J T_1^K\,. \end{aligned} \tag{38}$$

We make combinations which are even/odd under the Z_2; the gauge-invariant even combinations are

$$\begin{aligned} i_1^I &\equiv T_0^I\,, \\ i_2^{IJ} &\equiv T_2^I T_{-2}^J + T_2^J T_{-2}^I\,, \\ i_3^{IJ} &\equiv T_1^I T_{-1}^J + T_1^J T_{-1}^I\,, \\ i_4^{IJK} &\equiv T_2^I T_{-1}^J T_{-1}^K - T_{-2}^I T_1^J T_1^K\,. \end{aligned} \tag{39}$$

A product of two combinations odd under the Z_2 is even under the Z_2. The only such combination which cannot be written in terms of the already obtained even combinations is

$$i_5^{IJKL} \equiv \left(T_2^I T_{-2}^J - T_2^J T_{-2}^I\right)\left(T_1^K T_{-1}^L - T_1^L T_{-1}^K\right)\,. \tag{40}$$

The combinations i_1^I, i_2^{IJ}, i_3^{IJ}, i_4^{IJK}, i_5^{IJKL} completely parametrize the gauge-inequivalent orbits of the configuration space.

We now promote these to SO(3) invariants; that is, we look for SO(3) invariants which reduce to combinations (39), (40) on the gauge-fixed configuration space of equation (34). Once again, we use the flavor symmetry as a guide.

The combination with one free index *i.e.* i_1^I should be reproduced from operators with one free flavor index. One such operator is provided by the operator I_2^{IJ}, where one of the fields is taken to be V^1. Indeed, we find

$$I_2^{1I} = 3a i_1^I\,. \tag{41}$$

Hence, a knowledge of the invariant I_2^{IJ} allows us to reproduce the combinations with one free flavor index i_1^I.

The combinations with two free flavor indices *i.e.* i_2^{IJ}, i_3^{IJ} are both symmetric in IJ, and should be reproduced from SO(3) invariants with two symmetric free flavor indices IJ. Indeed, we find

$$I_2^{IJ} = \frac{1}{4} i_2^{IJ} + i_3^{IJ} + \ldots\,, \tag{42}$$

$$I_3^{1IJ} = \frac{a}{4}\left(i_2^{IJ} - 2 i_3^{IJ}\right) + \ldots\,, \tag{43}$$

where the ellipses indicate terms with (already determined) lower degree combinations like $i_1^I i_1^J$. These two invariants therefore determine i_2^{IJ}, i_3^{IJ}.

For i_4^{IJK}, we have JK symmetrized. This will be reproduced by an SO(3) invariant with three flavor indices IJK, where the JK indices are symmetrized. The flavor indices may be organized into various symmetries, and must map to SO(3) invariants with the same structure. These may be guessed to be

Tableau	Candidate SO(3) invariant
$\begin{array}{\|c\|c\|c\|}\hline I & J & K \\ \hline\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline I & J & K \\ \hline\end{array}$
$\begin{array}{\|c\|c\|}\hline J & K \\ \hline I \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline J & K \\ \hline I & 1 \\ \hline\end{array}$

To verify that this guess is correct, we evaluate the invariants after gauge fixing. We find

$$I_3^{IJK} = -\frac{i}{4}\left(i_4^{IJK} + i_4^{JIK} + i_4^{KIJ}\right) + \dots\,, \tag{44}$$

$$\tilde{I}_4^{IJK1} + \tilde{I}_4^{IKJ1} = -\left(\frac{3ai}{2}\right)\left(2i_4^{IJK} - i_4^{JIK} - i_4^{KIJ}\right) + \dots\,, \tag{45}$$

where the ellipses indicate terms with (already determined) lower-degree combinations. Hence, the invariants I_3^{IJK}, $\tilde{I}_4^{IJKL}$ indeed allow us to solve for i_4^{IJK}.

Finally, for i_5^{IJKL}, we have that IJ and KL are antisymmetrized. Once again, this invariant will be organized into various symmetries, and must map to SO(3) invariants with the same structure.

