# SEMI-CLASSICAL KINETIC THEORY FOR MASSIVE SPIN-HALF FERMIONS WITH LEADING-ORDER SPIN EFFECTS*

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We consider the quantum kinetic-theory description for interacting massive spin-half fermions using the Wigner function formalism. We derive a general kinetic theory description assuming that the spin effects appear at the classical and quantum level. To track the effect of such different contributions, we use the semi-classical expansion method to obtain the generalized dynamical equations including spin, analogous to the classical Boltzmann equation. This approach can be used to obtain a collision kernel involving local as well as non-local collisions among the microscopic constituent of the system and eventually, a framework of spin hydrodynamics ensuring the conservation of the energy-momentum tensor and total angular momentum tensor.

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## 1. Introduction

Relativistic fluid dynamics has been very successful in modeling the collective evolution of the strongly interacting matter produced in relativistic heavy-ion collision experiments [1–6]. Considering such success, an attempt has been made to incorporate the spin degrees of freedom within the hydrodynamic framework to explain the recent measurements of spin polarization of particles emitted in these processes [7–15]. Such a formalism of relativistic fluid dynamics with spin was first proposed in Ref. [16] and further developed in Refs. [17–35]; for reviews, see Refs. [36–38]. Other similar approaches used concepts of thermodynamic equilibrium [39], effective action [40–43], entropy current [32, 44–48], statistical operator [49], non-local collisions [50–59], chiral kinetic theory [50, 60, 61], and holographic duality [62–64].

The understanding of global polarization (along the direction of the total angular momentum) and local polarization (along the beam direction) phenomena has become a subject of very intense investigations recently [65–67]. While the theoretical approaches assuming spin-vorticity coupling [68] can explain the global polarization measurements [68–75], they fail to describe differential observables [76–80]; though there are some recent advances in this respect [81–85].

On general thermodynamic grounds [86], the spin polarization effects are expected to be quantified by an antisymmetric tensor $\omega^{\mu\nu}$ conjugated to the generators of the Lorentz transformations, which in local thermal equilibrium may be independent of the thermal vorticity [16–18]. The spin polarization tensor $\omega^{\mu\nu}$ has been considered as a hydrodynamic variable that manifests the effects of spin at a macroscopic level. The hydrodynamic approach discussed in Refs. [16–18] implicitly assumes that the spin is a separately conserved quantity. However, in general, a microscopic collision process may also give rise to a transfer of angular momentum between the orbital and spin parts, keeping their sum conserved. An example of such a process is a non-local collision [53, 54].

In the current work, following Refs. [53, 54, 87], we extend the kinetic theory framework presented in Ref. [36]. We use the Wigner function formalism to formulate a quantum kinetic theory for interacting spin-half Dirac particles [87, 88], and employ semi-classical approximation to derive transport equations of various components of the Wigner function [89–93]. However, considering a systematic $\hbar$ expansion of the Wigner function components, we assume that the spin polarization effects can be manifested at both leading and next-to-leading order, which goes beyond the situation discussed in Refs. [53, 54]. The present approach can be used in future investigations to derive quantum kinetic equations using mapping of the Wigner function components to a classical distribution function with a phase-space extended to spin [94–96]. Moreover, this approach can also be used to systematically include the effect of local and non-local collisions in the collision kernel [53, 54], which can be used to develop a spin hydrodynamic framework.

The structure of the paper is as follows: We begin with the description of the Wigner function formalism in Sec. 2. Using the semi-classical expansion approach, in Sec. 3, we derive the transport equations for the components of the Wigner function, while in Sec. 4, we obtain mass-shell conditions at the zeroth and first-order in $\hbar$. In Sec. 5, we derive kinetic equations for the scalar and axial-vector components, and in Sec. 6, we formulate a general quantum kinetic equation. We summarize our findings in Sec. 7 with possible future directions.

Notations and conventions: In this work, we use the Cartesian coordinate system with $x^\mu \equiv (t, \boldsymbol{x})$ and the mostly-minus metric convention[1]. The scalar product of two four-vectors $a$ and $b$ reads $a \cdot b = a^0 b^0 - \boldsymbol{a} \cdot \boldsymbol{b}$, where the three-vectors are denoted by bold font. We work with the convention $\epsilon^{0123} = +1$ for the Levi-Civita symbol $\epsilon^{\alpha\beta\gamma\delta}$. Also, for symmetrization and anti-symmetrization, we use the notation $A_{\{\mu\nu\}} = A_{\mu\nu} + A_{\nu\mu}$ and $A_{[\mu\nu]} = A_{\mu\nu} - A_{\nu\mu}$, respectively. We assume $c = k_\mathrm{B} = 1$ throughout while keeping $\hbar$ explicitly for our calculations. The dual form of the tensor $A^{\mu\nu}$ is defined as $\tilde{A}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} A_{\alpha\beta}$.

## 2. Wigner function and its quantum kinetic equation

In the case of spin-half massive particles and the absence of gauge fields, the Wigner function can be expressed as follows [87, 90]:

$$W_{\alpha\beta}(x,k) = \int \frac{\mathrm{d}^4 y}{(2\pi\hbar)^4}\, \mathrm{e}^{-\frac{i}{\hbar}k\cdot y} \left\langle : \bar{\psi}_\beta(x_+)\psi_\alpha(x_-) : \right\rangle , \tag{1}$$

where the Dirac field operator $\psi$ and its adjoint $\bar{\psi} \equiv \psi^\dagger \gamma^0$ are defined at two different points in spacetime $x_\pm \equiv x \pm y/2$ with $x$ and $y$ being the center and relative position, respectively. Here, the angle brackets indicate the ensemble average and the colon denotes normal ordering[2].

In the presence of interactions, the Dirac equation is expressed as [87]

$$(\not{p} - m)\psi(x) = \hbar\rho(x)\,, \tag{2}$$

where $\rho(x) = -(1/\hbar)\partial\mathcal{L}_\mathrm{I}/\partial\bar{\psi}$, with $\mathcal{L}_\mathrm{I}(x)$ denoting the interaction Lagrangian density, and $\not{p} = i\hbar\gamma^\mu\partial_\mu$.

From the Lagrangian density $\mathcal{L}(x) = \mathcal{L}_\mathrm{D}(x) + \mathcal{L}_\mathrm{I}(x)$ where

$$\mathcal{L}_\mathrm{D}(x) = \frac{1}{2}\bar{\psi}(x)\overleftrightarrow{\not{p}}\,\psi(x) - m\bar{\psi}(x)\psi(x) \tag{3}$$

is the Lagrangian density for the free Dirac field with mass $m$ and $\overleftrightarrow{\not{p}} \equiv \overrightarrow{\not{p}} - \overleftarrow{\not{p}}$, we can derive the following transport equation for the Wigner

[1] $g_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$.

