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ON THE DYNAMICS OF DELAYED AND NON-DELAYED FRACTIONAL-ORDER AND DISTRIBUTED-ORDER CONSUMER MODELS

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The aim of our article is to introduce and investigate the effect of memory on the consumer model under the influence of advertisements. This model describes the movement from potential buyers to buyers under the influence of advertisements. For the non-delayed model, the local stability of equilibria is investigated by using its characteristic equation. The theory of fractional differential equations (FDEs) is applied to determine the fractional-order values q at which the model undergoes Hopf bifurcation. For the delayed model, we introduced the fractional-order consumer model under the time-delay effect. The time-delay parameter makes the model dynamics richer, which explains the model's behavior more realistically. By considering the time-delay value as a bifurcation parameter beside the fractional-order q, the Hopf bifurcation is analyzed. We calculated the formula of the time-delay value that leads to Hopf bifurcation. Furthermore, for supporting the theoretical outcomes, we give some examples, which illustrate the influence of both the fractional order and the time delay on the behavior of the model. We also introduced the distributed order consumer model which is a generalization of the integer and fractional orders ones. We considered two different expressions for the weight function to illustrate the stability and to get the periodic solution. A good agreement between both theoretical analysis and simulation results is found.

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1. Introduction

Fractional calculus is a hot research area in science and engineering due to its capacity to provide more detailed descriptions of a variety of nonlinear phenomena [1]. The current state can be described by the fractional

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differential equation (FDE) as a process involving the history of the previous states (memory effects) [2, 3]. Therefore, the FDE is gaining immense enthusiasm in many real research topics such as secure communication [4], control engineering [5], neural network [6], and biology models such as epidemiological and ecological models which have long-range temporal memory [7–11].

The dynamics behavior of FDEs has been studied in recent years. Through studies of FDEs systems, many new and interesting results in different fields have been obtained, *e.g.*, a single neuron model with time delay [12], a novel predator–prey system with time delay [13], a new delayed incommensurate gene regulatory network [14], a self-sustained birhythmic system with two delays, multiplicative and additive colored noises [15], and hyperchaotic complex systems [16].

On the other hand, the theory of differential equations with time delay is used to study and analyze the behavior of the dynamical systems [17–22]. The time delays describe the effect of some hidden processes in time, which may be non-momentary, such as the incubation periods of the disease, response, sensitivity, and growth pattern [23–27]. Therefore, the study of FDEs with time delays provides a more clear and comprehensive explanation of the behavior of the system, as it takes into account both the influence of memory and time on the interaction between the elements of the system [28–32].

In the past years, there has been much interest in studying distributed-order systems due to their many applications in several scientific fields [33–37]. In [38], the stability and dynamics, including chaotic and hyperchaotic solutions of fractional and distributed order of different systems were discussed. In [39], the stability of a nabla discrete distributed-order dynamical system has been investigated. In [40], the boundedness and projective synchronization of distributed-order neural networks are studied. For more studies, see [41–46].

One of the attractive subjects in economic systems is the consumer model. This model represents the dynamics of converting market visitors from potential buyers to buyers as a result of their interaction. In [47], a very popular method of studying the customers dynamics using agent-based modeling is presented. In [48], Feichtinger considered the population in the market consisting of potential customers x_1 and buyers x_2 as follows:

$$\begin{cases} \dot{x}_1 = k_0 - \alpha x_1 x_2^2 + \beta x_2 \,, \\ \dot{x}_2 = \alpha x_1 x_2^2 - (\beta + \epsilon) x_2 \,, \end{cases} \tag{1.1}$$

where the constants k_0, ε, β, and α are described as:
$k_0 > 0$: the influx of people into the market,

$\varepsilon > 0$: buyers who leave the market forever,
$\beta \geq 0$: buyers who switch to a competing brand,
$\alpha > 0$: proportionality measuring of the advertising effectiveness.

Due to the economic importance of the diffusion model, many studies have been devoted to clarify their dynamics [49–52].

Depending on the goals, advertising may have an impact on a company's sales volume both immediately and over time. The rate at which prospective consumers become actual customers is related to the advertising's impact function (see Fig. 1).

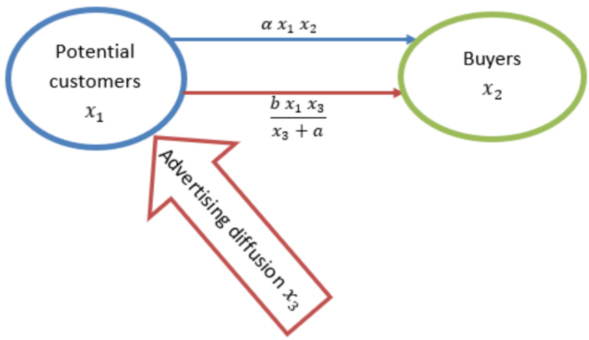


Fig. 1. The effect of advertising on markting.

The authors of [52] developed the diffusion model (1.1) to describe the effect of advertising on selling and converting potential buyers into buyers over time as follows:

$$\begin{cases} \dot{x}_1 = k_0 + \beta x_2 - \alpha x_1 x_2^2 - \frac{b x_1 x_3}{x_3 + a}\,, \\ \dot{x}_2 = \alpha x_1 x_2^2 - (\beta + \epsilon) x_2 + \frac{b x_1 x_3}{x_3 + a}\,, \\ \dot{x}_3 = d x_3 \left(1 - \frac{x_3}{k}\right)\,, \end{cases} \tag{1.2}$$

where
$x_3 \geq 0$: the dynamics of advertising diffusion,
a : the half-saturation,
b : the response rate of the potential buyers,
$k > 0,\ d \geq 0$: the logistics parameters.
By using the transformations,

$$t = \frac{t'}{n}\,, \qquad x_1 = \frac{n(n-\beta)}{k_0 \alpha} x\,, \qquad x_2 = \frac{k_0}{n-\beta} y\,, \qquad x_3 = z\,,$$

and

$$\gamma = \frac{\alpha k_0^2}{n(n-\beta)^2}\,, \qquad \beta' = \frac{\beta}{n}\,, \qquad b' = \frac{b}{n}\,, \qquad n = \beta + \epsilon\,, \qquad d' = \frac{d}{n}\,,$$

and for the sake of facilitating, we omit the primes, then model (1.2) takes the form

$$\begin{cases} \dot{x} = \gamma\left(1 - xy^2 + \beta(y-1)\right) - \frac{bxz}{a+z}\,, \\ \dot{y} = -y + xy^2 + \frac{bxz}{\gamma(a+z)}\,, \\ \dot{z} = dz - \frac{dz^2}{k}\,. \end{cases} \tag{1.3}$$

The behavior of most marketing models has memory (after-effect) [53–57]. We believe that both consumer reaction and advertising impact rates rely on both the present state and all past states. Therefore, fractional ordinary differential equations offer greater advantages than classical integer-order ones. First, we introduce the fractional version of model (1.3) as

$$\begin{cases} D^q x(t) = \gamma\left(1 - xy^2 + \beta(y-1)\right) - \frac{bxz}{a+z}\,, \\ D^q y(t) = -y + xy^2 + \frac{bxz}{\gamma(a+z)}\,, \\ D^q z(t) = dz - \frac{dz^2}{k}\,, \\ 0 < q \leq 1\,, \quad t \geq 0\,, \end{cases} \tag{1.4}$$

where D^q denotes the Caputo fractional derivative of the order of q.

Secondly, we state the delayed fractional consumer model (1.4) as

$$\begin{cases} D^q x(t) = \gamma\left(1 - xy^2 + \beta(y-1)\right) - \frac{bx(t-\tau)z}{a+z}\,, \\ D^q y(t) = -y + xy^2 + \frac{bx(t-\tau)z}{\gamma(a+z)}\,, \\ D^q z(t) = dz - \frac{dz^2}{k}\,, \\ 0 < q \leq 1\,, \quad t \geq 0\,. \end{cases} \tag{1.5}$$

We consider the existence of a time lag in the impact of the advertisement on the potential buyers class. It is presumable that prospective buyers take a τ amount of time to respond to the advertisement.

Finally, the distributed form of model (1.3) is introduced as

$$\begin{cases} D^{\phi(q)} x(t) = \gamma\left(1 - xy^2 + \beta(y-1)\right) - \frac{bxz}{a+z}\,, \\ D^{\phi(q)} y(t) = -y + xy^2 + \frac{bxz}{\gamma(a+z)}\,, \\ D^{\phi(q)} z(t) = dz - \frac{dz^2}{k}\,. \end{cases} \tag{1.6}$$

The distributed-order system (1.6) is considered as a generalization of systems with integer and fractional orders. If we take $\phi(q) = \delta(q-s)$ in model (1.6), the fractional order with order s can be given, where $\delta(q-s)$ is the Kronecker delta.

