

WIGNER AND HUSIMI PARTONIC DISTRIBUTIONS  
OF THE PION IN A CHIRAL QUARK MODEL\*

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Generalized transverse momentum distributions (GTMDs), the Wigner and the Husimi distributions of quarks in the pion are evaluated in a chiral quark model at the one-loop level. Analytic expressions are obtained for GTMDs, allowing for a qualitative discussion of their features, whereas the Wigner and the Husimi distributions are obtained with numerical integration of simple formulas. We explain the features of the Wigner distributions, in particular their non-positivity. In our model, the Husimi distributions, which are interpreted as coarse-grained Wigner distributions, are not mathematically positive-definite, but the magnitude of their negative values is tiny and occurs at large transverse momenta and impact parameters. Hence, as expected, coarse graining leads to better-behaved functions from the point of view of the probabilistic interpretation.

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## 1. Introduction

In this paper, we provide a nontrivial model example of the Wigner [1] and Husimi [2] quark distributions in the pion at a low non-perturbative quark-model energy scale [3], confronting the issue of positivity of these distributions. Various kinds of partonic distributions are related to each

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other by integrations or by Fourier transforms (for the complete “genealogical tree”, see, *e.g.*, Fig. 2 of [4]), in particular, the Wigner distribution is a Fourier transform of the forward generalized transverse momentum distribution (GTMD) (see also [5–10]). Naturally, positivity of partonic distributions is a necessary feature for their probabilistic interpretation. As a matter of fact, it is highly nontrivial, and requesting that both (the forward) GTMD and the Wigner distributions are positive definite relates to a difficult mathematical problem of having both a function and its Fourier transform positive definite [11].

Even in Quantum Mechanics, the Wigner distributions [12] are in general not positive definite. A way of curing this problem with appropriate Gaussian coarse graining was proposed by Husimi [13]. The construction leads to a positive-definite phase-space density. A few years ago, the concept of the Husimi distributions was introduced to the partonic physics by Hagiwara, Hatta, and Ueda [2, 14]. Using an example of the light-front quark model wave functions from [8], these authors have shown that the corresponding non-positive Wigner distribution is made positive via the Husimi coarse graining.

In this paper, we work in a covariant one-quark-loop framework of low-energy quantum field theoretical chiral quark models, complying to all the Lorentz and gauge covariance requirements. As a result, both the charge and momentum sum rules are satisfied. The model, supplemented with the QCD evolution, was found to reasonably reproduce wide-ranging properties of the pion, such as the parton distribution functions (PDFs) [15–17], the distribution amplitude [18], the generalized distribution functions (GPDs) [3], the generalized form factors [19], the quasi-distribution amplitude [20], the quasi- and pseudo-PDFs, GPDs, and the Ioffe-time distributions [21, 22], or the double distribution functions [23].

Chiral quark models are generically intended to model hadron structure at a low-resolution scale in a non-perturbative scheme where explicit gluonic degrees of freedom are absent. All results of this paper pertain to a low non-perturbative quark model scale of  $\sim 330$  MeV [3], which is defined as the scale where quarks are the only degrees of freedom. We do not perform the DGLAP evolution to higher scales, whereby gluons would be radiatively generated, as this is highly nontrivial for GTMDs or their Fourier transforms.

We find that in our model, both the Wigner and the Husimi distributions are not positive definite. However, the negative values in the Husimi distributions are very small compared to the Wigner distributions, so the problem, though present mathematically, is from a “practical” point of view significantly improved through the Husimi coarse graining.

The Wigner distributions in the light-front approach were recently studied in [24–27], where the non-positivity feature is also manifest.

## 2. Definitions

We denote a generic four-vector as  $a = (a^0, \mathbf{a}, a^3)$ , with the bold face indicating the transverse part  $\mathbf{a} = (a^1, a^2)$ . We also use  $a_T \equiv |\mathbf{a}|$ . The light-cone coordinates are  $a^\pm = (a^0 \pm a^3)/\sqrt{2}$ , whence  $a \cdot b = a^+ b^- + a^- b^+ - \mathbf{a} \cdot \mathbf{b}$ . The symmetric kinematic convention (the Breit frame) is used, where the momenta of the initial and final pions are, correspondingly,  $p \pm \frac{1}{2}\Delta$ . We also introduce a null vector  $n$ , whence

$$p = (p^0, \mathbf{0}, 0), \quad \Delta = (0, \mathbf{\Delta}, 0), \quad n = \frac{1}{p^0}(1, \mathbf{0}, -1). \quad (1)$$

The initial and final pions are on mass shell, hence

$$p^0 = \sqrt{m_\pi^2 + \frac{1}{4}\mathbf{\Delta}^2}, \quad (2)$$

where  $m_\pi$  is the pion mass. Covariantly,

$$n^2 = 0, \quad p^2 = m_\pi^2 - \frac{1}{4}t, \quad t = -\mathbf{\Delta}^2, \quad p \cdot n = 1, \quad p \cdot \Delta = 0, \quad \xi \equiv n \cdot \Delta = 0, \quad (3)$$

where the last condition reflects the vanishing skewness  $\xi$  for the considered forward case. In the partonic framework,

$$k \cdot n = x, \quad (4)$$

where in the so-called symmetric convention,  $k$  is the average (before and after the interaction) momentum of the probed quark (*cf.* Fig. 1).

