OFF-ENERGY-SHELL SCATTERING BY ADDITIVE INTERACTIONS UNDER AN APPROXIMATION SCHEME

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Inelastic scattering of charged hadronic systems is studied by considering a new approximation scheme to effective potential. A short-ranged electromagnetic interaction with the same range as the nuclear one is adapted to visualize the effect of such a potential model in treating the off-energy-shell scattering of the nuclear systems. Under this approximation, the Schrödinger equation admits an exact analytical solution and the related off-shell quantities are expressed in their maximal reduced form to make them amenable to numerical treatment. The nucleon–light nuclei system is studied and close agreement in numerical results with other works is found.

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1. Introduction

In our previous article, we have treated the charged hadron scattering under additive interaction, namely, short-ranged electromagnetic plus nuclear potential [1]. In practice, it is difficult to achieve an exact solution of the Schrödinger equation with two such local potentials. When the short-range local nuclear potential is replaced by a non-local separable interaction, one can get an exact solution of the concerned system and such problems have been advocated by a number of research groups [2–15]. In the recent past [16, 17], one of us (U.L.) treated the alpha–proton and alpha–alpha systems for a few lower partial waves by calculating approximate s-wave analytical solutions of the nuclear Hulthén plus the atomic Hulthén potential with different range of interactions in conjunction with the formalism of super-symmetric algebra. For simplicity of calculation and to obtain an exact solution, people may consider the same ranged electromagnetic and nuclear parts of the total interaction. This is no loss of generality as, in

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practical situation, the effect of such two-potentials is observed within the nuclear domain [1]. In Ref. [1], we achieved close agreement in scattering phase parameters for nucleon–nucleus and nucleus–nucleus elastic scattering under such approximation and thereby established our conjecture. The present text addresses itself to the study of inelastic scattering involving the off-shell Jost function, half-off-shell transition matrix (T-matrix), and off-shell extension function [3, 7, 9–15, 18–20] for motion in the nuclear Manning–Rosen plus atomic Hulthén potential [1, 21–27]. Here, we treat only the s-wave case as the all partial wave treatment involves inordinate mathematical complications.

2. Methods for off-shell solutions

In this section, we adapt two different approaches to the problem of construction of exact analytical expression for the off-shell Jost solution for motion in the Manning–Rosen plus Hulthén potential by exploiting the theory of ordinary differential equation in conjunction with the properties of the special functions of mathematical physics.

2.1. Differential equation approach

The nuclear Manning–Rosen potential is defined by [21–26]

$$V_{\mathrm{N}}(s) = \frac{1}{\delta^2}\left[\eta(\eta-1)\frac{\exp(-2s/\delta)}{[1-\exp(-s/\delta)]^2} - D\frac{\exp(-s/\delta)}{[1-\exp(-s/\delta)]}\right]. \tag{1}$$

The parameters η, D are dimensionless quantities and δ is the screening radius for nuclear potential having dimension of length. As an electromagnetic interaction, we adapt the screened atomic Hulthén potential [27]

$$V_{\mathrm{A}}(s) = E_0\frac{\exp(-s/b)}{1-\exp(-s/b)} \tag{2}$$

with E_0, the strength, and b, the screening radius of the potential. For our present analysis, we have considered the situation where $b=\delta$. Thus, the effective potential is given as

$$V_{\mathrm{eff}}(s) = \frac{1}{\delta^2}\left\{\eta(\eta-1)\frac{\exp(-2s/\delta)}{[1-\exp(-s/\delta)]^2} - \left(D-E_0\delta^2\right)\frac{\exp(-s/\delta)}{[1-\exp(-s/\delta)]}\right\}. \tag{3}$$

At a centre-of-mass energy $E=\xi^2+i\epsilon$, where ξ is the centre-of-mass momentum and ϵ is a small perturbation due to relative velocity if any [28], the off-shell Jost solution $f(\xi, p, s)$ for the above effective potential satisfies an inhomogeneous Schrödinger-like equation, written as

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}s^2} + \xi^2 - V_{\mathrm{eff}}(s)\right] f(\xi, p, s) = \left(\xi^2 - p^2\right) \exp(ips)\,. \tag{4}$$

At this stage, we have considered ϵ to be zero and having no effect on final calculations of physical observables. Introducing the following transformation:

$$f(\xi, p, s) = \delta^\eta \left[1 - \exp\left(-s/\delta\right)\right]^\eta \exp\left(i\xi s\right) \Theta(\xi, p, s)\,, \tag{5}$$

equation (4) takes the form

$$\begin{aligned}
&\exp(s/\delta)\delta^2(1 - \exp\left(-s/\delta\right))\Theta^{''}(\xi, p, s) + \Big\{2\eta\delta + 2i\xi\delta^2 \exp(s/\delta) \\
&\times(1 - \exp\left(-s/\delta\right))\Big\}\Theta^{'}(\xi, p, s) + \Big\{2i\xi\delta\eta - \eta + \left(D - E_0\delta^2\right)\Big\}\Theta(\xi, p, s) \\
&= \left(\xi^2 - p^2\right) \exp(i(p - \xi)s) \exp(s/\delta)(1 - \exp\left(-s/\delta\right))^{1-\eta}\delta^{2-\eta}\,,
\end{aligned} \tag{6}$$

where $\Theta(\xi, p, s)$ is a newly defined function of momenta and position. If we rewrite Eq. (6) by changing a new variable of the form $(1 - \exp\left(-s/\delta\right)) = r$ and substituting $\eta = \eta + 1$, it yields

$$\begin{aligned}
&r(1 - r)\frac{\mathrm{d}^2\Theta}{\mathrm{d}r^2} + \{2\eta + 2 - (3 + 2\eta - 2i\xi\delta)r\}\frac{\mathrm{d}\Theta}{\mathrm{d}r} - \big(1 + \eta - D \\
&+E_0\delta^2 - 2\eta i\xi\delta - 2i\xi\delta\big)\,\Theta = \delta^{1-\eta}\left(\xi^2 - p^2\right) r^{-\eta}(1 - r)^{i\delta(\xi - p) - 1}\,.
\end{aligned} \tag{7}$$