Using analytical and numerical methods, equilibria, periodic solutions, and their stability will be studied for each case; fractional, fractional with time delay, and distributed order of model (1.3).

The motivation of the study is to introduce three new versions of the consumer model (1.3) which are (1.4)–(1.6) models. We analyze the purchasing memory effect on sales. The dynamics and Hopf bifurcations of models (1.4) and (1.5) are studied analytically and numerically. A good agreement is found between them. The stability analysis of the equilibria of these two models is studied. We also study the stability analysis of fixed points, Hopf bifurcation, and periodic solutions of the distributed-order consumer model (1.6). The effect of changing the weight function of this model on its dynamics is investigated. We used the predictor–corrector method [58, 59] in numerical treatments for fractional versions and the spectral numerical method [60] for solving distributed-order one.

The paper is organized as follows. Section 2 contains a summary of basic information on fractional calculus. Section 3 deals with the fractional-order consumer model (1.4) and its dynamics and Hopf bifurcation. The fractional order with time-delay consumer model (1.5) is studied in detail in Section 4. In Section 5, the distributed-order consumer model (1.6) is introduced. The equilibrium points and their stability, periodic solutions, and Hopf bifurcation of this model are studied. Section 6 contains the summary of the results of this work.

2. Preliminaries

In this part, we give the mathematical basics of fractional calculus [61].

Definition 2.1 *The Caputo fractional derivative of the order of* $n-1 < q < n$ $(n \in N)$ *of a function* $f : R^+ \to R$ *is defined by*

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \int\limits_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{q+1-n}} \mathrm{d}\tau\,, \tag{2.1}$$

where the function $f(t)$ *has absolutely continuous derivatives up to the* $(n-1)$ *order.*

Specially, when $0 < q \leq 1$, we get

$$D^q f(t) = \frac{1}{\Gamma(1-q)} \int\limits_0^t \frac{f'(\tau)}{(t-\tau)^{q}} \mathrm{d}\tau\,. \tag{2.2}$$

We observe that the fractional derivatives include integration with non-local operators. Therefore, it can be used to consider the memory effect in many systems [62].

Let us consider a fractional-order system as

$$D^q x = f(x)\,, \qquad q \in (0,1]\,, \tag{2.3}$$

and suppose that E is its equilibrium point. Then, the stability of E can be analyzed according to the following theorem [63–65].

Theorem 2.1 *For the fractional-order system (2.3), its equilibrium point E is*

1. *locally asymptotically stable iff $|\arg(\lambda_i)| > \frac{q\pi}{2}$, $i = 1,2,3$;*
2. *stable iff $|\arg(\lambda_i)| \geq \frac{q\pi}{2}$, $i = 1,2,3$ and the eigenvalues with $|\arg(\lambda_i)| = \frac{q\pi}{2}$ have the same geometric and algebraic multiplicity;*
3. *unstable iff $\exists$ such that $|\arg(\lambda_i)| < \frac{q\pi}{2}$;*
4. *a saddle point iff some eigenvalues of λ_i satisfy $|\arg(\lambda_i)| < \frac{q\pi}{2}$ and some other eigenvalues satisfy $|\arg(\lambda_i)| > \frac{q\pi}{2}$.*

We state some basic concepts for distributed-order derivatives [66–68].

Definition 2.2 *The distributed derivative of a continuous variable x(t) is*

$$D^{\phi(q)}x(t) = \int\limits_{l-1}^{l} \phi(q) D^q x(t)\mathrm{d}q \approx \sum_{j=1}^{m} w\left(q_j\right) D^{q_j} x(t)\Delta\tau_j\,, \tag{2.4}$$

where $q \in (l-1, l]$, $0 = \tau_0 < \tau_1 < \cdots < \tau_m = 1$, $\Delta\tau_j = \tau_j - \tau_{j-1} = \frac{1}{m}$, $q_j = \frac{\tau_j + \tau_{j-1}}{2} = \frac{2j-1}{2m}$, $j = 1,2,\ldots,m, m \in \mathbb{N}$.

Definition 2.3 *In distributed-order non-autonomous systems of the following kind:*

$$D^{\phi(q)}x(t) = Ax + f(x,t)\,, \tag{2.5}$$

the constant vector $x_0 \in R^n$ represents the equilibrium point if and only if $Ax_0 + f\left(x_0, t\right) = 0$.

Theorem 2.2 *The zero solution of (2.5) is asymptotically stable if*

1. $\lim\limits_{\|x(t)\|\to 0} \frac{\|f(x(t))\|}{\|x(t)\|} = 0\,,$
2. $\left|\arg\lambda_i\left(-\frac{\theta_{l-1}}{\theta_l}\right)\right| > \frac{\pi\mu_l}{2}\,; \quad i = 1,2,\ldots,n\,; \quad l = 1,2,\ldots,m\,,$

where $\lambda\left(\frac{\theta_{l-1}}{\theta_l}\right)$ are the eigenvalues of the matrix $\frac{\theta_{l-1}}{\theta_l}$, $\theta_l \in \mathbb{R}^{n\times n}$, such that $\theta_0 = -A$, $\theta_l = I\Delta\tau_l w\left(q_l\right)$, $\mu_l = q_l - q_{l-1}$, $q_l = \frac{2l-1}{2m}$, and $(m+1)$ is the number of steps for $q \in (0,1]$.

3. Fractional-order consumer model (1.4)

Now, we study the dynamics of model (1.4).

Proposition 3.1 *Model (1.4) has two equilibrium points, the semi-trivial equilibrium point* $E_1(1,1,0)$ *and the non-trivial one* $E_2(a_0,1,k)$*, where* $a_0 = \frac{\gamma(a+k)}{\gamma(a+k)+bk}$.

For the stability analysis of $E_{1,2}$, the Jacobian matrix $J(E)$ of (1.4) at the equilibrium point $E(x^*,1,z^*)$ is

$$J(E) = \begin{pmatrix} -\frac{bz^*}{a+z^*} - \gamma & \gamma(\beta - 2x^*) & -\frac{abx^*}{(a+z^*)^2} \\ \frac{bz^*}{(a+z^*)\gamma} + 1 & 2x^* - 1 & \frac{abx^*}{(a+z^*)^2\gamma} \\ 0 & 0 & d - \frac{2dz^*}{k} \end{pmatrix}, \tag{3.1}$$

where, $x^* = 1,\ z^* = 0$ for E_1 and $x^* = a_0,\ z^* = k$ for E_2.

The characteristic equation of (3.1) is

$$\left(d - \frac{2dz^*}{k} - \lambda\right)\left(\lambda^2 + \Phi_1\lambda + \Phi_2\right) = 0\,, \tag{3.2}$$

where

$$\begin{aligned} \Phi_1 &= \frac{-2x^*(a+z^*) + \gamma(a+z^*) + a + bz^* + z^*}{a+z^*}, \\ \Phi_2 &= -\frac{(\beta-1)(\gamma(a+z^*) + bz^*)}{a+z^*}, \end{aligned}$$

which leads to the eigenvalues as follows:

$$\lambda_1 = d - \frac{2dz^*}{k}, \tag{3.3}$$

$$\lambda_{2,3} = \frac{-\Phi_1 \pm \sqrt{\Delta}}{2}, \qquad \Delta = \Phi_1^2 - 4\Phi_2\,. \tag{3.4}$$

Hence, the equilibrium point E_1 is unstable, because the first λ_1 is positive for integer and fractional consumer models (1.3) and (1.4). For E_2, the corresponding eigenvalues can be written as

$$\begin{aligned} \lambda_1 &= -d\,, \qquad \lambda_{2,3} = \frac{R \pm \sqrt{\Delta}}{2}, \qquad \Delta = R^2 - 4Q\,, \\ Q &= \frac{(bk + (a+k)\gamma)(1-\beta)}{a+k}, \\ R &= 1 - \gamma - bk\left(\frac{1}{a+k} + \frac{2}{bk + (a+k)\gamma}\right). \end{aligned} \tag{3.5}$$

If $\|\arg(\lambda_{2,3})\| > \frac{q\pi}{2}$, then the equilibrium point E_2 is stable for model (1.4), while E_2 is stable for integer consumer model (1.3) for the condition $\|\arg(\lambda_{2,3})\| > \frac{\pi}{2}$. This means that the region of stability for the fractional model is larger than in the integer case.

Now, we study the existence of Hopf bifurcation of model (1.4). In the case of $q = 1$, the Hopf bifurcation is related to the real part in the conjugate complex eigenvalues. However, in the fractional case, the stability of E is related to the sign of $P_{1,2}(q, \gamma) = \frac{q\pi}{2} - |\arg(\lambda_i)|$, $i = 1, 2$. Thus, the function $P_{1,2}(q, \gamma)$ has an effect comparable to the real part of eigenvalue in integer systems, therefore, the Hopf bifurcation condition can be extended to the fractional systems as follows [64, 69–71].

