IR AND UV LIMITS OF CDT AND THEIR RELATIONS TO FRG* **

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Causal Dynamical Triangulations (CDT) is a lattice theory of quantum gravity. It is shown how to identify the IR and the UV limits of this lattice theory with similar limits studied using the continuum, functional renormalization group (FRG) approach. The main technical tool in this study will be the so-called two-point function. It will allow us to identify a correlation length not directly related to the propagation of physical degrees of freedom.

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1. Introduction

Four-dimensional gravity is not perturbatively renormalizable. For many years it has been discussed if the theory could be defined as a unitary, non-perturbative quantum field theory. This putative theory could contain other terms than the classical Einstein–Hilbert terms in the action and these additional terms could make the theory UV well defined. This has been well understood since the seminal work of Stelle [1] where an R^2 term was added to the classical GR action. Unfortunately, it was not so clear how to ensure the unitarity of the corresponding quantum theory. A more general setup is known as the asymptotic safety scenario [2] where one, appealing to the Wilsonian renormalization group, tries to understand if the UV limit of a quantum gravity theory can be associated with a fixed point of the renormalization group. This fixed point could in principle be non-Gaussian,

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and starting with the seminal work of Reuter [3], a lot of evidence has been accumulated supporting the existence of such a non-Gaussian fixed point (see [4–6] for extensive reviews). The evidence for this non-Gaussian fixed point comes from solving the renormalization group equations using the so-called functional renormalization group technique (FRG) that goes back to Wetterich (see [7] for a review). However, "solving" FRG also means in this context a truncation of the full equation, and it can be difficult to judge how reliable the results are. Thus, it would be reassuring if one could verify the FRG results using an independent calculation and since we are discussing non-perturbative quantum field theory, the use of lattice quantum field theory is natural. In that case, one often has to use Monte Carlo simulations, and they also represent an approximation, but of a different kind than used in FRG. The purpose of this paper is to compare the two approaches.

The rest of the paper is organized as follows: first we recall how one can use the lattice field theory to test the existence of a putative non-perturbative UV fixed point. Here, we use a ϕ^4 theory in four dimensions as an example. Next, we discuss ways in which quantum gravity can be formulated as a lattice field theory, namely by the use of the so-called Dynamical Triangulations (DT) or Causal Dynamical Triangulations (CDT). In this section, we also discuss how to introduce the concept of a diffeomorphism invariant correlation length in quantum gravity, and in what way it implies finite-size scaling in the lattice quantum gravity theories. We then compare the lattice results (obtained by Monte Carlo simulations) with the simplest results obtained using the FRG approach. The final section contains a discussion of the results obtained so far.

2. Identifying fixed points in ϕ^4 lattice theory

Let us consider a ϕ^4 lattice field theory in four dimensions. The lattice action used is

$$S = \sum_n \left(\sum_{i=1}^{4} (\phi(n+e_i) - \phi(n))^2 + \mu_0 \phi_n^2 + \kappa_0 \phi^4(n) \right) , \tag{1}$$

where n denotes a lattice point, e_i is a unit vector in direction i, and the fields $\phi(n)$ and the coupling constants μ_0 and $\kappa_0 \geq 0$ are dimensionless, and the length of the lattice links is 1. The theory has a second-order phase transition line, starting at $\kappa_0 = \mu_0 = 0$ and extending to $\kappa_0 = \infty$. It separates the symmetric phase ($\langle\phi\rangle = 0$) from the broken phase ($\langle\phi\rangle \neq 0$). We will consider only the symmetric phase. For each value of μ_0 and κ_0, one has a correlation length $\xi(\mu_0, \kappa_0)$ defined by the exponential fall-off of the two-point function. It diverges when one approaches the second-order phase transition line. Rather than using μ_0, κ_0 as variables defining the theory, we will use ξ, κ_0. Then the phase transition line will be at $\xi^{-1} = 0$. The possible fixed points for the theory will be on this line and there can potentially be both IR and UV fixed points as shown in Fig. 1. For each value of the bare coupling constant κ_0, we can define a renormalized coupling constant $\kappa_{\rm R}(\kappa_0, \xi)$. It can be expressed in terms of bare four-point and bare two-point functions (see [9] for details). The theory will have a UV fixed point $\kappa_0^{\rm UV}$ if it is possible to find a path $(\xi, \kappa_0(\xi))$ in the ξ, κ_0 coupling constant plane such that

$$\kappa_{\rm R}(\kappa_0(\xi), \xi) = \kappa_{\rm R} \quad \text{for} \quad \xi \to \infty\,. \tag{2}$$

Such paths for different $\kappa_{\rm R}$ are illustrated in Fig. 1. Differentiating (2) w.r.t. ξ, we obtain

$$0 = \xi \frac{\mathrm{d}}{\mathrm{d}\xi} \kappa_{\rm R}(\kappa_0(\xi), \xi) = \xi \frac{\partial \kappa_{\rm R}}{\partial \xi}\bigg|_{\kappa_0} + \frac{\partial \kappa_{\rm R}}{\partial \kappa_0}\bigg|_{\xi} \; \xi \frac{\mathrm{d}\kappa_0}{\mathrm{d}\xi}\bigg|_{\kappa_{\rm R}}\,. \tag{3}$$

Introducing the bare and the renormalized β-functions

$$\beta_0(\kappa_0) = \xi \frac{\mathrm{d}\kappa_0}{\mathrm{d}\xi}\bigg|_{\kappa_{\rm R}}, \qquad \beta_{\rm R}(\kappa_{\rm R}) = -\xi \frac{\partial \kappa_{\rm R}}{\partial \xi}\bigg|_{\kappa_0} \tag{4}$$

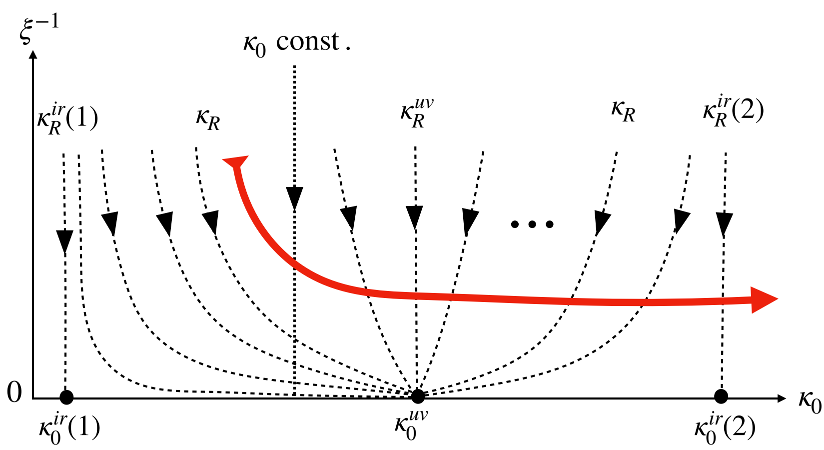

Fig. 1. The tentative ϕ^4 phase diagram with a UV fixed point and two IR fixed points. The dashed lines are paths where the renormalized ϕ^4 coupling constant $\kappa_{\rm R}$ is kept fixed, while on the dotted line, the bare coupling constant κ_0 is fixed. The thick red line illustrates the way the real renormalization group flow will be in a ϕ^4 theory with a fixed $\kappa_{\rm R}$. It will never reach the critical line where $\xi = \infty$ and, accordingly, there will not be a continuum quantum field theory with a fixed $\kappa_{\rm R} > 0$, as first shown in [8].

Eq. (3) can be written as[1]

$$\beta_{\mathrm{R}}(\kappa_{\mathrm{R}}) = \frac{\partial \kappa_{\mathrm{R}}}{\partial \kappa_0}\, \beta_0(\kappa_0)\,. \tag{5}$$

A typical $\beta_0(\kappa_0)$ is shown in Fig. 2, and solving (4) for a fixed κ_{R} close to the fixed point κ_0^{UV}, we obtain

$$\xi \frac{\mathrm{d}\kappa_0}{\mathrm{d}\xi}\bigg|_{\kappa_{\mathrm{R}}} = \beta_0'\left(\kappa_0^{\mathrm{UV}}\right)\left(\kappa_0 - \kappa_0^{\mathrm{UV}}\right), \quad i.e. \quad |\kappa_0(\xi) - \kappa_0^{\mathrm{UV}}| = c\left(\kappa_{\mathrm{R}}\right)\, \xi^{\beta_0'\left(\kappa_0^{\mathrm{UV}}\right)}, \tag{6}$$

i.e. $\kappa_0(\xi) \to \kappa_0^{\mathrm{UV}}$ for $\xi \to \infty$ if $\beta_0'(\kappa_0^{\mathrm{UV}}) < 0$. κ_0^{UV} thus serves as a UV fixed point. Similarly, solving (4) for κ_{R} as a function of ξ for fixed κ_0, it is seen from Fig. 1 that $\kappa_{\mathrm{R}}(\xi)$ flows to an IR fixed point $\kappa_{\mathrm{R}}^{\mathrm{IR}}$ for $\xi \to \infty$.

