LIMITING SPECTRAL DENSITY OF ELLIPTIC VOLATILITY SAMPLE COVARIANCE ENSEMBLE WITH STUDENT'S t TAILED VOLATILITY

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In this paper, we study an ensemble of random matrices called the Elliptic Volatility Model, which arises in finance as models of stock returns. This model consists of a product of independent matrices $X = \Sigma Z$, where Z is a T by S matrix of i.i.d. light-tailed variables with mean 0 and variance 1, and Σ is a diagonal matrix. In this paper, we take the randomness of Σ to be i.i.d. heavy-tailed. We obtain an explicit formula for the empirical spectral distribution of X^*X in the particular case when the elements of Σ are distributed as Student's t with parameter 3.

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1. Introduction

A key problem in random matrix theory is understanding eigenvalue properties when the matrix dimensions are large. There is a large body of work on properties of the Sample Covariance Matrix Ensemble. The eigenvalue distribution has been shown to be the Marchenko–Pastur in a very general case for the matrix of covariances of i.i.d. variables with a variance, and then similar results were extended to matrices with various correlation structures. In this paper, we explore a random matrix ensemble originating from financial mathematics where the entries are heavy-tailed and uncorrelated but not independent. This dependence structure and its somewhat heavy tails result in a different eigenvalue density in the limit of a large dimension. Unlike the Marchenko–Pastur distribution, the eigenvalue density has a heavy tail as well. A similar eigenvalue density has been observed in correlation matrices arising in diverse settings in multiple applied fields, including in calcium imaging data in various types of tissue [1, 2], machine learning [3], and finance [4]. The breadth of applications where such distributions are found may indicate a new universal phenomenon.

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In financial mathematics, a volatility process is commonly defined as

$$X_t = \sigma_t Z_t\,,$$

where Z_t are independent $\mathcal{N}(0,1)$ random variables (further, we will call them noise), and variables Z_t and σ_t are independent. The process σ_t is called volatility and can be modelled in multiple ways. For example, in the celebrated Black–Scholes model, centred log-returns of the price can be modelled as $X_t = \sigma_0\,(B_t - B_{t-1})$, where B_t is the Brownian motion, the volatility equals σ_0 for any t.

A random variable X is called heavy-tailed (or fat-tailed, Pareto-tailed, power-law tailed) if a power law can approximate its density $p(x)$ for the large x

$$p(x) \underset{|x|\to\infty}{\sim} \frac{c}{|x|^{\alpha+1}}\,. \tag{1.1}$$

For arbitrary α such tails are regularly varying, and α is referred to as tail exponent. A canonical example of a heavy-tailed distribution with tail exponent α is the Student's t distribution with α degrees of freedom, which we abbreviate as Student (α) throughout the manuscript. In his work [5], Mandelbrot argued using the example of cotton price changes that the empirical distribution of price changes is better approximated by an α-stable distribution other than normal, *i.e.* a distribution with tail parameter α. The log-returns of stock prices in developed countries are believed to follow a power law with exponent $\alpha \approx 3$ [6]. This tail property is called cubic law. Figure 1 illustrates the cubic law for log-returns of three examples of major companies.

The heaviness of the tails in stock log-returns is important for portfolio optimisation. If risky assets returns are i.i.d. and the second moment exists, investing equally into each asset, *i.e.* diversifying the portfolio reduces risks, and the distribution of portfolio returns can be approximated using the Central Limit Theorem (see *e.g.* [7]). The diversification strategy may not remain optimal for the distributions with heavier tails [8, 9]. For example, in the case of Cauchy distributed price changes, the diversified portfolio will have a similar risk distribution as the non-diversified because the sample mean of the i.i.d. Cauchy random variables has the Cauchy distribution with the same parameters. Considering even heavier tails would lead to the optimality of the non-diversification of the portfolio [10].

For the sequence of random matrices $\{\boldsymbol{X}_T\}$ of the size $S \times T$, where $\frac{S}{T} \to y > 0$, with i.i.d. entries with 0 mean and variance σ^2, the limiting spectral distribution of $\frac{\boldsymbol{X}_T \boldsymbol{X}_T^*}{T}$ exists, and is called the Marchenko–Pastur law. This holds for heavy-tailed variables as well with tail parameter $\alpha > 2$, and the limiting spectral distribution changes only for $\alpha < 2$. However, it is

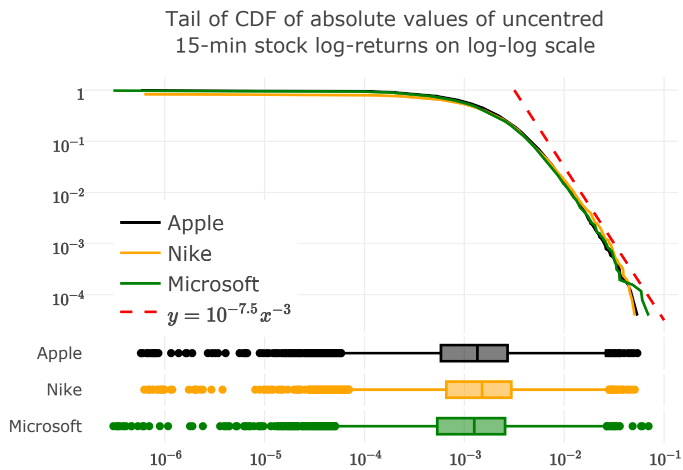

Fig. 1. Illustration of the cubic law. The tail $\bar{F}(x) = 1 - F(x)$, where $F(x)$ is empirical c.d.f. of returns of the chosen stock has slope ≈ 3 when plotted on a log–log scale. The box-whiskers plot displays the distribution of logarithms of log-returns. On the left, the plot for three major companies. Data taken from `https://polygon.io/`