Lemma 3.1 *For the fractional-order model (1.4), a Hopf bifurcation occurs around an equilibrium E at the bifurcation parameter $\gamma = \gamma_0$ and critical value $q = q_0$, if the following conditions are hold:*

1. *The Jacobian matrix has a pair of complex-conjugate eigenvalues $\lambda_{1,2} = \Phi(\gamma) \pm i\omega(\gamma)$;*
2. *$P_{1,2}(q, \gamma_0) = 0$;*
3. *$\frac{\partial P_{1,2}(q,\gamma)}{\partial q}|_{q=q_0} \neq 0$.*

For the equilibrium point $E_1(1, 1, 0)$, it is easy to get the characteristic equation as follows:

$$(d - \lambda)\left(\lambda^2 - (\gamma - 1)\lambda + \gamma(1 - \beta)\right) = 0\,. \tag{3.6}$$

Obviously, $\lambda_1 = d$ is a positive eigenvalue and $\lambda_{2,3}$ satisfy

$$\lambda_{2,3} = \frac{(\gamma - 1) \pm \sqrt{\Delta}}{2}\,, \qquad \Delta = \gamma(\gamma + 4\beta - 6) + 1\,. \tag{3.7}$$

Theorem 3.1 *A Hopf bifurcation occurs at E_1 if $\Delta < 0$ and the critical value q_0 is*

$$q_0 = \frac{2}{\pi} \tan^{-1}\left(\frac{\sqrt{|\Delta|}}{\gamma_0 - 1}\right)\,.$$

Proof. Let the bifurcation parameter be $\gamma = \gamma_0$, which leads to $\Delta < 0$. Therefore, the Jacobian matrix $J(E_1)$ has two conjugate complex eigenvalues.

We define

$$P_{1,2}(q, \gamma_0) = \frac{q\pi}{2} - |\arg(\lambda_i)| = \frac{q\pi}{2} - \tan^{-1}\left(\frac{\sqrt{|\Delta|}}{\gamma_0 - 1}\right) = 0\,, \tag{3.8}$$

which yields to

$$q_0 = \frac{2}{\pi} \tan^{-1} \left(\frac{\sqrt{|\Delta|}}{\gamma_0 - 1} \right) . \tag{3.9}$$

Finally,

$$\left. \frac{\partial P_{1,2}(q, \gamma)}{\partial q} \right|_{q=q_0} = \frac{\pi}{2} \neq 0 . \tag{3.10}$$

Therefore, a Hopf bifurcation appears at $E_1(1, 1, 0)$ when condition (3.9) is met. ■

For the equilibrium point $E_2(a_0, 1, k)$, the corresponding eigenvalues are given in equation (3.5). Obviously, $\lambda_1 = -d$ is a negative eigenvalue, and $\lambda_{2,3}$ satisfy

$$\lambda_{2,3} = \frac{R \pm \sqrt{\Delta}}{2} , \qquad \Delta = R^2 - 4Q . \tag{3.11}$$

Theorem 3.2 *A Hopf bifurcation exists at $E_2(a_0, 1, k)$ if $\Delta < 0$ and the critical value q_0 is*

$$q_0 = \frac{2}{\pi} \tan^{-1} \left(\frac{\sqrt{|\Delta|}}{R} \right) .$$

It can be proved in a similar manner as we did for Theorem 3.1.

Remark 3.1 *It is worth noting that the use of fractional derivatives gives a more realistic and deeper visualization, in addition to increasing the model parameters (fractional derivative order q), which increases the dynamics of the model.*

3.1. Numerical simulations

For supporting the theoretical analysis, we carry out some numerical simulations. Model (1.4) has 7 parameters, including order q of the derivative which can be chosen as follows: we consider γ as the bifurcation parameter and $(a, b, d, e, \beta) = (5, 5, 1, 0.1, 0.5)$. Regarding the non-trivial equilibrium, we have $E_2(\frac{5.1\gamma}{0.5+5.1\gamma}, 1, 0.1)$, and by simple calculations, we get the coefficients of (3.11) as

$$R = 1 - \gamma - 0.5 \left(0.196078 + \frac{2}{0.5 + 5.1\gamma} \right) , \qquad Q = 0.0980392(0.5 + 5.1\gamma) .$$

Then,

$$\begin{aligned} \Delta &= -0.196078 - 2\gamma + \left(-0.901961 + \gamma + \frac{1}{0.5 + 5.1\gamma} \right)^2 > 0 \\ &\Leftrightarrow 0.0430158 < \gamma < 3.54892 . \end{aligned}$$

From Theorem 3.2, we fix $\gamma = 0.5$. To show the impact of fractional order, we consider two cases:

Case 1: In this case, we consider the integer model (1.3). Figure 2 illustrates the behavior of model (1.3). We observe that the time series preserves their oscillation constantly.

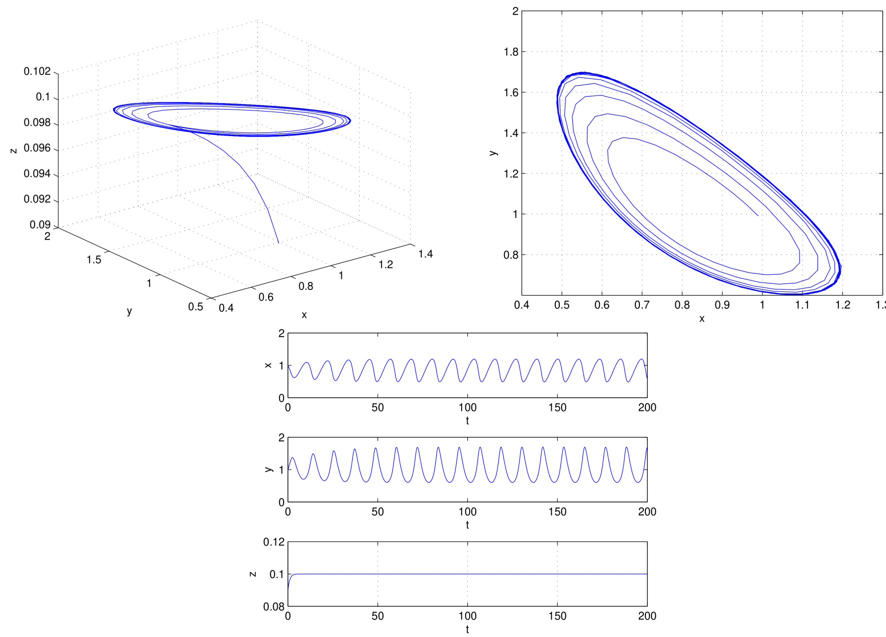


Fig. 2. The phase portrait and time series of model (1.3).

Case 2: In this case, we consider the fractional version (memory effect). By direct computation, we get $q_0 = 0.956838$.

Let $q = 0.94, 0.98$ and the initial value $(0.836066, 1.099, 0.091)^T$, we can easily check the stability of the non-trivial equilibrium point. This fact is depicted in the following discussion.

Figure 3 shows that when $q = 0.94 < q_0$, then the behavior of the model solutions tends to a fixed point and the time series gradually lose their oscillation. Thus, the equilibrium point E_2 of model (1.4) is locally asymptotically stable. Figure 4 shows that when the fractional order q crosses the critical value q_0, then the behavior of the solutions tends to an orbit and the time series preserves their oscillation constantly. Thus, a Hopf bifurcation will appear. In Fig. 5, we observe that for $q = 0.98 > q_0$, the behavior of the solutions is moving away from a fixed point to create periodic solutions and the oscillation of the time series gradually increases.

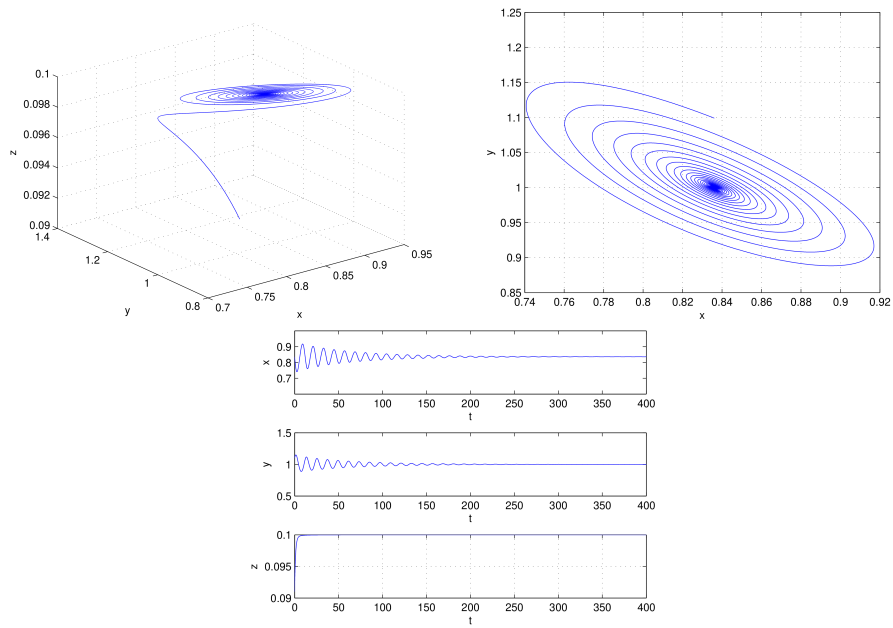


Fig. 3. The phase portrait and time series of model (1.4). The equilibrium E_2 is asymptotically stable when $q = 0.94 < q_0$.

