EXAMPLES OF STRONG ACTION OF HIGHLY RELATIVISTIC SPIN–GRAVITY COUPLING ON A SPINNING PARTICLE IN SCHWARZSCHILD'S BACKGROUND

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The Mathisson–Papapetrou equations are used to study the deviation of the spinning particle world lines and trajectories from the corresponding geodesic lines in Schwarzschild's background. The traditional form of these equations and their consequences in terms of the comoving tetrads are considered. Analytical and numerical calculations for the equatorial motions are performed. The circular, quasi-circular, and quasi-radial motions of the highly relativistic spinning particle are analyzed. Different illustrations are presented.

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1. Introduction

The history of research on the behavior of a spinning particle in general relativity is only one year "younger" than the history of the Dirac equation. Indeed, in 1929, papers [1–3] were published in which the generalization of Dirac's equation for the case of the curved spacetime was obtained. The next step was made in 1937 when the equations of motions of a classical (nonquantum) spinning particle were derived [4]. Later, the corresponding equations were rederived in [5–9] using alternative methods. Now, these equations are known as the Mathisson–Papapetrou (MP) or Mathisson–Papapetrou–Dixon (MPD) equations. In recent paper [10], the spin–gravity coupling is considered without mentioning the Dirac and MP equations.

To interpret the spin–gravity effects that follow from the MP equations in the Schwarzschild and Kerr field, the clear analogies with the spin–orbit and spin–spin interactions in electromagnetism were considered [11]. For this purpose, the corresponding expressions in the post-Newtonian approximation are analyzed. Now, the MP equations are often used to describe the behavior of very different spinning objects: from small particles such as electrons, protons, and neutrinos [12–17] to massive extended objects, pulsars, and black holes [18–22]. In recent paper [23], the spin–gravity effects are considered for a charged spinning particle in the charged black hole. It is important that the MP equations follow, in a certain sense, from the general relativistic Dirac equation as a classical approximation, the corresponding wide list of papers is presented in [24, 25].

A separate direction of research on the spin–gravity effects concerns the specificity of highly relativistic motions of a spinning particle in various gravitational fields. These effects are under consideration in Schwarzschild's, Kerr's [24], and the Schwarzschild–de Sitter backgrounds [26, 27]. It was established that in the highly relativistic region of the particle velocity, the spin–gravity coupling shows new important properties. In general, in this region, the deviation of the particle's motion from the geodesic motion becomes much larger than at low velocity.

Depending on the spin orientation and the direction of particle's motion relative to the source of the gravitational field, the spin–gravity coupling acts on the particle as some repulsive or attractive force. Different cases of strong repulsive action for a highly relativistic spinning particle are considered in [24, 26–28].

The main purpose of this paper is to investigate the strength of the highly relativistic spin–gravity coupling in Schwarzschild's background by estimating the acceleration of the spinning particle relative to the spinless one from the point of view of a comoving observer. That is, in comparison with the studies of spin–gravity effects in Schwarzschild's background by other authors, as well as with our previous studies, in this paper, we not only compare the shape of world lines and trajectories of the spinning and spinless particles, but also investigate the value of the acceleration of the spinning particle relative to the geodesic free fall. Another important goal of the research and our new result is the development of a computational procedure that makes it possible to describe the solutions of the original Mathisson equations in the equatorial plane of the Schwarzschild metric, which are appropriate for the analysis of the highly relativistic motions of a spinning particle.

The study of the strong spin–gravity coupling is important when discussing possible experiments for registration of spin–gravity effects, in particular, in the context of paper [10].

The paper is organized in the following way. In Section 2, the MP equations and some of their consequences are presented. The solutions of these equations for circular motions in Schwarzschild's background are considered in Section 3. In Section 4, the explicit form of the MP equations for the equatorial motions in this background is presented using the constants of motion. Section 5 is devoted to describe the method for choosing values of these constants that correspond to the motion of the particle's proper center of mass. In Section 6, the deviation of the spinning particle from the noncircular geodesic free fall in Schwarzschild's background in terms of its coordinates and velocity is studied. The acceleration of the spinning particle relative to the free falling spinless particle is estimated in Section 7. In Section 8, we consider an example of the effect of highly relativistic spin–gravity coupling on the motion of the spinning particle, which complements the results of Section 6. We conclude in Section 9.

2. Mathisson–Papapetrou equations

According to Mathisson's pioneer paper [4], the corresponding equations can be written as

$$\frac{D}{\mathrm{d}\tau}\left(mu^\lambda + u_\mu \frac{DS^{\lambda\mu}}{\mathrm{d}\tau}\right) = -\frac{1}{2}u^\pi S^{\rho\sigma} R^\lambda{}_{\pi\rho\sigma}\,, \tag{1}$$

$$\frac{DS^{\mu\nu}}{\mathrm{d}\tau} + u^\mu u_\sigma \frac{DS^{\nu\sigma}}{\mathrm{d}\tau} - u^\nu u_\sigma \frac{DS^{\mu\sigma}}{\mathrm{d}\tau} = 0\,, \tag{2}$$

$$S^{\lambda\nu} u_\nu = 0\,, \tag{3}$$

where $u^\lambda \equiv \mathrm{d}x^\lambda/\mathrm{d}\tau$ is the particle's 4-velocity, $S^{\mu\nu}$ is the antisymmetric tensor of spin, m is the constant of the particle's mass, and $D/\mathrm{d}\tau$ is the covariant derivative along u^λ. Here, and in the following, Greek indices run through 1, 2, 3, 4 and Latin indices run through 1, 2, 3; the signature of the metric $(-, -, -, +)$ and the units $c = G = 1$ are chosen. Naturally, for $S^{\mu\nu} = 0$, Eqs. (1)–(3) reduce to the geodesic equations. For $S^{\mu\nu} \neq 0$, the two subsets of ordinary differential Eqs. (1) and (2) cannot be integrated separately. However, in some important partial cases, one can obtain explicit solutions of subset (2) independently of subset (1). For example, when a spinning particle is moving in the equatorial plane of Schwarzschild's metric.