well-known that Marchenko–Pastur law does not approximate the spectrum of a stock returns correlation matrix, even though the returns have a tail parameter close to 3, *i.e.* a lot bigger than 2. This discrepancy occurs due to correlations or dependence between stocks. In [11], it is demonstrated that a factor model with any number of factors (a model with k rank one matrices added to an i.i.d. random matrix) does not approximate stock return correlation eigenvalues either.

A model that does approximate the stock returns correlation eigenvalues well is the Student-Wishart Elliptic Volatility Matrix [12]. In this paper, we will be concerned with models that generalise this. We introduce a definition here:

Definition 1.1. Let $T \times S$ random matrix $\boldsymbol{X}$ is an Elliptic Volatility Matrix (EVM) if

$$\boldsymbol{X} = (\sigma_t Z_{t,s})_{\substack{t \leq T \\ s \leq S}} , \tag{1.2}$$

where random variables $Z_{t,s}$ are independent identically distributed random variables with a finite variance and σ_ts are independent of $(Z_{t,s})_{\substack{t \leq T \\ s \leq S}}$ and whose empirical cumulative distribution function converges almost surely to $F(x)$, which is the c.d.f. of some heavy-tailed random variable σ with tail exponent α. We denote $F'(x) =: f(x)$. Furthermore, we define the Elliptic

Volatility Sample Covariance Ensemble (EVSCE) as the following random matrix ensemble:

$$\boldsymbol{A} := \frac{\boldsymbol{X}^*\boldsymbol{X}}{T}\,, \tag{1.3}$$

when $T, S \to +\infty$ and $\frac{T}{S} \to y$, where $0 < y < +\infty$. Denote $\boldsymbol{Z} := (Z_{t,s})_{\substack{t\leq T\\ s\leq S}}$ and $\boldsymbol{\Sigma} = \operatorname{diag}(\sigma_t)_{t\leq T}$.

Notice that in the above definition, $\boldsymbol{X} = \boldsymbol{\Sigma}\boldsymbol{Z}$ for the diagonal matrix $\boldsymbol{\Sigma}$ with $\boldsymbol{\Sigma}_{tt} = \sigma_t$ and then

$$\frac{\boldsymbol{X}^*\boldsymbol{X}}{T} = \frac{\boldsymbol{Z}^*\boldsymbol{\Sigma}^2\boldsymbol{Z}}{T}\,. \tag{1.4}$$

The Student-Wishart is defined with σ_ts i.i.d. Student's t distributed and Z_{ts} i.i.d. normally distributed.

The EVSCE has many limitations. It cannot fully describe the market data, as it is well-known that meaningful stock correlations *e.g.* stocks in similar industries, account for some of the largest eigenvalues [4]. The discrepancy between EVSCE and market data was demonstrated definitively in [13] using a copula method. Nevertheless, understanding the spectrum of EVSCE can be valuable as it helps elucidate mechanisms by which the large eigenvalues of a correlation matrix can arise via dependence and heavy tails in the distribution of the entries. Two limitations that could be relaxed in a future work are volatility clustering ("large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes" [5]) and the "leverage effect" (negative past returns tend to increase future volatilities and positive past returns tend to decrease future volatilities). The "leverage effect" could be studied via a study of dependence in the σ_ts. Volatility clustering is already accounted for in EVSCE as the spectrum of $\frac{\boldsymbol{X}^*\boldsymbol{X}}{T}$ is preserved under the permutations of the rows of $\boldsymbol{X}$ but a reasonable model design for σ_ts with dependence is left for future work.

The object of study in this paper is the empirical spectral measure of EVSCE when volatilities are Student(3) distributed. For a Hermitian $N\times N$ matrix $\boldsymbol{A}$ with eigenvalues $\lambda_1, \lambda_2 \dots \lambda_N$, the probability measure $\mu_{\boldsymbol{A}}$ is called its empirical spectral measure (ESM) if

$$\mu_{\boldsymbol{A}} := \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i}\,. \tag{1.5}$$

The corresponding c.d.f.

$$F^{\boldsymbol{A}}(x) := \frac{1}{N}\#\left\{j \leq N : \lambda_j \leq x\right\} \tag{1.6}$$

is called the empirical spectral distribution (ESD) of matrix $\boldsymbol{A}$. Here, $\#E$ denotes the cardinality of the set E. If for a given sequence of Hermitian matrices $\{\boldsymbol{A}_N\}$

$$F^{\boldsymbol{A}_N}(x) \to F(x)\,, \tag{1.7}$$

for all $x \in \mathbb{R}$ where $F(\cdot)$ is a c.d.f. of probability measure μ, $F(\cdot)$ is called limiting spectral distribution (LSD) of this sequence and $F'(\cdot)$ is called limiting spectral density.