Comparing Eq. (7) with the following standard differential equation for Gaussian hypergeometric function [29–32]

$$r(1 - r)\frac{\mathrm{d}^2\Theta}{\mathrm{d}r^2} + \{R - (1 + M + N)r\}\,\frac{\mathrm{d}\Theta}{\mathrm{d}r} - MN\Theta = r^{\sigma - 1}(1 - \rho r)^{\tau - 1}\,, \tag{8}$$

we obtain

$$M = 1 + \eta - i\xi\delta + \sqrt{\eta^2 + \eta + D - E_0\delta^2 - \xi^2\delta^2}\,, \tag{9}$$

$$N = 1 + \eta - i\xi\delta - \sqrt{\eta^2 + \eta + D - E_0\delta^2 - \xi^2\delta^2}\,, \tag{10}$$

$$R = 2 + 2\eta\,, \qquad \rho = 1\,, \qquad \sigma = 1 - \eta\,, \tag{11}$$

and

$$\tau = i\delta(\xi - p)\,. \tag{12}$$

Two linearly independent solutions of the homogeneous part of Eq. (8), namely $v_1(r)$ and $v_2(r)$ are given by [29, 30]

$$\begin{aligned} v_1(r) \;&=\; {}_2F_1(M,N;R;r) = \frac{\Gamma(R)}{\Gamma(M)\Gamma(N)} \sum_{n=0}^{\infty} \frac{\Gamma(M+n)\Gamma(N+n)}{\Gamma(R+n)} \frac{r^n}{n!}\,; \\ R \;&>\; 0\,, \end{aligned} \tag{13}$$

and

$$\begin{aligned} &v_2(r) = {}_2F_1(M,N;M+N-R+1;1-r)\,; \\ &M+N-R+1 \neq 0,-1,-2\ldots \end{aligned} \tag{14}$$

With the following transformation [29, 30] on $v_2(r)$:

$${}_2F_1(M,N;R;r) = (1-r)^{R-N-M}{}_2F_1(R-M,R-N;R;r)\,, \tag{15}$$

one gets the expression for $v_2(r)$ as

$$\begin{aligned} v_2(r) &= \left(1-\mathrm{e}^{-r/\delta}\right)^{-2\eta-1} \\ &\times {}_2F_1\left(N-R+1,M-R+1;M+N-R+1;\mathrm{e}^{-r/\delta}\right)\,. \end{aligned} \tag{16}$$

The particular solution [32] of Eq. (8) is written as

$$F_P(r) = \left(\xi^2-p^2\right)\delta^{1-\eta} \sum_{n=0}^{\infty} \frac{\Gamma(n+1-i\delta(\xi-p))}{\Gamma(1-i\delta(\xi-p))n!}\, f_{n+1-\eta}(M,N;R;r) \tag{17}$$

with

$$\begin{aligned} f_n(a,b;c;r) \;&=\; r^n \sum_{j=0}^{\infty} \frac{\Gamma(n+a+j)\Gamma(n+b+j)\Gamma(n)\Gamma(n+c-1)}{\Gamma(n+a)\Gamma(n+b)\Gamma(n+j+1)\Gamma(n+c+j)} r^j \\ &=\; \frac{r^n}{n(n+c-1)}{}_3F_2(1,n+a,n+b;n+1,n+c;r)\,. \end{aligned} \tag{18}$$

The complete expression for $f(\xi,p,s)$ is obtained from Eq. (5) in conjunction with Eqs. (9)–(18) as

$$\begin{aligned} f(\xi,p,s) \;=\; &\delta^{\eta+1}\left[1-\exp\left(-s/\delta\right)\right]^{\eta+1} \exp\left(i\xi s\right)\Big[A_1\,{}_2F_1(M,N;R; \\ &1-\exp\left(-s/\delta\right)) + A_2(1-\exp\left(-s/\delta\right))^{-2\eta-1} \\ &{}_2F_1(1-M^*,1-N^*;1-2i\xi\delta;\exp\left(-s/\delta\right)) \\ &\times\left(\xi^2-p^2\right)\delta^{1-\eta}\sum_{n=0}^{\infty}\frac{\Gamma(n+1-i\delta(\xi-p))}{\Gamma(1-i\delta(\xi-p))n!} \\ &\times f_{n+1-\eta}(M,N;R;1-\exp\left(-s/\delta\right))\Big]\,. \end{aligned} \tag{19}$$