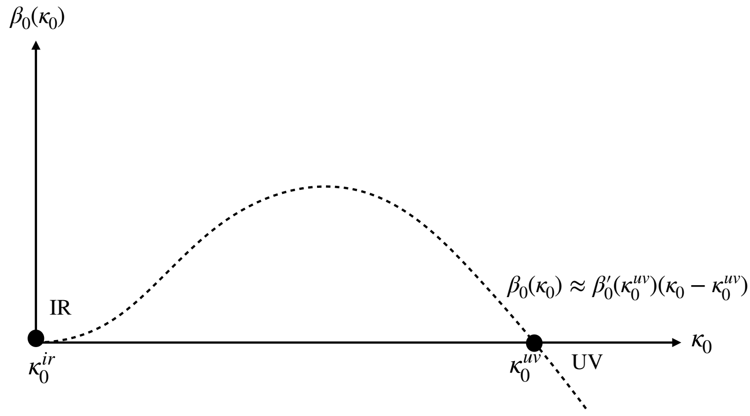

Fig. 2. The expected form of the ϕ^4 β-function *if* the ϕ^4 theory would have a UV fixed point.

In Eq. (1) we assumed the lattice spacing was 1. One can instead introduce a lattice spacing of length a in (1) and a will then act as an adjustable UV cut-off. At a UV fixed point, one can define a "continuum limit" where $a \to 0$ (and $\kappa_{\mathrm{R}} > 0$) in the following way: introduce a physical length ℓ_{ph} between lattice points n_1 and n_2, and a physical (renormalized) mass m_{R} by

$$\ell_{\mathrm{ph}}\left(n_1, n_2\right) = a|n_1 - n_2|\,, \qquad m_{\mathrm{R}} = \frac{1}{a\xi}\,. \tag{7}$$

[1] The $\beta_0(\kappa_0)$ function as defined by (3) is strictly speaking also a function of ξ, but for large ξ (the so-called scaling region), this dependence can be ignored. The same remarks are true for $\beta_{\mathrm{R}}(\kappa_{\mathrm{R}})$.

This ensures that the exponential decay of the continuum, renormalized two-point function is defined by $m_{\rm R}$ since we have

$$\mathrm{e}^{-m_{\rm ph}\ell_{\rm ph}(n_1,n_2)} = \mathrm{e}^{-|n_1-n_2|/\xi}\,, \tag{8}$$

and for fixed $m_{\rm R}$ (7) shows that $\xi \to \infty$ leads to $a \to 0$, *i.e.* a removal of the UV cut-off and the definition of a continuum quantum field theory with renormalized coupling constants $m_{\rm R}$ and $\kappa_{\rm R} > 0$.

The above discussion assumes that the lattice is infinite since we are discussing the limit where $\xi \to \infty$. In actual numerical lattice Monte Carlo simulations, we are forced to have a finite lattice consisting of $N_4 = L^4$ lattice points. The correlation length can then not be larger than $L = N_4^{1/4}$. However, assuming that N_4 is sufficiently large we have, according to (6),

$$\left|\kappa_0\left(N_4^{1/4}\right) - \kappa_0^{\rm UV}\right| = c(\kappa_{\rm R}) N_4^{\beta_0'(\kappa_0^{\rm UV})/4}\,, \tag{9}$$

which is a so-called finite-size scaling relation that we will use also in the case of lattice gravity. If we are in a regime of coupling constant space where finite-size scaling is valid, we could have replaced the ξ^{-1} axis with an $N_4^{-1/4}$ axis, or even an N_4^{-1} axis. The qualitative features of Fig. 1 would be unchanged. This is precisely what we will do in the case of lattice gravity, as will be explained below.

3. Lattice quantum gravity: CDT

Four-dimensional Dynamical Triangulations (DT) and four-dimensional Causal Dynamical Triangulations (CDT) provide lattice regularizations of 4d quantum gravity (see [10–13] for reviews). For an ordinary lattice field theory, such as the ϕ^4 theory discussed above, the lattice is fixed and the dynamics comes from the fields $\phi(n)$ living on the lattice points n. In DT and CDT, the dynamics comes from summing over different lattices. One considers 4d piecewise linear manifolds of a fixed topology, defined by gluing together identical building blocks of four-simplices with link length a, the a acting as a UV cut-off as in the case of the ϕ^4 lattice theory. Viewing the four-simplices as flat in the interior, a unique geometry is associated with each such piecewise linear manifold since geodesic distances between two points are well defined. At the same time, the gluing will define a four-dimensional triangulation, a lattice. The path integral of quantum gravity involves the integration over all geometries and it will now be represented as a sum over all such triangulations or lattices of the given topology. Action (1) used in the case of the ϕ^4 lattice theory represents the simplest discretized version of the continuum ϕ^4 action. For piecewise linear manifolds, there exists a beautiful geometric discretization of the Einstein–Hilbert action, where the curvature in the four-dimensional case lives on the triangles of the four-dimensional triangulation, the so-called Regge action [14]. In the case where the triangulation is constructed by gluing together identical building blocks, the Regge action becomes exceedingly simple since it will just depend linearly on the total number of four-simplices and the total number of triangles. We will use this simple action in the definition of the path integral. The final ingredient entering in CDT is that we assume that geometries have a proper time foliation that we implement in the following way. Let time be discretized. At each time t_i, we have a spatial slice $\Sigma(t_i)$ with a fixed spatial topology. Here, we consider the simplest case where the spatial topology is that of the three-sphere S^3. We triangulate each $\Sigma(t_i)$ by gluing together tetrahedra to form a triangulation with the topology of S^3. We then fill out the slab between $\Sigma(t_i)$ and $\Sigma(t_{i+1})$ by four-dimensional simplices, glued together in such a way that the topology of the slab is $S^3 \times [0, 1]$. These four-dimensional triangulations can share a tetrahedron, a triangle, a link or a vertex with the three-dimensional triangulation of $\Sigma(t_i)$, and they will then share a vertex, a link, a triangle, or a tetrahedron with the three-dimensional triangulation of $\Sigma(t_{i+1})$, respectively. The construction is shown in Fig. 3.

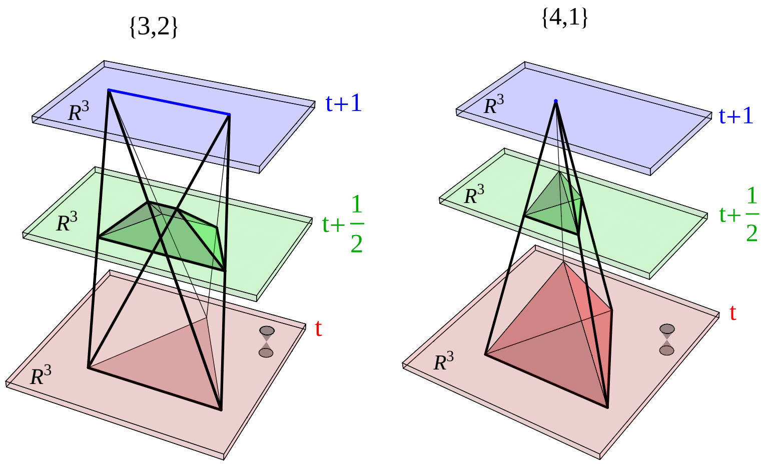

Fig. 3. The build-up of a CDT triangulation between the time-slab at t and at $t+1$. Shown is a so-called (3,2) four-simplex and a $(4,1)$ four-simplex.

In the path integral, we then sum over all possible 3d triangulations of the spatial slices $\Sigma(t_i)$s and all possible 4d triangulations that fill out the slabs. Finally, each such triangulation T_{L} is associated with a weight $e^{iS_{\mathrm{regge}}[T_{\mathrm{L}}]}$, where $S_{\mathrm{regge}}[T_{\mathrm{L}}]$ is the Regge action associated with T_{L}. The continuum path integral is then replaced by the following sum:

$$Z(G,\Lambda)=\int\mathcal{D}[g]\,\mathrm{e}^{iS_{\mathrm{eh}}[g;G,\Lambda)}\to Z_{\mathrm{L}}(k_2,k_4)=\sum_{T_{\mathrm{L}}}\mathrm{e}^{iS_{\mathrm{regge}}[T_{\mathrm{L}};k_2,k_4]}\,, \tag{10}$$

where the continuum Einstein–Hilbert action refers to the gravitational and cosmological coupling constants G and Λ, while the Regge action refers to the dimensionless lattice analogues k_2 and k_4.

A special property of the CDT setup is that for each Lorentzian triangulation T_{L}, we can perform a rotation to an Euclidean triangulation T_{E}, simply by changing the length assignment $l^2=-a^2$ of the time-like links connecting $\Sigma(t_i)$ and $\Sigma(t_{i+1})$ to $l^2=a^2$. Formally, this is a rotation to imaginary time, *i.e.* Euclidean time. The Regge action will then change in the standard way

$$iS_{\mathrm{regge}}[T_{\mathrm{L}}]\to -S_{\mathrm{regge}}[T_{\mathrm{E}}]\,,\quad i.e.\quad Z_{\mathrm{E}}[k_2,k_4]=\sum_{T_{\mathrm{E}}}\mathrm{e}^{-S_{\mathrm{regge}}[T_{\mathrm{E}};k_2,k_4]}\,. \tag{11}$$

In the following, we will always sum over this class of Euclidean triangulations and drop the subscript E. We then have a theory with Euclidean signature, like in the ϕ^4 case, but the class of geometries is smaller than the one provided by the full class of Euclidean triangulations since the triangulations T_{E} that enter in (11) still remember the time-slicing we imposed on the triangulations[2] T_{L}.