The main result of this paper is a computation of the limiting spectral density for the EVSCE with σ_t i.i.d. with Student(3) distribution and a general distribution Z_{ts} provided it has finite moments. This result is important as it can provide the solved "model" for further studies of the universality of such ensembles as GOE and GUE do for the Wigner universality. Our approach to the problem is via the Stieltjes transform which, for a probability measure $\mu(x)$ on the real line, is defined as

$$m_\mu(z) := \int\limits_{-\infty}^{\infty} \frac{1}{x-z} \mathrm{d}\mu(x)\,. \tag{1.8}$$

The statement [14, Theorem 4.3] provides a formula for the Stieltjes transform of the limiting density of the Elliptic Volatility Model (without any requirements on volatility empirical moments convergence), which can be reduced to (2.7) for the case of the Student(3) volatility. It is known that for the measure μ on the real line with density function $\rho(x)$ for $x \in \mathbb{R}$

$$\rho(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \operatorname{Im} m_\mu(x + \boldsymbol{i}\varepsilon)\,. \tag{1.9}$$

Using (2.7), we carefully follow the construction of the solution of a quartic polynomial to find the solution with the imaginary part in $\mathbb{C}^+$ and to show that it is unique. Our construction furthermore allows us to obtain the exact expression for the limit of its imaginary part when approaching the real line. This new approach to solving a self-consistent equation for a Stieltjes transform directly, using a carefully constructed solution whose imaginary part is then easy to understand, could be useful in finding explicit limiting densities in other random matrix ensembles.

Furthermore, we perform explicit data analytics to illustrate our results via simulations and to compare them to real-world financial returns data. Suppose that $S_t(\text{Open})$ and $S_t(\text{Close})$ denote the open and close prices of the stock on the t^{th} time interval. We are interested in log-return of the price on time interval t, defined as

$$X_t := \log \frac{S_t(\text{Close})}{S_t(\text{Open})}\,. \tag{1.10}$$

We directly study the distribution of returns at a given time t and compute its standard deviation as an estimate of σ_t. Then, we observe that the tail parameter of the σ_t is approximately 3.

2. Spectral properties of the Elliptic Volatility Matrix

The Stieltjes transform of limiting spectral distribution of matrix $\boldsymbol{A}$ can be obtained, using the following simplification of [14, Theorem 4.3].

Theorem 2.1. *Suppose that the entries of $\boldsymbol{Y}(n \times p)$ are complex random variables that are independent for each n and identically distributed for all n and satisfy $E(|Y_{11} - E(Y_{11})|^2) = 1$. Also, assume that $\boldsymbol{T} = \operatorname{diag}(\tau_1, \ldots, \tau_p)$, τ_i is real, and the empirical distribution function of $\{\tau_1, \ldots, \tau_p\}$ converges almost surely to a probability distribution function H as $n \to \infty$. Set $\boldsymbol{B} := \frac{1}{n}\boldsymbol{YTY}^*$. Assume also that $\boldsymbol{Y}$ and $\boldsymbol{T}$ are independent. When $p = p(n)$ with $p/n \to y > 0$ as $n \to \infty$, then, almost surely, $F^{\boldsymbol{B}_n}$, the ESD of the eigenvalues of $\boldsymbol{B}_n$, converges vaguely, as $n \to \infty$, to a (nonrandom) d.f. F, where for any $z \in \mathbb{C}^+ \equiv \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, its Stieltjes transform $s = s(z)$ is the unique solution in $\mathbb{C}^+$ to the equation*

$$s = \frac{1}{y\int \frac{\tau \mathrm{d}H(\tau)}{1+\tau s} - z}.$$

Remark 2.1. (1) There is no requirement for the moment convergence of the empirical spectral distribution of $\boldsymbol{T}$, thus H can have any regularly varying tail. (2) While in Lemma 2.1 and Theorem 2.2 we introduce an assumption of independence on σ_ts, we only use it for the application of Theorem 2.1, which does not require independence. Thus, this condition could potentially be relaxed for sequences of σ_t such that the empirical distribution function of $\{\tau_1, \ldots, \tau_p\}$ converges almost surely to a Student(3).

The Stieltjes transform for the EVSCE model was obtained in [12] in an integral form for a general Student's t distribution. Here, we obtain an explicit expression of the Stieltjes transform in the particular case of the Student(3). The result follows directly from the theorem given above.

Lemma 2.1. *For $\boldsymbol{X}$ as in Definition 1.1 with σ_t distributed as independent Student(3) for all t, the Stieltjes transform of the limiting spectral distribution is given by*

$$\frac{1}{s(z)} + z = \frac{1}{\left(1 + \sqrt{\frac{s(z)}{y}}\right)^2}. \tag{2.1}$$

Proof. Matching the notation in Theorem 2.1, we set $\boldsymbol{Y} := \boldsymbol{Z}^*$, $\boldsymbol{T} := \boldsymbol{\Sigma}^2$, $n := S$, and $p := T$, then the theorem gives the Stieltjes transform of the matrix

$$\frac{\boldsymbol{X}^*\boldsymbol{X}}{S} = \frac{T}{S}\boldsymbol{A} = \frac{\boldsymbol{Z}^*\boldsymbol{\Sigma}^2\boldsymbol{Z}}{S},$$

and in this case, $y := \lim_{T\to\infty}\frac{T}{S}$.