Relation (3) was introduced in the natural way in the framework of the general procedure of obtaining Eqs. (1)–(3) [4]. Namely, for the correct definition of the inner rotation, it is necessary to calculate the value $S^{\mu\nu}$ relative to the center of mass of a rotating body. Then relation (3) follows in a simple way [29]. However, an unusual situation arises in relativistic mechanics, which is impossible in classical mechanics. Indeed, in special relativity, the position of the center of mass of a rotating body depends on the reference frame [30]. That is, here we are not dealing with one center of mass, but with many centers, and all of them are located on the so-called disk of centers of mass [30]. It is important that all centers satisfy the same relation (3). Among these centers, there is only one, which is located in the geometrical center of the disk and can be named the proper center of mass. Here, the word "proper" means that this center is determined in the frame of reference where the axis of the body's rotation is at rest. All other (nonproper) centers of mass move around the proper center. The analogues situation takes place with a spinning particle in general relativity: Eqs. (1)–(3) have both a single solution that describes the motion of the proper center of mass and a family of solutions that have the circular or helical character and describe the nonproper centers [30, 31]. It means that in the study of Eqs. (1)–(3) in general relativity, it is important to select the solution for the proper center, because it is this solution that directly describes the propagation of the spinning particle in the curved spacetime. These circular or helical solutions do not appear if, instead of MP Eqs. (1)–(3), the MPD equations are used of the form [6, 7]

$$\frac{DP^\lambda}{\mathrm{d}\tau} = -\frac{1}{2}u^\pi S^{\rho\sigma} R^\lambda{}_{\pi\rho\sigma}\,, \tag{4}$$

$$\frac{DS^{\mu\nu}}{\mathrm{d}\tau} = 2P^{[\alpha}u^{\beta]}\,, \tag{5}$$

$$S^{\lambda\nu}P_\nu = 0\,, \tag{6}$$

where

$$P^\nu = mu^\nu + u_\lambda \frac{DS^{\nu\lambda}}{\mathrm{d}\tau} \tag{7}$$

is the particle 4-momentum. That is, in Eqs. (4)–(6) instead of (3) relation (6) is used. In general, when the second term on the right-hand side of Eq. (7) is not equal to 0, P_ν is not parallel to u_ν, and (6) does not coincide with (3). Nevertheless, in different physical situations, relation (6) is a good substitute for (3). However, in general, for the highly relativistic motions, there is a clear restriction on the use of (6). For example, it is shown that under condition (6), a spinning particle, which begins motion in Schwarzschild's field with a velocity less than the speed of light, can be accelerated to the superluminal velocity [32]. Another unphysical result under this condition follows from the analysis of the expression for the spinning particle 4-momentum through its 4-velocity: it is shown that for the highly relativistic tangential velocity, the values of the momentum components become imaginary [24]. Since in this paper we take into account the highly relativistic motions of a spinning particle, in the following, we will consider Eqs. (1)–(3).

Many results, analytical and numerical, on solutions of the MP equations under various conditions are presented in [33–37].

Both Eqs. (1)–(3) and (4)–(6) have a constant of motion

$$S^2 = \frac{1}{2}S_{\mu\nu}S^{\mu\nu}\,, \tag{8}$$

where $|S|$ is the absolute value of spin.

The tensor $S^{\mu\nu}$ and the 4-vector s_λ are used as a characteristic of the spin of the particle, where by definition

$$s_\lambda = \frac{1}{2}\varepsilon_{\lambda\mu\nu\sigma}\sqrt{-g}u^\mu S^{\nu\sigma}\,, \tag{9}$$

where $\varepsilon_{\lambda\mu\nu\sigma}$ and g are the Levi-Civita symbol and the determinant of the metric tensor, respectively. It follows from (9) that

$$s_\lambda u^\lambda = 0\,, \tag{10}$$

and Eq. (2) takes the form

$$\frac{Ds^\lambda}{\mathrm{d}\tau} = s_\mu \frac{Du^\mu}{\mathrm{d}\tau}u^\lambda, \tag{11}$$

i.e. the 4-vector of spin is Fermi transported [24].

In practical calculations, it is convenient to use the three-component value S_i as well, where by definition

$$S_i = \frac{1}{2u_4}\varepsilon_{ikl}\sqrt{-g}S^{kl}\,, \tag{12}$$

where ε_{ikl} is the spatial Levi-Civita symbol [24].

It is appropriate to consider a consequence which follows from Eqs. (1)–(3) in terms of the local tetrads values $\lambda^\mu_{(\nu)}$ that correspond to the comoving frame of reference, when $\lambda^\mu_{(4)} = u^\mu$ [24]. (Here and in the following, the local indices are placed in the parenthesis.) Then the local components of the spin 4-vector satisfy the condition $s_{(4)} = 0$ and the relation $s_{(\mu)}s^{(\mu)} = -S^2$ [29] (sign "$-$" is present on the right-hand side of this relation as a result of our choice of the metric tensor signature). If for convenience the first spatial local vector lies along the direction of spin, then

$$s_{(1)} \neq 0\,, \qquad s_{(2)} = 0\,, \qquad s_{(3)} = 0\,, \quad s_{(4)} = 0\,, \qquad s_{(1)}s^{(1)} = -S^2\,. \tag{13}$$