As stated above, the Regge action becomes very simple when one uses identical building blocks. In CDT we have, in a Wilsonian spirit, chosen to generalize the Regge action slightly by allowing different cosmological coupling constants associated with four-simplices of type (4,1) and type (3,2) shown in Fig. 3. The action then becomes

$$S[T]=-k_2N_2(T)+k_{32}N_{32}(T)+k_{41}N_{41}(T)\,, \tag{12}$$

where $N_2(T)$ is the number of triangles in T, $N_{32}(T)$ the number of $(3,2)$ plus $(2,3)$ four-simplices, and $N_{41}(T)$ the number of $(4,1)$ plus $(1,4)$ four-simples. The total number of four-simplices in T is $N_4(T)=N_{41}(T)+N_{32}(T)$. Using the so-called Dehn–Sommerville relations between the number of subsimples $N_i(T)$ of the order of i, where $N_0(T)$ is the number of vertices, we can write (12) as follows (see [10] for details):

$$S[T;k_0,\Delta,k_4]=-(k_0+6\Delta)N_0(T)+k_4N_4(T)+\Delta N_{41}(T)\,. \tag{13}$$

[2] The four-dimensional DT lattice gravity formulation pre-dates the CDT formulation [15, 16]. In the DT theory, one sums over the full class of Euclidean triangulations. In this way, one avoids introducing a time foliation. However, it is unclear how to relate the theory to a gravity theory with the Lorentzian signature. Also, it was not clear how to obtain an interesting continuum limit of the DT lattice theory, although this is still under investigation [17, 18].

This is the action we will use in the regularized path integral

$$Z[k_0,\Delta,k_4]=\sum_T \mathrm{e}^{-S[T;k_0,\Delta,k_4]}\,. \tag{14}$$

Here, k_0 is formally related to the a^2/G via Regge calculus, Δ affects the ratio between $(4,1)$ and $(3,2)$ four-simplices, while k_4 monitors N_4, the number of four-simplices.

3.1. Coupling constants and correlation lengths

The coupling constant k_4 in (13) plays a special role. This is seen by writing (14) as

$$Z[k_0,\Delta,k_4]=\sum_{N_4} \mathrm{e}^{-k_4N_4}Z[k_0,\Delta;N_4]\,, \tag{15}$$

where $Z[k_0,\Delta;N_4]$ denotes the partition function for a fixed N_4. It grows exponentially with N_4 and we can write

$$Z[k_0,\Delta,k_4]=\sum_{N_4} \mathrm{e}^{-\left(k_4-k_4^c(k_0,\Delta)\right)N_4}F(k_0,\Delta;N_4)\,, \tag{16}$$

where F is subleading as a function of N_4. We cannot perform the sum analytically and the only way to study the partition function is via Monte Carlo simulations, and in these studies we are interested in testing as large N_4 as possible. In principle, by changing k_4 in the neighborhood of $k_4^c(k_0,\Delta)$ we can monitor N_4. However, it is much more convenient to fix N_4 in the computer simulations. Then k_4 will play no active role, and to compensate for this, we perform independent computer simulations for different N_4. In reality, we are then studying $F(k_0,\Delta;N_4)$ where we can choose to view N_4 as a "coupling constant". This seems a little weird from the point of view of the ordinary lattice field theory where N_4 is simply the volume of spacetime. However, as we discussed in the case of the lattice ϕ^4 theory, the correlation length ξ played a dominant role when we wanted to study the continuum limit, and first we exchanged the bare mass for the correlation length, and next, when the volume N_4 was finite, we changed the maximal correlation length with $N_4^{1/4}$ and studied finite-size scaling in the limit $N_4\to\infty$. Thus, even in that case, one could (under the right circumstances) view N_4 as a coupling constant and we were interested in the limit where this coupling constant went to infinity. Here, in the case of gravity, we are of course also interested in the limit where N_4 goes to infinity, but the first obvious question is: how can this limit, $N_4\to\infty$, be related to a divergent correlation length?

In fact, this question forces us to take a step back and ask the following question: how does one define the concept of a correlation length in a theory of quantum gravity? In an ordinary QFT-like the ϕ^4 theory, one can define the correlation length as the exponential fall-off of the two-point function $\langle\phi(x)\phi(y)\rangle$, as discussed above. It will be a function of the (geodesic) distance between x and y (apart from lattice artifacts for small lattice distances, if we consider a lattice version of the ϕ^4 theory). However, in a theory of quantum gravity, we are integrating over all metrics, and it is the metric that determines the geodesic distance between two points x and y. One way to define a (non-local) two-point function that *is* a function of a geodesic distance is the following:

$$\begin{aligned} G_\phi(D) \;=\; & \int \mathcal{D}[g] \int \mathcal{D}\phi \, \mathrm{e}^{-S[g,\phi]} \\ & \times \iint \mathrm{d}^4x \, \mathrm{d}^4y \sqrt{g(x)}\sqrt{g(y)} \; \phi(x)\phi(y) \; \delta(D_\mathrm{g}(x,y) - D) \,. \end{aligned} \tag{17}$$

In (17) D_g denotes the geodesic distance between points x and y measured using the metrics g used in the path integral. This formula has been shown to work well for Euclidean two-dimensional gravity coupled to conformal fields [19]. We are here going to apply it in the very simplest case where instead of fields $\phi(x)$ we just use the unit function $1(x)$. We then write

$$G(D) = \int \mathcal{D}[g] \, \mathrm{e}^{-S[g]} \iint \mathrm{d}^4x \, \mathrm{d}^4y \sqrt{g(x)}\sqrt{g(y)} \; \delta(D_\mathrm{g}(x,y) - D) \,. \tag{18}$$

Let $\langle V_4\rangle$ denote the average four-volume of our ensemble of geometries we use in the path integral and let d_H denote the Hausdorff dimension of the ensemble of geometries. Then, under quite general conditions, one can show [20] that the two-point function $G(D)$ falls off exponentially as

$$G(D) = f(D) \, \mathrm{e}^{-c\, D/\langle V_4\rangle^{1/d_\mathrm{H}}} \,, \qquad D \gg \langle V_4\rangle^{1/d_\mathrm{H}} \,, \tag{19}$$

where $f(D)$ is subleading. The above expressions are readily translated to the lattice formulation with V_4 replaced by N_4 and D being replaced by the shortest graph distance n in a triangulation, and we write

$$G(n) = f(n) \, \mathrm{e}^{-c\, n/\langle N_4\rangle^{1/d_\mathrm{H}}} \,, \qquad n \gg \langle N_4\rangle^{1/d_\mathrm{H}} \,. \tag{20}$$

The intuition behind the fall-off is illustrated in Fig. 4: the number of triangulations where two points are separated by a distance n is a decreasing function of n. The derivation in [21–23] for two dimensions and in [20] for four dimensions is for Euclidean quantum gravity. *I.e.* in the lattice version it is for DT. In the case of CDT one has to modify the proof due to the special role of the time direction. We will omit the details here.

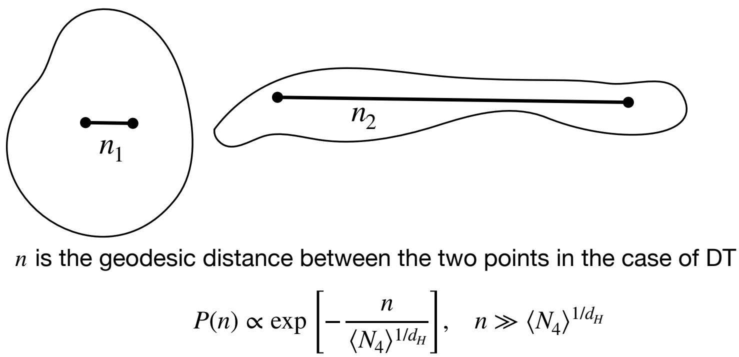

Fig. 4. Typical shape of a universe when n is small and when n is large.

From Eq. (20) it is seen that $\langle N_4\rangle^{1/d_{\rm H}}$ plays the role of a correlation length for the two-point function $G(n)$. Let k_0, Δ be fixed. If k_4 can be chosen such that $\langle N_4\rangle$ is very large, we expect that most observables will have the same value in the grand canonical ensemble with that chosen value of k_4 and in the canonical ensemble where we fix $N_4 = \langle N_4\rangle_{k_4}$, and thus the interpretation of $N_4^{1/d_{\rm H}}$ as a correlation length for the ensemble of fluctuating geometries is a natural analogy to $N_4^{1/4}$ being the correlation length for the ensemble of lattice $\phi(x)$ field configurations when the dimensionless mass parameter is chosen such that the correlation length is equal to the linear size of the lattice. In the standard finite-size scaling scenario, one chooses N_4 and then adjusts the bare mass parameter such that the correlation length is equal to $N_4^{1/4}$ and the critical surface is reached for $N_4 \to \infty$. In practical applications, one does not actually measure correlation length, but uses convenient scaling variables to observe finite-size scaling, taking for granted that such scaling is only observed when the correlation length is comparable to $N_4^{1/4}$. In our CDT case, we will use the same philosophy: if we observe finite-size scaling for some observables, when comparing measurements for systems with different N_4, we will take it as a sign that $N_4^{1/d_{\rm H}}$ can be used as the correlation length and that the critical surface is reached when $N_4 \to \infty$. What is different in our case is that: (1) we cannot separate the correlation length from the (average) size of the system and (2) the existence of the two-point function $G(n)$ with a divergent correlation length does not imply that we have propagating degrees of freedom associated with this two-point function.