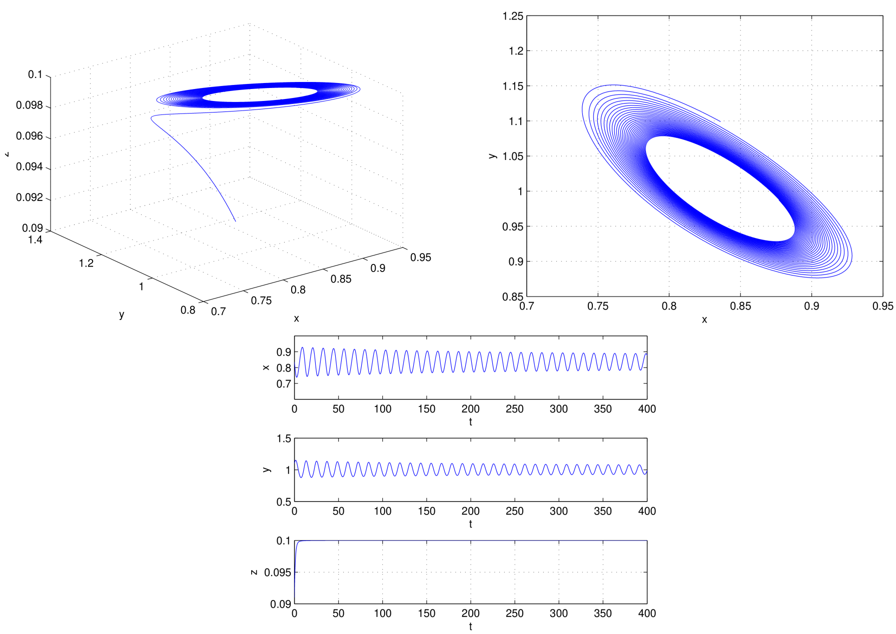


Fig. 4. The phase portrait and time series of model (1.4) at $q_0 = 0.956838$. The solution of model (1.4) approaches orbits and the time series preserve their oscillation constantly.

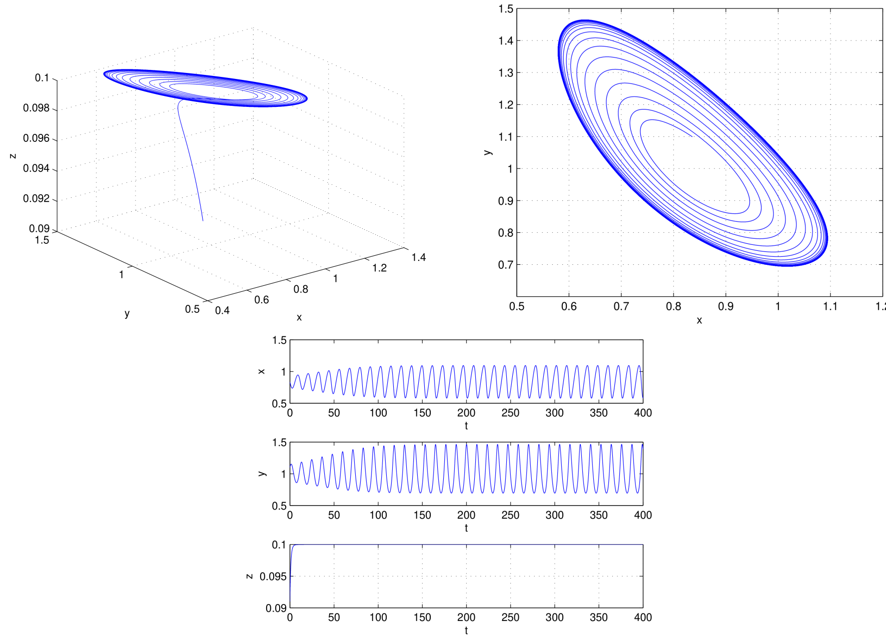


Fig. 5. The phase portrait and time series of model (1.4). A periodic oscillation bifurcates from the equilibrium E_2 when $q = 0.98 > q_0$.

Comparing the two cases, we conclude that the fractional derivative has a clear effect on changing the stability of the model. This refers to the effect of memory on shopping and sales behavior.

4. Delayed fractional-order consumer model (1.5)

At the point $E(x, y, z^*)$, model (1.5) can be rewritten using the theory of series as

$$\begin{cases} D^q x(t) = & \gamma\left(1 - xy^2 + \beta(y-1)\right) \\ & -b\left(\frac{z^*}{a+z^*} + \frac{a(z-z^*)}{(a+z^*)^2} - \frac{a(z-z^*)^2}{(a+z^*)^3}\right) x(t-\tau) + O\left((z-z^*)^3\right) , \\ D^q y(t) = & -y + xy^2 \\ & +\frac{b}{\gamma}\left(\frac{z^*}{a+z^*} + \frac{a(z-z^*)}{(a+z^*)^2} - \frac{a(z-z^*)^2}{(a+z^*)^3}\right) x(t-\tau) + O\left((z-z^*)^3\right) , \\ D^q z(t) = & dz - \frac{dz^2}{k} . \end{cases} \tag{4.1}$$

Using the transformation

$$u(t) = x(t) - x^* , \qquad v(t) = y(t) - 1 , \qquad w(t) = z(t) - z^* ,$$

the linearized system (4.1) has the form

$$\begin{cases} \dot{u} = -\gamma u - \frac{bz^*}{a+z^*}u(t-\tau) + \gamma(\beta - 2x^*)v - \frac{abx^*}{(a+z^*)^2}w\,, \\ \dot{v} = u + \frac{bz^*}{\gamma(a+z^*)}u(t-\tau) + (2x^*-1)v + \frac{abx^*}{\gamma(a+z^*)^2}w\,, \\ \dot{w} = w\left(d - \frac{2dz^*}{k}\right). \end{cases} \tag{4.2}$$

The Jacobian matrix of (4.2) can be obtained as

$$J(E) = \begin{pmatrix} -\gamma - \frac{bz^*\mathrm{e}^{-\lambda\tau}}{a+z^*} & \gamma(\beta - 2x^*) & -\frac{abx^*}{(a+z^*)^2} \\ 1 + \frac{bz^*\mathrm{e}^{-\lambda\tau}}{(a+z^*)\gamma} & 2x^* - 1 & \frac{abx^*}{(a+z^*)^2\gamma} \\ 0 & 0 & d - \frac{2dz^*}{k} \end{pmatrix},$$

and its characteristic equation is

$$F(\lambda,\tau) := (d_1 - \lambda^q)\left(\lambda^{2q} + c_1\lambda^q + c_0 + (l_1\lambda^q + l_0)\,\mathrm{e}^{-\lambda\tau}\right), \tag{4.3}$$

where $d_1 = d(\frac{2z^*}{k}-1)$, $c_1 = \gamma - 2x^* + 1$, $c_0 = \gamma(1-\beta)$, $l_1 = \frac{bz^*}{a+z^*}$, $l_0 = l_1(1-\beta)$. One gets $\lambda_1 = d_1$ and $\lambda_{2,3}$ satisfy the following equation:

$$\lambda^{2q} + c_1\lambda^q + c_0 + (l_1\lambda^q + l_0)\,\mathrm{e}^{-\lambda\tau} = 0\,, \tag{4.4}$$

Thus, we discuss the roots of equation (4.4).

The case of $\tau = 0$ has been discussed in the previous section. Now, we consider $\tau \neq 0$ and suppose that $\lambda = i\omega = \omega(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$, $\omega > 0$ is a root of equation (4.4).