When, in addition, the second local spatial vector is oriented along the particle's motion, then it follows from Eq. (2) that

$$\gamma_{(k)(1)(4)} = 0\,, \tag{14}$$

where $\gamma_{(k)(1)(4)}$ are the Ricci coefficients of rotation [24]. At the same conditions from Eqs. (1)–(3), we have [24]

$$ma_{(1)} + s_{(1)}R_{(1)(4)(2)(3)} \;=\; 0\,, \tag{15}$$

$$ma_{(2)} + s_{(1)}\left(R_{(2)(4)(2)(3)} - \dot a_{(3)} - a_{(2)}\gamma_{(2)(3)(4)}\right) \;=\; 0\,, \tag{16}$$

$$ma_{(3)} + s_{(1)}\left(R_{(3)(4)(2)(3)} + \dot a_{(2)} - a_{(3)}\gamma_{(2)(3)(4)}\right) \;=\; 0\,, \tag{17}$$

where $a_{(i)} \equiv \gamma_{(1)(4)(4)}$ and $R_{(i)(4)(2)(3)}$ are the corresponding local components of the Riemann tensor; a dot denotes differentiation with respect to the particle's proper time τ. The local 3-vector $a_{(i)}$ has the direct physical meaning as the acceleration of the spinning particle relative to the free motion of a spinless particle. At $s_{(1)} = 0$, *i.e.* for a spinless particle, according to (15)–(17), $a_{(i)} = 0$.

3. MP equations in Schwarzschild's metric and their solutions for circular motions

We use the Schwarzschild metric in the standard coordinates $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$, and $x^4 = t$. Then the nonzero components of the metric tensor $g_{\mu\nu}$ are

$$\begin{aligned} g_{11} &= -(1-2\alpha)^{-1}\,, \qquad g_{22} = -r^2\,, \\ g_{33} &= -r^2 \sin^2\theta\,, \qquad g_{44} = 1 - 2\alpha\,, \qquad \alpha \equiv M r^{-1}\,, \end{aligned} \tag{18}$$

where M is Schwarzschild's mass.

In the following, we will consider Eqs. (1)–(3) for the spinning particle motion in the plane $\theta = \pi/2$, when spin is orthogonal to this plane and $S_1 = 0$, $S_3 = 0$, $S_2 \neq 0$. Then from (2), we obtain

$$S_2 = rS\,, \tag{19}$$

where S is the constant that is present in (8). We stress that Eq. (19) is valid if the spin is aligned with the coordinate θ, in other words, with the z axis. Without any loss in generality, we choose the orientation of the particle's spin such that $S_\theta \equiv S_2 > 0$.

It is easy to check that Eqs. (1)–(3) have the simple partial solution with $u^1 \neq 0$, $u^4 \neq 0$, $u^2 = 0$, and $u^3 = 0$ which describes the radial motion of a spinning particle. According to this solution, for any value of the particle's spin, this motion coincides exactly with the radial motion of a spinless particle that is described by the correspondent geodesic lines in Schwarzschild's metric. It means that in the case of the radial motion, any influence of the spin–gravity coupling on the particle world line is absent. Another situation takes place for the circular motion of a spinning particle in Schwarzschild's background.

Equations (1)–(3) admit the circular motions with constant orbital velocity, when $u^1 = 0$, $u^2 = 0$, $u^3 = \text{const.} \neq 0$, and $u^4 = \text{const.} \neq 0$. Indeed, with relation (19), it follows from Eq. (1) that

$$\begin{aligned} &m\left(\Gamma^1_{33} u^3 u^3 + \Gamma^1 u^4 u^4\right) + 3\alpha r^{-2} u^3 u_4 S_2 \\ &+(1-3\alpha)\left(u^3 u^3 - \alpha r^{-2} u^4 u^4\right) u_3 u_4 S_2 = 0\,, \end{aligned} \tag{20}$$

where $\Gamma^{\pi}_{\rho\sigma}$ is the Christoffel symbol. This equation together with the relation

$$g_{33}u^3u^3 + g_{44}u^4u^4 = 1 \tag{21}$$

(which is the partial case of the known general relation for the 4-velocity components $u_\mu u^\mu = 1$) determines the dependence of the orbital velocity on the radial coordinate r and the spin component S_2. Then from (20) and (21), we obtain the equation for the tangential component of the 4-velocity $u_\perp = ru^3$

$$u_\perp^3\beta(1-3\alpha)^2 - u_\perp^2(1-2\alpha)(1-3\alpha) + u_\perp\beta\alpha(2-3\alpha) + \alpha(1-2\alpha) = 0\,, \tag{22}$$

where

$$\beta \equiv \frac{S_2}{mr^2}(1-2\alpha)^{1/2}\left(1+u_\perp^2\right)^{1/2}. \tag{23}$$

In the trivial case when $M = 0$ (and $\alpha = 0$) according to (18), we have from (22)

$$u_\perp = \frac{1}{\beta} = \frac{mr}{Su_4}\,, \tag{24}$$

i.e. the expression for the known so-called Weyssenhoff's circular orbits in the Minkowski spacetime [31].

For $M \neq 0$, it follows from (22) and (23) that the value $u_\perp^2 \equiv y$ satisfies the equation

$$y^4d_4 + y^3d_3 + y^2d_2 + yd_1 + d = 0\,, \tag{25}$$

where

$$\begin{aligned}
d_4 &= \varepsilon^2\alpha^2(1-2\alpha)(1-3\alpha)^4\,,\\
d_3 &= \varepsilon^2\alpha^2(1-2\alpha)(1-3\alpha)^2\left(1-2\alpha+3\alpha^2\right)\,,\\
d_2 &= \varepsilon^2\alpha^3(1-2\alpha)(2-3\alpha)\left(2-10\alpha+15\alpha^2\right) - (1-2\alpha)^2(1-3\alpha)^2\,,\\
d_1 &= \varepsilon^2\alpha^4(1-2\alpha)(2-3\alpha)^2 + 2\alpha(1-2\alpha)^2(1-3\alpha)\,,\\
d &= -\alpha^2(1-2\alpha)^2\,,
\end{aligned} \tag{26}$$

and the notation

$$\varepsilon \equiv \frac{S}{mM} \tag{27}$$

is used.