These are guessed to be

Tableau	Candidate SO(3) invariant
$\begin{array}{\|c\|c\|}\hline I & K \\ \hline J & L \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline I & K \\ \hline J & L \\ \hline\end{array}$
$\begin{array}{\|c\|c\|}\hline I & K \\ \hline J \\ \cline{1-1} L \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline I & K \\ \hline J & 1 \\ \hline L \\ \cline{1-1}\end{array}$
$\begin{array}{\|c\|}\hline I \\ \hline J \\ \hline K \\ \hline L \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 1 \\ \hline I \\ \hline J \\ \hline K \\ \hline L \\ \hline\end{array}$

Evaluating these invariants on the gauge-fixed configuration, we find that indeed i_5^{IJKL} can be written as a combination of $\tilde{I}_4^{IJKL}$, $\tilde{I}_5^{IJKL1}$, I_6^{IJLK11}.

The parameters of the gauge-fixed configuration space with the enhanced U(1) symmetry are therefore in 1–1 correspondence with the invariants (that is, every parameter of the gauge-fixed configuration with the enhanced U(1) symmetry can be found as a polynomial in the invariants)

$$I_2^{IJ}\,,\qquad I_3^{IJK}\,,\qquad \tilde{I}_4^{IJKL}\,,\qquad \tilde{I}_5^{IJKLM}\,,\qquad I_6^{IJLKM}\,.$$

More generally, the eigenvalues of V^1 are all different, and we have

$$\begin{aligned} V^1 &= \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix}, \\ V^I &= \begin{pmatrix} V^I_{11} & V^I_{12} & V^I_{13} \\ V^I_{21} & V^I_{22} & V^I_{23} \\ V^I_{31} & V^I_{32} & V^I_{33} \end{pmatrix} \quad \text{for} \quad I > 1\,. \end{aligned} \tag{46}$$

The parameters a, b are reproduced from I_2^{11}, I_3^{111} following the analysis for a single field.

This parametrization of the configuration space preserves two Z_2 symmetries

$$\begin{aligned} (Z_2)_A &: V^I_{12} \to -V^I_{12}, V^I_{13} \to -V^I_{13}\,, \\ (Z_2)_B &: V^I_{12} \to -V^I_{12}, V^I_{23} \to -V^I_{23}\,. \end{aligned} \tag{47}$$

To find the moduli space, we should find the combinations of V^I_{ij} which are gauge-invariant under these discrete symmetries. Once again, the remaining symmetry is simple enough that we can just do this by inspection. We find that the polynomials invariant under the two discrete symmetries are generated by

$$V^I_{11}\,,\qquad V^I_{22}\,,\qquad V^I_{12}V^J_{12}\,,\qquad V^I_{13}V^J_{13}\,,\qquad V^I_{23}V^J_{23}\,,\qquad V^I_{12}V^J_{13}V^K_{23}\,.$$

We now promote these to SO(3) invariants; that is, we look for SO(3) invariants which reduce to these combinations on the configuration space of equation (46). We first check whether the invariants we have found are sufficient to do this.

We start with invariants with one flavor index I. We find

$$I_2^{1I} = aV^I_{11} + bV^I_{22} + (a+b)\left(V^I_{11} + V^I_{22}\right)\,, \tag{48}$$

$$I_3^{11I} = a^2V^I_{11} + b^2V^I_{22} + (a+b)^2\left(-V^I_{11} - V^I_{22}\right) \tag{49}$$

which can be used to solve for V^I_{11}, V^I_{22} in terms of I_2^{1I}, I_2^{11I}. This inversion fails only if $(b+2a) = 0, (2b+a) = 0$ or $a = b$, *i.e.* when two eigenvalues in V^1 are equal, which we have already assumed to not be the case.

Similarly, it is easy to show that the invariants I_2^{IJ}, I_3^{IJ1}, $\tilde{I}_4^{I1J1}$ (symmetric in IJ) are sufficient to reconstruct $V_{12}^I V_{12}^J$, $V_{13}^I V_{13}^J$, $V_{23}^I V_{23}^J$.

The invariant $V_{12}^I V_{13}^J V_{23}^K$ can be organized in several symmetries, as in the analysis above. We find that these are reproduced by I_3^{IJK}, $\tilde{I}_4^{IJK1}$, I_6^{I1J1K1}.