To determine the unknown constants A_1 and A_2, we shall apply the boundary conditions at $s = 0$ and $s \to \infty$. One gets the off-shell Jost function $f(\xi, p)$ [7] by considering the limiting behaviour of the off-shell Jost solution $f(\xi, p, s)$ at the origin. Thus, for $s \to 0$, one gets

$$A_2 = \frac{f(\xi, p)}{\delta^{\eta+1} \mathcal{J}(\xi)}\,, \tag{20}$$

where the on-shell Jost function $\mathcal{J}(\xi)$ [33] is

$$\mathcal{J}(\xi) = \delta^{\eta} \frac{\Gamma(2+2\eta)\Gamma(1-2i\xi\delta)}{\Gamma(M)\Gamma(N)}\,. \tag{21}$$

To obtain the other constant A_1, we use the limit when $s \to \infty$. The quantity $F_P(s)$ in Eq. (19) is related to the regular Manning–Rosen plus Hulthén Green's function $G^{(R)}(s, s')$ as

$$\left(\xi^2 - p^2\right) \delta^2 \left[1 - \exp\left(-s/\delta\right)\right]^{\eta+1} \exp\left(i\xi s\right) \sum_{n=0}^{\infty} \frac{\Gamma(n+1-i\delta(\xi-p))}{\Gamma(1-i\delta(\xi-p))n!}$$
$$\times f_{n+1-\eta}(M, N; R; 1 - \exp\left(-s/\delta\right)) = \left(\xi^2 - p^2\right) \int_0^s G^{(R)}\left(s, s'\right) \exp\left(ips\right) \mathrm{d}s' \tag{22}$$

with

$$G^{(R)}\left(s, s'\right) = \frac{1}{\mathcal{J}(\xi)} \left[\phi(\xi, s) f\left(\xi, s'\right) - \phi\left(\xi, s'\right) f(\xi, s)\right]\,. \tag{23}$$

Here, $\phi(\xi, s)$ and $f(\xi, s)$ are the regular and irregular solutions of the Manning–Rosen plus Hulthén potential [1]

$$\phi(\xi, s) = \delta^{\eta+1} \left[1 - \exp\left(-s/\delta\right)\right]^{\eta+1} \exp\left(i\xi s\right) {}_2F_1(M, N; R; 1 - \exp\left(-s/\delta\right)) \tag{24}$$

and

$$f(\xi, s) = \left[1 - \exp\left(-s/\delta\right)\right]^{-\eta} \exp\left(i\xi s\right) {}_2F_1(1-M^*, 1-N^*; 1-2i\xi\delta; \exp\left(-s/\delta\right))\,. \tag{25}$$

Fuda and Whiting [34], and Laha and Bhoi [35] were able to prove that only the particular solution of Eq. (4) gives the off-shell Jost solution as

$$f(\xi, p, s) = \left(\xi^2 - p^2\right) \int_s^{\infty} G^{(I)}\left(s, s'\right) \exp\left(ips'\right) \mathrm{d}s' \tag{26}$$

with the irregular Green's function [36] for the Manning–Rosen plus Hulthén potential

$$G^{(I)}\left(s,s'\right)=\frac{1}{\mathcal{J}(\xi)}\left[\phi\left(\xi,s'\right)f(\xi,s)-\phi(\xi,s)f\left(\xi,s'\right)\right]. \tag{27}$$

Under the limit $s\to\infty$, Eq. (19) together with Eqs. (22)–(27) yields

$$\begin{aligned}A_1&=\frac{i(p-\xi)}{\mathcal{J}(\xi)}\frac{\Gamma(1-\eta)\Gamma(1-i(\xi+p)\delta)}{\Gamma(1-\eta-i(\xi+p)\delta)}\\&\times{}_3F_2(M-1-2\eta,N-1-2\eta,-i(\xi+p)\delta;1-2i\xi\delta,1-\eta-i(\xi+p)\delta;1).\end{aligned} \tag{28}$$

In evaluating the above constant we have applied the following standard integral [29, 30, 37]

$$\begin{aligned}&\int\limits_0^s z^{\rho-1}(s-z)^{\sigma-1}{}_2F_1(\alpha,\beta;\gamma;cz)dz=\frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho+\sigma)}s^{\rho+\sigma-1}\times\\&{}_3F_2(\alpha,\beta,\rho;\gamma,\rho+\sigma;cs)\end{aligned} \tag{29}$$

with Re $\sigma>0$, Re $\rho>0$, $\mathrm{Re}(\gamma+\sigma-\alpha-\beta)>0$.

Having the constants A_1 and A_2, one obtains the compact expression for $f(\xi,p,s)$ as

$$\begin{aligned}f(\xi,p,s)&=\delta^{\eta+1}\left[1-\exp\left(-s/\delta\right)\right]^{\eta+1}\exp\left(i\xi s\right)\\&\times\left[\frac{i(p-\xi)}{\mathcal{J}(\xi)}\frac{\Gamma(1-\eta)\Gamma(1-i(\xi+p)\delta)}{\Gamma(1-\eta-i(\xi+p)\delta)}\right.\\&\times{}_3F_2(M-1-2\eta,N-1-2\eta,-i(\xi+p)\delta;1-2i\xi\delta,1-\eta-i(\xi+p)\delta;1)\\&\times{}_2F_1(M,N;R;1-\exp\left(-s/\delta\right))+\frac{f(\xi,p)}{\delta^{\eta+1}\mathcal{J}(\xi)}(1-\exp\left(-s/\delta\right))^{-2\eta-1}\\&\times{}_2F_1(1-M^*,1-N^*;1-2i\xi\delta;\exp\left(-s/\delta\right))\\&\left.+\left(\xi^2-p^2\right)\delta^{1-\eta}\sum_{n=0}^{\infty}\frac{\Gamma(n+1-i\delta(\xi-p))}{\Gamma(1-i\delta(\xi-p))n!}f_{n+1-\eta}(M,N;R;1-\exp\left(-s/\delta\right))\right].\end{aligned} \tag{30}$$