In the partial case when $\varepsilon = 0$, *i.e.* for a spinless particle, Eq. (25) for $\alpha \neq 2$ takes the form of the second-order algebraic equation for y

$$y^2(1-3\alpha)^2 - 2y\alpha(1-3\alpha) + \alpha^2 = 0\,. \tag{28}$$

From (28), we have

$$y = \alpha(1-3\alpha)\,. \tag{29}$$

Then, because $u_\perp = \sqrt{y}$, we write

$$u_\perp = \pm\sqrt{\alpha}(1-3\alpha)^{1/2}\,. \tag{30}$$

Equation (30) is the known expression that follows from the geodesic equations in Schwarzschild's metric for the circular motions of a spinless particle (the two signs in (30) correspond to the orbital motions with different directions by the angle φ).

Let us consider the solutions of Eqs. (22) and (25) for $M \neq 0$ and $S \neq 0$, *i.e.* $\alpha \neq 0$, $\varepsilon \neq 0$. In the partial case when $r = 3M$, Eq. (25) takes the form

$$y^2\varepsilon^2 + y\varepsilon^2 - 3 = 0\,. \tag{31}$$

The positive root of this equation is

$$y = -0.5 + \sqrt{0.25 + 3\varepsilon^{-2}}\,. \tag{32}$$

Then the corresponding solution of Eq. (22) is

$$u_\perp = -\sqrt{-0.5 + \sqrt{0.25 + 3\varepsilon^{-2}}} \tag{33}$$

(one can check that for the above choice of the sign of the spin 3-vector component $S_2 > 0$, when according to (25) $\beta > 0$, Eq. (22) is satisfied only for $u_\perp < 0$, as reflected in (33)). It is necessary to take into account the physical condition for a spinning test particle [11]

$$\frac{|S|}{mr} \ll 1\,. \tag{34}$$

Then according to (33) and (34),

$$u_\perp \approx -3^{1/4}\varepsilon^{-1/2}(1 + O(\varepsilon))\,. \tag{35}$$

and

$$u_\perp^2 \gg 1\,. \tag{36}$$

It means that in the circular orbit with $r = 3M$, the orbital velocity of the spinning particle is highly relativistic. The physical meaning of Eq. (35) is clear: the smaller the value of spin, the more ultrarelativistic speed is necessary for the particle's motion in this orbit. It is easy to check that, by choosing $S_2 < 0$, from the corresponding equations follows the expression for $u_\perp$ which differs from (35) only in sign.

Note that if instead of Eq. (1) its shortened form

$$m\frac{D}{d\tau}u^\lambda = -\frac{1}{2}u^\pi S^{\rho\sigma} R^\lambda{}_{\pi\rho\sigma} \tag{37}$$

is taken into account (Eq. (37) is often considered to study of the spin–gravity effects in the linear spin approximation), then the algebraic equation which determines the dependence of $u_\perp$ on r for the circular orbits is

$$u_\perp^2(1-3\alpha) - 3u_\perp\beta\alpha - \alpha = 0\,. \tag{38}$$

It is easy to check that in the particular case when $r = 3M$, the expression which follows from (37) coincides exactly with (33). It means that the solution which describes the circular motion in the orbit with $r = 3M$ is common for the exact MP Eqs. (1)–(3) and their linear spin approximation. As a result, this particular solution is common for the MP equation both under relations (3) and (6).

It is known from the geodesic equations in Schwarzschild's metric that a spinless particle with a nonzero mass of any velocity close to the velocity of light, starting in the tangential direction from the position $r = 3M$, will fall on the horizon surface within a finite proper time, whereas, a spinning particle will remain indefinitely on the circular orbit with $r = 3M$ due to the spin–gravity coupling. This coupling compensates for the usual ("geodesic") attraction. Let us estimate the force of this coupling. For this purpose, we use Eqs. (15)–(17). In the cases of the equatorial particle motions, it is not difficult to calculate that

$$\begin{aligned}
R_{(1)(4)(2)(3)} &= 0\,, && (39)\\
R_{(2)(4)(2)(3)} &= -\frac{3M}{r^2}u^1u^3\left(u_4u^4-1\right)^{-1/2}(1-2\alpha)^{-1/2}\,, && (40)\\
R_{(3)(4)(2)(3)} &= -\frac{3M}{r}\left(u^3\right)^2u^4\left(u_4u^4-1\right)^{-1/2}(1-2\alpha)^{1/2}\,. && (41)
\end{aligned}$$

Using the known expressions for Ricci's coefficients of rotation, for all possible equatorial circular orbits, we obtain

$$\begin{aligned}
\gamma_{(2)(3)(4)} &= -r^{-1}(1-3\alpha)|u_\perp|u^4\,, && (42)\\
\gamma_{(2)(4)(4)} &= 0\,, \qquad \gamma_{(3)(4)(4)} = \text{const.} \neq 0\,. && (43)
\end{aligned}$$

By the last equation, according to the notation for $a_{(i)}$ under Eq. (17), we have

$$\dot a_{(2)} = 0\,, \qquad \dot a_{(3)} = 0\,. \tag{44}$$

Then it follows from (39)–(41) that for the circular motions, when $u^1 = 0$, the single nonzero component of the acceleration is $a_{(3)}$. Note that according to the choice of the orientation of the spatial local vectors in Eqs. (13)–(17), for the circular motion, the component $a_{(3)}$ corresponds to the radial direction. Taking into account (17), (41), and (35), we obtain the expression for the absolute value $|a_{(3)}|$ of the spinning particle acceleration in the orbit with $r = 3M$

$$|a_{(3)}| = 3^{-3/2} M^{-1} (1 + O(\varepsilon)) . \tag{45}$$

Let us compare (45) with the known classical acceleration in the Newtonian theory of gravitation

$$a_{\text{class}} = \frac{M}{r^2} . \tag{46}$$

It follows from (45) and (46) for $r = 3M$ that

$$|a_{(3)}| = \sqrt{3}\, a_{\text{class}} (1 + O(\varepsilon)) . \tag{47}$$

Note that a_{class} shows the value of the gravitational interaction in the static regime, while (45) reveals the property of this interaction in the highly relativistic region.