By setting $\lambda = i\omega$, we get

$$(i\omega)^{2q}c_1(i\omega)^q + c_0 + (l_1(i\omega)^q + l_0)\,(\cos(\omega\tau) - i\sin(\omega\tau)) = 0\,.$$

From DeMoivre's theorem

$$i^{jq} = \cos\left(jq\frac{\pi}{2}\right) - i\sin\left(jq\frac{\pi}{2}\right)\,,$$

we get the real and imaginary parts as follows:

$$\begin{aligned} A_1\cos(\omega\tau) + A_2\sin(\omega\tau) &= C_1\,, \\ A_2\cos(\omega\tau) + A_1\sin(\omega\tau) &= C_2\,, \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} A_1 &= l_0 + l_1\omega^q\cos\left(q\frac{\pi}{2}\right)\,, \qquad A_2 = l_1\omega^q\sin\left(q\frac{\pi}{2}\right)\,, \\ C_1 &= c_0 + c_1\omega^q\cos\left(q\frac{\pi}{2}\right) + \omega^{2q}\sin\left(2q\frac{\pi}{2}\right)\,, \\ C_2 &= c_1\omega^q\sin\left(q\frac{\pi}{2}\right) + \omega^{2q}\cos\left(2q\frac{\pi}{2}\right)\,. \end{aligned}$$

By solving equations (4.5), we obtain

$$P(\omega^q) = \omega^{4q} + B_3\omega^{3q} + B_2\omega^{2q} + B_1\omega^q + B_0\,, \tag{4.6}$$

where

$$\begin{aligned} B_3 &= 2c_1 \sin\left(3q\frac{\pi}{2}\right)\,, \\ B_2 &= c_1^2 - l_1^2 + 4c_0\cos\left(q\frac{\pi}{2}\right)\sin\left(q\frac{\pi}{2}\right)\,, \\ B_1 &= 2(c_0c_1 - l_0l_1)\cos\left(q\frac{\pi}{2}\right)\,, \\ B_0 &= (c_0 - l_0)(c_0 + l_0)\,. \end{aligned} \tag{4.7}$$

Compared to the integer-order model in [52], we observe that the characteristic equation has been reduced from 4^{th} degree to 2^{nd} degree, which cannot be obtained for equation (4.6). Since it is difficult to determine the positive real roots of equation (4.6) analytically, we can consider that equation (4.6) has at least one positive root if $B_0 < 0$.

Moreover, by setting $q = 1$ in equation (4.6), one obtains the results of the integer-order model. Thus, if we set $\omega^q = \Omega$, we get

$$P(\Omega) = \Omega^4 + B_3\Omega^3 + B_2\Omega^2 + B_1\Omega + B_0\,. \tag{4.8}$$

Then, we discuss the real roots of equation (4.8). Therefore, we assume that equation (4.8) has four real roots $\Omega_s (s = 1, 2, \ldots, 4)$. Then, from (4.5), we can obtain

$$\tau_s^j = \frac{1}{\omega_s}\left(\tan^{-1}\left(\frac{A_2C_1 - A_1C_2}{A_1C_1 + A_2C_2}\right) + j\pi\right)\,, \qquad j = 0, 1, 2, \ldots, \tag{4.9}$$

then $\lambda = \pm i\omega_s$ are a pair of purely imaginary roots of equation (4.3) and τ_s^j are the corresponding critical delay values.

Lemma 4.1 *Let $\lambda = \eta(\tau) \pm i\omega(\tau)$ be a solution of equation (4.4) such that $\eta(\tau_s^{(j)}) = 0$ and $\omega(\tau_s^{(j)}) = \omega_s$, then $\frac{\mathrm{d}(\mathrm{Re}\lambda(\tau))}{\mathrm{d}\tau}|_{\tau=\tau_s^j} \neq 0$ and $\frac{\mathrm{d}(\mathrm{Re}\lambda(\tau))}{\mathrm{d}\tau}|_{\tau=\tau_s^j}$ has the same sign of $\omega_s^q P'(\omega_s)$.*

Proof. Assume that equation (4.4) is reformulated as

$$\Phi_1(\lambda^q) + \mathrm{e}^{-\lambda\tau}\Phi_2(\lambda^q) = 0\,, \tag{4.10}$$

where, $\Phi_1(\lambda^q) = \Phi_{1_r}(\lambda^q) + i\Phi_{1_i}(\lambda^q) = \lambda^{2q} + c_1\lambda^q + c_0$, $\Phi_2(\lambda^q) = \Phi_{2_r}(\lambda^q) + i\Phi_{2_i}(\lambda^q) = l_1\lambda^q + l_0$.

By differentiating equation (4.10) with respect to τ, we get

$$\left(q\left(\Phi_1'(\lambda^q)+\Phi_2'(\lambda^q)\,\mathrm{e}^{-\lambda\tau}\right)\lambda^{q-1}-\tau\Phi_2(\lambda^q)\mathrm{e}^{-\lambda\tau}\right)\frac{\mathrm{d}\lambda(\tau)}{\mathrm{d}\tau}-\lambda\Phi_2(\lambda^q)\,\mathrm{e}^{-\lambda\tau}=0\,, \tag{4.11}$$

which leads to

$$\Gamma=\frac{q\left(\Phi_1'(\lambda^q)+\Phi_2'(\lambda^q)\,\mathrm{e}^{-\lambda\tau}\right)\lambda^{q-1}}{\lambda\Phi_2(\lambda^q)\,\mathrm{e}^{-\lambda\tau}}-\frac{\tau}{\lambda}\,, \tag{4.12}$$

where, $\Gamma=[\frac{\mathrm{d}\lambda(\tau)}{\mathrm{d}\tau}]^{-1}$.

By using equation (4.10), we can rewrite (4.12) as follows:

$$\Gamma=q\lambda^{q-2}\left(-\frac{\Phi_1'(\lambda^q)}{\Phi_1(\lambda^q)}+\frac{\Phi_2'(\lambda^q)}{\Phi_2(\lambda^q)}\right)-\frac{\tau}{\lambda}\,, \tag{4.13}$$

hence

$$\begin{aligned}\Gamma|_{\tau=\tau_s^j}&=q(i\omega_s)^{q-2}\left(-\frac{\Phi_1'\left((i\omega_s)^q\right)}{\Phi_1\left((i\omega_s)^q\right)}+\frac{\Phi_2'\left((i\omega_s)^q\right)}{\Phi_2\left((i\omega_s)^q\right)}\right)-\frac{\tau}{i\omega_s}\\
&=q(i\omega_s)^{q-2}\left(-\frac{\Phi_1'\left((i\omega_s)^q\right)\overline{\Phi_1\left((i\omega_s)^q\right)}}{\Phi_1\left((i\omega_s)^q\right)\overline{\Phi_1\left((i\omega_s)^q\right)}}+\frac{\Phi_2'\left((i\omega_s)^q\right)\overline{\Phi_2\left((i\omega_s)^q\right)}}{\Phi_2\left((i\omega_s)^q\right)\overline{\Phi_2\left((i\omega_s)^q\right)}}\right)-\frac{\tau}{i\omega_s}\\
&=q(i\omega_s)^{q-2}\left(-\frac{\Phi_1'\left((i\omega_s)^q\right)\overline{\Phi_1\left((i\omega_s)^q\right)}}{\left|\Phi_1\left((i\omega_s)^q\right)\right|^2}+\frac{\Phi_2'\left((i\omega_s)^q\right)\overline{\Phi_2\left((i\omega_s)^q\right)}}{\left|\Phi_2\left((i\omega_s)^q\right)\right|^2}\right)-\frac{\tau}{i\omega_s}\,.\end{aligned}$$

From equations (4.6) and (4.10), we have

$$\begin{aligned}P(i\omega_s^q)&=\left|\Phi_1\left((i\omega_s)^q\right)\right|^2-\left|\Phi_2\left((i\omega_s)^q\right)\right|^2=0\\
&=\left(\Phi_{1_r}(\lambda^q)\right)^2+\left(\Phi_{1_i}(\lambda^q)\right)^2-\left(\Phi_{2_r}(\lambda^q)\right)^2-\left(\Phi_{2_i}(\lambda^q)\right)^2\,,\end{aligned} \tag{4.14}$$