3.2. The CDT phase diagram

The Monte Carlo simulations using (14) reveal that there are a number of different phases in CDT, called A, B, C_{b}, and C_{dS} [24]. We have coupling constants k_0, Δ and we have N_4. The corresponding three-dimensional phase diagram is shown in Fig. 5. It should be compared to the ϕ^4 phase diagram shown in Fig. 1. Only in the so-called de Sitter phase C_{dS} do we observe finite-size scaling when $N_4 \to \infty$. Thus, only this phase will have our interest. We view the other phases as lattice artifacts. In Fig. 6, we show the surface corresponding (approximately) to $N_4 = \infty$. The only part of this surface that we view as a critical surface is the part corresponding to phase C_{dS}.

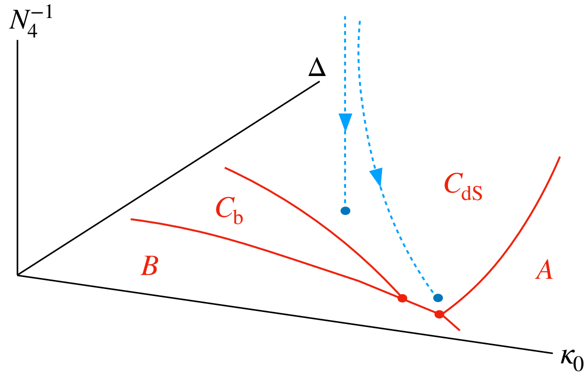

Fig. 5. The CDT phase diagram where N_4^{-1} is also included. Criticality can only occur when $N_4^{-1} = 0$. The straight vertical line corresponds to keeping the bare lattice coupling constants κ_0, Δ fixed, while the other line illustrates the flow when the renormalized coupling constants are fixed and one has to change the lattice coupling constants when approaching the critical surface.

Let us now discuss how we observe finite-size scaling in phase C_{dS} [25]. In the Monte Carlo simulations, we have direct access to the three-volume $N_3(i)$, the number of three-simplices at time-slice i. For a fixed N_4, we can now measure $\langle N_3(i)\rangle$ and $\langle N_3(i_1)N_3(i_2)\rangle$. For $N_3(i)$, we observe for fixed k_0, Δ that

$$\langle N_3(i)\rangle_{N_4} \propto N_4 \frac{1}{\omega N_4^{1/4}} \cos^3\left(\frac{i}{\omega N_4^{1/4}}\right) , \tag{21}$$

see Fig. 7. ω depends on k_0 and Δ, but is independent of N_4 for N_4 sufficiently large. Equation (21) shows finite size scaling with the Hausdorff dimension $d_{\mathrm{H}} = 4$. If we introduce *scaling variables*

$$s_i = \frac{i}{N_4^{1/4}} , \qquad n_3(s_i) = \frac{N_3(i)}{N_4^{3/4}} , \tag{22}$$

we can write

$$\langle n_3(s) \rangle = \frac{3}{4\omega} \cos^3 \left(\frac{s}{\omega} \right) . \tag{23}$$

Similarly, the correlations behave like

$$\begin{aligned} \langle \Delta N_3(i_1) \Delta N_3(i_2) \rangle &= \Gamma N_4 F \left(\frac{i_1}{\omega N_4^{1/4}}, \frac{i_2}{\omega N_4^{1/4}} \right) , \\ \Delta N_3(i) &= N_3(i) - \langle N_3(i) \rangle . \end{aligned} \tag{24}$$

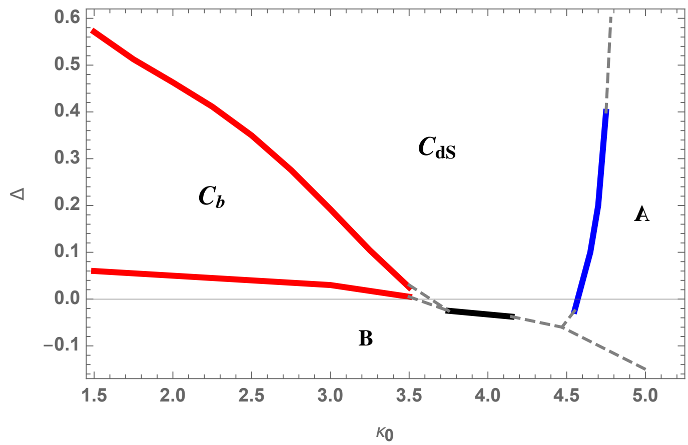

Fig. 6. The CDT phase diagram. In phase A, different time slices seem not to couple. In phase B, the time extension of the universe is only one time-slice. In phase C_{b}, the time extension of the universe is larger, but it does not scale when N_4 is increased. Only phase C_{dS} seems to represent a four-dimensional universe.

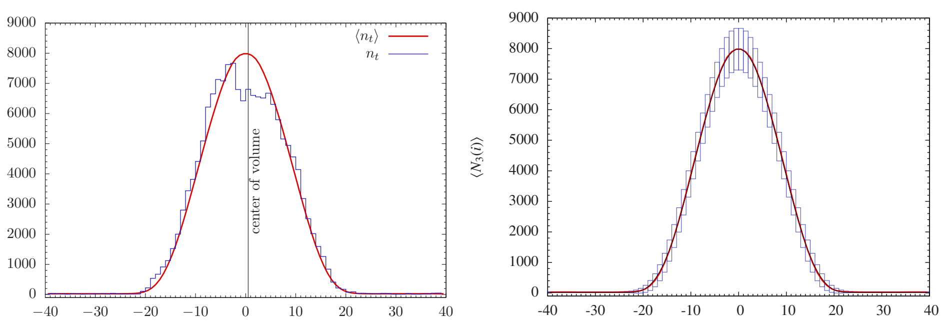

Fig. 7. Left panel: A plot of a single $N_3(i)$ configuration for $N_4 = 362\,000$, as well as the average of many configurations. Right panel: the plot of $\langle N_3(i) \rangle$ together with magnitudes $\sqrt{\langle \Delta N_3(i) \Delta N_3(i) \rangle}$ of the fluctuations of $N_3(i)$.

Expressed in scaled variables we have

$$\langle \Delta n_3(s_1) \Delta n_3(s_2) \rangle = \frac{\mathit{\Gamma}}{\sqrt{N_4}} \, F\left(\frac{s_1}{\omega}, \frac{s_2}{\omega}\right) . \tag{25}$$

Equations (23) and (25) are very well described by the following effective action:

$$S_{\rm eff}[k_0, \mathit{\Delta}] = \frac{1}{\mathit{\Gamma}} \sum_i \left(\frac{\left(N_3(i+1) - N_3(i)\right)^2}{N_3(i)} + \delta \, N_3^{1/3}(i) \right) \tag{26}$$

or, expressed in scaling variables (with $\mathrm{d}s_i = 1/N_4^{1/4}$)

$$S_{\rm eff}[k_0, \mathit{\Delta}] = \frac{\sqrt{N_4}}{\mathit{\Gamma}} \int\limits_{-\pi\omega/2}^{\pi\omega/2} \mathrm{d}s \, \left(\frac{\dot{n}_3^2(s)}{n_3(s)} + \delta \, n_3^{1/3}(s) \right) , \qquad \int\limits_{-\pi\omega/2}^{\pi\omega/2} \mathrm{d}s \, n_3(s) = 1 . \tag{27}$$

The solution to the "classical" eom associated with $S_{\rm eff}$ is precisely (23) provided δ and ω are related as follows:

$$\frac{\delta}{\delta_0} = \left(\frac{\omega_0}{\omega}\right)^{8/3} , \qquad \delta_0 = 9 \left(2\pi^2\right)^{2/3} , \qquad \omega_0 = \frac{3}{\sqrt{2}} \frac{1}{\delta_0^{3/8}} . \tag{28}$$

If $\delta = \delta_0$, (23) represents a "round" S^4 sphere with four-volume 1. We will denote (23) $n_3^{\rm cl}(s)$ and the data are then well described by $n_3^{\rm cl}(s)$ and Gaussian fluctuations around $n_3^{\rm cl}(s)$.

In the computer simulations producing these results we have kept k_0 and $\mathit{\Delta}$ fixed and varied N_4, that is, we have followed a straight blue path shown in Fig. 5. The effective action describing our data close to the surface $N_4 = \infty$ contains two effective coupling constants $\mathit{\Gamma}$ and δ. For $k_0, \mathit{\Delta}$ in the interior of phase $C_{\rm dS}$, $\mathit{\Gamma}$, and δ will depend on $k_0, \mathit{\Delta}$, but will be independent of N_4 for N_4 sufficiently large. However, how large N_4 has to be before $\mathit{\Gamma}(k_0, \mathit{\Delta}, N_4)$ and $\delta(k_0, \mathit{\Delta}, N_4)$ becomes independent of N_4 will depend on k_0 and $\mathit{\Delta}$. We will now compare these lattice gravity results to the simplest FRG results.