The detailed analysis of the circular motions in the small neighborhood of $r = 3M$, as well as for other values of r, is presented in [24]. It is shown that in the region of $2M < r < 3M$, there are the highly relativistic circular orbits with values $u_\perp$ which, similarly to (35), are proportional to $\varepsilon^{-1/2}$. These orbits are caused by the strong repulsive action of the spin–gravity coupling, as in the case of the above-considered orbit with $r = 3M$. It is remarkable that the absolute values of Riemann's tensor components (40) and (41) significantly depend on the velocity and become much greater for the highly relativistic velocity as compared to the low one. Just this dependence determines the physical reason for the great deviation of the spinning particle from the geodesic motion [24, 26].

Our results on the region of existence of highly relativistic circular orbits for the spinning particle in Schwarzschild's background complement those from [35–37]. In these papers, the situations are under consideration when the influence of spin on circular orbits is reduced to only small corrections and the velocity of the particle is not highly relativistic.

Note that in [38], the intensity of the electromagnetic radiation of a charged spinning particle on the highly relativistic circular orbits in Schwarzschild's background is estimated.

4. Equations for equatorial motions with constants of energy and angular momentum

As above, we use metric (18) and relations (19) with $S_2 > 0$. Due to symmetry of Schwarzschild's metric, Eqs. (1)–(3) have the constants of motion, the energy E, and the angular momentum J [39, 40]

$$
\begin{aligned}
E &= mu_4 + g_{44}u_\mu \frac{DS^{4\mu}}{ds} + \frac{1}{2}S^{\mu4}g_{44,\mu}\,, &(48)\\
J &= -mu_3 - g_{33}u_\mu \frac{DS^{3\mu}}{ds} - \frac{1}{2}S^{\mu3}g_{33,\mu}\,. &(49)
\end{aligned}
$$

In the following, it is appropriate to use in Eqs. (1)–(3) the dimensionless quantities z_i associated with the particle's coordinates and velocity

$$
z_1 = \frac{r}{M}\,, \qquad z_2 = \varphi\,, \qquad z_3 = u^1\,, \qquad z_4 = Mu^3\,. \tag{50}
$$

Then from (1)–(3), using constants of motion (48) and (49), we obtain the four first-order differential equations for the values z_i

$$
\begin{aligned}
\dot z_1 &= z_3\,, \qquad \dot z_2 = z_4\,, &(51)\\
\dot z_3 &= \frac{z_3^2}{z_1} + (z_1 - 3)\left(2z_4^2 + \frac{1}{z_1^2}\right) - \frac{a}{\varepsilon}z_1 z_4 &\\
&\quad + \frac{b}{\varepsilon z_1}\left[z_3^2 + \left(1 - \frac{2}{z_1}\right)(1 + z_1^2 z_4^2)\right]^{1/2}, &(52)\\
\dot z_4 &= -\frac{z_3 z_4}{z_1} + \frac{1 + z_1^2 z_4^2}{z_1 z_3}\left(z_4 - \frac{3z_4}{z_1} - \frac{a}{\varepsilon}\right) &\\
&\quad + \frac{1}{z_1 z_3 \varepsilon}(1 + bz_4)\left[z_3^2 + \left(1 - \frac{2}{z_1}\right)(1 + z_1^2 z_4^2)\right]^{1/2}, &(53)
\end{aligned}
$$

here a dot denotes the usual derivatives with respect to the dimensionless argument $x \equiv \tau/M$. The right-hand sides of Eqs. (52) and (53) contain the dimensionless parameters a and b which are proportional to the constant of motion E and J by the definition

$$
a \equiv \frac{E}{m}\,, \qquad b \equiv \frac{J}{mM}\,. \tag{54}
$$

The particle's spin is present in the value ε from the right-hand sides of Eqs. (52) and (53), where notation (27) is used.

For comparison, we write the geodesic equations for a spinless particle which is moving in the plane $\theta = \pi/2$ of Schwarzschild's metric in terms of the values z_i from (50)

$$
\begin{aligned}
\dot z_1 &= z_3\,, \qquad \dot z_2 = z_4\,,\\
\dot z_3 &= z_4^2(z_1 - 3) - \frac{1}{z_1^2}\,,\\
\dot z_4 &= -2\frac{z_3 z_4}{z_1}\,.
\end{aligned} \tag{55}
$$

At the fixed initial values of z_i, different values of the parameters a and b in Eqs. (52) and (53) correspond to the motions of different centers of mass of the spinning particle. It is important to know which numerical values of these parameters are suitable for describing the motion of the proper center of mass.

5. Choosing parameters a and b

Because a and b are the constants of motions, their values are the same for the all time of the particle's motion, including any small time interval after the start of motion, when the deviations of the functions z_i from their initial values $z_i(0)$ are very small. Then it is appropriate to introduce values

$$\xi_1 \equiv \frac{z_1 - z_1(0)}{z_1(0)}\,, \qquad \xi_2 \equiv \frac{z_3 - z_3(0)}{z_3(0)}\,, \qquad \xi_3 \equiv \frac{z_4 - z_4(0)}{z_4(0)}\,, \tag{56}$$

and to consider Eqs. (51)–(53) in the linear approximation in $|\xi_i| \ll 1$ (we emphasize that Eqs. (51)–(53) are exact in spin, and here the linear approximation applies only to small displacements of quantities z_i relative to their initial values). In this approximation, we obtain

$$\dot{\xi}_1 = (\xi_2 + 1)\frac{z_3(0)}{z_1(0)}\,, \tag{57}$$

$$\dot{\xi}_2 = \xi_1 \frac{z_1(0)}{z_3(0)} F_{1,Z_1} + \xi_2 F_{1,Z_3} + \xi_3 \frac{z_4(0)}{z_3(0)} F_{1,Z_4} + \frac{F_1(0)}{z_3(0)}\,, \tag{58}$$

$$\dot{\xi}_3 = \xi_1 \frac{z_1(0)}{z_4(0)} F_{2,Z_1} + \xi_2 F_{2,Z_3} \frac{z_3(0)}{z_4(0)} + \xi_3 F_{2,Z_4} + \frac{F_2(0)}{z_4(0)}\,, \tag{59}$$

where F_1 and F_2 are the right-hand sides of Eqs. (52) and (53), respectively, and F_{1,Z_1}, F_{1,Z_3}, F_{1,Z_4}, F_{2,Z_1}, F_{2,Z_3}, and F_{2,Z_4} are the corresponding partial derivatives with respect to z_1, z_3, z_4. The values of all these functions in (57) and (58) are taken for the initial time, *i.e.* for $x = 0$.