2.2. Integral transform method

For physical boundary condition, Eq. (4) takes the form [38, 39]

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}s^2}+\xi^2-V_{\mathrm{eff}}(s)\right]\psi^{(+)}(\xi,p,s)=\left(\xi^2-p^2\right)\sin(ps). \tag{31}$$

According to Refs. [34] and [35], the particular solution of the above equation also represents the off-shell physical solutions $\psi^{(+)}(\xi, p, s)$ written as

$$\psi^{(+)}(\xi, p, s) = \frac{(\xi^2 - p^2)}{2i}\left[\bar{G}^{(+)}(s, p) - \bar{G}^{(+)}(s, -p)\right], \tag{32}$$

where

$$\bar{G}^{(+)}(s, p) = \int_0^\infty G^{(+)}\left(s, s'\right)\exp\left(ips'\right)\mathrm{d}s'\,. \tag{33}$$

The quantities $\bar{G}^{(+)}(s, -p)$ and $\bar{G}^{(+)}(s, p)$ are related by

$$\bar{G}^{(+)}(s, -p) = \left[\bar{G}^{(+)}(s, p)\right]_{p \to -p}. \tag{34}$$

To evaluate $\bar{G}^{(+)}(s, p)$ in Eq. (33), we follow the ordinary differential equation approach. The inhomogeneous differential equation [36, 40] satisfied by $G^{(+)}(s, s')$ is

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}s^2} + \xi^2 - V_{\text{eff}}(s)\right] G^{(+)}\left(s, s'\right) = \delta\left(s - s'\right). \tag{35}$$

We use the same kind of transformation in the above equation as in Eq. (5)

$$G^{(+)}\left(s, s'\right) = \delta^\eta \left[1 - \exp\left(-s/\delta\right)\right]^\eta \exp\left(i\xi s\right) \varOmega\left(s, s'\right) \tag{36}$$

to have

$$\begin{aligned}
&\exp(s/\delta)\delta^2(1 - \exp\left(-s/\delta\right))\varOmega'' + \Big\{2\eta\delta + 2i\xi\delta^2\exp(s/\delta) \\
&\times(1 - \exp\left(-s/\delta\right))\Big\}\varOmega' + \Big\{2i\xi\delta\eta - \eta + \left(D - E_0\delta^2\right)\Big\}\varOmega \\
&= \exp(-i\xi s)\exp(s/\delta)(1 - \exp\left(-s/\delta\right))^{1-\eta}\delta^{2-\eta}\delta\left(s - s'\right).
\end{aligned} \tag{37}$$

Taking the Hankel transform of $\varOmega(s, s')$ with respect to s' in Eq. (37) and changing the independent variable by $y = (1 - \exp\left(-s/\delta\right))$ and substituting $\eta = \eta + 1$, we get

$$\begin{aligned}
&y(1 - y)\frac{\mathrm{d}^2\bar{\varOmega}}{\mathrm{d}y^2} + \{2\eta + 2 - (3 + 2\eta - 2i\xi\delta)y\}\frac{\mathrm{d}\bar{\varOmega}}{\mathrm{d}y} - (1 + \eta - D \\
&+ E_0\delta^2 - 2\eta i\xi\delta - 2i\xi\delta)\bar{\varOmega} = \delta^{2-\eta}y^{1-\eta}(1 - y)^{i\delta(\xi - p) - 1},
\end{aligned} \tag{38}$$

with

$$\bar{\varOmega}(s, p) = H\left\{\varOmega\left(s, s'\right); s' \to p\right\}. \tag{39}$$

Comparison of Eqs. (8) and (38) yields

$$\bar{\Omega}(s,p) = \Big[B_1\, {}_2F_1(M,N;R;1-\exp(-s/\delta))$$
$$+B_2\,(1-\exp(-s/\delta))^{-2\eta-1}{}_2F_1(1-M^*,1-N^*;1-2i\xi\delta;\exp(-s/\delta))$$
$$+\delta^{1-\eta}\sum_{n=0}^{\infty}\frac{\Gamma(n+1-i\delta(\xi-p))}{\Gamma(1-i\delta(\xi-p))n!}\times f_{n+1-\eta}(M,N;R;1-\exp(-s/\delta))\Big]\,. \quad (40)$$

In Eq. (40), B_1 and B_2 are two unknown constants which will be determined from the boundary conditions at $s=0$ and $s=\infty$. At $s=0$, $\bar{G}^{(+)}(s,p)=0$ and we obtain $B_2=0$. The s-wave physical Green's function for the Manning–Rosen plus Hulthén potential is written as [36]

$$G^{(+)}\left(s,s'\right) = -\frac{\phi(\xi,s_<)f(\xi,s_>)}{\mathcal{J}(\xi)}\,. \quad (41)$$