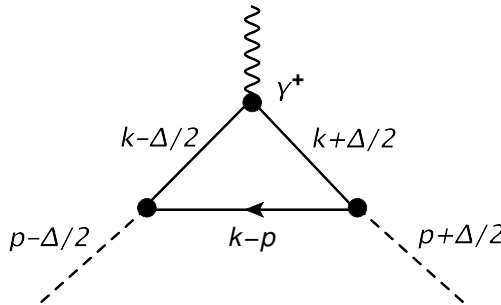


Fig. 1. One-quark-loop diagram for the evaluations of the quark forward GTMDs in chiral quark models. Condition (4) is understood and the transverse momentum  $\mathbf{k}$  is fixed, hence the loop integration is only over  $k^-$ . The dashed lines indicate the pion states and the solid line is the quark propagator. The distribution of antiquarks is given by the crossed diagram.

The leading-twist chirally even generalized transverse-momentum distributions (GTMDs) at zero skewness are defined via the following matrix elements:

$$\delta_{ab}\delta_{\alpha\beta}\mathcal{G}^{I=0}(x, \mathbf{\Delta}, \mathbf{k}) + i\epsilon^{abc}\tau_{\alpha\beta}^c\mathcal{G}^{I=1}(x, \mathbf{\Delta}, \mathbf{k}) = \frac{1}{2}\int\frac{dz^-d^2\mathbf{z}}{(2\pi)^3}e^{ixp^+z^- - i\mathbf{k}\cdot\mathbf{z}} \\ \times \left\langle \pi^b\left(p - \frac{1}{2}\mathbf{\Delta}\right) \left| \bar{\psi}_\alpha\left(-\frac{\mathbf{z}}{2}\right) \gamma^+ \mathcal{L} \psi_\beta\left(\frac{\mathbf{z}}{2}\right) \right| \pi^a\left(p + \frac{1}{2}\mathbf{\Delta}\right) \right\rangle \Big|_{z^+=0}, \quad (5)$$

where the superscripts in  $\mathcal{G}$  indicate the isospin combinations,  $\psi$  stands for the quark field, indices  $\alpha$  and  $\beta$  represent the quark flavor,  $a, b$  denote the isospin of the pions, while  $c$  is the isospin of the probing operator. The symbol  $\mathcal{L}$ , that makes the expression gauge invariant, is the staple-shaped [28] Wilson line extending along the light-cone coordinate  $z^-$ . A definition similar to (5) holds also for the gluons, not considered here.

The good isospin combinations of GTMDs from Eq. (5) are related to GTMDs of quarks and antiquarks as follows:

$$\mathcal{G}^{q,\bar{q}}(x, \mathbf{\Delta}, \mathbf{k}) = \frac{1}{2} [\mathcal{G}^{I=0}(x, \mathbf{\Delta}, \mathbf{k}) \pm \mathcal{G}^{I=1}(x, \mathbf{\Delta}, \mathbf{k})]. \quad (6)$$

By general arguments of the Lorentz covariance, the function  $\mathcal{G}^q(x, \mathbf{\Delta}, \mathbf{k})$  has the support  $x \in [0, 1]$ , whereas  $\mathcal{G}^{\bar{q}}(x, \mathbf{\Delta}, \mathbf{k})$  has the support  $x \in [-1, 0]$ . It complies with the convention that

$$\int d^2\mathbf{k} \mathcal{G}^q(x, \mathbf{\Delta} = 0, \mathbf{k}) = q(x), \\ \int d^2\mathbf{k} \mathcal{G}^{\bar{q}}(-x, \mathbf{\Delta} = 0, \mathbf{k}) = -\bar{q}(x), \quad (7)$$

where  $q(x)$  and  $\bar{q}(x)$  are the (positive) parton distribution functions (PDFs) of, correspondingly, quarks and antiquarks with the support  $x \in [0, 1]$ .

### 3. GTMDs in chiral quark models at the one-loop level

For simplicity, from now on we work in the strict chiral limit of the vanishing current quark mass. The model used in this paper is a nonlinear realization of the chiral quark model, with the Lagrangian density

$$\mathcal{L}(x) = \bar{\psi}(x) \left[ i\not{\partial} - \omega e^{-i\gamma_5\tau^a\phi^a(x)} \right] \psi(x). \quad (8)$$

Here,  $\omega$  denotes the (constituent) quark mass following from the dynamical chiral symmetry breaking,  $\tau_a$  are the Pauli isospin matrices, and  $\phi^a$  is the pion field, which is a Goldstone boson according to the Nambu–Jona-Lasinio mechanism. The pion decay constant  $f = 86$  MeV in the chiral limit.