According to the theory of differential equations, the general solution of linear Eqs. (57)–(59) is determined by the combination of $\mathrm{e}^{\lambda_i x}$ $(i = 1, 2, 3)$, where λ_i are the solutions of the third-order algebraic equation

$$\lambda^3 + C_2\lambda^2 + C_1\lambda + C_0 = 0\,, \tag{60}$$

where

$$C_2 = -F_{1,Z_3} - F_{2,Z_4}\,, \tag{61}$$

$$C_1 = -F_{1,Z_1} - F_{1,Z_4} F_{2,Z_3}\,, \tag{62}$$

$$C_0 = F_{2,Z_4} F_{1,Z_1} - F_{1,Z_4} F_{2,Z_1}\,. \tag{63}$$

The motions of the proper center of mass do not have the property of oscillations with the correspondent frequency, in contrast to motions of the nonproper centers. The case without these oscillations is realized when one of the three roots of Eq. (60) is equal to 0 and the two other roots coincide. Then according to (60), the values C_i have to satisfy the two relations

$$C_0 = 0\,, \qquad C_2^2 - 4C_1 = 0\,. \tag{64}$$

From (64), we obtain the two algebraic equations for a and b

$$k_1 a^2 + k_2 b^2 + k_3 ab + k_4 a + k_5 b + k_6 \;=\; 0\,, \tag{65}$$
$$p_1 a^2 + p_2 b^2 + p_3 ab + p_4 a + p_5 b + p_6 \;=\; 0\,, \tag{66}$$

where there are expressions for k_i and p_i

$$\begin{aligned}
k_1 &= -\frac{z_1^2 z_4^2 + 1}{\varepsilon^2 z_3^2}\,, \qquad k_2 = \frac{1}{\varepsilon^2 z_1^2 z_3^2}\left(1 + z_3^2 - 3z_1^{-1} - z_1^2 z_4^2\right)\,,\\
k_3 &= \frac{N z_4}{\varepsilon^2 z_3^2}\left[z_1 z_4^2 + 3z_1^{-1} - 1 + \left(1 - 2z_1^{-1}\right)\left(1 + z_1^2 z_4^2\right)\right]\,,\\
k_4 &= \frac{z_4}{\varepsilon z_3^2}\left[4\left(1 - z_1^2 z_4^2\right)\left(1 - 3z_1^{-1}\right) - \left(6z_1^{-1} - z_3^2 - z_1^2 z_4^2 + 1\right)\right]\\
&\quad + \frac{N}{\varepsilon^2 z_3^2}\left(1 + z_3^2 - 3z_1^{-1} - z_1 z_4^2\right) - \frac{z_4}{\varepsilon}\\
&\quad + \frac{z_4}{\varepsilon z_1 z_3^2}(z_1 - 3)\left(1 + 3z_1^2 z_4^2\right) + \frac{N}{\varepsilon^2 z_3^2} z_1 z_4^2 (z_1 - 2)\\
&\quad + \frac{2z_4}{\varepsilon z_1 z_3^2}\left[z_1\left(2z_1^2 z_4^2 - z_3^2 - 1\right) + 6\right]\,,\\
k_5 &= \frac{N z_4^2}{\varepsilon z_3^2}\left(1 - 2z_1^{-1}\right)\left(6z_1^{-1} + z_3^2 + z_1^2 z_4^2 - 1\right)\\
&\quad - \frac{N}{\varepsilon}\left(1 + z_3^2 - 3z_1^{-1} - z_1 z_4^2\right)\\
&\quad \times \left[z_4^2 z_3^{-2}\left(1 - 3z_1^{-1}\right) + z_1^{-2} - z_1^{-2} z_3^{-2}\left(1 - 3z_1^{-1}\right)\right]\\
&\quad - \frac{N}{\varepsilon z_1^3 z_3^2}\left[z_3^2 + \left(1 - 2z_1^{-1}\right)\left(1 + 2z_1^2 z_4^2\right)\right]\\
&\quad \times \left[z_1\left(2z_1^2 z_4^2 - z_3^2 - 1\right) + 6\right]\,,\\
k_6 &= \frac{4z_4^2}{z_1^2 z_3^2}(z_1 - 3)\left[6 + z_1\left(z_3^2 + z_1^2 z_4^2 - 1\right)\right]\\
&\quad + \frac{4N z_4 (z_1 - 3)}{\varepsilon z_1 z_3^2}\left(-1 - z_3^2 + 3z_1^{-1} + z_1 z_4^2\right)
\end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{z_1^2 z_3}\left[z_1\left(2z_1^2 z_4^2 - z_3^2 - 1\right) + 6\right] \\
& \times\left[-z_1^{-1} z_3 + z_1^{-2} z_3^{-1}(z_1 - 3)\left(1 + 3z_1^2 z_4^2\right)\right] \\
& -\frac{N z_4}{\varepsilon z_1^2 z_3^2}(z_1 - 2)\left[6 + z_1\left(2z_1^2 z_4^2 - z_3^2 -\right)\right]\ ; \\
p_1 \ &= \ -\frac{4}{\varepsilon^2 z_3^2}\,, \quad p_2 = \frac{N^2}{\varepsilon^2 z_1^2 z_3^2}\left(1 - 2z_1^{-1}\right)^2\,, \\
p_3 \ &= \ -\frac{4N z_4}{\varepsilon^2 z_3^2}\left(1 - 2z_1^{-1}\right)\,, \\
p_4 \ &= \ \frac{8z_4^3 z_1}{\varepsilon z_3^2}(z_1 - 3) + \frac{12 z_4}{\varepsilon} + \frac{16 z_4}{\varepsilon z_1 z_3^2}(z_1 - 3)\,, \\
p_5 \ &= \ \left[\frac{2N z_3}{\varepsilon z_1} + \frac{2N}{\varepsilon z_1 z_3}\left(1 - 2z_1^{-1}\right)\left(1 + 2z_1^2 z_4^2\right)\right] \\
& \times\left[\frac{z_3}{z_1} + \frac{z_1 - 3}{z_1^2 z_3}\left(1 + 3z_1^2 z_4^2\right) + \frac{N z_4}{\varepsilon z_3}(z_1 - 2)\right] \\
& -\frac{4N z_3}{\varepsilon z_1}\left[-\frac{z_3}{z_1} + \frac{z_1 - 3}{z_1^2 z_3}\left(1 + 3z_1^2 z_4^2\right) + \frac{N z_4}{\varepsilon z_3}(z_1 - 2)\right] \\
& -\frac{8N}{\varepsilon z_1^2}\left[z_3^2 + \left(1 - 2z_1^{-1}\right)\left(1 + 2z_1^2 z_4^2\right)\right] \\
& -\frac{4N}{\varepsilon z_1^2}\left(1 + z_3^2 - \frac{3}{z_1} - z_1 z_4^2\right) \\
& -\frac{16N z_4^2}{\varepsilon z_1 z_3^2}(z_1 - 3)\left(1 + z_1^2 z_4^2\right)\left(1 - 2z_1^{-1}\right) \\
& -\frac{4N z_4^2}{\varepsilon z_3}(z_1 - 2)\left[\frac{z_3}{z_1} + \frac{z_1 - 3}{z_1^2 z_3}\left(1 + z_1^2 z_4^2\right)\right. \\
& \left. + \frac{N}{\varepsilon z_1 z_3 z_4}\left(1 + z_1^2 z_4^2\right)\left(1 - 2z_1^{-1}\right)\right], \\
p_6 \ &= \ -7\frac{z_4^4}{z_3^2}(z_1 - 3)^2 - 10\frac{N z_4^3}{\varepsilon z_3^2}(z_1 - 2)(z_1 - 3) \\
& + \frac{N^2 z_4^2}{\varepsilon^2 z_3^2}(z_1 - 2)^2 - 10\frac{z_4^2 (z_1 - 3)^2}{z_1^2 z_3^2} \\
& -\frac{10}{z_1^2} + \frac{42}{z_1^3} - 34\frac{z_4^2}{z_1}(z_1 - 3) - 6(z_1 - 2)\frac{N z_4}{\varepsilon z_1} \\
& -14\frac{N z_4}{\varepsilon z_1^2 z_3^2}(z_1 - 2)(z_1 - 3) + 5\frac{z_3^2}{z_1^2} + (z_1 - 3)^2 z_1^{-4} z_3^{-2} + 8z_4^2\,, \quad (67)
\end{aligned}
$$