Let $s_0(z)$ be the limiting Stieltjes transform of $\frac{\boldsymbol{X}^*\boldsymbol{X}}{S}$, and $s(z)$ the limiting Stieltjes transform of $\frac{\boldsymbol{X}^*\boldsymbol{X}}{T}$. Then

$$y s_0(yz) = s(z)\,.$$

By Theorem 2.1

$$s_0(z) = \frac{1}{y\int\frac{\tau \mathrm{d}H(\tau)}{1+\tau s_0(z)} - z}\,.$$

Therefore,

$$s(z) = y s_0(yz) = \frac{y}{y\int\frac{\tau \mathrm{d}H(\tau)}{1+\tau s_0(yz)} - yz} = \frac{1}{\int\frac{\tau \mathrm{d}H(\tau)}{1+\tau s_0(yz)} - z} = \frac{1}{\int\frac{\tau \mathrm{d}H(\tau)}{1+\frac{\tau}{y}s(z)} - z}\,. \tag{2.2}$$

We will rewrite equation (2.2) for the case when the volatility has re-normalised Student(ν) with $\nu > 2$ degrees of freedom.

The probability density function of the standard Student(ν) is

$$g_\nu(t) = \frac{\mathit{\Gamma}\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\mathit{\Gamma}\left(\frac{\nu}{2}\right)}\left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2}\,. \tag{2.3}$$

It has 0 mean and variance $\frac{\nu}{\nu-2}$. The density of re-normalised Student(ν) (standard Student(ν) divided by $\sqrt{\frac{\nu}{\nu-2}}$) is

$$f_\nu(t) := \sqrt{\frac{\nu}{\nu-2}}g_\nu\left(\sqrt{\frac{\nu}{\nu-2}}t\right) = \frac{\mathit{\Gamma}\left(\frac{\nu+1}{2}\right)}{\sqrt{(\nu-2)\pi}\,\mathit{\Gamma}\left(\frac{\nu}{2}\right)}\left(1+\frac{t^2}{\nu-2}\right)^{-(\nu+1)/2}\,. \tag{2.4}$$

The diagonal elements of $\boldsymbol{\Sigma}^2$ are distributed as the squared re-normalised Student's t distributed random variable, therefore the empirical distribution of diagonal elements of $\boldsymbol{\Sigma}^2$ has limiting density $h_\nu(\tau)$ that we will find below. Let $F_\nu(\cdot)$ be the c.d.f. of re-normalised Student's t distribution, and $H_\nu(\cdot)$ be the c.d.f. of the diagonal elements of $\boldsymbol{\Sigma}^2$. For $\tau > 0$, it holds

$$H_\nu(\tau) = F_\nu\left(\sqrt{\tau}\right) - F_\nu\left(-\sqrt{\tau}\right)\,.$$

Thus,

$$\begin{aligned} h_\nu(\tau) &= \frac{1}{2\sqrt{\tau}}\left(f_\nu\left(\sqrt{\tau}\right)+f_\nu\left(-\sqrt{\tau}\right)\right) \\ &= \frac{1}{\sqrt{\tau}}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{(\nu-2)\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{\tau}{\nu-2}\right)^{-(\nu+1)/2} \end{aligned} \tag{2.5}$$

for $\tau > 0$. Particularly, for $\nu = 3$, we can compute

$$h_3(\tau) = \frac{2}{\pi}(1+\tau)^{-2} \times \frac{1}{\sqrt{\tau}}\,. \tag{2.6}$$

Equation (2.2) yields

$$\frac{1}{s(z)}+z = \int_0^{+\infty} \frac{\tau h_3(\tau)\mathrm{d}\tau}{1+\tau\frac{s(z)}{y}} = \frac{2}{\pi}\int_0^{+\infty} \frac{\sqrt{\tau}}{(1+\tau)^2\left(1+\tau\frac{s(z)}{y}\right)}\mathrm{d}\tau = \frac{1}{\left(1+\sqrt{\frac{s(z)}{y}}\right)^2} \tag{2.7}$$

where the principal branch cut of the square root is taken. □

While the tail asymptotic of the Stieltjes transform is given in equation (11) of [12], in the following corollary we offer a simple proof in the case of Student(3) for volatilities:

Corollary 2.1. *Let $\rho(x)$ be the limiting density of eigenvalues in the EVSCE with i.i.d. Student(3)-distributed σ_ts. Then the tail asymptotic is given by*

$$\lim_{x\to\infty}\frac{\rho(x)}{x^{2.5}} = \frac{2}{\sqrt{y}\pi}\,. \tag{2.8}$$