4. FRG

In the FRG approach, one attempts to calculate an effective action as a function of a scale k. In actual calculations, one uses a trial action with adjustable coefficients and tries to determine their k dependence. Their behavior in the IR is then obtained for $k \to 0$, while the behavior in the UV is revealed for $k \to \infty$. The simplest effective action considered is the

Einstein–Hilbert action where the gravitational constant G and the cosmological constant Λ are functions of the scale k that enters the FRG

$$\Gamma_k[g_{\mu\nu}] = \frac{1}{16\pi G_k}\int \mathrm{d}^4x\sqrt{g(x)}\,\left(-R(x)+2\Lambda_k\right) . \tag{29}$$

In (29), $\Gamma_k[g_{\mu\nu}]$ is written for the Euclidean signature of spacetime. In the seminal work of Reuter [3], it was found that there is a UV fixed point for the "running" coupling constants G_k and Λ_k. More elaborate calculations have not changed this conclusion (see [4–6] for details). Since the scale k has the dimension of mass, we can write

$$G_k := g_k/k^2\,, \qquad g_k \to g_*\,, \qquad \Lambda_k := \lambda_k k^2\,, \qquad \lambda_k \to \lambda_*\,, \tag{30}$$

where g_k and λ_k are dimensionless coupling constants that approach their UV fixed point values g_* and λ_* for $k\to\infty$, and one might try to compare g_k, λ_k to suitable dimensionless lattice gravity coupling constants. In particular, we have for the dimensionless combination $G_k\Lambda_k$

$$G_k\Lambda_k = g_k\lambda_k \to g_*\lambda_* \quad \text{for} \quad k\to\infty\,. \tag{31}$$

It has been argued [26] that the dimensionless combination $G\Lambda$ is the only relevant coupling in the truncation (29), or even for a more general class of truncations [27], and both in [27] and [26], a β-function for $\eta=\sqrt{G\Lambda}$ is found. In [26], it is even provided as an explicit rational function of η, shown in Fig. 8 that should be compared to Fig. 2. Around the UV fixed point, they behave qualitatively in the same way. The only difference is that the β-function shown in Fig. 2 is for the bare lattice coupling constant κ_0, while the β-function shown in Fig. 8 is for the continuum, renormalized coupling constant η. The FRG is an equation for continuum, renormalized fields, and

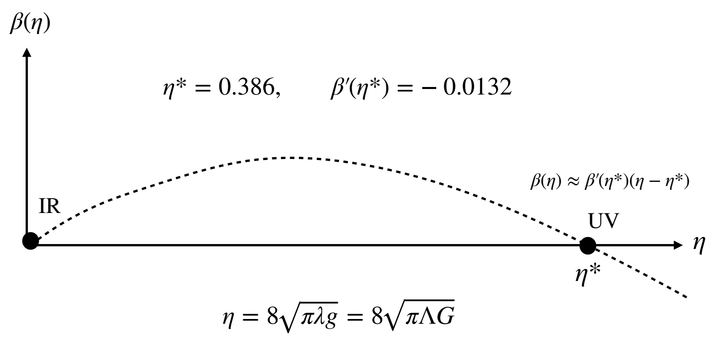

Fig. 8. A qualitative picture of the β-function provided in [26].

coupling constants. According to Eq. (5), the β-functions for the bare and the renormalized coupling constants agree qualitatively, and one can show that they are identical to lowest non-trivial order at a UV fixed point.

We now treat (29) as a standard effective action[3] and find the extremum for $\Gamma_k[g_{\mu\nu}]$. It is the (Euclidean) de Sitter universe with cosmological constant Λ_k, *i.e.* a four-sphere, S^4, with radius $R_k = 3/\sqrt{\Lambda_k}$. This four-sphere has the four-volume

$$V_4(k) = \frac{8\pi^2}{3} R_k^4 = \frac{8\pi^2}{3} \frac{81}{\lambda_k^2} \frac{1}{k^4} \to \frac{8\pi^2}{3} \frac{81}{\lambda_*^2} \frac{1}{k^4} \quad \text{for} \quad k \to \infty \,. \tag{32}$$

In order to compare the FRG effective action with the CDT effective action, we will further restrict the effective action to only include global fluctuations where $V_4(k)$ is kept fixed rather then Λ_k and write the corresponding minisuperspace action using a proper time metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 + r^2(t)\mathrm{d}\Omega_3^2 \,, \qquad V_3(t) = r^3(t) \int \mathrm{d}\Omega_3 = 2\pi^2 r^3(t) \,. \tag{33}$$

The effective action for $r(t)$, or more conveniently $V_3(t)$, is then

$$S_{\text{eff}} = -\frac{1}{24\pi G_k} \int \mathrm{d}t \left(\frac{\dot{V}_3^2}{V_3} + \delta_0 V_3^{1/3} \right) , \qquad \int \mathrm{d}t \; V_3(t) = V_4(k) \,. \tag{34}$$

One can now study fluctuations around this solution and compare them to the fluctuations observed in CDT[4]. Introducing dimensionless variables $v_3 = V_3/V_4^{3/4}$ and $s = t/V_4^{1/4}$, we can write

$$S_{\text{eff}} = -\frac{1}{24\pi} \frac{\sqrt{V_4(k)}}{G_k} \int \mathrm{d}s \; \left(\frac{\dot{v}_3^2}{v_3} + \delta_0 v_3^{1/3} \right) , \qquad \int \mathrm{d}s \, v_3(s) = 1 \,. \tag{35}$$

Here, s and $v_3(s)$ will be of the order of $O(1)$ and the "classical" solution to the eom, $v_3^{\text{cl}}(s)$ is the four-sphere with volume 1. We note that the

[3] In the actual FRG calculations, one often makes the decomposition $g_{\mu\nu} = g^{\text{B}}_{\mu\nu} + h_{\mu\nu}$, where $g^{\text{B}}_{\mu\nu}$ is a fixed background metric (*i.e.* a fixed de Sitter metric) that is fixed even when the scale k is changing. From first principles, the effective action can only depend on $g_{\mu\nu}$, not the arbitrary choice $g^{\text{B}}_{\mu\nu}$. Our treatment here is the most naive implementation of what is suggested in [28, 29], namely that the background one should use for a given scale k should be the one that satisfies the equations of motion at that scale. In [29] it is called the choice of self-consistent background geometries.

[4] In [30] it is shown that when calculating fluctuations for "global" quantities like the three-volume, only constant modes contribute when space is compact. These modes are precisely the modes used when calculating fluctuations in the minisuperspace approximation.

fluctuations around $v_3^{\mathrm{cl}}(s)$ will, for a given k, be governed by the effective coupling constant

$$g^2_{\mathrm{eff}}(k) = \frac{24\pi\, G_k}{\sqrt{V_4(k)}} = \frac{4}{\sqrt{6}}\, \Lambda_k G_k \approx 1.63\, \lambda_k g_k\,. \tag{36}$$

In the FRG analysis, $\lambda_k g_k$ is an increasing function of k, but even at the UV fixed point it is not large. Thus, somewhat surprising, simple Gaussian fluctuations around $v_3^{\mathrm{cl}}(s)$ seems to be a good approximation all the way to the UV fixed point. This might explain the related observation mentioned above for CDT.

5. Comparing CDT and FRG

We want to compare the lattice effective action (27) and the FRG effective action (35)[5,6]. Let us for the moment ignore that $\delta \neq \delta_0$. Then it is natural to identify

$$\frac{\sqrt{N_4}}{\Gamma(\kappa_0, \Delta, N_4)} = \frac{\sqrt{V_4(k)}}{24\pi G_k} \approx \frac{1}{1.63\, \lambda_k g_k}\,. \tag{37}$$

Recall the discussion for the ϕ^4 theory. A renormalized coupling constant κ_{R} could take values between $\kappa_{\mathrm{R}}^{\mathrm{IR}}$ and $\kappa_{\mathrm{R}}^{\mathrm{UV}}$. In the ϕ^4 theory, these values were obtained from the bare coupling constants as shown in Fig. 1. However, they could equally well be obtained by solving the renormalization group equation using the $\beta_{\mathrm{R}}(\kappa_{\mathrm{R}})$. This β-function would look more or less like the β-function shown in Fig. 2, just with κ_0 replaced by κ_{R}. The renormalized running coupling constant would then run between $\kappa_{\mathrm{R}}^{\mathrm{IR}}$ and $\kappa_{\mathrm{R}}^{\mathrm{UV}}$, and any value in this range will qualify as the renormalized coupling constant, defined from the bare lattice coupling constants when the continuum limit of the lattice theory is defined by approaching the lattice UV fixed point. This is the way we will view (37): the r.h.s. is a renormalized coupling constant and the l.h.s. expresses how it is defined in terms of lattice coupling constants,

[5] A first such comparison was done in [31]. However, at that time the so-called bifurcation phase C_{b} had not been discovered. It was viewed as part of phase C_{dS}.

[6] The alert reader might have noticed a disturbing sign difference between (27) and (35). We will argue that it is a good thing. Our lattice theory provides a regularization of the path integral and is finite. On the other hand, the effective action (35) is sick since the kinetic term has a wrong sign. This is why Hartle and Hawking made a further analytic continuation [32]. Using (Euclidean) conformal time, they made an analytic continuation of the conformal factor such that the kinetic term changed a sign. Using proper time instead of conformal time, this analytic continuation leads precisely from (35) to (27). Thus, the CDT version of the effective action *is* the Hartle–Hawking effective action.

i.e. in terms of k_0, $\mathit{\Delta}$, and the lattice correlation length $N_4^{1/4}$ (for notational simplicity, we will use N_4 instead of $N_4^{1/4}$ below). In this context, the scale k that appears in the continuum renormalization group equation just becomes a parametrization that determines how η_k "runs" between η^{IR} and $\eta^{\mathrm{UV}} = \eta_*$ shown in Fig. 8.