With the nonlinear realization, one maintains chiral symmetry without introducing the  $\sigma$  field of the linear model, which results in simpler results for the off-forward case (not studied here). From Eq. (8), we read out the quark–pion vertex  $i\omega/f \gamma_5 \tau^a$ .

At the one-quark-loop level, which corresponds to the leading- $N_c$  quark-model calculation, GTMDs are evaluated from the diagram of Fig. 1. From definition (5) transformed into the momentum space, the Feynman rules yield the following expression:

$$G^q(x, \Delta, \mathbf{k}) = \frac{N_c \omega^2}{f^2} \int \frac{dk^+ dk^-}{(2\pi)^4} \delta(k^+ - x) \text{Tr} \left[ \gamma^+ S_{k-\frac{q}{2}} \gamma_5 S_{k-p} \gamma_5 S_{k+\frac{q}{2}} \right], \quad (9)$$

where

$$S_k = \frac{i}{\not{k} - \omega + i\epsilon} \quad (10)$$

denotes the quark propagator and the trace is over the Dirac space. The evaluation of the trace and the standard one-loop reduction yields the basic structure

$$G^q(x, \Delta, \mathbf{k}) = \frac{1}{2} [I_- + I_+ + (1-x)J]. \quad (11)$$

The one-loop functions appearing above are defined as

$$\begin{aligned} I_s &= -i \frac{N_c \omega^2}{4\pi^4 f^2} \int dk^+ dk^- \frac{\delta(k \cdot n - x)}{[(k-p)^2 - \omega^2 + i\epsilon] \left[ \left(k + s\frac{q}{2}\right)^2 - \omega^2 + i\epsilon \right]}, \\ J &= -i \frac{N_c \omega^2 t}{4\pi^4 f^2} \int dk^+ dk^- \\ &\quad \times \frac{\delta(k \cdot n - x)}{[(k-p)^2 - \omega^2 + i\epsilon] \left[ \left(k + \frac{q}{2}\right)^2 - \omega^2 + i\epsilon \right] \left[ \left(k - \frac{q}{2}\right)^2 - \omega^2 + i\epsilon \right]}, \end{aligned} \quad (12)$$

where  $s = \pm 1$ . Note that the definition of  $J$  includes the factor of  $t$ . The arguments of all the functions in the above equations are  $(x, \Delta, \mathbf{k})$ . However, the distributive structure of Eq. (11) is generic and holds also for the Wigner and the Husimi distributions discussed later on, or for the GPDs as derived in [3]. The loop functions  $I_s$  and  $J$  are evaluated explicitly in Appendix A.

#### 4. Regularization

The evaluation of GPDs, which are integrals of GTMDs over  $d^2\mathbf{k}$ , requires regularization since the two-point function  $I_s$  is logarithmically divergent. However, the need for regularization is physically motivated also for finite quantities, in order to separate the hard momenta, not treated in

low-energy models, and the soft momenta, crucial for the dynamics. The spectral regularization [29] used in this paper, or the Pauli–Villars regularization, may be carried out in an elegant way on the formulas for the loop functions. The key feature here is that the product of the Klein–Gordon propagators written in the Schwinger representation, appearing in the one-loop functions such as (12) (*cf.* Appendix A), contains generically the factor  $\exp[-(\alpha_1 + \alpha_2 + \dots + \alpha_n)\omega^2]$ , where  $\alpha_i$  are the Schwinger parameters. Then, regularizations involving distributions of the quark mass  $\omega$  lead to simple (analytic) expressions<sup>1</sup>.

The spectral regularization [3, 29] amounts to the evaluation of the quark loop integrals according to the prescription

$$\mathcal{A}^{\text{SQM}} = \int_C d\omega \rho(\omega) \mathcal{A}, \quad (13)$$

where  $\mathcal{A}$  stands for an unregularized amplitude,  $\rho(\omega)$  is a properly chosen spectral density function, and  $C$  is a contour of integration in the complex  $\omega$  plane (*cf.* Fig. 1 in [29]). In SQM, one has a possibility to implement exactly the vector meson dominance in the pion electromagnetic form factor, which is successful phenomenologically. More details are provided in Appendix B.

## 5. Quark GTMD

With the formulas from Appendix A, it is straightforward to write down the explicit formulas for the forward quark GTMD in SQM

$$\begin{aligned} \mathcal{G}^q(x, \Delta, \mathbf{k}) = & \frac{3M_V^3}{\pi} \left[ \frac{1}{(4\mathbf{k}_+^2 + M_V^2)^{5/2}} + \frac{1}{(4\mathbf{k}_-^2 + M_V^2)^{5/2}} \right] \\ & + \frac{3M_V^3(1-x)\Delta^2}{2\pi \mathbf{k} \cdot \Delta} \left[ \frac{1}{(4\mathbf{k}_+^2 + M_V^2)^{5/2}} - \frac{1}{(4\mathbf{k}_-^2 + M_V^2)^{5/2}} \right], \quad (14) \end{aligned}$$

with the shorthand notation

$$\mathbf{k}_{\pm} = \mathbf{k} \pm \frac{1}{2}(1-x)\Delta. \quad (15)$$

The form of  $\mathcal{G}^q$  exhibits explicitly the scaling

$$\mathcal{G}^q(x, \Delta, \mathbf{k}) = \mathcal{G}^q[(1-x)\Delta, \mathbf{k}], \quad (16)$$