[2] Note that alternative definitions of the Wigner function without normal ordering have also been considered in the literature, *e.g.* in Ref. [97], where it has been argued that different definitions of the Wigner function may give rise to different results when the chiral anomaly is involved. However, we do not discuss such a situation here.

function (1) [87]:

$$\left[\gamma\cdot k+\frac{\not{p}}{2}-m\right]W(x,k)=\hbar\,\mathcal{C}\left[W(x,k)\right]\,. \tag{4}$$

Here, the collision term $\mathcal{C}\left[W(x,k)\right]$ is defined as [87]

$$\mathcal{C}\left[W(x,k)\right]\equiv\int\frac{\mathrm{d}^4y}{(2\pi\hbar)^4}\,\mathrm{e}^{-\frac{i}{\hbar}k\cdot y}\left\langle:\rho(x_-)\bar{\psi}(x_+):\right\rangle\,. \tag{5}$$

We would like to point out here that, in global equilibrium, the collision term (5) must vanish regardless of its form [87, 88]. Here, we consider that the collision term describes the system away from equilibrium and gives rise to quantum corrections to the leading-order Wigner function, appearing at the $\hbar$ order or higher.

The Wigner function $W(x,k)$ is a matrix in the Dirac space, therefore, we can express it in terms of the generators of the Clifford algebra as

$$\begin{aligned}W(x,k)\;=\;&\frac{1}{4}\left[\mathbf{1}\,\mathcal{F}(x,k)+i\,\gamma^5\,\mathcal{P}(x,k)+\gamma^\mu\,\mathcal{V}_\mu(x,k)\right.\\&\left.+\,\gamma^5\,\gamma^\mu\,\mathcal{A}_\mu(x,k)+\Sigma^{\mu\nu}\,\mathcal{S}_{\mu\nu}(x,k)\right]\,,\end{aligned} \tag{6}$$

with $\Sigma^{\mu\nu}\equiv(1/2)\sigma^{\mu\nu}\equiv(i/4)[\gamma^\mu,\gamma^\nu]$ being the Dirac spin operator.

Since the Wigner function is a complex matrix of the order of 4, it has 16 independent components: $\mathcal{F}(x,k)$, $\mathcal{P}(x,k)$, $\mathcal{V}_\mu(x,k)$, $\mathcal{A}_\mu(x,k)$, and $\mathcal{S}_{\mu\nu}(x,k)$, which can be obtained by calculating the trace of $W(x,k)$ after multiplying first by the matrices: $\Gamma_\mathcal{X}\in\{\mathbf{1},-i\gamma_5,\gamma^\mu,\gamma^\mu\gamma_5,2\Sigma^{\mu\nu}\}$, where $\mathcal{X}\in\{\mathcal{F},\mathcal{P},\mathcal{V},\mathcal{A},\mathcal{S}\}$, respectively.

Under Lorentz transformations, the expansion coefficients of the Wigner function, $\mathcal{F}$, $\mathcal{P}$, $\mathcal{V}_\mu$, $\mathcal{A}_\mu$, and $\mathcal{S}_{\mu\nu}$ transform as a scalar, pseudo-scalar, vector, axial-vector, and tensor, respectively [90]. The coefficients $\mathcal{F}$ and $\mathcal{P}$ have the interpretation of mass and pseudo-scalar condensate, respectively, whereas, $\mathcal{V}_\mu$ and $\mathcal{A}_\mu$ are known as the fermion number current density and the polarization density, respectively. Since $\mathcal{S}_{\mu\nu}$ is antisymmetric, it has six independent components having the physical interpretation of electric and magnetic dipole moments.

Using the representation of the Wigner function in terms of the generators of the Clifford algebra (6) in the kinetic equation (4) gives rise to kinetic equations for different coefficients of the Wigner function [98], which, after separating the real and imaginary parts, yields two sets of equations for $\mathcal{F},\mathcal{P},\mathcal{V}^\mu,\mathcal{A}^\mu$, and $\mathcal{S}^{\mu\nu}$, where the real parts are

$$k^\mu \mathcal{V}_\mu - m\,\mathcal{F} \;=\; \hbar\,\mathcal{D}_{\mathcal{F}}\,, \tag{7}$$

$$\frac{\hbar}{2}\,\partial^\mu \mathcal{A}_\mu + m\,\mathcal{P} \;=\; -\hbar\,\mathcal{D}_{\mathcal{P}}\,, \tag{8}$$

$$k_\mu \mathcal{F} - \frac{\hbar}{2}\,\partial^\nu \mathcal{S}_{\nu\mu} - m\,\mathcal{V}_\mu \;=\; \hbar\,\mathcal{D}_{\mathcal{V},\mu}\,, \tag{9}$$

$$-\frac{\hbar}{2}\,\partial_\mu \mathcal{P} + k^\beta \overset{\star}{\mathcal{S}}_{\mu\beta} + m\,\mathcal{A}_\mu \;=\; -\hbar\,\mathcal{D}_{\mathcal{A},\mu}\,, \tag{10}$$

$$\frac{\hbar}{2}\,\partial_{[\mu}\mathcal{V}_{\nu]} - \epsilon_{\mu\nu\alpha\beta}k^\alpha \mathcal{A}^\beta - m\,\mathcal{S}_{\mu\nu} \;=\; \hbar\,\mathcal{D}_{\mathcal{S},\mu\nu}\,, \tag{11}$$

while the imaginary parts are expressed as

$$\hbar\;\partial^\mu \mathcal{V}_\mu \;=\; 2\hbar\,\mathcal{C}_{\mathcal{F}}\,, \tag{12}$$

$$k^\mu \mathcal{A}_\mu \;=\; \hbar\,\mathcal{C}_{\mathcal{P}}\,, \tag{13}$$

$$\frac{\hbar}{2}\partial_\mu \mathcal{F} + k^\nu \mathcal{S}_{\nu\mu} \;=\; \hbar\,\mathcal{C}_{\mathcal{V},\mu}\,, \tag{14}$$

$$k_\mu \mathcal{P} + \frac{\hbar}{2}\partial^\beta \overset{\star}{\mathcal{S}}_{\mu\beta} \;=\; -\hbar\,\mathcal{C}_{\mathcal{A},\mu}\,, \tag{15}$$

$$k_{[\mu}\mathcal{V}_{\nu]} + \frac{\hbar}{2}\epsilon_{\mu\nu\alpha\beta}\partial^\alpha \mathcal{A}^\beta \;=\; -\hbar\,\mathcal{C}_{\mathcal{S},\mu\nu}\,. \tag{16}$$

In Eqs. (7)–(16), $\mathcal{D}_{\mathcal{X}} = \Re\,\mathrm{Tr}\big[\mathit{\Gamma}_{\mathcal{X}}\mathcal{C}[W(x,k)]\big]$ and $\mathcal{C}_{\mathcal{X}} = \Im\,\mathrm{Tr}\big[\mathit{\Gamma}_{\mathcal{X}}\mathcal{C}[W(x,k)]\big]$.

Note that since Eq. (12) has $\hbar$ on both sides, one can argue that this equation can be considered at the leading order (zeroth order in $\hbar$), however, in this work we consider Eq. (12) at the first order in $\hbar$ [90], keeping $\hbar$ on both sides of the equation.