where

$$N = \left[z_3^2 + \left(1 - 2z_1^{-1}\right)\left(1 + z_1^2 z_4^2\right)\right]^{-1/2} .$$

Note that the values of z_1, z_3, and z_4 in (67) are taken for $x = 0$.

6. An example of the noncircular motion

Let us consider a case of the spinning particle motion when the initial values of the particle velocity components are close to the corresponding values for the circular motion in the region of $r > 3M$. Then Eq. (22) has the three real roots for $u_\perp$. The two of them due to (34) are close to (30), *i.e.* in the corresponding orbits, the influence of the spin–gravity coupling is small, whereas another situation takes place with the third root of equation (22). As in some typical case, we put in Eq. (22) $\varepsilon = 10^{-4}$ and $\alpha = 0.05$ (the last value corresponds to the circular orbit with $r = 20M$ or in notation (50) to $z_1 = 20$). Then after the calculation of the pointed out third root of Eq. (22), we obtain

$$z_4 = 23.6230243983586 . \tag{68}$$

[The positive sign of the value z_4 in (68) is associated with our choosing $S_2 > 0$.] Note that according to notation (50), there is a relation $z_1 z_4 = u_\perp$. Then by (68) for $r = 20M$, we have $u_\perp^2 \gg 1$, *i.e.* the velocity on the circular orbit is highly relativistic.

From (48) and (49) for the numerical values $\varepsilon = 10^{-4}$, $z_1 = 20$ and (68), we find the values

$$a = 0.002383148065 , \tag{69}$$

$$b = -0.03442184393 \tag{70}$$

for the spinning particle motion in the corresponding circular orbit.

Now we consider noncircular motions of the spinning particle. For this purpose, we use the corresponding solutions of Eqs. (51)–(53). As an example, below we consider the case when the particle with $\varepsilon = 10^{-4}$ starts from the position $z_1(0) = 20$, $z_2(0) = 0$, $z_3(0) = 0.02$, and $z_4(0)$ that is equal to (68). It means that these initial values of z_i are different from the corresponding values in the above-considered circular orbit only by the small nonzero value of $z_3(0)$, *i.e.* of the radial component of the particle velocity. We find from the solutions of algebraic Eqs. (65) and (66) the appropriate values of the parameters a and b in Eqs. (52) and (53)

$$a = 0.002383148074 , \tag{71}$$

$$b = -0.03442184377 . \tag{72}$$

Values (71) and (72) are very close to the corresponding values from (69) and (70). Figures 1–6 show the dependence of the radial coordinate, the angle φ, and the radial and tangential velocities, respectively, on the particle proper time in comparison with the corresponding solutions of geodesic Eqs. (55). In addition, Fig. 5 shows the form of the trajectories of the spinning and spinless particles at the same initial conditions and for the same time interval. According to the graphs in Figs. 1–5, the differences of the corresponding values for spinning and spinless particles become significant. In this example, the effect of the spin–gravity coupling is attractive: in Fig. 5, the spinless particle goes far away from the Schwarzschild source, whereas the distance of the spinning particle from this source is constant in this time interval.