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<sup>1</sup> Note that such a prescription is equivalent to subtractions, hence it may promptly affect the positivity property of the parton distributions.

where the arguments in the right-hand side indicate that  $\mathcal{G}^q$  is a function of the combination  $(1-x)\Delta$  (and  $\mathbf{k}$ ) only. This feature is specific to the one-loop model in the chiral limit.

In Fig. 2, we plot  $\mathcal{G}^q$  as a function of  $k_T$  and  $(1-x)\Delta_T$  for a selected value of the angle between  $\mathbf{k}$  and  $\Delta$ , namely  $\phi = \pi/4$  (for other values of  $\phi$  the results are qualitatively similar). We note that  $\mathcal{G}^q$  is positive near the origin, but at larger values of  $\Delta_T$ , it assumes negative values, indicated with a lighter (blue) color.

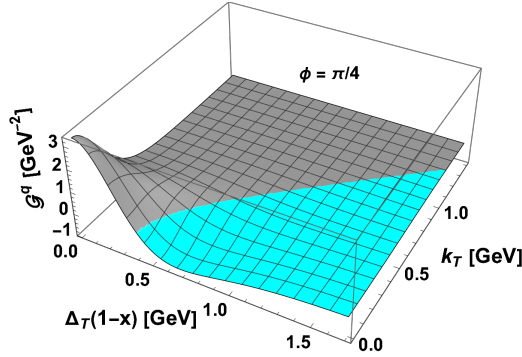


Fig. 2. Quark forward ( $\xi = 0$ ) GTMD of the pion in SQM, plotted as a function of the quark transverse momentum  $k_T$  and the combination  $(1-x)\Delta_T$  for a sample value of the angle  $\phi$  between  $\mathbf{k}$  and  $\Delta$ . The lighter (blue) color indicates negative values of the distribution.

For the special case of  $\Delta = \mathbf{0}$ , Eq. (14) reduces to the  $k_T$ -unintegrated PDF and equals to

$$\mathcal{G}^q(x, \mathbf{0}, \mathbf{k}) = \frac{6M_V^3}{\pi (4k^2 + M_V^2)^{5/2}}, \quad (17)$$

in agreement with [30]. The forward ( $\xi = 0$ ) GPD is equal to [29]

$$G^q(x, \Delta) = \int d^2\mathbf{k} \mathcal{G}^q(x, \Delta, \mathbf{k}) = \frac{M_V^2 (M_V^2 + t(x-1)^2)}{(M_V^2 - t(x-1)^2)^2}, \quad (18)$$

and the corresponding forward impact parameter distribution

$$\begin{aligned} q(x, \mathbf{b}) &= \int \frac{d^2\Delta}{(2\pi)^2} e^{-i\Delta \cdot \mathbf{b}} G^q(x, \Delta) \\ &= \frac{M_V^2}{2\pi(1-x)^2} \left[ \frac{b_T M_V}{1-x} K_1 \left( \frac{b_T M_V}{1-x} \right) - K_0 \left( \frac{b_T M_V}{1-x} \right) \right], \end{aligned} \quad (19)$$

where  $K_0$  and  $K_1$  are modified Bessel functions (we correct a global minus sign typo of [30]). This function is *not* positive definite at small values of  $b_T \leq (1-x)/M_V z_0$ , with  $z_0 \simeq 0.596$  fulfilling  $K_0(z_0) = z_0 K_1(z_0)$ , and actually at  $b_T \rightarrow 0$  diverges logarithmically.

The electromagnetic and gravitational form factors are [19]

$$\begin{aligned} F_V(t) &= \int_0^1 dx G^q(x, \Delta) = \frac{M_V^2}{M_V^2 - t}, \\ \theta_2(t) &= 2 \int_0^1 dx x G^q(x, \Delta) = \frac{M_V^2}{t} \log \left( \frac{M_V^2}{M_V^2 - t} \right). \end{aligned} \quad (20)$$

Clearly,  $F_V(0) = \theta_2(0) = 1$ , expressing the charge and momentum (mass) sum rules.