## 3. Semi-classical expansion

In general, quantum kinetic equations (7)–(16) are quite complicated due to the couplings between different components of the Wigner function. However, employing semi-classical expansion, we can decrease the complexity by breaking Eqs. (7)–(16) into a number of independent equations. The form of Eqs. (7)–(16) indicate that we can search for the solutions for various components of the Wigner function in the form of a series expansion in $\hbar$, $\mathcal{X} = \sum_n \hbar^n \mathcal{X}^{(n)}$. Similarly, for the collision terms $\mathcal{C}_{\mathcal{X}}$ and $\mathcal{D}_{\mathcal{X}}$, we write $\mathcal{C}_{\mathcal{X}} = \sum_n \hbar^n \mathcal{C}_{\mathcal{X}}^{(n)}$ and $\mathcal{D}_{\mathcal{X}} = \sum_n \hbar^n \mathcal{D}_{\mathcal{X}}^{(n)}$. Below, we analyse Eqs. (7)–(16) up to second-order in $\hbar$; for the extension to the third-order, see Appendix A.

### 3.1. Zeroth order

In the leading order, *i.e.* the zeroth order in $\hbar$, the real parts give [90]

$$k^\mu \mathcal{V}^{(0)}_\mu - m\mathcal{F}^{(0)} = 0\,, \tag{17}$$
$$m\mathcal{P}^{(0)} = 0\,, \tag{18}$$
$$k_\mu \mathcal{F}^{(0)} - m\mathcal{V}^{(0)}_\mu = 0\,, \tag{19}$$
$$k^\beta \overset{\star}{\mathcal{S}}{}^{(0)}_{\mu\beta} + m\mathcal{A}^{(0)}_\mu = 0\,, \tag{20}$$
$$\epsilon_{\mu\nu\alpha\beta} k^\alpha \mathcal{A}^{\beta(0)} + m\mathcal{S}^{(0)}_{\mu\nu} = 0\,, \tag{21}$$

while the imaginary parts yield [90]

$$k^\mu \mathcal{A}^{(0)}_\mu = 0\,, \tag{22}$$
$$k^\nu \mathcal{S}^{(0)}_{\nu\mu} = 0\,, \tag{23}$$
$$k_\mu \mathcal{P}^{(0)} = 0\,, \tag{24}$$
$$k_{[\mu} \mathcal{V}^{(0)}_{\nu]} = 0\,. \tag{25}$$

From Eqs. (17)–(25), we conclude that $\mathcal{F}^{(0)}$ and $\mathcal{A}^{(0)}_\mu$ can be assumed as the basic independent coefficients in terms of which all other components of the Wigner function can be expressed, provided $\mathcal{A}^{(0)}_\mu$ satisfies orthogonality condition (22).

### 3.2. First order

In the first order in $\hbar$, the real parts give

$$k^\mu \mathcal{V}^{(1)}_\mu - m\mathcal{F}^{(1)} = \mathcal{D}^{(0)}_{\mathcal{F}}\,, \tag{26}$$
$$\frac{1}{2}\partial^\mu \mathcal{A}^{(0)}_\mu + m\mathcal{P}^{(1)} = -\mathcal{D}^{(0)}_{\mathcal{P}}\,, \tag{27}$$
$$k_\mu \mathcal{F}^{(1)} - \frac{1}{2}\partial^\nu \mathcal{S}^{(0)}_{\nu\mu} - m\mathcal{V}^{(1)}_\mu = \mathcal{D}^{(0)}_{\mathcal{V},\mu}\,, \tag{28}$$
$$-\frac{1}{2}\partial_\mu \mathcal{P}^{(0)} + k^\beta \overset{\star}{\mathcal{S}}{}^{(1)}_{\mu\beta} + m\mathcal{A}^{(1)}_\mu = -\mathcal{D}^{(0)}_{\mathcal{A},\mu}\,, \tag{29}$$
$$\frac{1}{2}\partial_{[\mu} \mathcal{V}^{(0)}_{\nu]} - \epsilon_{\mu\nu\alpha\beta} k^\alpha \mathcal{A}^{\beta(1)} - m\mathcal{S}^{(1)}_{\mu\nu} = \mathcal{D}^{(0)}_{\mathcal{S},\mu\nu}\,, \tag{30}$$

and the imaginary parts yield

$$\partial^\mu \mathcal{V}_\mu^{(0)} = 2\mathcal{C}_{\mathcal{F}}^{(0)}\,, \tag{31}$$

$$k^\mu \mathcal{A}_\mu^{(1)} = \mathcal{C}_{\mathcal{P}}^{(0)}\,, \tag{32}$$

$$\frac{1}{2}\partial_\mu \mathcal{F}^{(0)} + k^\nu \mathcal{S}_{\nu\mu}^{(1)} = \mathcal{C}_{\mathcal{V},\mu}^{(0)}\,, \tag{33}$$

$$k_\mu \mathcal{P}^{(1)} + \frac{1}{2}\partial^\beta \overset{\star}{\mathcal{S}}{}_{\mu\beta}^{(0)} = -\mathcal{C}_{\mathcal{A},\mu}^{(0)}\,, \tag{34}$$

$$k_{[\mu}\mathcal{V}_{\nu]}^{(1)} + \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\partial^\alpha \mathcal{A}^{\beta(0)} = -\mathcal{C}_{\mathcal{S},\mu\nu}^{(0)}\,. \tag{35}$$

From Eq. (32), we can immediately see that, due to the presence of the collisions, the first-order axial-vector coefficient $\mathcal{A}^{(1)}$ is not orthogonal to $k$, *cf.* Eq. (22).

### *3.3. Second order*

We need to have second-order ($\mathcal{O}(\hbar^2)$) equations of motion to derive the transport equations for the first-order ($\mathcal{O}(\hbar)$) coefficients of the Wigner function. Hence, in the second order, the real parts give

$$k^\mu \mathcal{V}_\mu^{(2)} - m\mathcal{F}^{(2)} = \mathcal{D}_{\mathcal{F}}^{(1)}\,, \tag{36}$$

$$\frac{1}{2}\partial^\mu \mathcal{A}_\mu^{(1)} + m\mathcal{P}^{(2)} = -\mathcal{D}_{\mathcal{P}}^{(1)}\,, \tag{37}$$

$$k_\mu \mathcal{F}^{(2)} - \frac{1}{2}\partial^\nu \mathcal{S}_{\nu\mu}^{(1)} - m\mathcal{V}_\mu^{(2)} = \mathcal{D}_{\mathcal{V},\mu}^{(1)}\,, \tag{38}$$

$$-\frac{1}{2}\partial_\mu \mathcal{P}^{(1)} + k^\beta \overset{\star}{\mathcal{S}}{}_{\mu\beta}^{(2)} + m\mathcal{A}_\mu^{(2)} = -\mathcal{D}_{\mathcal{A},\mu}^{(1)}\,, \tag{39}$$

$$\frac{1}{2}\partial_{[\mu}\mathcal{V}_{\nu]}^{(1)} - \epsilon_{\mu\nu\alpha\beta}k^\alpha \mathcal{A}^{\beta(2)} - m\mathcal{S}_{\mu\nu}^{(2)} = \mathcal{D}_{\mathcal{S},\mu\nu}^{(1)}\,, \tag{40}$$

while the imaginary parts yield

$$\partial^\mu \mathcal{V}_\mu^{(1)} = 2\mathcal{C}_{\mathcal{F}}^{(1)}\,, \tag{41}$$

$$k^\mu \mathcal{A}_\mu^{(2)} = \mathcal{C}_{\mathcal{P}}^{(1)}\,, \tag{42}$$

$$\frac{1}{2}\partial_\mu \mathcal{F}^{(1)} + k^\nu \mathcal{S}_{\nu\mu}^{(2)} = \mathcal{C}_{\mathcal{V},\mu}^{(1)}\,, \tag{43}$$

$$k_\mu \mathcal{P}^{(2)} + \frac{1}{2}\partial^\beta \overset{\star}{\mathcal{S}}{}_{\mu\beta}^{(1)} = -\mathcal{C}_{\mathcal{A},\mu}^{(1)}\,, \tag{44}$$

$$k_{[\mu}\mathcal{V}_{\nu]}^{(2)} + \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\partial^\alpha \mathcal{A}^{\beta(1)} = -\mathcal{C}_{\mathcal{S},\mu\nu}^{(1)}\,. \tag{45}$$