The quantities $s_>$ and $s_<$ have the usual meaning. For the limit $s\to\infty$, we combine Eqs. (36),(40) and (41) with the judicious application of Eq. (29) to obtain

$$B_1 = \frac{1}{i(\xi+p)\mathcal{J}(\xi)}\frac{\Gamma(1-\eta)\Gamma(1-i(\xi+p)\delta)}{\Gamma(1-\eta-i(\xi+p)\delta)}$$
$$\times {}_3F_2(M-1-2\eta,N-1-2\eta,-i(\xi+p)\delta;1-2i\xi\delta,1-\eta-i(\xi+p)\delta;1)\,. \quad (42)$$

From Eqs.(36),(40) and (42) with $B_2=0$, we have

$$\bar{G}^{(+)}(s,p) = \delta^{\eta+1}\left[1-\exp(-s/\delta)\right]^{\eta+1}\exp(i\xi s)$$
$$\times\Big[\frac{1}{i(\xi+p)\mathcal{J}(\xi)}\frac{\Gamma(1-\eta)\Gamma(1-i(\xi+p)\delta)}{\Gamma(1-\eta-i(\xi+p)\delta)}$$
$$\times {}_3F_2(M-1-2\eta,N-1-2\eta,-i(\xi+p)\delta;1-2i\xi\delta,1-\eta-i(\xi+p)\delta;1)$$
$$\times {}_2F_1(M,N;R;1-\exp(-s/\delta))+\delta^{1-\eta}\sum_{n=0}^{\infty}\frac{\Gamma(n+1-i\delta(\xi-p))}{\Gamma(1-i\delta(\xi-p))n!}$$
$$\times f_{n+1-\eta}(M,N;R;1-\exp(-s/\delta))\Big]\,. \quad (43)$$

The expression for $\bar{G}^{(+)}(s,-p)$ is obtained by replacing p with $-p$ in Eq. (43). Therefore, by utilizing Eqs. (32) and (43), one is able to write an expression for the off-shell physical solution for motion in the Manning–Rosen plus Hulthén potential. There exists a relation between the off-shell physical and Jost solutions [14, 41] expressed as

$$\psi^{(+)}(\xi,p,s) = \frac{\pi p}{2}T_{\mathrm{h}}\left(\xi,p,\xi^2\right)f(\xi,s)+\frac{1}{2i}\Big[f(\xi,p,s)-f(\xi,-p,s)\Big]\,, \quad (44)$$

where $T_{\rm h}(\xi,p,\xi^2)$ stands for the half-off-shell T-matrix written as

$$T_{\rm h}\left(\xi,p,\xi^2\right) = \frac{f(\xi,p)-f(\xi,-p)}{i\pi p f(\xi)}\,. \tag{45}$$

The quantity $f(\xi,p)$ represents the off-shell Jost function for the Manning–Rosen plus Hulthén potential. The off-shell Jost function $f(\xi,p)$ is obtained from $f(\xi,p,s)$ as

$$\begin{aligned} f(\xi,p) &= \lim_{s\to 0} f(\xi,p,s) \\ &= \delta^{\eta+2}\left(\xi^2-p^2\right)\frac{\Gamma(\eta+2)\Gamma(-i(\xi+p)\delta)}{\Gamma(\eta+2-i(\xi+p)\delta)} \\ &\quad \times {}_3F_2(M,N,\eta+2;R,\eta+2-i(\xi+p)\delta;1)\,. \end{aligned} \tag{46}$$

Exploiting two times the following transformation [42]:

$$\begin{aligned} {}_3F_2(a_1,a_2,a_3;b_1,b_2;1) &= \frac{\Gamma(b_2)\Gamma(b_1+b_2-a_1-a_2-a_3)}{\Gamma(b_2-a_3)\Gamma(b_1+b_2-a_1-a_2)} \\ \times {}_3F_2(b_1-a_1,b_1-a_2,a_3;b_1,b_1+b_2-a_1-a_2;1)\,, \end{aligned} \tag{47}$$

Eq. (46) reads

$$\begin{aligned} f(\xi,p) &= \delta^{\eta}\frac{\Gamma(\eta+2)\Gamma(1+i(\xi-p)\delta)\Gamma(1-i(\xi+p)\delta)}{\Gamma(i(\xi-p)\delta-\eta+M)\Gamma(2\eta+2-i(\xi+p)\delta)-M)} \\ &\quad \times {}_3F_2(2\eta+2-M,N,\eta;2\eta+2,2\eta+2-i(\xi+p)\delta-M;1)\,. \end{aligned} \tag{48}$$

Equations (46) and (48) are equivalent. However, Eq. (48) is the most suitable one for checking limiting values and numerical treatment. In the on-shell limit *i.e.* $p\to\xi$, $f(\xi,p)=f(\xi)$.

2.3. Off-shell extension function

The half-shell T-matrix may be rewritten as

$$T_{\rm h}\left(\xi,p,\xi^2\right) = \left(\frac{\xi}{p}\right)\frac{|f(\xi,p)|2\sin\mathit{\Delta}(\xi,p)\,{\rm e}^{i\delta(\xi)}}{\pi p|f(\xi)|}\,, \tag{49}$$

where $\mathit{\Delta}(\xi,p)$ is the quasi-phase and the quantity $\delta(\xi)$ stands for the scattering phase shift. After some algebraic manipulation, one can get

$$T_{\rm h}\left(\xi,p,\xi^2\right) = T_{\rm h}\left(\xi,\xi,\xi^2\right)H(\xi,p)\,, \tag{50}$$

where $T_{\rm h}(\xi,\xi,\xi^2)$ is the on-shell T-matrix and is expressed as

$$T_{\rm h}\left(\xi,\xi,\xi^2\right) = -\frac{2}{\pi\xi}\sin\delta(\xi)\,{\rm e}^{i\delta(\xi)}\,. \tag{51}$$