## 6. Wigner distributions

The Wigner distribution is the Fourier transform of the forward GTMD from the momentum space,  $\Delta$ , into the impact parameter space,  $\mathbf{b}$ ,

$$\mathcal{W}^q(x, \mathbf{b}, \mathbf{k}) = \int \frac{d^2 \Delta}{(2\pi)^2} e^{-i\Delta \cdot \mathbf{b}} \mathcal{G}^q(x, \Delta, \mathbf{k}). \quad (21)$$

The marginal distributions are the impact-parameter distribution

$$\int d^2 \mathbf{k} \mathcal{W}^q(x, \mathbf{b}, \mathbf{k}) = q(x, \mathbf{b}), \quad (22)$$

and the TMD distribution

$$\int d^2 \mathbf{b} \mathcal{W}^q(x, \mathbf{b}, \mathbf{k}) = T^q(x, \mathbf{k}) = \mathcal{G}^q(x, \mathbf{0}, \mathbf{k}). \quad (23)$$

The normalization from the double integration yields the quark PDF

$$\int d^2 \mathbf{b} d^2 \mathbf{k} \mathcal{W}^q(x, \mathbf{b}, \mathbf{k}) = \int d^2 \mathbf{k} \mathcal{G}^q(x, \mathbf{0}, \mathbf{k}) = q(x). \quad (24)$$

From Eq. (16) it follows that in our model,  $\mathcal{W}^q$  obeys the scaling

$$\mathcal{W}^q(x, \mathbf{b}, \mathbf{k}) = \frac{1}{(1-x)^2} \mathcal{W}^q[\mathbf{b}/(1-x), \mathbf{k}]. \quad (25)$$



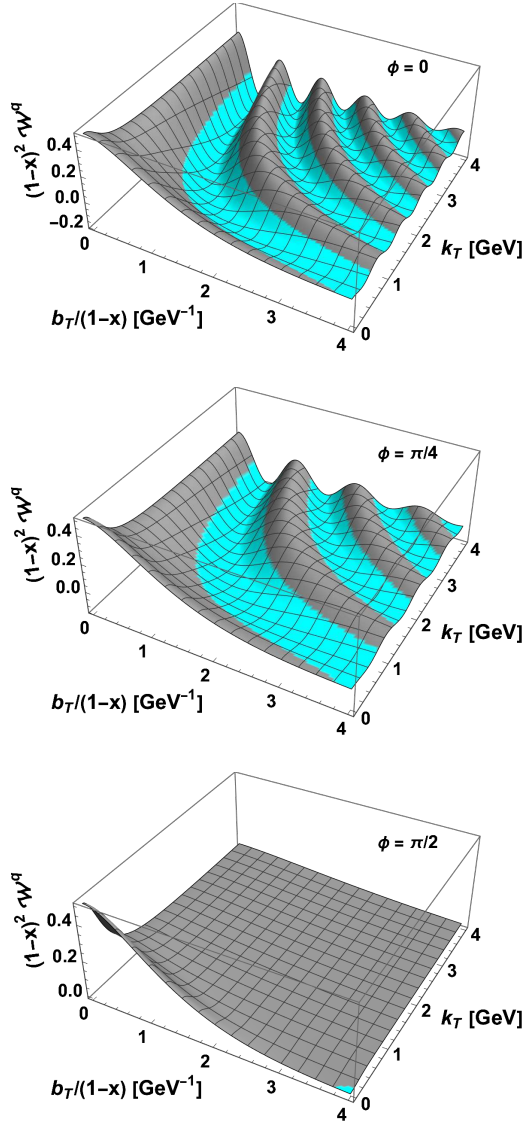


Fig. 3. The quark Wigner distribution of the pion in SQM multiplied with  $(1-x)^2$ , plotted as a function of the quark transverse momentum  $k_T$  and the combination  $b_T/(1-x)$  for three sample values of the angle  $\phi$  between  $\mathbf{b}$  and  $\Delta$ . The lighter (blue) color indicates negative values of the distributions.

With Eqs. (A.6) and (A.8) we get the following semi-analytic formula in SQM:

$$\begin{aligned}
 (1-x)^2 \mathcal{W}^q(x, \mathbf{b}, \mathbf{k}) &= \frac{1}{\pi^2} e^{-B_T M_V} (B_T M_V + 1) \cos(2B_T k_T \cos \phi) \\
 &\quad - \left[ \frac{\partial^2}{\partial b_T^2} + \frac{1}{b_T} \frac{\partial}{\partial b_T} \right] \int_{-1}^1 dz \frac{M_V^3}{2\pi^2 u^5} e^{-\sqrt{2} B_T u} \\
 &\quad \times \left[ B_T u \left( 2B_T u + 3\sqrt{2} \right) + 3 \right] \cos(2B_T k_T z \cos \phi) , \quad (26)
 \end{aligned}$$

where

$$u = \sqrt{M_V^2 + 4k_T^2 (1 - z^2)}, \quad B_T = \frac{b_T}{1 - x}.$$

The first line in Eq. (26) originates from the two-point functions, whereas the integral over the parameter  $z$  comes from the three-point functions specified in Appendix A. The appropriate differentiation with respect to  $b_T$  brings down the factor of  $\Delta^2$  present in the integrand of the definition of the Wigner transform (21). The transverse momentum and impact parameter marginal distributions are checked to be given by Eq. (17) and Eq. (19) respectively.