## 4. Mass-shell conditions

### 4.1. Zeroth order

From Eq. (18), we trivially obtain the leading-order pseudo-scalar component, $\mathcal{P}^{(0)}=0$, while from Eq. (19), we find the expression for the leading-order vector coefficient in terms of the leading-order scalar coefficient as [18, 90, 99]

$$\mathcal{V}_\mu^{(0)} = \frac{k_\mu}{m}\mathcal{F}^{(0)} \,. \tag{46}$$

Multiplying Eq. (25) by $k^\mu$ and then using Eqs. (17) and (19), we obtain constraint equation for the leading-order vector coefficient

$$\left(k^2 - m^2\right)\mathcal{V}_\mu^{(0)} = 0 \,. \tag{47}$$

Inserting Eq. (46) into Eq. (17), we get analogous constraint equation for the leading-order scalar coefficient [99]

$$\left(k^2 - m^2\right)\mathcal{F}^{(0)} = 0 \,. \tag{48}$$

From Eq. (21), one can find the definition of the leading-order tensor coefficient and its dual in terms of leading-order axial-vector coefficient [18, 90], respectively, as

$$\mathcal{S}_{\mu\nu}^{(0)} = -\frac{1}{m}\epsilon_{\mu\nu\alpha\beta}k^\alpha\mathcal{A}^\beta_{(0)} \,, \tag{49}$$

$$\overset{\star}{\mathcal{S}}{}^{\mu\nu}_{(0)} = \frac{1}{m}k^{[\mu}\mathcal{A}^{\nu]}_{(0)} \,. \tag{50}$$

Using Eq. (50) in Eq. (20) and employing Eq. (22), one can get the constraint equation for $\mathcal{A}_\mu^{(0)}$ [99]

$$\left(k^2 - m^2\right)\mathcal{A}_\mu^{(0)} = 0 \,. \tag{51}$$

One should note here that, due to our assumption that polarization effects can come from both the leading and first order in $\hbar$, $\mathcal{A}_\mu^{(0)}$ does not vanish which implies that the zeroth-order tensor coefficient $\mathcal{S}_{\mu\nu}^{(0)}$ in Eq. (49) does not vanish either. We would like to stress that in this respect, our analysis goes beyond the approach of Refs. [53, 54] and assumes that spin does not have to be only a dissipative effect, having both classical and quantum counterparts.

Similarly to Eq. (49), we can arrive at the expression

$$\mathcal{A}^\rho_{(0)} = -\frac{k_\lambda}{m}\overset{\star}{\mathcal{S}}{}^{\rho\lambda}_{(0)} = -\frac{1}{2m}\epsilon^{\rho\lambda\alpha\beta}k_\lambda\mathcal{S}^{(0)}_{\alpha\beta} \,. \tag{52}$$

Putting Eq. (52) into Eq. (21) and then using Eq. (23), we get the constraint equation for $\mathcal{S}^{(0)}_{\mu\nu}$

$$\left(k^2 - m^2\right) \mathcal{S}^{(0)}_{\mu\nu} = 0 \,. \tag{53}$$

Therefore, for a non-trivial solution to exist, all the leading-order coefficients have to satisfy the on-shell condition, *i.e.*, $k^2 = m^2$, where $k$ denotes the kinetic momentum.

Hence, in the leading order, we obtain Eqs. (46), (49), and (50), in terms of independent quantities $\mathcal{F}^{(0)}$ and $\mathcal{A}^{(0)}_\nu$ [18, 90] where on-shell conditions for $\mathcal{F}^{(0)}$ and $\mathcal{A}^{(0)}_\nu$ (Eq. (48) and Eq. (51), respectively), lead to [99]

$$\mathcal{F}^{(0)} = \delta\left(k^2 - m^2\right) F^{(0)} \,, \qquad \mathcal{A}^{(0)}_\mu = \delta\left(k^2 - m^2\right) A^{(0)}_\mu \,, \tag{54}$$

with $F^{(0)}$ and $A^{(0)}_\mu$ being arbitrary scalar and axial-vector functions, respectively, which are non-singular at $k^2 = m^2$ and need to be determined by the kinetic equations. One can easily verify at this point that Eqs. (46), (49), and (50) satisfy Eqs. (17)–(25) if the axial-vector component of the zeroth order fulfills the orthogonality condition (22).

### *4.2. First order*

From Eqs. (27), (28), and (30), one obtains the first-order contributions to the pseudo-scalar, vector, and tensor components of the Wigner function, respectively, as

$$\mathcal{P}^{(1)} = -\frac{1}{2m}\left[\partial^\mu \mathcal{A}^{(0)}_\mu + 2\mathcal{D}^{(0)}_{\mathcal{P}}\right] , \tag{55}$$

$$\mathcal{V}^{(1)}_\mu = \frac{1}{m}\left[k_\mu \mathcal{F}^{(1)} - \frac{1}{2}\partial^\nu \mathcal{S}^{(0)}_{\nu\mu} - \mathcal{D}^{(0)}_{\mathcal{V},\mu}\right] , \tag{56}$$

$$\mathcal{S}^{(1)}_{\mu\nu} = \frac{1}{2m}\left[\partial_{[\mu}\mathcal{V}^{(0)}_{\nu]} - 2\epsilon_{\mu\nu\alpha\beta}k^\alpha \mathcal{A}^\beta_{(1)} - 2\mathcal{D}^{(0)}_{\mathcal{S},\mu\nu}\right] , \tag{57}$$

where the dual form of $\mathcal{S}^{(1)}_{\mu\nu}$ is obtained by contracting with the Levi-Civita tensor

$$\overset{\star}{\mathcal{S}}{}^{(1)}_{\mu\beta} = \frac{1}{m}\left[\frac{1}{4}\epsilon_{\mu\beta\sigma\rho}\partial^{[\sigma}\mathcal{V}^{\rho]}_{(0)} + k_{[\mu}\mathcal{A}^{(1)}_{\beta]} - \frac{1}{2}\epsilon_{\mu\beta\sigma\rho}\mathcal{D}^{\sigma\rho}_{\mathcal{S}(0)}\right] . \tag{58}$$