Proof. First, we observe that since the branch cut of the square root is principal, thus has a positive real part

$$\left|\frac{1}{1+\sqrt{\frac{s(z)}{y}}}\right| \le 1\,. \tag{2.9}$$

Thus for large x, equation (2.1) implies that

$$\frac{1}{s(x+i0^+)} = -x+o(x) \tag{2.10}$$

which yields that $\operatorname{Re} s(x+i0^+) = -1/x + o(1/x)$ as well as that $|s(x+i0^+)| = \frac{1}{x}+o(1/x)$, which furthermore implies that $\operatorname{Im} s(x+i0^+) = o(1/x)$. This implies that $\arg(\sqrt{s(x+i0^+)})$ is near $\pi/2$. Now from equation (2.1), we see that

$$\begin{aligned}\frac{-\operatorname{Im} s(x+i0^+)}{|s(x+i0^+)|^2} &= \operatorname{Im} \frac{1}{1+\frac{s}{y}+2\sqrt{\frac{s(x+i0^+)}{y}}} = -2\operatorname{Im}\sqrt{\frac{s(x+i0^+)}{y}} + o\left(\frac{1}{x}\right)\\ &= \frac{2}{\sqrt{yx}} + o\left(\frac{1}{\sqrt{x}}\right) \qquad (2.11)\end{aligned}$$

yielding that $\operatorname{Im} s(x+i0^+) = \frac{2}{x^{2.5}\sqrt{y}} + o(1/x^{2.5})$ and via equation (1.9), we obtain the corollary. □

2.1. Derivation of the limiting density when $\nu = 3$

Here, we offer a derivation of the limiting density for EVSCE with Student(3) volatilities.

Theorem 2.2. *Let*

$$\begin{aligned} q \;:=\; & y^6(x-1)^6 + 6y^5(x-1)^3\left(x^2+4x+1\right)\\ & +3y^4\left(5x^4+16x^3+30x^2+16x+5\right) + 3y^2\left(5x^2+2x+5\right)\\ & +4y^3\Big(6x^{3/2}\sqrt{3y^3(x-1)^3+9y^2\left(x^2+7x+1\right)+9y(x-1)+3}\\ & +5x^3+12x^2-12x-5\Big) + 6y(x-1)+1\,, \qquad (2.12)\end{aligned}$$

and let w_* be given by

$$\begin{aligned} 12x^2 w_* :=\; & -y^2\left(x^2+10x+1\right) - 2\sqrt[3]{q} + 2y(x-1) + 1\\ & - \frac{2\left(y^4(x-1)^4 + 4y^3\left(x^3+3x^2-3x-1\right) + 6y^2(x+1)^2 + 4y(x-1) + 1\right)}{\sqrt[3]{q}}\,. \end{aligned} \qquad (2.13)$$

Furthermore, let

$$\begin{aligned} A &:= -\frac{y^2\left(x^2+10x+1\right)+2y(x-1)+1}{2x^2}\,,\\ B &:= -\frac{4y^3(1+x)}{x^2}\,,\\ C &:= \frac{y^4(x+1)^2\left(x^2-14x+1\right)+4y^3(x-1)(x+1)^2+6y^2(x+1)^2+4y(x-1)+1}{16x^4} \end{aligned} \qquad (2.14)$$

and let $R^{\pm} \in \mathbb{R}$ be given by

$$\begin{aligned} R^+ &:= 2w_* - A\,, \\ R^- &:= -2w_* - A\,. \end{aligned} \tag{2.15}$$

For $\boldsymbol{X}$ as in Definition 1.1 with σ_t distributed as independent Student(3) for all t, the limiting density of eigenvalues for $x > 0$ of $\boldsymbol{A}$ as in (1.3) is given by

$$\rho(x) = \frac{1}{2\pi}\sqrt{-R^- - \frac{2B}{\sqrt{R^+}}}\,. \tag{2.16}$$

Proof of Theorem 2.2. By equation (2.7), the limiting density $\rho(x) = \lim_{\eta\downarrow 0} \operatorname{Im} s_*$, where s_* has a positive imaginary part and is the solution of the equation derived above in (2.1). To find the solution s_* we rewrite the equation as follows:

$$\sqrt{\frac{4s}{y}} = \frac{s}{sz+1} - \left(\frac{s}{y} + 1\right)\,. \tag{2.17}$$

Now, we square both sides and multiply through by the denominator to obtain a quartic polynomial

$$Q(s) := \frac{4s(sz+1)^2}{y} - \left(s - \left(\frac{s}{y}+1\right)(sz+1)\right)^2 = 0\,. \tag{2.18}$$

When we do this, we will introduce spurious solutions. We will first demonstrate that these spurious solutions are real for all values of $z > 0$ and $y > 1$.

The spurious solutions will satisfy the following equation:

$$\sqrt{\frac{4s}{y}} = -\left(\frac{s}{sz+1} - \left(\frac{s}{y}+1\right)\right) \tag{2.19}$$

equivalent to

$$\sqrt{\frac{4s}{y}} - \frac{s}{y} = -\frac{s}{sz+1} + 1\,. \tag{2.20}$$

We notice that the LHS is a parabola in $\sqrt{\frac{s}{y}}$ with zeros at 0 and 2, and maximum at 1. The RHS is 1 at 1 and is strictly decreasing to $1 - \frac{1}{z}$ as $s \to \infty$. Thus, there are two real solutions to equation (2.20) in the interval $(0, 2)$ for any $z > 0$.

The quartic equation was first solved by Cardano and Ferrari in 1540. Here, we follow a more modern construction of the solution to a quartic polynomial using a resolvent cubic equation, see *e.g.* Theorem 4 in [15].