Recalling again the discussion surrounding Fig. 1, we have two ways to approach the critical surface $N_4 = \infty$: (1) we can keep $k_0, \mathit{\Delta}$ fixed. Then the renormalized coupling, *i.e.* $\lambda_k g_k$ should flow to an IR fixed point and (2) we keep the renormalized coupling $\lambda_k g_k$ fixed while approaching the critical surface $N_4 = \infty$. This is only possible if we also change the bare coupling constants $k_0, \mathit{\Delta}$, and *if* it is possible to take $N_4 \to \infty$ while keeping $\lambda_k g_k$ fixed, the bare couplings $k_0, \mathit{\Delta}$ should flow to a UV fixed point. If it is not possible, then there is no UV fixed point[7].

5.1. The IR limit

Let us first study case (1): we keep the bare coupling constants $k_0, \mathit{\Delta}$ fixed and located in the interior of phase C_{dS}. As already mentioned, this implies that $\mathit{\Gamma}(k_0, \mathit{\Delta}, N_4)$ (and $\delta(k_0, \mathit{\Delta}, N_4)$) will be independent of N_4 for sufficiently large N_4. From (37) it follows that when we approach the critical surface $N_4 = \infty$, then $\lambda_k g_k \to 0$. Thus, $\lambda_k g_k = 0$ should be an IR fixed point. Does this agree with the FRG picture? If both λ_k and g_k go to zero, we precisely approach the so-called Gaussian fixed point of the renormalization group flow and in fact lowest order perturbation theory tells us (*e.g.* see the linear approximation to Eq. (74) in [4])

$$g_k = g_{k_0} \frac{k^2}{k_0^2}\,, \qquad \lambda_k = \left(\lambda_{k_0} - \frac{g_{k_0}}{8\pi}\right) \frac{k_0^2}{k^2} + \frac{g_{k_0}}{8\pi} \frac{k^2}{k_0^2}\,, \qquad k \approx k_0\,, \quad g_{k_0}, \lambda_{k_0} \ll 1\,. \tag{38}$$

When $k \to 0$, then $\lambda_k \to \infty$ unless $g_{k_0} = 8\pi\, \lambda_{k_0}$, in which case we start out precisely at the unique renormalization group trajectory that leads to the Gaussian fixed point. Unless that is the case, naive lowest order perturbation theory will become invalid for $k \to 0$, since $\lambda_k \to \infty$. However, it has been argued [27] that using a somewhat more general setup, called Essential Quantum Einstein Gravity, instead of the simple effective action (29), one obtains the Gaussian fixed point as the end point of a whole class of renormalization group trajectories for $k \to 0$. This makes the Gaussian fixed point a natural IR fixed point.

In addition, the $k \to 0$ limit has been studied using FRG for time-foliated spacetimes [33]. This setup is closer to the CDT approach and a new IR fixed point was found with the property that a whole set of renormalization

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[7] Recall that this was actually the case in the ϕ^4 theory in four dimensions.

group trajectories starting out at the UV fixed point will converge to this fixed point for $k \to 0$

$$(g_k, \lambda_k) \to \left(0, \frac{1}{2}\right) \quad \text{for} \quad k \to 0\,. \tag{39}$$

More precisely, it was found that

$$g_k \propto \frac{k^4}{\tilde{k}^4}\,, \qquad \lambda_k - \frac{1}{2} \propto \frac{k}{\tilde{k}} \quad \text{for} \quad k \to 0\,, \tag{40}$$

where $\tilde{k}$ is some fixed, small scale. This scenario is also compatible with the CDT limit for $N_4 \to \infty$ and k_0, Δ fixed. This IR fixed point is different from the Gaussian fixed point since the approach to the Gaussian fixed point can be parametrized by a classical gravitational coupling constant $g_k/k^2 = G_k \to G_0$, while $\Lambda_k \to 0$ as k^2. For the other IR fixed point we have $G_k \propto k^2 \to 0$, and also $\Lambda_k \propto k^2 \to 0$.

Again, it is instructive to compare it to the ϕ^4 lattice theory. Keeping the bare coupling κ_0 fixed and increasing the correlation length ξ to infinity (or, in a finite-size scaling setup, $N_4 \to \infty$), we end up at a critical line associated to the IR fixed point: the renormalized coupling constant flows to its IR fixed point value when we approach the $\xi = \infty$ line. Similarly here, keeping the bare coupling constants k_0, Δ fixed and in the interior of the C_{dS} region, the renormalized ΛG flows to its IR or Gaussian fixed point, and the whole interior C_{dS} region is thus associated with this IR or Gaussian fixed point.

5.2. The ultraviolet limit

We now turn to scenario (2) and try to localize a putative lattice UV fixed point. We thus keep the r.h.s. of Eq. (37) fixed and try to find paths $N_4 \to (k_0(N_4), \Delta(N_4))$ such the l.h.s. of (37) stays fixed for $N_4 \to \infty$. From the behavior of $\Gamma(k_0, \Delta, N_4)$ discussed above, such a path has to lead to the boundary of the C_{dS} phase region since $\Gamma(k_0, \Delta, N_4)$ stays finite for any k_0, Δ in the interior of the C_{dS} phase. More precisely, we only see a substantial increase of $\Gamma(k_0, \Delta, N_4)$ when k_0, Δ approaches the A–C_{dS} boundary, see Figs. 5 and 6. This is thus where a possible UV fixed point has to be located. However, before we discuss this in more detail, we have to deal with the fact that $\delta \neq \delta_0$ in Eq. (27), since $\delta(k_0, \Delta)$ also increases a lot when we get close to the A–C_{dS} boundary, *i.e.* according to (28), $\omega(k_0, \Delta)$ decreases, implying that the time-extension of the four-dimensional computer universe shrinks.

5.2.1. Dealing with $\delta \neq \delta_0$

The measured values of ω in the lattice simulations are in general different from the value ω_0 dictated by GR. Since we explicitly break the symmetry between space and time in our lattice regularization, we also have the freedom to scale space-like links and time-like links differently in order to obtain continuum results compatible with the spacetime symmetry present in GR. Denote the length of the time-like links by a_t and the length of the space-like links by $a_s \equiv a$. The continuum three-volume of a spatial slice at time $t_i \equiv a_t i$, consisting of $N_3(t_i)$ tetrahedra will then be $V_3(t_i) \propto N_3(t_i) a^3$. Similarly, the continuum four-volume of N_4 four-simplices will be $V_4 \propto N_4 a_t a^3$. Strictly speaking the situation is somewhat more complicated for the four-simplices. We refer to [10, 34] for details. However, for notational simplicity, we will simply write

$$V_4 = N_4 a_t a^3 \,, \qquad V_3 = N_3 a^3 \,. \tag{41}$$

Then Eq. (26), where a was chosen to be 1, can be rewritten as

$$S = \frac{1}{\Gamma} \sum_i \left(\frac{(N_3(t_i + a_t) - N_3(t_i))^2}{N_3(t_i)} + \delta\, N_3^{1/3}(t_i) \right) \qquad (t_i \equiv a_t i) \tag{42}$$

$$= \frac{a_t}{a^3 \Gamma} \sum_i a_t \left(\frac{(V_3(t_i + a_t) - V_3(t_i))^2 / a_t^2}{V_3(t_i)} + \frac{a^2}{a_t^2}\, \delta\, V_3^{1/3}(t_i) \right) , \tag{43}$$

$$\to \frac{1}{24\pi G} \int \mathrm{d}t \left(\frac{\dot{V}_3^2}{V_3} + \tilde{\delta}\, V_3^{1/3} \right) , \qquad \tilde{\delta} = \frac{a^2}{a_t^2}\, \delta \,, \quad 24\pi G = \frac{a^3}{a_t}\, \Gamma \,, \tag{44}$$

and

$$\sum_i N_3(i) = N_4 \to \int \mathrm{d}t\, V_3(t) = V_4 \,, \qquad V_4 = a_t a^3 N_4 \,, \tag{45}$$

where $\tilde{\delta}$ and ω are related as in (28): $\tilde{\delta}\, \omega^{8/3} = \delta_0 \omega_0^{8/3}$. If $\omega \neq \omega_0$, the lattice configurations are the "deformed" spheres because the time extension $N_t a_t = \omega N_4^{1/4} a_t$ does not match the spatial extension $N_3^{1/3} a$, when we write $N_4 = N_t N_3$ and $a_t = a$. We can correct that by writing

$$a_t = \left(\frac{\omega_0}{\omega} \right)^{4/3} a \,. \tag{46}$$

From Eq. (44) it then follows that $\tilde{\delta} = \delta_0$ for the round S^4 and thus this choice of a_t leads to an action S_{eff} given in (27) that we can identify with the FRG effective action (35) for some value of the scale parameter k.