Our final result is then that every point on the gauge-invariant configuration space can be reproduced by a knowledge of the invariants

$$I_2^{IJ}, \qquad I_3^{IJK}, \qquad \tilde{I}_4^{IJKL}, \qquad I_5^{IJKLM}, \qquad \tilde{I}_5^{IJKLM}, \qquad I_6^{IJLKM}. \tag{50}$$

Hence the theorem described in the introduction ensures that any gauge-invariant polynomial in this theory can be generated by these invariants.

4. SU(3) with sextets

We now consider a theory with a SU(3) symmetry and fields in the sextet representation V_{ij}^I.

We can form an infinite set of invariants by contracting these fields with the epsilon tensor ϵ^{ijk}. We now find an independent set of tensors by finding a set that can parametrize the gauge-fixed configuration space.

4.1. Counting candidate invariants

We follow the counting procedure described in the previous section. For each candidate tableau in the first column (shown as a list of the number of boxes per row) we use the program LiE to find the number of invariants (shown in the second column) with that symmetry. We then subtract any invariants that can be written as a product of smaller invariants; these are shown in the following column of the table. The net number of independent invariants after subtraction is shown in the last column.

Tableau	# Invts	Subtract	Net
(2)	0	0	0
(1,1)	0	0	0
(3)	1	0	1
(2,1)	0	0	0
(1,1,1)	0	0	0
(4)	0	0	0
(3,1)	0	0	0
(2,2)	0	0	0
(2,1,1)	0	0	0
(1,1,1,1)	0	0	0
(5)	0	0	0
(4,1)	0	0	0
(3,2)	0	0	0
(3,1,1)	0	0	0
(2,2,1)	0	0	0
(2,1,1,1)	0	0	0
(1,1,1,1,1)	0	0	0
(6)	1	$(3) \otimes (3)$	0
(5,1)	0	0	0
(4,2)	1	$(3) \otimes (3)$	0
(3,3)	0	0	0
(4,1,1)	0	0	0
(3,2,1)	0	0	0
(2,2,2)	1	0	1
(3,1,1,1)	0	0	0
(2,2,1,1)	0	0	0
(2,1,1,1,1)	0	0	0
(1,1,1,1,1,1)	1	0	1

We define the invariants

$$\begin{aligned} I_3^{IJK} &\equiv \epsilon^{ikm}\epsilon^{jln}V^I_{ij}V^J_{kl}V^K_{mn}\,, && (51)\\ I_6^{IJKLMN} &= \epsilon^{abc}V^I_{ad}V^J_{be}V^K_{cf}\ \epsilon^{eim}\epsilon^{fkn}\epsilon^{djl}V^L_{ij}V^M_{kl}V^N_{mn}\,. && (52)\end{aligned}$$

4.2. Construction of invariants

We begin by considering a single field. By a gauge transformation, we can bring it to the form

$$V^1 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}\,. \tag{53}$$

This form of the field preserves two U(1) symmetries; the first is

$$\begin{aligned} V_{11} &\to e^{2i\alpha}V_{11}\,, \qquad V_{22}\to e^{-2i\alpha}V_{22}\,, \qquad V_{33}\to V_{33}\,,\\ V_{13} &\to e^{i\alpha}V_{13}\,, \qquad V_{23}\to e^{-i\alpha}V_{23}\,, \qquad V_{12}\to V_{12}\,, \end{aligned} \tag{54}$$

and the second is

$$\begin{aligned} V_{11} &\to e^{2i\alpha}V_{11}\,, \qquad V_{33}\to e^{-2i\alpha}V_{33}\,, \qquad V_{22}\to V_{22}\,,\\ V_{12} &\to e^{i\alpha}V_{12}\,, \qquad V_{23}\to e^{-i\alpha}V_{23}\,, \qquad V_{13}\to V_{13}\,. \end{aligned} \tag{55}$$

These symmetries alter the eigenvalues without changing the form; the first one takes $a\to e^{2i\alpha}a$, $b\to e^{-2i\alpha}b$, $c\to c$, and the second takes $a\to e^{2i\alpha}a$, $c\to e^{-2i\alpha}c$, $b\to b$. The only gauge-invariant combination is the product abc.

We find

$$I_3^{111} = 6abc\,. \tag{56}$$

This invariant, therefore, reproduces the gauge-fixed configuration space for a single field, and is, therefore, a complete set of invariants for a single sextet of SU(3).