Using Eqs. (18) and (58) in Eq. (29) and subsequently using Eqs. (32) and (46), we get constraint condition for the first-order axial-vector coefficient as

$$\left(k^2 - m^2\right) \mathcal{A}^{(1)}_\mu = k_\mu \mathcal{C}^{(0)}_{\mathcal{P}} - \frac{1}{2}\epsilon_{\mu\beta\sigma\rho}k^\beta \mathcal{D}^{\sigma\rho}_{\mathcal{S}(0)} + m\mathcal{D}^{(0)}_{\mathcal{A},\mu} \,. \tag{59}$$

To obtain the constraint equation satisfied by the first-order scalar coefficient, we contract Eq. (28) with $k^\mu$ and use Eqs. (26) and (49), getting

$$\left(k^2 - m^2\right) \mathcal{F}^{(1)} = k^\mu \mathcal{D}^{(0)}_{\mathcal{V},\mu} + m \mathcal{D}^{(0)}_{\mathcal{F}} \,. \tag{60}$$

Multiplying Eq. (35) by $k_\mu$ and then using Eq. (26), as well as Eqs. (21) and (28) yields the constraint equation for the first-order vector coefficient

$$\left(k^2 - m^2\right) \mathcal{V}^\rho_{(1)} = m \mathcal{D}^\rho_{\mathcal{V}(0)} + k^\rho \mathcal{D}_{\mathcal{F}(0)} - k_\lambda \mathcal{C}^{\lambda\rho}_{\mathcal{S}(0)} \,. \tag{61}$$

On the other hand, multiplying Eq. (34) by $k_\rho$, we get

$$k^2 \mathcal{P}^{(1)} + \frac{1}{2} k_\rho \partial_\lambda \overset{\star}{\mathcal{S}}{}^{\rho\lambda}_{(0)} = -k_\rho \mathcal{C}^\rho_{\mathcal{A}(0)} \,. \tag{62}$$

Subtracting Eq. (27) multiplied by $m$ from Eq. (62), and using in the resulting formula Eq. (20), we get the constraint equation for the first-order pseudo-scalar coefficient as

$$\left(k^2 - m^2\right) \mathcal{P}^{(1)} = -k_\rho \mathcal{C}^\rho_{\mathcal{A}(0)} + m \mathcal{D}^{(0)}_{\mathcal{P}} \,. \tag{63}$$

Combining Eqs. (18), (29), and (30), after some straightforward algebraic manipulations, gives us the following equation:

$$m \partial^{[\rho} \mathcal{V}^{\lambda]}_{(0)} - \epsilon^{\alpha\rho\lambda\sigma} \epsilon_{\alpha\gamma\delta\beta} k_\sigma k^\beta \mathcal{S}^{\gamma\delta}_{(1)} + 2 \epsilon^{\rho\lambda\sigma\alpha} k_\sigma \mathcal{D}^{(0)}_{\mathcal{A},\alpha} - 2m^2 \mathcal{S}^{\rho\lambda}_{(1)} = 2m \mathcal{D}^{\rho\lambda}_{\mathcal{S}(0)} \,. \tag{64}$$

Contracting Levi-Civita tensors and using Eq. (33) in Eq. (64), and subsequently using Eq. (19), one arrives at the constraint equation for the first-order tensor coefficient $\mathcal{S}^{\rho\lambda}_{(1)}$

$$\left(k^2 - m^2\right) \mathcal{S}^{\rho\lambda}_{(1)} = k^{[\rho} \mathcal{C}^{\lambda]}_{\mathcal{V}(0)} + m \mathcal{D}^{\rho\lambda}_{\mathcal{S}(0)} - \epsilon^{\rho\lambda\sigma\alpha} k_\sigma \mathcal{D}^{(0)}_{\mathcal{A},\alpha} \,. \tag{65}$$

Equations (59)–(61), (63), and (65) are the mass-shell conditions for all the first-order coefficients of the Wigner function. From these equations, one can observe that in the collisionless limit, all the first-order coefficients remain on-shell. Furthermore, unlike in Refs. [53, 54], the above equations assume the most general structure of the interactions by considering all the collision terms to be non-vanishing, *i.e.* $\mathcal{D}_\mathcal{X} \neq 0$ and $\mathcal{C}_\mathcal{X} \neq 0$.

## 5. Kinetic equations for scalar and axial-vector components

Combining Eqs. (31) and (46), we find the kinetic equation to be satisfied by the leading-order scalar coefficient

$$k^\mu \partial_\mu \mathcal{F}^{(0)} = 2m \, \mathcal{C}^{(0)}_{\mathcal{F}} \,, \tag{66}$$

while plugging Eq. (56) into Eq. (41), we obtain the kinetic equation for the first-order scalar coefficient

$$k^\mu \partial_\mu \mathcal{F}^{(1)} = 2m\, \mathcal{C}^{(1)}_{\mathcal{F}} + \partial^\mu \mathcal{D}^{(0)}_{\mathcal{V},\mu} \,. \tag{67}$$

Using Eq. (27) together with Eq. (50) in Eq. (34), we get the kinetic equation to be satisfied by the leading-order axial-vector coefficient

$$k^\beta \partial_\beta \mathcal{A}^{(0)}_\mu = 2m\, \mathcal{C}^{(0)}_{\mathcal{A},\mu} - 2k_\mu \mathcal{D}^{(0)}_{\mathcal{P}} \,. \tag{68}$$

Finally, using Eq. (58) and Eq. (A.11) in Eq. (44), we get the kinetic equation to be satisfied by the first-order axial-vector coefficient

$$k^\beta \partial_\beta \mathcal{A}^{(1)}_\mu = 2m\, \mathcal{C}^{(1)}_{\mathcal{A},\mu} - 2k_\mu \mathcal{D}^{(1)}_{\mathcal{P}} - \frac{1}{2}\epsilon_{\mu\beta\gamma\delta}\partial^\beta \mathcal{D}^{\gamma\delta}_{\mathcal{S}(0)} \,. \tag{69}$$

So far, we have discussed mass-shell conditions and kinetic equations satisfied by different components of the Wigner function without imposing any special assumptions. At this point, it is useful to highlight the important differences between the present work and Refs. [53, 54]. The crucial assumption considered in Refs. [53, 54] is that the spin effects appear at the first order in $\hbar$ in the semi-classical expansion of the Wigner function. Such an assumption is physically motivated when one considers relatively small spin polarization effects arising through scatterings in a vortical medium. Within the quantum kinetic theory approach, the polarization effects appear through the axial-vector component $\mathcal{A}^\mu$ [53, 54, 100]. The assumption that the spin polarization effect is at least of the order of $\hbar$ implies that $\mathcal{A}^{(0)}_\mu$ can be considered to be zero, and consequently, the tensor component of the Wigner function $\mathcal{S}^{(0)}_{\mu\nu} = 0$ (see Eq. (49)). Moreover, the leading-order pseudo-scalar component $\mathcal{P}^{(0)}$ is always vanishing (18). Furthermore, for the consistency of the framework, it has been argued that the leading-order collision terms involving the pseudo-scalar, axial-vector, and tensor components must vanish, *i.e.*, $\mathcal{C}^{(0)}_{\mathcal{P}} = 0$; $\mathcal{C}^{\mu(0)}_{\mathcal{A}} = 0$; $\mathcal{C}^{\mu\nu(0)}_{\mathcal{S}} = 0$; $\mathcal{D}^{(0)}_{\mathcal{P}} = 0$; $\mathcal{D}^{\mu(0)}_{\mathcal{A}} = 0$; $\mathcal{D}^{\mu\nu(0)}_{\mathcal{S}} = 0$ [53, 54]. Such constraints on the collision terms also affect the on-shell conditions for various components of the Wigner function. It can be shown in this case that $\mathcal{A}^{(1)}_\mu$ not only is on-shell, see Eq. (59), but also remains orthogonal to momentum, $k^\mu \mathcal{A}_\mu = \mathcal{O}(\hbar^2)$ [53, 54]. Similarly, due to the assumptions that $\mathcal{A}^{(0)}_\mu$ and the collision term $\mathcal{D}^{(0)}_{\mathcal{P}}$ are vanishing, the pseudo-scalar component $\mathcal{P}$ is at least of the order of $\hbar^2$, see Eq. (55).