Throughout this proof, we use Mathematica to assist with labour-intensive computations, and our Mathematica notebook is available upon request. We know from algebra that a quartic polynomial has exactly two complex solutions if and only if its discriminant is negative. As we have shown that for $z > 0$, Q has real solutions, we deduce that when the discriminant is positive, Q has 4 real solutions and thus no solution with positive imaginary part. To find the spectral edge, it suffices to find $z > 0$ where the discriminant is negative. Taking the discriminant of Q, we obtain

$$\begin{aligned}\mathrm{Disc}(Q) \;=\; & -\frac{256}{y^6}\left(y^3z^6 - 3y^3z^5 + 3y^3z^4 - y^3z^3 + 3y^2z^5\right.\\ & \left.+21y^2z^4 + 3y^2z^3 + 3yz^4 - 3yz^3 + z^3\right) \end{aligned} \tag{2.21}$$

yielding the following equation, after division by common factors,

$$1 - 3y + 3y^2 - y^3 + 3yz + 21y^2z + 3y^3z + 3y^2z^2 - 3y^3z^2 + y^3z^3 = 0\,. \tag{2.22}$$

For $y > 0$, this equation has the following solutions:

$$z \;=\; \frac{\left(\sqrt[3]{y} - 1\right)^3}{y}\,, \tag{2.23}$$

$$\text{or} \quad z \;=\; \frac{3y^{2/3} - 3\sqrt[3]{y} + 2y - 2}{2y} \pm i\frac{3\sqrt{3}\left(\sqrt[3]{y} + 1\right)}{2y^{2/3}}\,. \tag{2.24}$$

Noting that (2.24) has a non-zero imaginary part for all $y > 0$, we deduce that (2.23) yields the spectral edge.

Now we proceed to construct the solution of the quartic with positive imaginary part. First, we transform the quartic Q into a monic depressed quartic $\tilde{Q}$ via

$$\tilde{Q}(s) = -\frac{y^2}{z^2}Q\left(s - \frac{1}{4}\left(-\frac{2y}{z} - 2y + \frac{2}{z}\right)\right) = s^4 + As^2 + Bs + C\,, \tag{2.25}$$

where we have set A, B, C as in (2.14). We now construct and solve the resolvent cubic equation

$$P(w) := (2w - A)(w^2 - C) - \frac{B^2}{4} = 0\,. \tag{2.26}$$

Recalling equations (2.12) and (2.13) for q and w_*, we note that w_* is a real solution of (2.26). We notice that $w_* \in \mathbb{R}$ whenever $q \in \mathbb{R}$ and $q \in \mathbb{R}$ whenever

$$3y^3(z-1)^3 + 9y^2\left(z^2 + 7z + 1\right) + 9y(z-1) + 3 > 0\,. \tag{2.27}$$

We notice that this inequality is identical to (2.22) and is satisfied whenever z is above the spectral edge. Thus, q and w_* are real whenever z is above the spectral edge.

Let $R^{\pm}$ be as in (2.15). Then for $j,k \in \{0,1\}$, the four solutions of the depressed quartic equation $\tilde{Q}$ are given by

$$2s = (-1)^j \sqrt{R^+} + (-1)^k \sqrt{R^- + (-1)^{j+1} \frac{2B}{\sqrt{R^+}}}\,. \tag{2.28}$$

We will prove that $R^+ > 0$ for z above the spectral edge using the established fact that exactly two of the solutions are real.

Suppose for contradiction that $R^+ < 0$. We take the standard branch cut of the square root along the negative x-axis, with $\sqrt{-1} = i$, making $\sqrt{R^+}$ purely imaginary with positive imaginary part. Recall also that $B < 0$ for $z, y > 0$, making $\operatorname{Im}\left(\frac{2B}{\sqrt{R^+}}\right) > 0$. Then if

$$(-1)^j \sqrt{R^+} + (-1)^k \sqrt{R^- + (-1)^{j+1} \frac{2B}{\sqrt{R^+}}} \in \mathbb{R}\,,$$

we must have $j = k$, which would yield two real solutions. This would imply that the two solutions with $j \neq k$ must be complex conjugates. However, taking the complex conjugate of the solution with $j = 0 = k-1$, we check that its conjugate does not equal the solution with $j = 1 = k+1$

$$\overline{\sqrt{R^+} - \sqrt{R^- - \frac{2B}{\sqrt{R^+}}}} = -\sqrt{R^+} - \sqrt{R^- + \frac{2B}{\sqrt{R^+}}} \neq -\sqrt{R^+} + \sqrt{R^- + \frac{2B}{\sqrt{R^+}}}\,, \tag{2.29}$$

where for the last statement, we recall that $R^- \in \mathbb{R}$, while $\frac{2B}{\sqrt{R^+}}$ would be purely imaginary. Thus by contradiction, we have established that $R^+ > 0$ and thus $\sqrt{R^+} > 0$.