We now consider multiple fields, We can bring these to the form

$$\begin{aligned} V^1 &= \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}\,,\\ V^I &= \begin{pmatrix} V^I_{11} & V^I_{12} & V^I_{13} \\ V^I_{12} & V^I_{22} & V^I_{23} \\ V^I_{13} & V^I_{23} & V^I_{33} \end{pmatrix} \quad \text{for} \quad I>1\,, \end{aligned} \tag{57}$$

where we have used the U(1) symmetries to further simplify the form of the first field.

The form of the configuration space still preserves an SO(3) symmetry, and each sextet of SU(3) decomposes to a $1+5$ of SO(3). Fortunately, we have already analyzed this system in the previous section, and so we can write down the invariants. The only new invariant is the singlet, which is the trace of the matrix. Combining with the previously derived SO(3) invariants for a dimension-5 field, the SO(3) invariant combinations for this theory are

$$\begin{aligned}
i_1^I &\equiv \mathrm{Tr}\left(V^I\right) , \\
i_2^{IJ} &\equiv \mathrm{Tr}\left(V^I \cdot V^J\right) , \\
i_3^{IJK} &\equiv \mathrm{Tr}\left(V^I \cdot V^J \cdot V^K\right) , \\
\tilde{i}_4^{IJKL} &\equiv \mathrm{Tr}\left(V^{[I} \cdot V^{J]} \cdot V^{[K} \cdot V^{L]}\right) , \\
\tilde{i}_5^{IJKLM} &\equiv \mathrm{Tr}\left(V^{[I} \cdot V^J \cdot V^K \cdot V^L \cdot V^{M]}\right) , \\
i_6^{IJLKM} &\equiv \mathrm{Tr}\left(V^{[I} \cdot V^J \cdot V^{K]} \cdot V^{[L} \cdot V^{M]}\right) .
\end{aligned} \tag{58}$$

We now find SU(3) invariants which reproduce these combinations on the configuration space (57).

From the invariant that we have already defined, we obtain

$$\begin{aligned}
I_3^{11I} &= 2a^2 i_1^I , \\
I_3^{1IJ} &= -a i_2^{IJ} + \ldots , \\
I_3^{IJK} &= 2 i_3^{IJK} + \ldots
\end{aligned} \tag{59}$$

which reproduces all combinations with one, two, or three free flavor indices.

For the remaining combinations, we need to consider invariants containing six fields contracted with 4 epsilon tensors. The structure of the combinations above suggests that we should look at combinations where there are two pairs of three fields, where the three fields are antisymmetrized. We find

$$I_6^{[IJ]1[KL]1} = -a^2 \tilde{i}_4^{IJKL} , \tag{60}$$

$$I_6^{[IJKLM]1} = 6a \tilde{i}_5^{IJKLM} , \tag{61}$$

$$I_6^{[IJK][LM]1} = 4a i_6^{IJKLM} . \tag{62}$$

We thus find that every point on the gauge-fixed configuration space can be reproduced by a knowledge of the invariants

$$I_3^{IJK} , \qquad I_6^{IJKLMN} . \tag{63}$$

Hence the theorem described in the introduction ensures that any gauge-invariant polynomial in this theory can be generated by these invariants.

5. Summary and conclusion

We have discussed a new method to efficiently find a set of independent invariant tensors in gauge theories. We have done this by using a theorem, familiar from supersymmetric field theories, that relates D-flat directions to the invariants in a gauge theory. Specifically, this theorem asserts that the constant configurations, identified by complex gauge transformations, are in 1–1 correspondence with the gauge-invariant operators in the theory.

We have shown that this provides a straightforward method to find the independent invariant tensors. We have explicitly applied these methods to three gauge theories — SO(3) with fields in the symmetric tensor representation, SU(2) with a dimension-4 representation, and SU(3) with matter in the sextet — and in each case, we have found the set of independent polynomial invariants. This shows the practicality of the approach.

Our methods are general, and as far as we can see, can be applied to any group with any matter content. The immediate ones which would be interesting to analyze are exceptional groups with matter in the (anti)-fundamental representation. Knowing the invariant tensors would also help in looking for dual pairs in supersymmetric gauge theories with exceptional gauge groups.

We hope to return to this topic in future work.

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