The novelty of the current study in comparison to Refs. [53, 54] is that we have assumed that the polarization effects can also appear at the leading-order of the semi-classical approximation, *i.e.* $\mathcal{A}^{(0)}_\mu \neq 0$. The physical motivation behind our assumption is that the polarization effect can be manifested even at the classical level, and thus polarization can be generated in the presence as well as in the absence of specific collision processes in a vortical fluid. We also assume that all the collision terms are, in general, non-vanishing, hence, in contrast to Refs. [53, 54], we consider $\mathcal{P}^{(1)} \neq 0$, $(k^2-m^2)\mathcal{A}^{(1)}_\mu \neq 0$ and $k^\mu \mathcal{A}_\mu = \mathcal{O}(\hbar)$. Such a different treatment of the axial-vector component of the Wigner function and various collision terms may give rise to significantly different on-shell conditions and kinetic equations for various components of the Wigner function.

## 6. General kinetic equation and its classical counterpart

In this section, we combine the zeroth and first-order kinetic equations for $\mathcal{F}$ and $\mathcal{A}_\mu$. This may be used to develop a hydrodynamic framework where spin effects arise at both $\hbar^0$ and $\hbar^1$ orders.

Combining Eqs. (66) and (67), we get the kinetic equation for the scalar coefficient as

$$k^\mu \partial_\mu \tilde{\mathcal{F}} = 2m\, \tilde{\mathcal{C}}_{\mathcal{F}}\,, \tag{70}$$

where

$$\begin{aligned} \tilde{\mathcal{F}} &= \mathcal{F}^{(0)} + \hbar\, \mathcal{F}^{(1)}\,, \\ \tilde{\mathcal{C}}_{\mathcal{F}} &= \mathcal{C}^{(0)}_{\mathcal{F}} + \hbar \left( \mathcal{C}^{(1)}_{\mathcal{F}} + \frac{1}{2m} \partial^\mu \mathcal{D}^{(0)}_{\mathcal{V},\mu} \right)\,. \end{aligned} \tag{71}$$

Similarly, from Eqs. (68) and (69), we obtain the following kinetic equation for axial-vector component:

$$k^\beta \partial_\beta \tilde{\mathcal{A}}_\mu = 2m\, \tilde{\mathcal{C}}_{\mathcal{A},\mu}\,, \tag{72}$$

where

$$\tilde{\mathcal{A}}_\mu = \mathcal{A}^{(0)}_\mu + \hbar\, \mathcal{A}^{(1)}_\mu\,, \tag{73}$$

$$\tilde{\mathcal{C}}_{\mathcal{A},\mu} = \mathcal{C}^{(0)}_{\mathcal{A},\mu} + \hbar\, \mathcal{C}^{(1)}_{\mathcal{A},\mu} - \frac{k_\mu}{m} \left( \mathcal{D}^{(0)}_{\mathcal{P}} + \hbar \mathcal{D}^{(1)}_{\mathcal{P}} \right) - \frac{\hbar}{4m} \epsilon_{\mu\beta\gamma\delta} \partial^\beta \mathcal{D}^{\gamma\delta}_{\mathcal{S}(0)}\,. \tag{74}$$

It is useful to introduce spin as an additional phase-space variable in the distribution function as [36, 53, 94–96]

$$\mathfrak{f}(x,k,\mathfrak{s}) = \frac{1}{2} \left( \tilde{\mathcal{F}}(x,k) - \mathfrak{s} \cdot \tilde{\mathcal{A}}(x,k) \right)\,, \tag{75}$$

where $\mathfrak{s}^\alpha$ is the spin four-vector. Using Eq. (75), one can also obtain $\tilde{\mathcal{F}}(x,k)$ and $\tilde{\mathcal{A}}(x,k)$

$$\int \mathrm{d}S(k)\,\mathfrak{f}(x,k,\mathfrak{s}) \;=\; \tilde{\mathcal{F}}(x,k)\,, \tag{76}$$

$$\int \mathrm{d}S(k)\,\mathfrak{s}^\mu\;\mathfrak{f}(x,k,\mathfrak{s}) \;=\; \tilde{\mathcal{A}}^\mu(x,k)\,, \tag{77}$$

where the spin measure [53, 54]

$$\int \mathrm{d}S(k) \equiv \frac{1}{\pi}\sqrt{\frac{k^2}{3}}\int \mathrm{d}^4\mathfrak{s}\,\delta(\mathfrak{s}\cdot\mathfrak{s}+3)\,\delta(k\cdot\mathfrak{s})\,, \tag{78}$$

satisfies the following identities [53, 54]:

$$\begin{aligned}\int \mathrm{d}S(k) \;&=\; 2\,,\\ \int \mathrm{d}S(k)\,\mathfrak{s}^\mu \;&=\; 0\,,\\ \int \mathrm{d}S(k)\,\mathfrak{s}^\mu\mathfrak{s}^\nu \;&=\; -2\left(g^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right).\end{aligned} \tag{79}$$

The relation between $\tilde{\mathcal{F}}(x,k)$ and $\mathfrak{f}(x,k,\mathfrak{s})$ (76) can be easily established using the above identities. But it is rather non-trivial to express $\tilde{\mathcal{A}}^\mu(x,k)$ in terms of $\mathfrak{f}(x,k,\mathfrak{s})$. Using Eq. (75) on the left-hand side of Eq. (77), it can be shown that

$$\int \mathrm{dS}(k)\,\mathfrak{s}^\mu\,\mathfrak{f}(x,k,\mathfrak{s}) = \tilde{\mathcal{A}}^\mu(x,k) - \frac{k^\mu}{k^2}\left(k\cdot\tilde{\mathcal{A}}\right)\,. \tag{80}$$