So given computer data $N_4, \omega, \mathit{\Gamma}$, we can associate a corresponding continuum, round S^4 with four-volume V_4 and gravitation constant G via

$$(N_4, \omega, \mathit{\Gamma}) \to (V_4(k), \omega_0, G_k)\,, \tag{47}$$

where

$$V_4(k) = \left(\frac{\omega_0}{\omega}\right)^{4/3} N_4 a^4\,, \qquad 24\pi G_k = \left(\frac{\omega}{\omega_0}\right)^{4/3} \mathit{\Gamma}\, a^2 \tag{48}$$

and, in particular,

$$\frac{\sqrt{N_4}}{\mathit{\Gamma}} = \frac{\omega^2}{\omega_0^2}\,\frac{\sqrt{V_4(k)}}{24\pi G_k} \qquad \text{or} \qquad \frac{\omega^2 \mathit{\Gamma}(k_0, \mathit{\Delta}, N_4)}{\omega_0^2\sqrt{N_4}} \simeq 1.63\, \lambda_k g_k\,. \tag{49}$$

From what is said above, it is clear that the only chance to satisfy (49) for $N_4 \to \infty$ is by approaching the A–$C_{\rm dS}$ transition line from the $C_{\rm dS}$ side. We will discuss this below.

5.2.2. Scaling at the UV limit

As discussed in the numerical results section below, the observed dependence on $\mathit{\Delta}$ is weak in the region of interest, and for notational simplicity we will omit most references to $\mathit{\Delta}$ in the following. In this way, the critical surface $N_4 = \infty$ becomes a critical line, precisely as was the case for the ϕ^4 theory. This line is then naturally associated with the IR fixed point of the FRG, in the same way as the critical line for the ϕ^4 theory was associated with an IR fixed point of the renormalized coupling $\kappa_{\rm R}$. We want to investigate if there should be a lattice UV fixed point on the critical line.

Approaching a point $(k_0, N_4 = \infty)$ on the critical line, we have for all k_0 different from such a UV critical point $k_0^{\rm UV}$ that

$$\mathit{\Gamma}(k_0, N_4) \;\to\; \mathit{\Gamma}(k_0) < \infty\,, \quad \text{for} \quad N_4 \to \infty\,. \tag{50}$$

$$\omega(k_0, N_4) \;\to\; \omega(k_0)\,, \qquad 0 < \omega(k_0) < \infty \quad \text{for} \quad N_4 \to \infty\,. \tag{51}$$

The putative UV fixed point $k_0^{\rm UV}$ has to be located at the A–$C_{\rm dS}$ transition line and we observe numerically that $\mathit{\Gamma}(k_0) \to \infty$ and $\omega(k_0) \to 0$ for $k_0 \to k_0^{\rm UV}$. It is thus natural to assume that close to $k_0^{\rm UV}$ we can have the following critical behavior:

$$\begin{aligned} \mathit{\Gamma}(k_0) \;&\propto\; \frac{1}{|k_0^{\rm UV} - k_0|^{\alpha}}\,, \qquad \omega(k_0) \propto |k_0^{\rm UV} - k_0|^{\beta}\,, \\ \omega^2(k_0)\mathit{\Gamma}(k_0) \;&\propto\; \frac{1}{|k_0^{\rm UV} - k_0|^{\alpha - 2\beta}}\,. \end{aligned} \tag{52}$$

We further assume that for a finite N_4, there is a *pseudo-critical point* $k_0^{\mathrm{UV}}(N_4) < k_0^{\mathrm{UV}}$, where $\omega^2(k_0, N_4)\Gamma(k_0, N_4)$ has a maximum for fixed N_4, and that this pseudo-critical point approaches k_0^{UV} for $N_4 \to \infty$ as

$$k_0^{\mathrm{UV}}(N_4) = k_0^{\mathrm{UV}} - \frac{c}{N_4^{1/4\nu_{\mathrm{UV}}}} \qquad \left(i.e. \quad \xi \propto \frac{1}{|k_0^{\mathrm{UV}} - k_0^{\mathrm{UV}}(\xi)|^{\nu_{\mathrm{UV}}}} \right) . \tag{53}$$

This implies that

$$\Gamma\left(k_0^{\mathrm{UV}}(N_4)\right) \propto N_4^{\alpha/4\nu_{\mathrm{UV}}}, \qquad \omega\left(k_0^{\mathrm{UV}}(N_4)\right) \propto N_4^{-\beta/4\nu_{\mathrm{UV}}}, \tag{54}$$

as well as

$$\omega^2\left(k_0^{\mathrm{UV}}(N_4)\right)\Gamma\left(k_0^{\mathrm{UV}}(N_4)\right) \propto N_4^{(\alpha-2\beta)/4\nu_{\mathrm{UV}}} . \tag{55}$$

From Eq. (49) it follows that we have to have

$$\alpha - 2\beta \geq 2\nu_{\mathrm{UV}} \tag{56}$$

and if that is the case, the following path in the bare lattice coupling constant space will lead us to the putative UV fixed point while keeping $\lambda_k g_k$ fixed

$$k_0(N_4) = k_0^{\mathrm{UV}} - \frac{c}{N_4^{1/2(\alpha-2\beta)}} . \tag{57}$$

The situation is illustrated in Fig. 9.

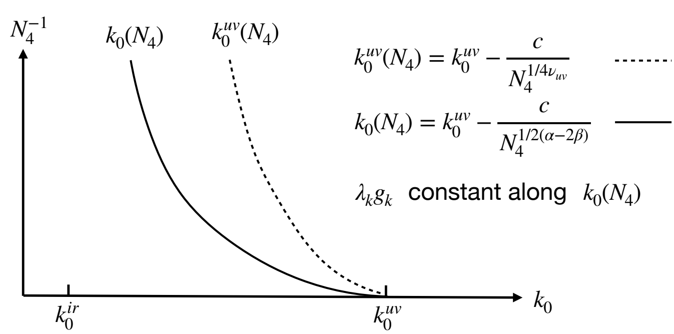

Fig. 9. The tentative CDT phase diagram (k_0, N_4^{-1}) (with coupling constant Δ ignored). Pseudo-criticality appears along the dotted line $k_0^{\mathrm{UV}}(N_4)$ and the solid line $k_0(N_4)$, where $\lambda_k g_k$ is constant is shown to the left of $k_0^{\mathrm{UV}}(N_4)$. The critical line is $N_4^{-1} = 0$.

5.2.3. The enigmatic relation between a and k

k is a scale of dimension mass that appears in the FRG. The dimensionless coupling constant $\lambda_k g_k$ runs to the UV fixed point value $\lambda_* g_*$ for $k \to \infty$. Similarly, the inverse lattice spacing a^{-1} is a UV cut-off scale that can be taken to infinity when the bare lattice coupling constants are approaching a UV lattice fixed point: the "continuum limit" can be taken in such a way that the renormalized couplings are finite and non-trivial when $a \to 0$. It is thus natural also to think about k as a kind of UV cut-off[8] such that $k \propto a^{-1}$ is close to the UV fixed point. In our simple model we can address this. Recall that in our discussion so far, k only played a role as a parametrization of the renormalized coupling constant $\lambda_k g_k$. Using (48), (54), and $V_4(k) \propto \Lambda_k^{-2}$, we find that

$$a \propto \frac{1}{\sqrt{\Lambda_k}}\, N_4^{-\frac{1}{4}\left(1+\frac{\beta}{3\nu_{\mathrm{UV}}}\right)}\,, \quad i.e. \quad a \propto \frac{1}{k}\, N_4^{-\frac{1}{4}\left(1+\frac{\beta}{3\nu_{\mathrm{UV}}}\right)} \quad \text{for} \quad k \to \infty\,. \tag{58}$$

Thus, the lattice spacing a scales to zero for a fixed value of k (*i.e.* a fixed value of $\lambda_k g_k$) when we approach the critical surface $N_4 = \infty$. It also follows from (54) and (46) that:

$$a_t \propto \frac{1}{\sqrt{\Lambda_k}}\, N_4^{-\frac{1}{4}\left(1-\frac{\beta}{\nu_{\mathrm{UV}}}\right)} \quad i.e. \quad a_t \propto \frac{1}{k}\, N_4^{-\frac{1}{4}\left(1-\frac{\beta}{\nu_{\mathrm{UV}}}\right)} \quad \text{for} \quad k \to \infty\,. \tag{59}$$

This slower decrease of a_t is a reflection of the fact that we, when approaching the A–C_{dS} transition line, have to rescale our lattice four-spheres that become increasingly "contracted" in the time direction, in order to match the round four-spheres of the FRG. Thus, under the assumptions

$$\frac{\alpha - 2\beta}{4\nu^{\mathrm{UV}}} > \frac{1}{2}\,, \qquad \frac{\beta}{4\nu^{\mathrm{UV}}} < \frac{1}{4}\,, \tag{60}$$

we can reach a UV fixed point where we can also take the continuum limit a, $a_t \to 0$ for $N_4 \to \infty$.

While (58) and (59) tell us that $a \propto 1/k$, this is unfortunately just a dimensional relation. The real content of (58) is that for fixed k, *i.e.* for fixed $\lambda_k g_k$, the lattice UV cut-off a goes to zero when the correlation length ξ (related to N_4) goes to infinity. In this sense, it corresponds to relation (7) for the ϕ^4 lattice theory, k playing the role of m_{R}.

[8] In the FRG community, k is often talked about as a UV cut-off, but it is also often emphasized that this should not be taken too literal, since formally, no UV cut-off is introduced explicitly when formulating the FRG.

6. Numerical results

This section will be rather short since the purpose of this paper has not been to discuss technical details of how to obtain the numerical results. For information about that we refer to [34] and references therein.

In Fig. 10, we show the measurements of $\omega(k_0, \Delta)$, $\Gamma(k_0, \Delta)$, and $\omega^2(k_0, \Delta)\Gamma(k_0, \Delta)$ for a fixed N_4. One observes the increase of Γ and $\omega^2\Gamma$ and the decrease of ω when moving towards the A–C_{dS} transition line. Also, the insensitivity to Δ is seen.