therefore,

$$\begin{aligned}\Gamma|_{\tau=\tau_s^j}&=\frac{q(i\omega_s)^{q-2}}{|\Phi_2((i\omega_s)^q)|^2}\left(-\Phi_1'((i\omega_s)^q)\overline{\Phi_1((i\omega_s)^q)}+\Phi_2'((i\omega_s)^q)\overline{\Phi_2((i\omega_s)^q)}\right)\\
&\quad-\frac{\tau}{i\omega_s}\,,\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}\mathrm{Re}\left\{\Gamma|_{\tau=\tau_s^j}\right\} &= \frac{q(\omega_s)^{q-2}}{|\Phi_2((i\omega_s)^q)|^2} \\ &\quad \times \mathrm{Re}\Big\{i^{q-2}\Big(-\Phi_1'((i\omega_s)^q)\overline{\Phi_1((i\omega_s)^q)} + \Phi_2'((i\omega_s)^q)\overline{\Phi_2((i\omega_s)^q)}\Big)\Big\} \\ &= \frac{q(\omega_s)^{q-2}}{|\Phi_2((i\omega_s)^q)|^2} \\ &\quad \times \mathrm{Im}\Big\{i^{q-1}\Big(-\Phi_1'((i\omega_s)^q)\overline{\Phi_1((i\omega_s)^q)} + \Phi_2'((i\omega_s)^q)\overline{\Phi_2((i\omega_s)^q)}\Big)\Big\} \\ &= \frac{q(\omega_s)^{q-2}}{|\Phi_2((i\omega_s)^q)|^2}\left(\Phi_{1_r}'(\lambda^q)\Phi_{1_r}(\lambda^q) + \Phi_{1_i}'(\lambda^q)\Phi_{1_i}(\lambda^q)\right. \\ &\quad \left.-\Phi_{2_r}'(\lambda^q)\Phi_{2_r}(\lambda^q) - \Phi_{2_i}'(\lambda^q)\Phi_{2_i}(\lambda^q)\right) \\ &= \frac{q(\omega_s)^{q-2}}{|\Phi_2((i\omega_s)^q)|^2}\frac{P'(\omega^q)}{2}\,,\end{aligned}$$

therefore,

$$\mathrm{Re}\left\{\left[\frac{\mathrm{d}(\lambda(\tau))}{\mathrm{d}\tau}\right]^{-1}\Bigg|_{\tau=\tau_s^j}\right\} = \frac{q(\omega_s)^{q-2}}{|\Phi_2((i\omega_s)^q)|^2}\frac{P'(\omega^q)}{2}\,, \tag{4.16}$$

then

$$\mathrm{sign}\left\{\left[\frac{\mathrm{d}(\mathrm{Re}\lambda(\tau))}{\mathrm{d}\tau}\right]^{-1}\Bigg|_{\tau=\tau_s^j}\right\} = \frac{q}{2\omega_s^2|\Phi_2((i\omega_s)^q)|^2}\ \mathrm{sign}\left(\omega_s^q P'(\omega^q)\right)\,, \tag{4.17}$$

hence, we conclude that $\{[\frac{\mathrm{d}(\mathrm{Re}\lambda(\tau))}{\mathrm{d}\tau}]^{-1}|_{\tau=\tau_s^j}\}$, $\{[\frac{\mathrm{d}(\mathrm{Re}\lambda(\tau))}{\mathrm{d}\tau}]|_{\tau=\tau_s^j}\}$, and $\omega_s^q P'(\omega^q)$ have the same sign. ■

Theorem 4.1 *Let τ_s^j be defined by equation (4.9), then a Hopf bifurcation occurs when $\tau = \tau_s^j$, and a family of periodic solutions generates from point E at τ passes through the critical value τ_s^j.*

Proof. From the above discussion and Lemma 4.1, we get the proof of the theorem. ■

4.1. Numerical simulations

In this subsection, we present some numerical calculations to illustrate and support our analytical results developed for the fractional order of the consumer model (1.5) using the time-delay effect. We select the parameters as $\gamma = 0.2, \beta = 0.5, a = 0.35, b = 0.3, d = 1, k = 0.8$.

Case 1: In this case, we consider that there is no time delay $\tau = 0$, thus, we find that the periodic solution disappeared, see Fig. 6.

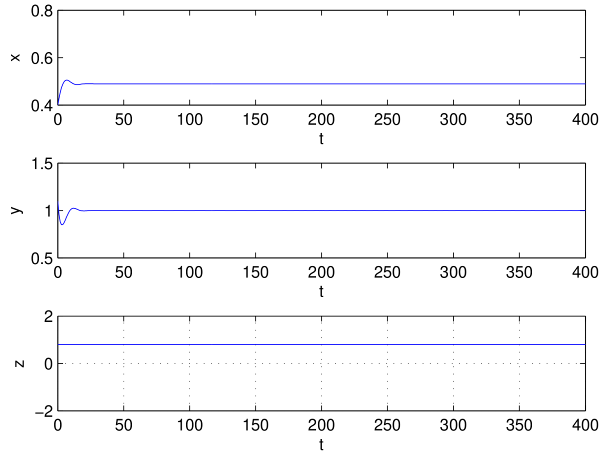


Fig. 6. The time series when $q = 0.95$, $\tau = 0$.

Case 2: We consider that there is a time delay $\tau \neq 0$, and we need to determine the critical value of τ corresponding to the derivative order q for which model (1.5) undergoes Hopf bifurcation. Thus, equation (4.6) can be written as

$$P(\omega^q) = \omega^{4q} + B_3\omega^{3q} + B_2\omega^{2q} + B_1\omega^q + B_0\,, \tag{4.18}$$

where

$$\begin{aligned} B_3 &= 0.442553\ \sin\left(3\frac{\pi q}{2}\right)\,, \\ B_2 &= 0.00540946 + 0.2\ \sin(q\pi)\,, \\ B_1 &= 0.000701444\ \cos\left(q\frac{\pi}{2}\right)\,, \\ B_0 &= -0.000888469\,. \end{aligned} \tag{4.19}$$

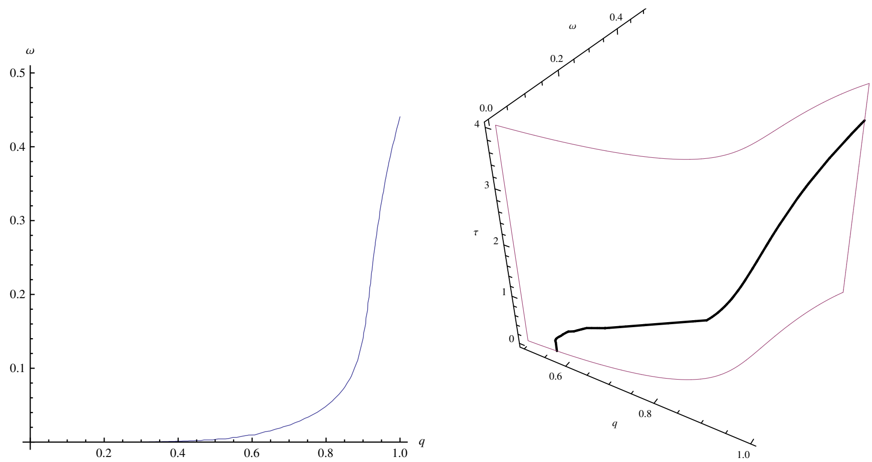
From Lemma 4.1, we get

$$\text{sign}\left\{\left[\frac{\mathrm{d}(\mathrm{Re}\lambda(\tau))}{\mathrm{d}\tau}\right]\Bigg|_{\tau=\tau_s^j}\right\} > 0\,.$$

By selecting different values for the fractional derivative order q, we get the corresponding τ as in Table 1. The bifurcation diagram in Fig. 7 shows the effect of the order of q on the values of τ_0 and ω_0. The value of ω_0 increases as the order q increases as in Fig. 7 (a). Also, the critical value τ_0 increases obviously as the order q increases as shown in Fig. 7 (b), which indicates the sensitivity of the critical value τ_0 to the change of the order of q.

Table 1. The corresponding values of fractional order q, eigenvalues ω, and time delay τ.

q	ω_0	τ_0
0.95	0.364007	2.91154
0.975	0.409723	3.15545
0.985	0.423543	3.24682
0.995	0.435395	3.33769

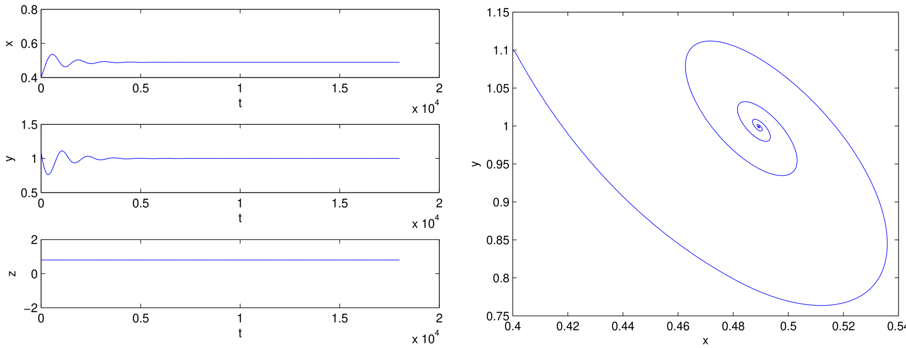


(a) (q, ω_0) curve (b) (q, ω_0, τ_0) curve

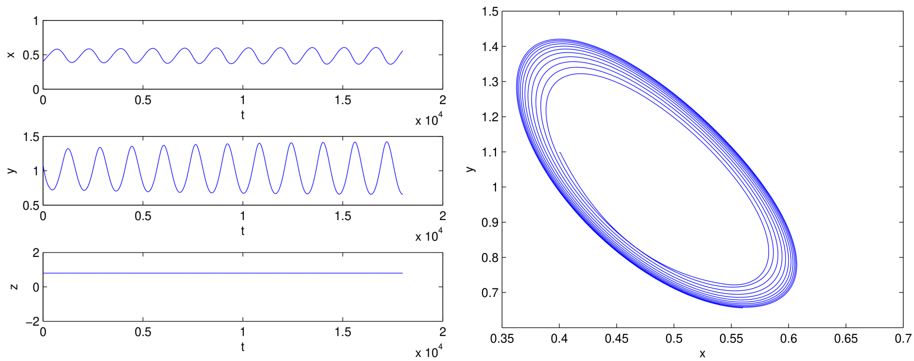
Fig. 7. Bifurcation diagram for the fractional order q, the eigenvalue ω_0, and the critical value τ_0.