Thus, $2\operatorname{Im} s = \operatorname{Im}\left(\sqrt{R^- \pm \frac{2B}{\sqrt{R^+}}}\right)$. We recall again that the solutions form exactly one conjugate pair, implying that one of $R^- \pm \frac{2B}{\sqrt{R^+}}$ is positive and the other is negative. As $B < 0$, we deduce that $R^- + \frac{2B}{\sqrt{R^+}} < 0$, yielding that

$$\operatorname{Im} s = \frac{1}{2}\sqrt{-R^- - \frac{2B}{\sqrt{R^+}}}\,. \tag{2.30}$$

We notice that $-R^- - \frac{2B}{\sqrt{R^+}}$ is continuous in z as a complex variable for z strictly above the spectral edge, thus the identity $\rho(x) = \frac{1}{\pi}\lim_{\eta\downarrow 0} \operatorname{Im} s(x+i\eta)$ yields the desired result. $\square$

3. Comparison of EVSCE and historical data

We conduct research on S&P500 15-minute intervals of stock-returns from January 2020 to October 2022. Data were obtained from https://polygon.io/.

3.1. Data preparation: removing the "market mode" and re-normalisation

The return of the stock s over the time interval t is calculated in the following way:

$$\boldsymbol{x}'_{ts} := \log\left(\frac{\text{Close price}_s(t)}{\text{Open price}_s(t)}\right), \tag{3.1}$$

where Close $\text{price}_s(t)$ and Open $\text{price}_s(t)$ denote Close and Open price of the stock s on the time interval t respectively. If there were no sales of the stock s on the time interval t and, consequently, Open and Close prices can not be determined, we assume the value of $\boldsymbol{x}'_{ts}$ to be equal to 0. Denote the matrix $\boldsymbol{X}' := (\boldsymbol{x}'_{ts})_{\substack{t\le T\\ s\le S}}$. Below, we describe the procedure of re-normalisation and "market mode" removal.

For a $T\times S$ matrix $\boldsymbol{Y}$ re-normalization is conducted in the following way:

— For each entry of the matrix $\boldsymbol{Y}$ subtract the empirical mean of the entries of its column.

— Divide each entry of the matrix you got in the previous step by the empirical standard deviation of the entries of its column.

This way, the ts element of the re-normalized matrix will be given by

$$y'_{ts} := \frac{y_{ts}-\bar{y}_s}{\sigma_s},$$

$$\text{where}\quad \bar{y}_s := \frac{1}{T}\sum_{t=1}^{T} y_{ts} \quad\text{and}\quad \sigma_s := \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}(y_{ts}-\bar{y}_s)^2}. \tag{3.2}$$

The "market mode" causes the overwhelming majority of entries of the matrix $\frac{\boldsymbol{X}^*_{\text{data}}\boldsymbol{X}_{\text{data}}}{T}$ to be positive and drives its maximal eigenvalue. It also causes the maximal eigenvalue of $\frac{\boldsymbol{X}^*_{\text{data}}\boldsymbol{X}_{\text{data}}}{T}$ to be significantly larger than the typical maximal eigenvalue EVSCE with Student(3)-distributed σ_ts.

For the second step, we apply standard PCA to separate the "market mode" of $\boldsymbol{X}_{\text{data}}$ and re-normalize the result. To "clear" $T \times S$ matrix $\boldsymbol{Y}$ using the S-component vector $\boldsymbol{y}_0$, such that $\|\boldsymbol{y}_0\| = 1$, we replace each row $\boldsymbol{y}_t$ of the matrix $\boldsymbol{Y}$ with

$$\boldsymbol{y}'_t := \boldsymbol{y}_t - \langle \boldsymbol{y}_t, \boldsymbol{y}_0 \rangle \, \boldsymbol{y}_0 \,. \tag{3.3}$$

To separate the "market mode", we apply the procedure of "clearing" to the matrix $\boldsymbol{X}_{\text{data}}$ using the vector $\boldsymbol{x}_{\max}$, where $\boldsymbol{x}_{\max}$ is the eigenvector of $\frac{\boldsymbol{X}^*_{\text{data}} \boldsymbol{X}_{\text{data}}}{T}$ corresponding the maximal eigenvalue $\lambda_{\max}$. We obtain the matrix $\boldsymbol{X}'_{\text{cl}}$, and after applying re-normalization procedure to $\boldsymbol{X}'_{\text{cl}}$, we obtain the matrix $\boldsymbol{X}_{\text{cl}}$.

Note, that the eigenvalues of $\frac{\boldsymbol{X}^*_{\text{data}} \boldsymbol{X}_{\text{data}}}{T}$ and $\frac{(\boldsymbol{X}'_{\text{cl}})^* \boldsymbol{X}'_{\text{cl}}}{T}$, apart from $\lambda_{\max}$, are matching, and $\frac{(\boldsymbol{X}'_{\text{cl}})^* \boldsymbol{X}'_{\text{cl}}}{T}$ has 0 instead of $\lambda_{\max}$. Nevertheless, after re-normalization, eigenvalues can shift depending on the sample variances of columns of the matrix $\boldsymbol{X}'_{\text{cl}}$. Further, we compare the spectrum of the matrix $\frac{\boldsymbol{X}^*_{\text{cl}} \boldsymbol{X}_{\text{cl}}}{T}$ with the spectrum of the matrix $\frac{\boldsymbol{X}^* \boldsymbol{X}}{T}$, where the matrix $\boldsymbol{X}$ is obtained from the Elliptic Volatility Model and has the same size as $\boldsymbol{X}_{\text{cl}}$.

3.2. Data analytics: comparing EVSCE and market data

First, we observe that data suggests that the distribution of empirical standard deviations of the rows of "cleared" returns is heavy-tailed (see figure 2) and the tail parameter is approximately 3.