Equation (77) is valid only when $k\cdot\tilde{\mathcal{A}}=0$. From Eqs. (22) and (32), one can observe that $k\cdot\tilde{\mathcal{A}}=\hbar\,\mathcal{C}_{\mathcal{P}}^{(0)}$. Thus, the collision term $\mathcal{C}_{\mathcal{P}}^{(0)}$ must vanish to obtain Eq. (77). In Refs. [53, 54], it was considered that $\mathcal{C}_{\mathcal{P}}^{(0)}=0$ since the spin polarization effects are assumed to be at least of the order of $\hbar$. In the present investigation, one can still consider that $\mathcal{C}_{\mathcal{P}}^{(0)}$ vanishes. From Eqs. (10) and (13), one obtains

$$k^\mu\partial_\mu\mathcal{P} = 2m\,\mathcal{C}_{\mathcal{P}} + 2k^\mu\mathcal{D}_{\mathcal{A},\mu}\,. \tag{81}$$

Using the semi-classical expansion of $\mathcal{P}$, $\mathcal{C}_{\mathcal{P}}$, and $\mathcal{D}_{\mathcal{A},\mu}$ with the condition $\mathcal{P}^{(0)}=0$, the above equation gives

$$m\,\mathcal{C}_{\mathcal{P}}^{(0)} + k^\mu\mathcal{D}_{\mathcal{A},\mu}^{(0)} = 0\,. \tag{82}$$

If we consider that the collision term $\mathcal{D}^{(0)}_{\mathcal{A},\mu}$ is such that it satisfies $k^\mu\mathcal{D}^{(0)}_{\mathcal{A},\mu}=0$, then Eq. (82) gives $\mathcal{C}^{(0)}_{\mathcal{P}} = 0$ [3]. In that case, Eqs. (76) and (77) provide a well-defined prescription to obtain $\tilde{\mathcal{F}}(x,k)$ and $\tilde{\mathcal{A}}^\mu(x,k)$ in terms of the distribution function $\mathfrak{f}(x,k,\mathfrak{s})$ [4]. The inversion relations from $\tilde{\mathcal{F}}(x,k)$ and $\tilde{\mathcal{A}}^\mu(x,k)$ to $\mathfrak{f}(x,k,\mathfrak{s})$ are crucial because, in the quantum kinetic theory approach, energy-momentum and the spin tensors are expressed in terms of various independent components of the Wigner function. Therefore, with the help of inversion relations (76) and (77), those macroscopic currents can be written in terms of $\mathfrak{f}(x,k,\mathfrak{s})$ [16–18, 37]. From Eqs. (70), (72), and (75), we get the general Boltzmann equation to be solved[5]

$$k^\mu\partial_\mu\mathfrak{f}(x,k,\mathfrak{s}) = m\,\mathfrak{C}(\mathfrak{f})\,, \tag{83}$$

where $\mathfrak{C}(\mathfrak{f})=\tilde{\mathcal{C}}_{\mathcal{F}} - \mathfrak{s}\cdot\tilde{\mathcal{C}}_{\mathcal{A}}$ is the collision term that explicitly takes into account the effect of spin. Within the quasiparticle approximation, one may choose a generalized representation of the distribution function $\mathfrak{f}(x,k,\mathfrak{s}) = m\delta(k^2-M^2)f(x,k,\mathfrak{s})$ [53, 54]. Here, the on-shell singularity for the quasiparticle has been expressed as $\delta(k^2-M^2)$ and the function $f(x,k,\mathfrak{s})$ is a function without singularity. Furthermore, $M$ denotes quasiparticle mass, which includes quantum corrections to the bare mass $m$. These quantum corrections (off-shell corrections) can primarily originate from the spin dependence. Note that these off-shell contributions can appear on both sides of the above equation. If the off-shell terms from both sides of Eq. (83) cancel, then one will be left with the on-shell Boltzmann equation involving the distribution function $f(x,k,\mathfrak{s})$ [53, 54]. To check this cancellation of off-shells correction, one has to calculate the generalized collision term $\mathfrak{C}(\mathfrak{f})$ involving a spin-dependent distribution function in the presence of local and non-local collisions. Explicit expressions of such spin-dependent collision kernels have been put forward in Refs. [53, 54] where one only considers the spin contribution at the order of $\hbar$. The explicit expression of $\mathfrak{C}(\mathfrak{f})$ in

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[3] Note that the collision term $\mathcal{D}^{(0)}_{\mathcal{A},\mu}$ transforms as an axial-vector under the Lorentz transformation. Microscopically, such axial-vector nature of $\mathcal{D}^{(0)}_{\mathcal{A},\mu}$ may originate from spin ($\mathfrak{s}^\mu$) which transforms like an axial vector. Therefore, one may express $\mathcal{D}^{(0)}_{\mathcal{A},\mu}$ in terms of the spin four-vector $\mathfrak{s}^\mu$. Since $k\cdot\mathfrak{s}=0$ due to the Dirac delta function $\delta(k\cdot\mathfrak{s})$ in the expression of generalized spin measure (78), one may expect that $k^\mu\mathcal{D}^{(0)}_{\mathcal{A},\mu}=0$ for a generic collision term.

[4] One may also define, alternatively, the distribution function (75) as $\mathfrak{f}(x,k,\mathfrak{s}) = \frac{1}{2}(\tilde{\mathcal{F}}(x,k) - \mathfrak{s}\cdot\tilde{\mathcal{A}}_\perp(x,k))$, where $\tilde{\mathcal{A}}_\perp(x,k)$ is the orthogonal component of $\tilde{\mathcal{A}}(x,k)$ with respect to $k^\mu$, *i.e.* $\tilde{\mathcal{A}}^\mu_\perp(x,k) = \tilde{\mathcal{A}}^\mu(x,k) - \frac{k^\mu}{k^2}(k\cdot\tilde{\mathcal{A}})$. In that case, Eqs. (76) and (77) are trivially satisfied without any constraint on the collision terms.

[5] Note that Boltzmann equation (83) derived here assumes that spin effects are at least of the leading and first-order in $\hbar$ which is not the case derived in Refs. [53, 54].

Eq. (83) that incorporates spin effects in both the leading and the first order in $\hbar$ including local and non-local collision terms needs to be scrutinized in detail which we leave for future studies.

## 7. Summary and conclusions

In this work, we have extended the formalism presented in Ref. [36] using the Wigner function formalism and employed semi-classical expansion to derive generalized quantum kinetic equations for the components of the Wigner function of massive spin-half Dirac particles. In our calculations, we have considered that the spin polarization effects may arise from both the zeroth and first-order contributions in the $\hbar$ expansion. We have derived a general quantum kinetic equation for the independent Wigner function components and used the ansatz for the generalized phase-space distribution function, including the spin degrees of freedom, to obtain its classical counterpart. We have shown that the latter has the same Boltzmann-like form as the one found in Refs. [53, 54], and the distribution function can be mapped back to the components of the Wigner function. In the present manuscript, we have not given explicit expressions of the collision kernel. However, we expect that using some suitable approximation of the collision kernel, one can develop a general spin-hydrodynamic formalism in a way presented in Ref. [56] ensuring the conservation of the energy-momentum and total angular momentum tensors.

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## Appendix A

*Third-order kinetic equations for the Wigner function coefficients*

In this appendix, we derive the third-order kinetic equations for the coefficients of the Wigner function. Mass-shell conditions and transport equations for $\mathcal{F}^{(2)}$ and $\mathcal{A}_\mu^{(2)}$ are also shown.