In the following discussion, we try to show the effect of time delay at different values of the fractional order. For $q = 0.95$, we get the critical value of time delay $\tau_0 = 2.91154$, when $\tau = 2.45 < \tau_0$, then the behavior of the model solutions tends to a fixed point and the time series gradually lose their oscillation. Thus, the equilibrium point is locally asymptotically stable. If $\tau = 4.55 > \tau_0$, the solution moves away from the equilibrium to create periodic orbits, which is also evident in the time series as shown in Fig. 8. For $q = 0.975$, we get the critical value of time delay $\tau_0 = 3.15545$, when $\tau = 3.05 < \tau_0$, then the behavior of the model solution tends to a fixed point and the time series gradually lose their oscillation. Thus, the equilibrium point is locally asymptotically stable. For $\tau = 4.55 > \tau_0$, the solutions are moving away from the equilibrium to create periodic orbits, as shown in Fig. 9. The same results occurred for different values of the order of q.

Remark 4.1 *The time-delay parameter has an effect on the dynamical behavior of the fractional-order model as shown in Cases 1 and 2.*



(a) $\tau = 2.45$



(b) $\tau = 4.55$

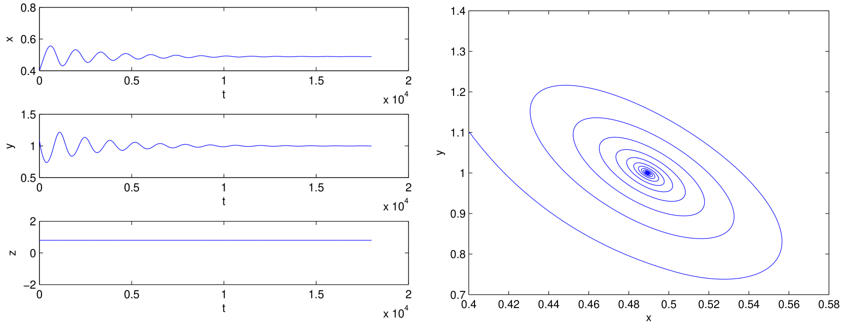
Fig. 8. The phase portrait and time series at $q_0 = 0.95$.

5. Dynamics of distributed-order consumer model (1.6)

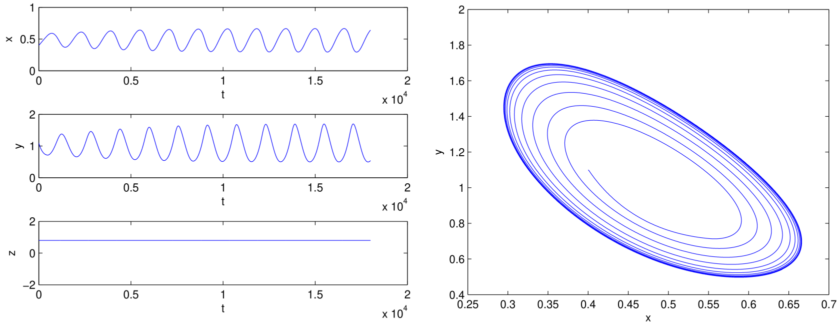
The stability analysis and the periodic solutions of the distributed-order consumer model (1.6) are investigated.

5.1. Stability analysis of equilibrium points

In this subsection, we calculate the equilibria and their stability. To get the equilibria as in Proposition 3.1, one can use Definition 2.3. Using Theorem 2.2, the stability of the zero solution of model (1.6) is studied. Now, we test the two conditions of Theorem 2.2. Let the general equilibrium point



(a) $\tau = 3.05$



(b) $\tau = 4.55$

Fig. 9. The phase portrait and time series at $q_0 = 0.975$.

be $(x^*, 1, z^*)^T$, and using the transformation

$$u(t) = x(t) - x^*, \qquad v(t) = y(t) - 1, \qquad w(t) = z(t) - z^*,$$

the model (1.6) becomes

$$\dot{X} = AX + F(X) + O\left(X^4\right), \tag{5.1}$$

where

$$X = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \qquad A = \begin{pmatrix} -\frac{bz^*}{a+z^*} - \gamma & \gamma(\beta - 2x^*) & -\frac{abx^*}{(a+z^*)^2} \\ \frac{bz^*}{(a+z^*)\gamma} + 1 & 2x^* - 1 & \frac{abx^*}{(a+z^*)^2\gamma} \\ 0 & 0 & d - \frac{2dz^*}{k} \end{pmatrix},$$

$$F(X) = \begin{bmatrix} -\chi_{21} - \gamma\chi_{22} \\ \frac{\chi_{21}}{\gamma} + \chi_{22} \\ \frac{d\left({z^*}^2 - w^2\right)}{e} \end{bmatrix},$$

$$\chi_{21} = \frac{b\left(az^*(x^*(a+w)+uw)+aw(au-wx^*)+x^*{z^*}^2(2a+w)+x^*{z^*}^3\right)}{(a+z^*)^2(a+w+\mathrm{z1})},$$
$$\chi_{22} = v(u(v+2)+vx^*),$$

$$\begin{aligned}\lim_{||x(t)||\to 0}\frac{||F(x(t))||}{||x(t)||} &= \lim_{||x(t)||\to 0}\frac{\sqrt{\left(\frac{\chi_1}{\gamma}+\chi_2\right)^2+(\chi_1+\gamma\chi_2)^2+\chi_3^2}}{\sqrt{u^2+v^2+w^2}} \\ &\le \lim_{||x(t)||\to 0}\left(\frac{\xi}{k^2(a+z^*)^4(a+w+z^*)^2\gamma^2}\right)^{\frac{1}{2}}\frac{||x(t)||^4}{||x(t)||} \\ &\le \lim_{||x(t)||\to 0}\left(\frac{\xi}{k^2(a+z^*)^4(a+w+z^*)^2\gamma^2}\right)^{\frac{1}{2}}||x(t)||^3 = 0,\end{aligned} \tag{5.2}$$

where $\xi = \max\{a^2\gamma^2, a^4\gamma^2, a^2\gamma^2, ab\gamma, ab^2, a^2b^2\}$. Therefore, the first condition of Theorem 2.2 is hold.

For testing the second condition, let $m = 30$, $\phi = \Gamma(1-\alpha)$, $l = 1$, and $\tau_m = \frac{1}{m}$, then,

$$-\frac{\theta_0}{\theta_1} = 0.0336633\begin{pmatrix} -\frac{bz^*}{a+z^*}-\gamma & \gamma(\beta-2x^*) & -\frac{abx^*}{(a+z^*)^2} \\ \frac{bz^*}{(a+z^*)\gamma}+1 & 2x^*-1 & \frac{abx^*}{(a+z^*)^2\gamma} \\ 0 & 0 & d-\frac{2dz^*}{k}\end{pmatrix}. \tag{5.3}$$

Thus, the characteristic equation is

$$\left(d-\frac{2dz^*}{k}-\lambda\right)\left(\lambda^2+\Phi_1\lambda+\Phi_2\right) = 0, \tag{5.4}$$

where

$$\Phi_1 = \frac{-2x^*(a+z^*)+\gamma(a+z^*)+a+bz^*+z^*}{a+z^*},$$
$$\Phi_2 = -\frac{(\beta-1)(\gamma(a+z^*)+bz^*)}{a+z^*}.$$

Then, the eigenvalues of (5.3) are

$$\lambda_1 = d-\frac{2dz^*}{k}, \tag{5.5}$$

$$\lambda_{2,3} = \frac{-\Phi_1\pm\sqrt{\Delta}}{2}, \qquad \Delta = \Phi_1^2-4\Phi_2. \tag{5.6}$$

Hence, the equilibrium point $E_1(1,1,0)$ is unstable, because the first λ_1 is positive.