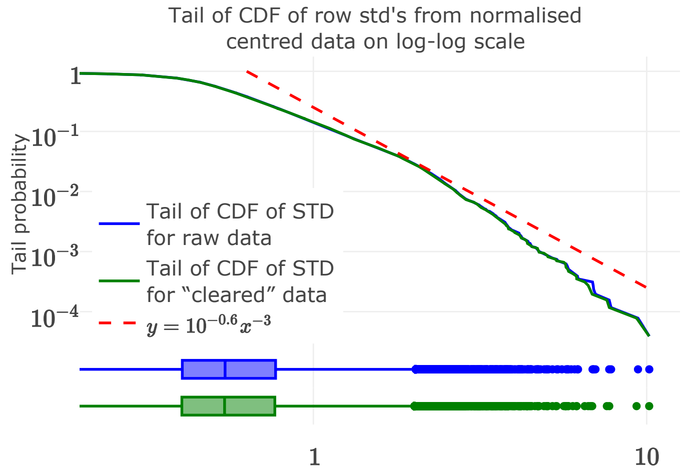

Fig. 2. Cubic law for standard deviations of return vectors at a fixed time. Data taken from https://polygon.io/

Figure 3 (top left) shows that the histogram of spectrum of the matrix $\frac{\boldsymbol{X}_{\mathrm{cl}}^{*}\boldsymbol{X}_{\mathrm{cl}}}{T}$ is well approximated by the limiting spectrum of EVSCE with σ_ts i.i.d. as Student(3). Figure 3 (top right) shows that it is not well approximated by the spectrum of EVSCE with normally distributed volatility. In EVSCE, the heaviness of the tail of the limiting spectrum depends on the heaviness of the distribution of the volatility as shown in equation (11) of [12] and Corollary 2.1. Figure 3 (bottom) shows the histograms of eigenvalues for the data and the simulation EVSCE with σ_ts i.i.d. as Student(3), where the entries of $\boldsymbol{X}$ are randomly shuffled. The Marchenko–Pastur law is plotted as well, and we see that the shuffled data approximates the Marchenko–Pastur law well. This is a control to verify that the dependence and correlation structures in the two data sets cause the heavy tails in the corresponding spectral measures.

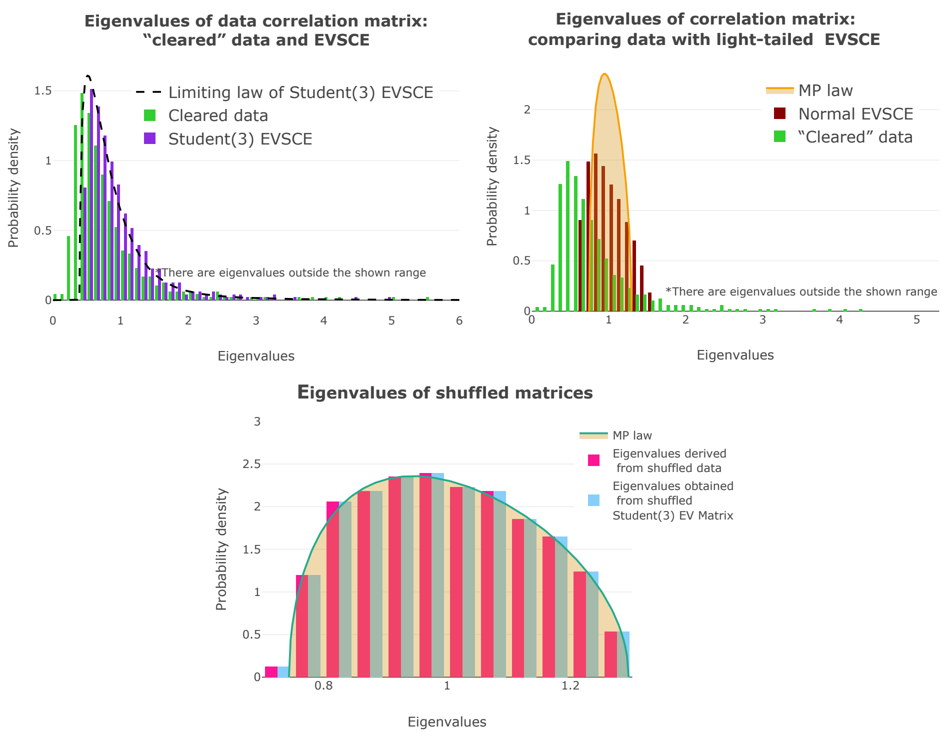

Fig. 3. Top left: Histograms of spectrum of simulated Student(3) EVSCE and matrix $\frac{\boldsymbol{X}_{\mathrm{cl}}^{*}\boldsymbol{X}_{\mathrm{cl}}}{T}$, and the limit obtained in Theorem 2.2. Top right: Comparison of the spectrum of $\frac{\boldsymbol{X}_{\mathrm{cl}}^{*}\boldsymbol{X}_{\mathrm{cl}}}{T}$, randomly generated EVSCE with normally distributed σ_ts and the Marchenko–Pastur law. Bottom: Comparison of the spectrum of covariance matrix of shuffled data and similarly shuffled EVM to the Marchenko–Pastur law. Data taken from `https://polygon.io/`

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