The quark Wigner distribution of the pion obtained in our model is plotted in Fig. 3. We clearly note an oscillatory character, following directly from the form of the argument of the cosine functions in Eq. (26). The contribution of the two-point functions  $I_s$  contains  $\cos(2B_T k_T \cos \phi)$ , hence its zeros are at hyperbolas in the  $b_T - k_T$  plane at locations

$$b_T k_T = \frac{(2j+1)\pi}{4 \cos \phi} (1-x), \quad j \in \mathbb{Z}. \quad (27)$$

The contribution of the three-point loop function  $J$  contains an integral of  $\cos(2zB_T k_T \cos \phi)$ , with the rest of the integrand sharply peaked at  $z = \pm 1$ , which results in a condition close to Eq. (27). The behavior of Eq. (27) is clearly seen in Fig. 3. In particular, we note that the period of the oscillations increases as  $1/\cos \phi$ , and no oscillations occur at the boundaries  $b_T = 0$  or  $k_T = 0$ , or when  $\mathbf{b}$  and  $\mathbf{k}$  are perpendicular, *i.e.*,  $\phi = \pi/2$ .

## 7. Husimi distributions

Consider the smeared distribution

$$\mathcal{H}^q(x, \mathbf{b}, \mathbf{k}) = \frac{\Lambda^2}{\pi^2 \lambda^2} \int d^2 \mathbf{b}' d^2 \mathbf{k}' e^{-(\mathbf{b}-\mathbf{b}')^2/\lambda^2 - (\mathbf{k}-\mathbf{k}')^2 \Lambda^2} \mathcal{W}^q(x, \mathbf{b}', \mathbf{k}') , \quad (28)$$

where  $\lambda$  and  $\Lambda$  are the smearing parameters (with dimension of length). In Quantum Mechanics, the choice  $\Lambda = \lambda$  warrants the positivity of (28), which then becomes the celebrated Husimi distribution. Therefore, we follow [2, 14] and fix  $\Lambda = \lambda$  in our field-theoretic model as well. The normalization is

$$\int d^2\mathbf{b} d^2\mathbf{k} \mathcal{H}^q(x, \mathbf{b}, \mathbf{k}) = q(x). \quad (29)$$

Note that  $\mathcal{H}^q$  does not exhibit scaling analogous to Eq. (25) (unless we also scaled  $\lambda$  with  $(1-x)$  and kept a separate  $\Lambda$  unscaled, which we do not do). Since the scale in our model is the vector meson mass  $M_V$ , we show the case  $\lambda = M_V$ , while other values lead to qualitatively similar results. The procedure for obtaining the Husimi distributions in our model is outlined in Appendix A.

The results for  $x = 0.8$  and  $x = 0.2$  are plotted, correspondingly, in Figs. 4 and 5. We take  $\phi = \pi/4$ , as the results for other angles are qualitatively similar. We notice that in SQM, the Husimi distributions are not strictly positive, as at larger values of  $b_T$  ( $b_T \gtrsim 2 \text{ GeV}^{-1} \simeq 0.5 \text{ fm}$ ), we find broad regions with negative values. However, contrary to the case of the Wigner distributions of Fig. 3, the distributions in these regions are very shallow compared to the value of the function at the origin. One might say that the positivity breaking has been cured to a large extent by the application of the Husimi coarse-graining procedure, although mathematically the problem does persist. The results are qualitatively similar at various values of  $x$ , as can be seen by comparing Figs. 4 and 5.

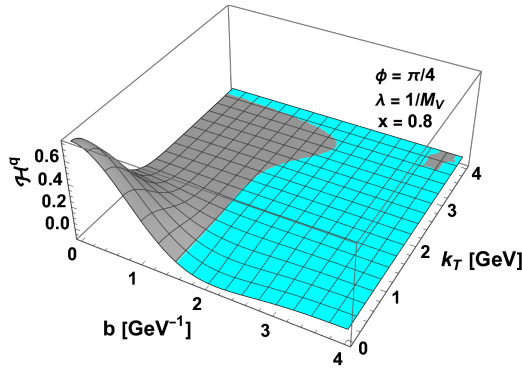


Fig. 4. The Husimi distributions of quarks in the pion in SQM, plotted as a function of the quark transverse momentum  $k_T$  and the transverse coordinate  $b_T$  for the angle  $\phi = \pi/4$  between  $\mathbf{b}$  and  $\Delta$ ,  $x = 0.8$ , and a sample value of the smearing scale  $\lambda = 1/M_V$ . The lighter color indicates negative values of the distributions.

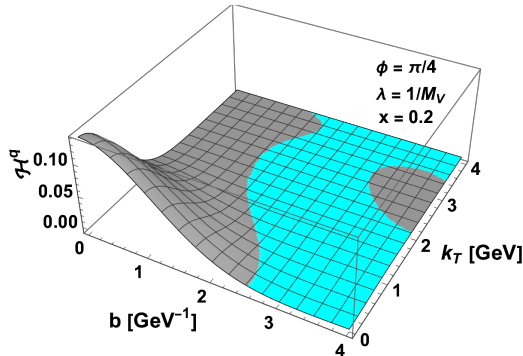


Fig. 5. The same as in Fig. 4, but for  $x = 0.2$ .