For $E_2(a_0, 1, k)$, the corresponding eigenvalues can be written as

$$\begin{aligned}\lambda_1 &= -d\,, \qquad \lambda_{2,3} = \frac{R \pm \sqrt{\Delta}}{2}\,, \qquad \Delta = R^2 - 4Q\,,\\ R &= 1 - \gamma - bk\left(\frac{1}{a+k} + \frac{2}{bk + (a+k)\gamma}\right)\,,\\ Q &= \frac{(bk + (a+k)\gamma)(1-\beta)}{a+k}\,. \end{aligned} \tag{5.7}$$

If $\|\arg(\lambda_{2,3})\| > \frac{\mu_1 \pi}{2}$, then the equilibrium point $E_2(a_0, 1, k)$ is stable.

The previous condition is held for the other values of l

$$\left|\arg \lambda_i \left(-\frac{\theta_{l-1}}{\theta_l}\right)\right| = \pi > \frac{\pi \mu_l}{2} = \frac{\pi}{60}\,, \qquad i = 1, 2, 3, \dots\,; \quad l = 2, 3, 4, \dots, m\,. \tag{5.8}$$

Therefore, the solution of model (1.6) is asymptotically stable.

5.2. Periodic solutions of model (1.6)

For the numerical simulations of the distributed-order consumer model (1.6), we select two functions for $\phi(q)$ and the parameters values as follows: $a = 3$, $b = 2$, $\beta = 0.1$, $d = 1$, $k = 0.1$, and γ is a varying parameter with the initial point $E(0.9,\ 0.99,\ 0.1)$.

5.2.1. $\phi(q) = \Gamma(1-q)$

Figure 10 shows the behavior of model (1.6) when $\phi(q) = \Gamma(1-q)$. In Fig. 10 (a)–(b), we find that the behavior tends to a fixed point and the time series gradually loses its oscillation at $\gamma = 0.1$. Thus, the equilibrium point of model (1.6) is locally asymptotically stable, while in Fig. 10 (c)–(d), the solution approaches the orbit and the time series preserves its oscillations constantly at $\gamma = 0.4$.

5.2.2. $\phi(q) = \delta(q - q(m)) + 4\delta(q - q(m-1))$

The behavior of the solution of (1.6) with this function is shown in Fig. 11. In Fig. 11 (a)–(b), it is clear that the time series preserves its oscillation constantly at $\gamma = 0.7$. From Fig. 11 (c)–(d), we see that the solution approaches the equilibrium point at $\gamma = 1$. Thus, this equilibrium point is locally asymptotically stable.

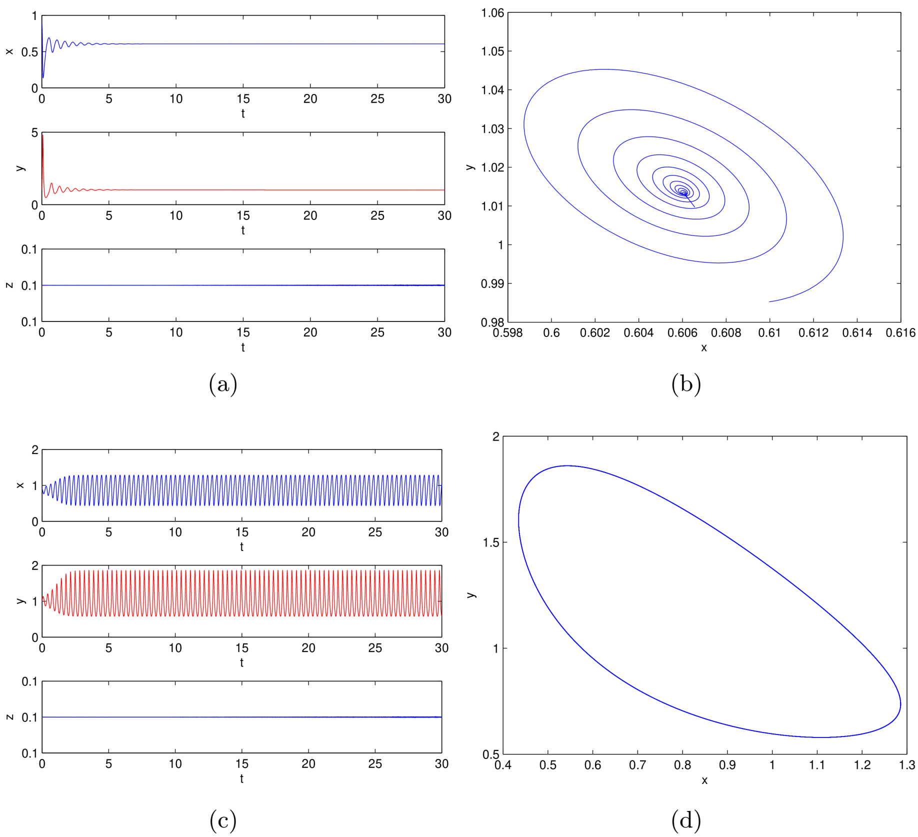
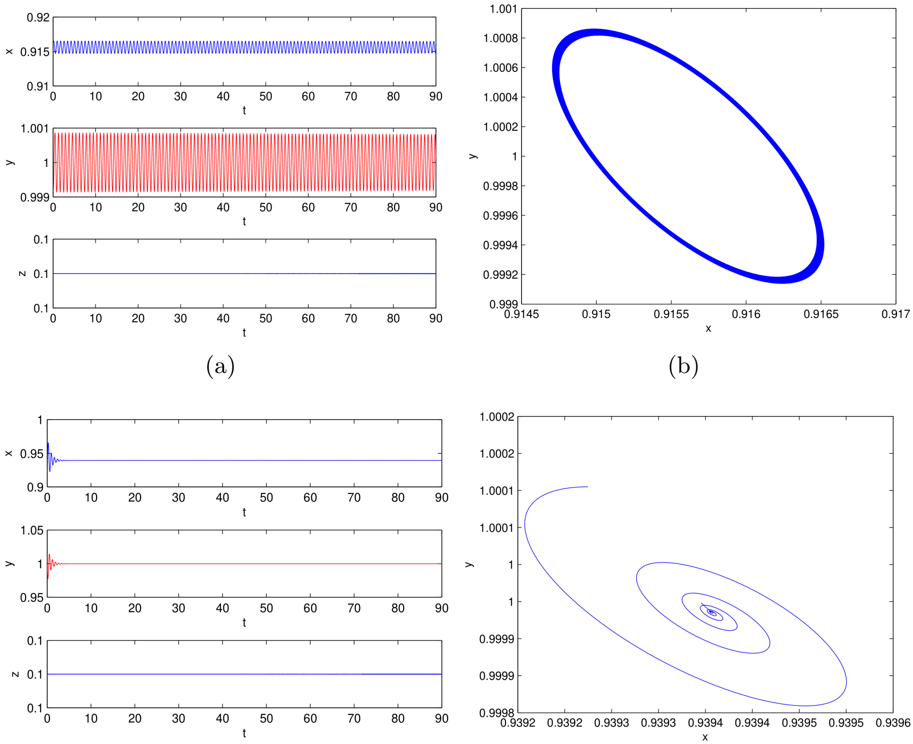


Fig. 10. The phase portrait and time series at $\phi(q) = \Gamma(1-q)$ and: (a), (b) $\gamma = 0.1$, and (c), (d) $\gamma = 0.4$.

6. Conclusion

The powerful theory of fractional-order non-linear differential equations has been used in these investigations. We have stated three new versions of the consumer model (1.3), which appeared in many applications.

These versions are the fractional-order model (1.4), the fractional-order model with time delay (1.5), and the distributed one (1.6). The dynamics of our models including the equilibrium points and their stability, Hopf bifurcations, and periodic solutions are studied numerically as well as analytically (see Figs. 3–11). A good agreement is found between both analytical and numerical results. It is shown that the fractional-order consumer model (1.4) has a Hopf bifurcations at the critical value q_0 which is stated in Theorems 3.1 and 3.2, and Lemma 3.1. Its fixed prints and their stability are studied, see Fig. 3.



(a) (b)

(c) (d)

Fig. 11. The phase portrait and time series at $\phi(q) = \delta(q-q(m)) + 4\delta(q-q(m-1))$ and: (a), (b) $\gamma = 0.7$, and (c), (d) $\gamma = 1$.

The fractional-order model (1.5) with time delay, which can reflect the memory and response behavior of customers more accurately, is investigated. It is illustrated that the critical value of the time delay τ_2 is sensitive to the change of the fractional-order q, see Table 1. The periodic solutions and Hop bifurcations of this model are discussed, see Figs. 7, 8, 9. For the distributed-order consumer model (1.6), we have investigated the stability conditions of its equilibria. Figures 10 and 11 describe the periodic solutions and their stability of this model for two different weight functions.

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