The off-shell extension function $H(\xi,p)$ reads

$$H(\xi,p) = \left(\frac{\xi}{p}\right)^{-1} \frac{|f(\xi,p)|}{|f(\xi)|} \frac{\sin \varDelta(\xi,p)}{\sin \delta(\xi)} . \tag{52}$$

For $p \to \xi$, $\varDelta(\xi,\xi) = \delta(\xi)$ and $H(\xi,\xi) = 1$. The off-shell quantities are used in many particle systems. In this connection, one may consider the off-shell T-matrix as an important quantity for the study of scattering theory because its on-shell limit is directly related to the scattering amplitude. The half-off-shell T-matrix related to the scattering phase shifts can be expressed in terms of the on- and off-shell Jost function as already mentioned in Eq. (45). Thus, having the compact analytical expressions for the on- and off-shell Jost functions one will be in a position to calculate the half-off-shell T-matrix and off-shell extension function. In the next section, we will compute scattering phase shift, half-shell T-matrix, and off-shell extension function for the α–^{3}H system.

3. Results and discussions

We apply our formalism to compute the half-shell T-matrix, off-shell extension function, and scattering phase shift for the α–^{3}H system which are depicted in Figs. 1 (left), 1 (right), and 2 respectively. For our calculation, we use $\hbar^2/2\mu = 12.0954$ MeV fm^2, $E_0\delta = 0.2381$ fm^{-1} for the system under consideration. The α–^{3}H system is unbound in its $1/2^+$ state so we give free running to our parameters in the numerical program to have proper values of the phase shifts. The best-fitted parameters for the Manning–Rosen and Hulthén potential are $\eta = 0.005$, $D = 1.954$, and $\delta = 0.57$ fm. In Fig. 1 (left), we present our results for the half-off-shell T-matrix as a function of off-shell momenta for two different laboratory energies. These numbers show that both Re $T_{\mathrm{h}}(\xi,p,\xi^2)$ and Im $T_{\mathrm{h}}(\xi,p,\xi^2)$ oscillate but approach zero as p becomes large. The function $f(\xi,p)$ also tends to zero as p increases. In Fig. 1 (right) for $p \to \xi$, $H(\xi,\xi) = 1$. Interestingly, for low values of laboratory energies, our potential shows large off-shell effects which is in conformity with the observations of earlier works [8, 43, 44] related to various kinds of potentials. These indicate that the off-shell behaviour of the potential in Eq. (4) is quite acceptable. This means that the action of the potential in producing a half-off-shell T-matrix $T_{\mathrm{h}}(\xi,p,\xi^2)$ depends also on p. It is well known that the phase of the half-shell transition matrix is the scattering phase shift. The phase parameters calculated from the half-shell T-matrix are depicted in Fig. 2 which are in reasonable agreement with those of Spiger and Tombrello [45].

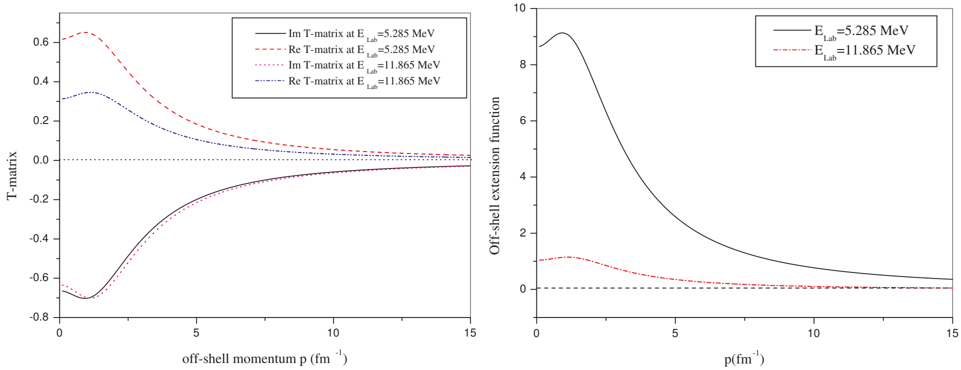

Fig. 1. Half-shell T-matrix and off-shell extension function for α–^{3}H system. Left: $T_{\mathrm{h}}(\xi, p, \xi^2)$for $\frac{1}{2}^+$ state. The zero line is represented by black dotted one. Right: off-shell extension function for $\frac{1}{2}^+$ state. The zero line is represented by black dashed one.