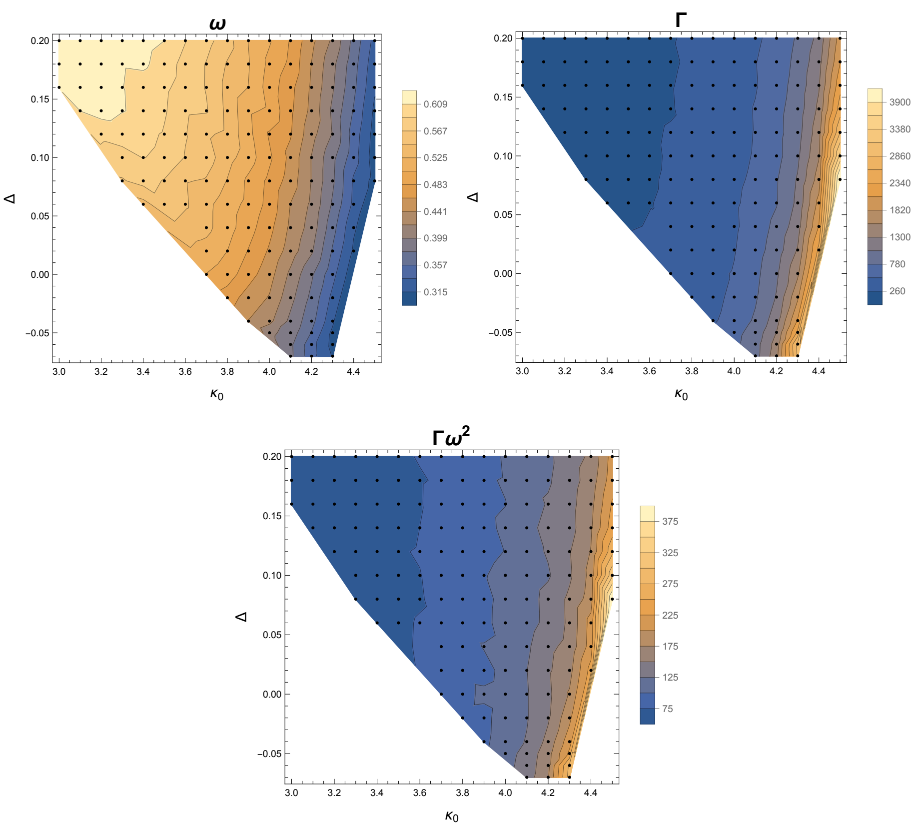

Fig. 10. Contour plots of ω (top left), Γ (top right), and $\omega^2\Gamma$ (bottom) as functions of the CDT coupling constants k_0, Δ. Points where actual measurements were done are denoted as black dots in the plots.

Figure 11 shows the measurements of the same observables at the pseudo-critical points $k_0^{\mathrm{UV}}(N_4)$. Close to $k_0^{\mathrm{UV}}(N_4)$, the change of the observables as a function of k_0 is fast, a fact that is not so visible in Fig. 10.

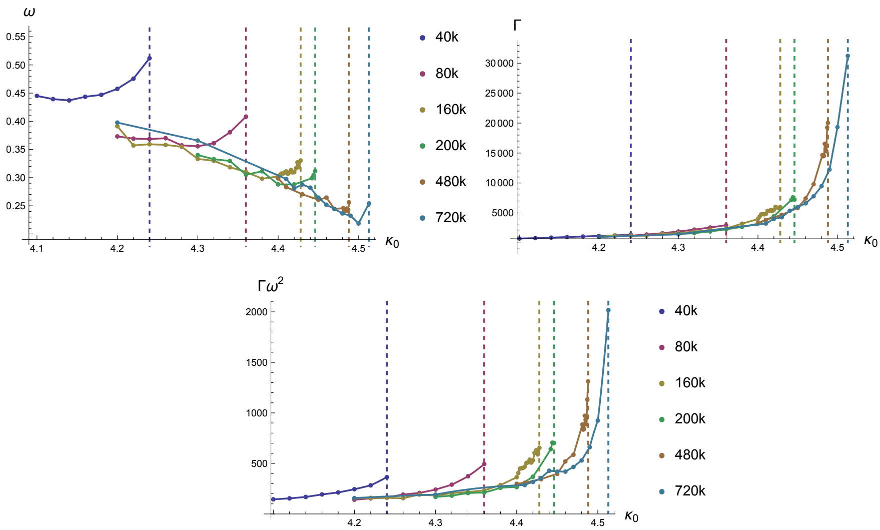

Fig. 11. Dependence of ω (top left), Γ (top right), and $\Gamma\omega^2$ (bottom) on k_0 for fixed $\Delta = 0$ measured for the number of (4,1)-simplices being 40 000, 80 000, 160 000, 200 000, 480 000, 720 000 (denoted by different colors). Positions of k_0 closest to the pseudo-critical points $\kappa_0^{\mathrm{UV}}(N_4)$ are denoted by dashed lines.

These measurements provide us with both $k_0^{\mathrm{UV}}(N_4)$, $\Gamma(k_0^{\mathrm{UV}}(N_4))$, $\omega(k_0^{\mathrm{UV}}(N_4))$, and $\omega^2(k_0^{\mathrm{UV}}(N_4))\Gamma(k_0^{\mathrm{UV}}(N_4))$, and we can then determine the critical exponents defined in Eqs. (53)–(55). The determination of the exponents $\alpha/4\nu^{\mathrm{UV}}$, $\beta/4\nu^{\mathrm{UV}}$, and $(\alpha - 2\beta)/4\nu^{\mathrm{UV}}$ is shown in Fig. 12 and the results are

$$\frac{\beta}{4\nu_{\mathrm{UV}}} = 0.23 \pm 0.02\,, \qquad \frac{\alpha}{4\nu_{\mathrm{UV}}} = 1.00 \pm 0.02\,, \qquad \frac{\alpha - 2\beta}{4\nu_{\mathrm{UV}}} = 0.54 \pm 0.04\,. \tag{61}$$

What is striking about these results is that they are very close to the limit (60). Thus, we can say that the data allows for the existence of a UV fixed point, but it cannot be used as strong evidence for such a fixed point.

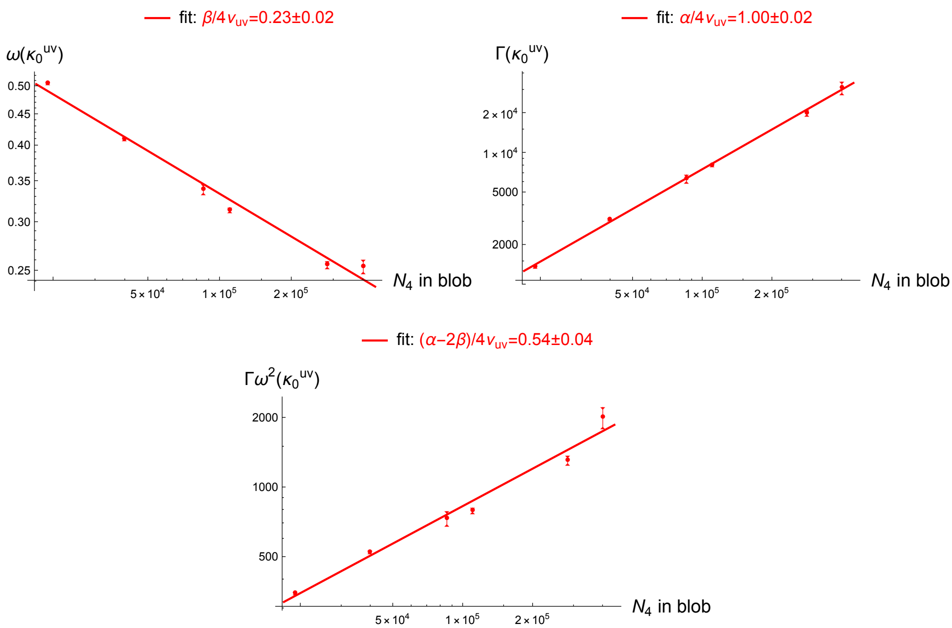

Fig. 12. Critical scaling of ω (top left), Γ (top right), and $\Gamma\omega^2$ (bottom) measured closest to the pseudo-critical points $\kappa_0^{\mathrm{UV}}(N_4)$ (see Fig. 11) for fixed $\Delta = 0$. Fits of Eqs. (54)–(55) are depicted by solid lines. The figures show scaling as a function of the N_4 volume contained in the S^4 "blob" (not all four-simplices are contained in the S^4 part. Some can be in the so-called stalk, see Fig. 7).

7. Discussion

We have tried to relate the simplest FRG flow to the CDT effective action for the scale factor of the universe. By using the analogy to the ϕ^4 lattice theory we argued that the $N_4 \to \infty$ limit of CDT, when in the C_{dS} phase, could be viewed as the critical surface associated with the Gaussian fixed point or an IR fixed point of the FRG theory. Again inspired by the ϕ^4 lattice theory we then searched for a CDT UV fixed point by studying the flow of the lattice coupling constants when the corresponding continuum coupling constants were kept fixed. Rather frustratingly the numerical accuracy is not yet good enough to decide if such a UV fixed point exists in CDT.

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