The problem of the lack of strict positivity of the Husimi distributions in our field-theoretic model may possibly be traced back to the subtractive nature of regularization, which also led to the non-positivity of the marginal impact parameter distribution of Eq. (19). We note that a similar situation was also encountered in the impact-parameter behavior of the double parton distributions [23], where an analogous conflict between regularization and positivity was faced. Further studies of this intriguing and genuine field theoretic issue of regularization *vs.* positivity are left for future research.

## 8. Conclusions

Several merits of the model study presented in this paper should be underlined. First, the model is simple but nontrivial, leading to a rich set of results for various pion properties. In particular, the framework allows to obtain the quark GTMD of the pion analytically. This GTMD complies with all the formal requirements, in particular the resulting GPDs possess polynomiality, with the charge and momentum sum rules satisfied, although no factorization in the kinematic variables is present in the model. The corresponding Wigner distribution contains one numerical integration, and the Husimi distribution — two, with simple integrands. In the obtained Wigner distribution, we can clearly see the origin and the pattern of the oscillations leading to the breaking of positivity. In the Husimi distributions, these oscillations are smoothed by coarse-graining. Although the Husimi distributions in our model are not strictly positive, the negative values are tiny and appear at larger values of the impact parameter,  $b_T \gtrsim 0.5$  fm.

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## Appendix A

### Loop functions

In this appendix we work with Euclidean momenta, corresponding to their Minkowski counterparts, but denoted with the same symbols. An effective way [3] to evaluate the one-loop integrals with a momentum constraint is to use the Schwinger representation of the Klein–Gordon propagator

$$\frac{1}{k^2 + \omega^2} = \int_0^\infty d\alpha e^{-\alpha(k^2 + \omega^2)}. \quad (\text{A.1})$$

The two-point function from Eq. (12) is then expressed as

$$\begin{aligned} I_s &= \frac{4N_c \omega^2}{f^2} \int \frac{dk_0 dk_3}{(2\pi)^2} \int \frac{d\xi}{2\pi} e^{i\xi(k \cdot n - x)} \int_0^\infty d\alpha \int_0^\infty d\beta e^{-\alpha[(k-p)^2 + \omega^2] - \beta[(k+s\Delta/2)^2 + \omega^2]} \\ &= \frac{N_c}{4\pi^3 f^2} \int_0^\infty d\alpha \int_0^\infty d\beta e^{-\frac{(2\mathbf{k}(\alpha+\beta) + \beta s \boldsymbol{\Delta})^2}{4(\alpha+\beta)} - (\alpha+\beta)\omega^2} \delta[x(\alpha + \beta) - \alpha], \end{aligned} \quad (\text{A.2})$$

where the intermediate steps follow exactly [3]. In SQM, with Eq. (B.2), the procedure yields

$$\begin{aligned} I_s &= \frac{M_V^3}{4\pi^{3/2}} \int_0^\infty d\alpha \int_0^\infty d\beta (\alpha + \beta)^{3/2} e^{-\frac{(2\mathbf{k}(\alpha+\beta) + \beta s \boldsymbol{\Delta})^2}{4(\alpha+\beta)} - \frac{1}{4}(\alpha+\beta)M_V^2} \delta[x(\alpha + \beta) - \alpha] \\ &= \frac{6M_V^3}{\pi ([2\mathbf{k} + s(1-x)\boldsymbol{\Delta}]^2 + M_V^2)^{5/2}}. \end{aligned} \quad (\text{A.3})$$

For the three-point function from Eq. (12), an analogous calculation yields

$$\begin{aligned}
J &= -\frac{4N_c \omega^2 t}{f^2} \int \frac{dk_0 dk_3}{(2\pi)^2} \int \frac{d\xi}{2\pi} e^{i\xi(k \cdot n - x)} \int_0^\infty d\alpha \int_0^\infty d\beta \int_0^\infty d\gamma \\
&\quad \times e^{-\alpha[(k-p)^2 + \omega^2] - \beta[(k+\Delta/2)^2 + \omega^2] - \gamma[(k-\Delta/2)^2 + \omega^2]} \\
&= -\frac{N_c \omega^2 t}{4\pi^3 f^2} \int_0^\infty dh h \int_0^1 dy \int_0^1 dz \theta(1-y-z) \delta(1-y-z-x) \\
&\quad \times e^{-h[\mathbf{k}^2 + \omega^2 + \boldsymbol{\Delta} \cdot \mathbf{k}(y-z) + \frac{1}{4} \boldsymbol{\Delta}^2 (y+z)^2]}, \tag{A.4}
\end{aligned}$$

where  $h = \alpha + \beta + \gamma$ ,  $y = \beta/h$ , and  $z = \gamma/h$ . In SQM,