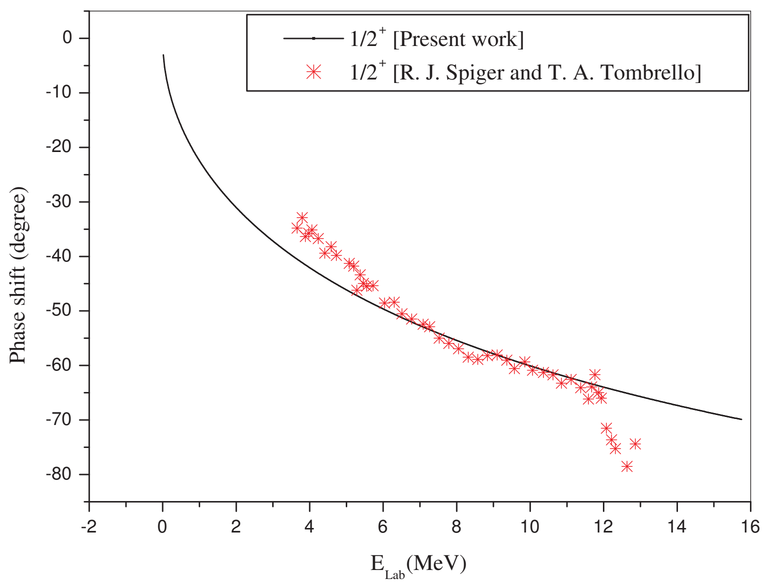

Fig. 2. Phase shift for $\frac{1}{2}^+$ state of α–^{3}H system as a function of E_{lab}.

4. Conclusions

The theoretical investigation of the (p–p) Bremsstrahlung is closely related to the study of the half-off-shell nucleon-nucleon T-matrix. Therefore, the expression for the T-matrix facilitates us to make the best possible use of the available information about the two-nucleon wave function in coordinate space. The present text deals with three-parameter central nuclear potential instead of several parameter interactions with the inclusion of spin-orbit and tensor interactions. With this simple potential model, our phase parameters agree quite well with the earlier works [45–47] except at very low energies. This may be due to improper accountability of the electromagnetic interaction in this energy range. The behaviours of the half-shell transition matrices computed with the potential in Eq. (4) and those of Laha and Talukdar [8], Haidenbauer and Plessas [43], Sahoo *et al.* [44], Behera *et al.* [48], and Khirali *et al.* [49] indicate that low-energy part of the two-nucleon potentials appears to be nearly the same, indicating that one may have a common low-momentum nucleon–nucleon potential. From the foregoing discussion it is noticed that our conjecture works quite satisfactorily with respect to the off-shell behaviour of the nucleon–nucleon potential under consideration. The present method can be applicable to the case of an arbitrary exponential type of nuclear local potential.

REFERENCES

[1] P. Sarkar, B. Khirali, U. Laha, P. Sahoo, «Exact solution of two-potential system under same range approximation and its implication», *Int. J. Mod. Phys. E* **30**, 2150066 (2021).

[2] Y. Yamaguchi, «Two-nucleon Problem When the Potential Is Nonlocal but Separable. I», *Phys. Rev.* **95**, 1628 (1954).

[3] H. van Haeringen, R. van Wageningen, «Analytic T matrices for Coulomb plus rational separable potentials», *J. Math. Phys.* **16**, 1441 (1975).

[4] J. Haidenbauer, W. Plessas, «Separable approximations of two-body interactions», *Phys. Rev. C* **27**, 63 (1983).

[5] W. Schweiger, W. Plessas, L.P. Kok, H. van Haeringen, «Separable representation of the nuclear proton–proton interaction», *Phys. Rev. C* **27**, 515 (1983).

[6] B. Talukdar, U. Laha, T. Sasakawa, «Green's function for motion in Coulomb-modified separable nonlocal potentials», *J. Math. Phys.* **27**, 2080 (1986).

[7] H. Van Haeringen, «Charged Particle Interactions. Theory and Formulas», *Coulomb Press Leyden*, Leiden, Netherlands 1985.

[8] U. Laha, B. Talukdar, «Half-shell T matrix for Coulomb-modified Graz separable potential», *Pramana — J. Phys.* **36**, 289 (1991).

[9] U. Laha, «Off-shell Jost solution for a Coulomb-like potential», *Phys. Rev. A* **74**, 012710 (2006).

[10] U. Laha, «An integral transform of Green's function, off-shell Jost solution and t-matrix for Coulomb–Yamaguchi potential in coordinate representation», *Pramana — J. Phys.* **72**, 457 (2009).

[11] U. Laha, «Coulomb-modified Nuclear Scattering-off the Energy Shell», *Lambert Academic Publishing*, Saarbrücken 2010.

[12] U. Laha, «Integral transform of the Coulomb Green's function by the Hankel function and off-shell scattering», *Phys. Rev. C* **88**, 064001 (2013).

[13] U. Laha, J. Bhoi, «An Integral Transform of Coulomb Green's Function via Sturmian Representation and Off-Shell Scattering», *Few-Body Syst.* **54**, 1973 (2013).

[14] U. Laha, J. Bhoi, «Off-shell Jost solutions for Coulomb and Coulomb-like interactions in all partial waves», *J. Math. Phys.* **54**, 013514 (2013).

[15] U. Laha, J. Bhoi, «Hadron–Hadron Scattering Within the Separable Model of Interactions», *OmniScriptum Publishing Group*, Beau Bassin 2018.

[16] J. Bhoi, U. Laha, «Supersymmetry-inspired low-energy α–p elastic scattering phases», *Theor. Math. Phys.* **190**, 69 (2017).

[17] J. Bhoi, U. Laha, «Hulthén potential models for α–α and α–He^3 elastic scattering», *Pramana — J. Phys.* **88**, 42 (2017).

[18] H.P. Noyes, «New Nonsingular Integral Equation for Two-Particle Scattering», *Phys. Rev.* **15**, 798 (1965).

[19] B. Talukdar, S. Saha, S.R. Bhattaru, D.K. Ghosh, «Off-shell extension function from a generalised phase-function method», *Z. Phys. A* **312**, 121 (1983).