In the $\hbar^3$ order, comparing the real and imaginary parts of the coefficients of the Wigner function in the Clifford-algebra basis, we arrive at two sets of equations, where the real part gives

$$k^\mu \mathcal{V}_\mu^{(3)} - m\mathcal{F}^{(3)} = \mathcal{D}_{\mathcal{F}}^{(2)} , \tag{A.1}$$

$$\frac{1}{2}\partial^\mu \mathcal{A}_\mu^{(2)} + m\mathcal{P}^{(3)} = -\mathcal{D}_{\mathcal{P}}^{(2)} , \tag{A.2}$$

$$k_\mu \mathcal{F}^{(3)} - \frac{1}{2}\partial^\nu \mathcal{S}_{\nu\mu}^{(2)} - m\mathcal{V}_\mu^{(3)} = \mathcal{D}_{\mathcal{V},\mu}^{(2)} , \tag{A.3}$$

$$-\frac{1}{2}\partial_\mu \mathcal{P}^{(2)} + k^\beta \overset{\star}{\mathcal{S}}{}_{\mu\beta}^{(3)} + m\mathcal{A}_\mu^{(3)} = -\mathcal{D}_{\mathcal{A},\mu}^{(2)} , \tag{A.4}$$

$$\frac{1}{2}\partial_{[\mu} \mathcal{V}_{\nu]}^{(2)} - \epsilon_{\mu\nu\alpha\beta} k^\alpha \mathcal{A}^{\beta(3)} - m\mathcal{S}_{\mu\nu}^{(3)} = \mathcal{D}_{\mathcal{S},\mu\nu}^{(2)} , \tag{A.5}$$

and from the imaginary parts, we obtain

$$\partial^\mu \mathcal{V}_\mu^{(2)} = 2\mathcal{C}_{\mathcal{F}}^{(2)} , \tag{A.6}$$

$$k^\mu \mathcal{A}_\mu^{(3)} = \mathcal{C}_{\mathcal{P}}^{(2)} , \tag{A.7}$$

$$\frac{1}{2}\partial_\mu \mathcal{F}^{(2)} + k^\nu \mathcal{S}_{\nu\mu}^{(3)} = \mathcal{C}_{\mathcal{V},\mu}^{(2)} , \tag{A.8}$$

$$k_\mu \mathcal{P}^{(3)} + \frac{1}{2}\partial^\beta \overset{\star}{\mathcal{S}}{}_{\mu\beta}^{(2)} = -\mathcal{C}_{\mathcal{A},\mu}^{(2)} , \tag{A.9}$$

$$k_{[\mu} \mathcal{V}_{\nu]}^{(3)} + \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\partial^\alpha \mathcal{A}^{\beta(2)} = -\mathcal{C}_{\mathcal{S},\mu\nu}^{(2)} . \tag{A.10}$$

From Eqs. (37), (38), and (40), we obtain the second-order contributions to the pseudo-scalar, vector, and tensor components of the Wigner function, respectively,

$$\mathcal{P}^{(2)} = -\frac{1}{2m}\left[\partial^\mu \mathcal{A}_\mu^{(1)} + 2\mathcal{D}_{\mathcal{P}}^{(1)}\right] , \tag{A.11}$$

$$\mathcal{V}_\mu^{(2)} = \frac{1}{m}\left[k_\mu \mathcal{F}^{(2)} - \frac{1}{2}\partial^\nu \mathcal{S}_{\nu\mu}^{(1)} - \mathcal{D}_{\mathcal{V},\mu}^{(1)}\right] , \tag{A.12}$$

$$\mathcal{S}_{\mu\nu}^{(2)} = \frac{1}{2m}\left[\partial_{[\mu} \mathcal{V}_{\nu]}^{(1)} - 2\epsilon_{\mu\nu\alpha\beta} k^\alpha \mathcal{A}_{(2)}^\beta - 2\mathcal{D}_{\mathcal{S},\mu\nu}^{(1)}\right] . \tag{A.13}$$

Again, the dual form of $\mathcal{S}^{(2)}_{\mu\nu}$ can be expressed as

$$\overset{\star}{\mathcal{S}}{}^{(2)}_{\mu\beta} = \frac{1}{m}\left[\frac{1}{4}\epsilon_{\mu\beta\gamma\delta}\partial^{[\gamma}\mathcal{V}^{\delta]}_{(1)} + k_{[\mu}\mathcal{A}^{(2)}_{\beta]} - \frac{1}{2}\epsilon_{\mu\beta\gamma\delta}\mathcal{D}^{\gamma\delta}_{\mathcal{S}(1)}\right]. \tag{A.14}$$

Contracting Eq. (38) with $k^\mu$ and using Eqs. (33) and (36), we get constraint equation for the second-order scalar coefficient

$$\left(k^2 - m^2\right)\mathcal{F}^{(2)} = \frac{1}{4}\partial^\mu\partial_\mu\mathcal{F}^{(0)} + k^\mu\mathcal{D}^{(1)}_{\mathcal{V},\mu} + m\mathcal{D}^{(1)}_{\mathcal{F}} - \frac{1}{2}\partial^\nu\mathcal{C}^{(0)}_{\mathcal{V},\nu}. \tag{A.15}$$

Using Eqs. (55) and (A.14) in Eq. (39), we obtain the constraint equation for the second-order axial-vector coefficient

$$\begin{aligned}\left(k^2 - m^2\right)\mathcal{A}^{(2)}_\mu &= \frac{1}{4}\partial_\mu\partial^\alpha\mathcal{A}^{(0)}_\alpha + \frac{1}{2}\partial_\mu\mathcal{D}^{(0)}_{\mathcal{P}} + k_\mu\mathcal{C}^{(1)}_{\mathcal{P}} \\ &\quad + \frac{1}{2}\epsilon_{\mu\beta\gamma\delta}k^\beta\left(\partial^\gamma\mathcal{V}^\delta_{(1)} - \mathcal{D}^{\gamma\delta}_{\mathcal{S}(1)}\right) + m\mathcal{D}^{(1)}_{\mathcal{A},\mu}.\end{aligned} \tag{A.16}$$

Combining Eqs. (A.6) and (A.12), we get the kinetic equation for $\mathcal{F}^{(2)}$ as

$$k^\mu\partial_\mu\mathcal{F}^{(2)} = 2m\mathcal{C}^{(2)}_{\mathcal{F}} + \partial^\mu\mathcal{D}^{(1)}_{\mathcal{V},\mu}. \tag{A.17}$$

Finally, to arrive at the kinetic equation for $\mathcal{A}^{(2)}_\mu$, we put Eqs. (A.2) and (A.14) into Eq. (A.9) getting

$$k^\beta\partial_\beta\mathcal{A}^{(2)}_\mu = 2m\mathcal{C}^{(2)}_{\mathcal{A},\mu} - 2k_\mu\mathcal{D}^{(2)}_{\mathcal{P}} - \frac{1}{2}\epsilon_{\mu\beta\gamma\delta}\partial^\beta\mathcal{D}^{\gamma\delta}_{\mathcal{S}(1)}. \tag{A.18}$$

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