$$\begin{aligned}
J &= -\frac{M_V^3 t}{4\pi^{3/2}} \int_0^\infty dh h^{5/2} \int_0^1 dy \int_0^1 dz \theta(1-y-z) \delta(1-y-z-x) \\
&\quad \times e^{-h[\mathbf{k}^2 + \frac{1}{4} M_V^2 + \boldsymbol{\Delta} \cdot \mathbf{k}(y-z) + \frac{1}{4} \boldsymbol{\Delta}^2 (y+z)^2]} \\
&= \frac{3M_V^3 \boldsymbol{\Delta}^2}{\pi \mathbf{k} \cdot \boldsymbol{\Delta}} \left[ \frac{1}{(4\mathbf{k}_-^2 + M_V^2)^{5/2}} - \frac{1}{(4\mathbf{k}_+^2 + M_V^2)^{5/2}} \right] \tag{A.5}
\end{aligned}$$

in the notation of Eq. (15) used in the last line.

The Fourier transform of  $I_s$  is elementary

$$\begin{aligned}
\int \frac{d^2 \boldsymbol{\Delta}}{(2\pi)^2} e^{-i\boldsymbol{\Delta} \cdot \mathbf{b}} I_s(x, \boldsymbol{\Delta}, \mathbf{k}) &= \int \frac{d^2 \boldsymbol{\Delta}'}{(2\pi)^2} \frac{6M_V^3 e^{-i\mathbf{b} \cdot (\boldsymbol{\Delta}' - \frac{2s\mathbf{k}}{1-x})}}{\pi [M_V^2 + \boldsymbol{\Delta}'^2 (1-x)^2]^{5/2}} \\
&= \frac{B_T M_V + 1}{\pi^2 (1-x)^2} e^{-B_T (M_V - 2is\mathbf{k}_T)}, \tag{A.6}
\end{aligned}$$

with  $\boldsymbol{\Delta}' = \boldsymbol{\Delta} + 2s\mathbf{k}/(1-x)$  and the notation of Eq. (6) used. For the case of the three-point function, it is useful to express it via the integral representation

$$J/\boldsymbol{\Delta}^2 = \frac{16M_V^3}{\pi^{3/2}} \int_0^\infty dh \int_{-1}^1 dz h^{5/2} e^{-h(4\mathbf{k}^2 - 4z\mathbf{k} \cdot \boldsymbol{\Delta} + \boldsymbol{\Delta}^2 + M_V^2)} \tag{A.7}$$

(for simplification, here we get rid of factor  $\boldsymbol{\Delta}^2$ , which is later restored via differentiation with respect to  $\mathbf{b}$ ). Carrying out first the Fourier transform from  $\boldsymbol{\Delta}$  to  $\mathbf{b}$ , and then integrating over  $h$  yields the formula

$$\begin{aligned}
 (1-x)^2 \int \frac{d^2 \Delta}{(2\pi)^2} J / \Delta^2 &= \int_{-1}^1 dz \frac{M_V^3}{\pi^2 u^5} e^{-\sqrt{2} B_T u} \\
 &\times \left[ B_T u (2 B_T u + 3\sqrt{2}) + 3 \right] \cos(2 B_T k_T z \cos \phi) \quad (\text{A.8})
 \end{aligned}$$

in the notation of Eq. (6). The one-dimensional integral over  $z$  is left to be done numerically.

To evaluate the Fourier transforms needed for the Husimi distributions, it is convenient to rewrite Eq. (A.3) in the integral representation

$$I_s = \frac{4M_V^3}{\pi^{3/2}} \int_0^\infty dh h^{3/2} e^{-h(4k_s^2 + M_V^2)}, \quad s = \pm 1. \quad (\text{A.9})$$

Then the Gaussian integrations over  $\mathbf{b}'$ ,  $\mathbf{k}'$ , and  $\mathbf{q}$  are carried out in a straightforward way, leaving a numerical integration over  $h$ . In the case of the three-point function  $J$ , a completely analogous procedure is carried out on expression (A.7), with the integrals over the  $h$  and  $z$  remaining as numerical.

## Appendix B

### *Spectral regularization*

The spectral function is [29]

$$\rho(\omega) = \frac{1}{2\pi i} \frac{1}{\omega} \frac{1}{(1 - 4\omega^2/M_V^2)^{5/2}}, \quad (\text{B.1})$$

and the closed quark line is associated with the integral over a suitably chosen contour  $C$ . The form of (B.1) implements vector meson dominance in the pion electromagnetic form factor. For the present applications, we need the formula

$$\int_C d\omega \omega^2 \rho(\omega) e^{-A\omega} = \frac{A^{3/2} M_V^5}{24\sqrt{\pi}} e^{-\frac{AM_V^2}{4}}, \quad A \geq 0. \quad (\text{B.2})$$

In SQM, there is the following relation between the pion decay constant and the vector meson mass:

$$f^2 = \frac{N_c M_V^2}{24\pi^2}. \quad (\text{B.3})$$

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