[20] G.C. Sett, B. Talukdar, «Phase-function method for velocity-dependent potentials», *Pramana — J. Phys.* **33**, 381 (1989).

[21] M.F. Manning, N. Rosen, «Exact Solutions of the Schrödinger Equation», *Phys. Rev.* **44**, 953 (1933).

[22] S.-H. Dong, J. García-Ravelo, «Exact solutions of the s-wave Schrödinger equation with Manning–Rosen potential», *Phys. Scr.* **75**, 307 (2007).

[23] W.-C. Qiang, K. Li, W.-L. Chen, «New bound and scattering state solutions of the Manning–Rosen potential with the centrifugal term», *J. Phys. A: Math. Theor.* **42**, 205306 (2009).

[24] B. Khirali, A.K. Behera, J. Bhoi, U. Laha, «Regular and Jost states for the S-wave Manning–Rosen potential», *J. Phys. G: Nucl. Part. Phys.* **46**, 115104 (2019).

[25] B. Khirali, A.K. Behera, J. Bhoi, U. Laha, «Scattering with Manning–Rosen potential in all partial waves», *Ann. Phys.* **412**, 168044 (2020).

[26] B. Khirali, U. Laha, A.K. Behera, P. Sahoo, «Off-shell T-matrix for the Manning–Rosen potential», *Pramana — J. Phys.* **95**, 179 (2021).

[27] L. Hulthén, «On the characteristic solutions of the Schrödinger deuteron equation», *Arkiv for Matematik, Astronomi och Fysik* **29B**, 1 (1942).

[28] J.V. Leeuwen, A. Reiner, «On the calculation of the t-matrix for potentials with a hard core», *Physica* **27**, 99 (1961).

[29] L.J. Slater, «Generalized hypergeometric functions», *Cambridge Univ. Press*, London 1966, `http://cds.cern.ch/record/106061`

[30] A. Erdelyi, «Higher Transcendental Functions», *McGraw-Hill*, New York 1953.

[31] W. Magnus, F. Oberhettinger, «Formulas and Theorems for the Special Functions of Mathematical Physics», *Chelsea Publishing Company*, 1949.

[32] A.W. Babister, «Transcendental Functions Satisfying Non-homogeneous Linear Differential Equations», *MacMillan*, New York 1967.

[33] R. Jost, «Über die falschen Nullstellen der Eigenwerte der S-Matrix», *Helv. Phys. Acta* **20**, 256 (1947), https://www.e-periodica.ch/digbib/view?pid=hpa-001:1947:20::559#258

[34] M.G. Fuda, J.S. Whiting, «Generalization of the Jost Function and Its Application to Off-Shell Scattering», *Phys. Rev. C* **8**, 1255 (1973).

[35] U. Laha, J. Bhoi, «On- and off-shell Jost functions and their integral representations», *Pramana — J. Phys.* **86**, 947 (2015).

[36] R.G. Newton, «Scattering Theory of Waves and Particles, 2nd ed.», *Springer Verlag*, New York 1982.

[37] I.S. Gradshteyn, I.M. Ryzhik, «Tables of Integrals, Series and Products», *Academic Press*, London 2000.

[38] O.P. Bahethi, M.G. Fuda, «The T Matrix for the Hulthén Potential», *J. Math. Phys.* **12**, 2076 (1971).

[39] P. Sahoo, U. Laha, B. Khirali, «Hulthén off-shell transition matrix for nuclear systems», *Chinese J. Phys.* **73**, 561 (2021).

[40] U. Laha, «On the integral representations of the Jost function and Coulomb off-shell Jost solution», *Pramana — J. Phys.* **67**, 357 (2006).

[41] P. Sahoo, U. Laha, B. Khirali, A.K. Behera, «Half-Shell T-Matrix for α–p and α–^{12}C Scattering», *Braz. J. Phys.* **51**, 1478 (2021).

[42] W.N. Bailey, «Generalized Hypergeometric Series», *Cambridge University Press*, London 1935.

[43] J. Haidenbauer, W. Plessas, «Separable representation of the Paris nucleon–nucleon potential», *Phys. Rev. C* **30**, 1822 (1984).

[44] P. Sahoo, U. Laha, «Nucleon–nucleus inelastic scattering by Manning–Rosen distorted nonlocal potential», *Canadian J. Phys.* **100**, 1 (2021).

[45] R.J. Spiger, T.A. Tombrello, «Scattering of He^3 by He^4 and of He^4 by Tritium», *Phys. Rev.* **163**, 964 (1967).

[46] A.K. Behera, P. Sahoo, B. Khirali, U. Laha, «Fredholm determinants for the Hulthén-distorted separable potential», *Pramana — J. Phys.* **95**, 1 (2021).

[47] J. Bhoi, R. Upadhyay, U. Laha, «Parameterization of Nuclear Hulthén Potential for Nucleus–Nucleus Elastic Scattering», *Commun. Theor. Phys.* **69**, 203 (2018).

[48] A.K. Behera, U. Laha, P. Sahoo, J. Bhoi, «Hulthén Half-off-Shell T Matrix-Application to n–p and n–d Systems», *J. Korean Phys. Soc.* **76**, 782 (2020).

[49] B. Khirali, A.K. Behera, J. Bhoi, U. Laha, «On- and off-shell Jost functions for the Manning–Rosen potential», *Phys. Scr.* **95**, 075308 (2020).