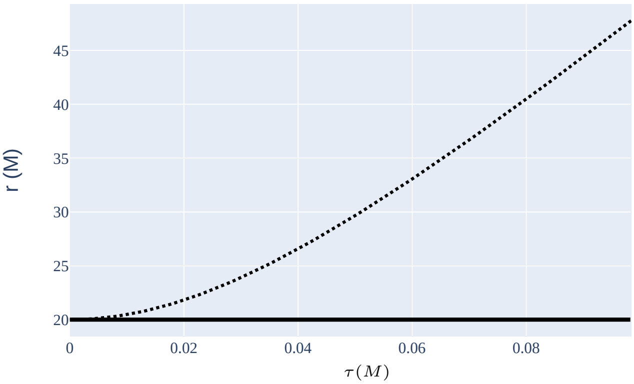

Fig. 1. Radial coordinate *vs.* proper time for the spinning (solid line) and the spinless particle (dotted line) with the same initial values of the coordinates and velocity.

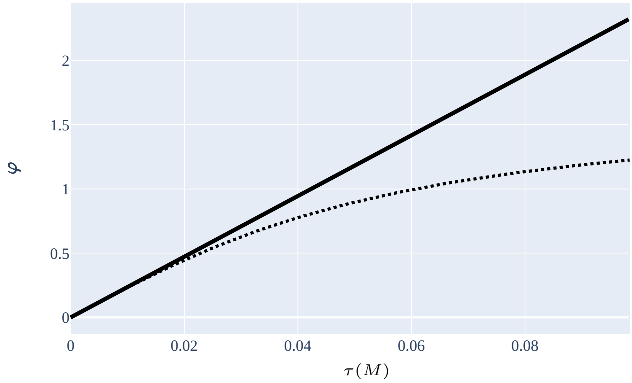

Fig. 2. Angle φ *vs.* proper time for the spinning (solid line) and the spinless particle (dotted line).

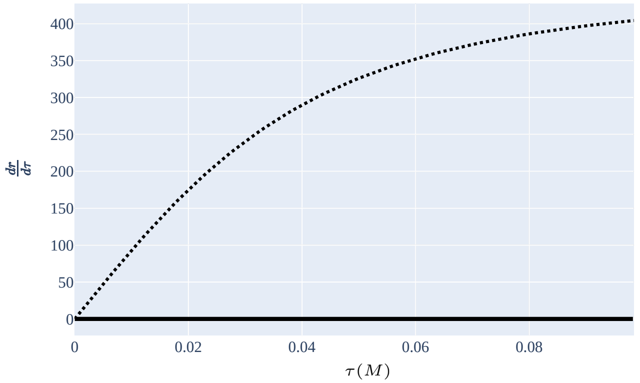

Fig. 3. Radial velocity *vs.* proper time for the spinning (solid line) and the spinless particle (dotted line).

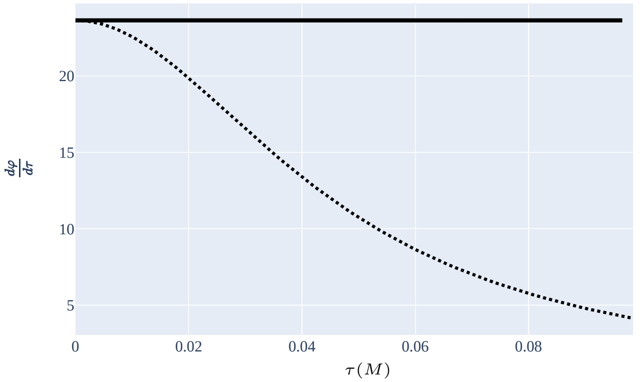

Fig. 4. Angular velocity *vs.* proper time for the spinning (solid line) and the spinless particle (dotted line).

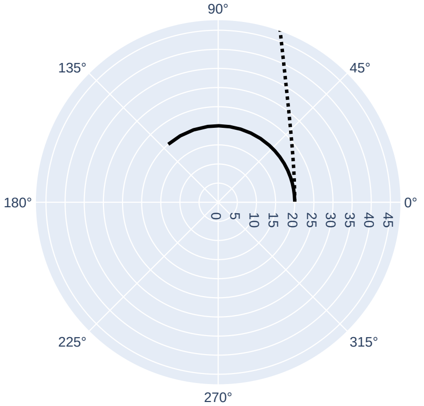

Fig. 5. Trajectories in the polar coordinates of the spinning (solid line) and the spinless particle (dotted line).

7. Acceleration of a spinning particle in Schwarzschild's background relative to free falling spinless particle

Let us consider Eqs. (15)–(17) in the case of the spinning particle motion in the plane $\theta = \varphi/2$ of Schwarzschild's metric (18). In these equations, we use (39)–(41) and the expression

$$\gamma_{(2)(3)(4)}) = \frac{(E - mu_4)u^4}{(u_4u^4 - 1)|S|}\,, \tag{73}$$

where, as above, E and m are the energy and mass of the spinning particle, respectively; $|S|$ is the absolute value of the particle's spin.

According to (15) and (39), we have $a_{(1)} = 0$. The values $a_{(2)}$ and $a_{(3)}$ can be found as solutions of the two differential Eqs. (16) and (17). Similarly as in Section 3, where dimensionless quantities (50) are used, here we rewrite Eqs. (16) and (17) in terms of z_i. Then it follows from these equations that

$$\dot z_5 = f_1 z_6 + f_2\,, \tag{74}$$

$$\dot z_6 = -f_1 z_5 + f_3\,, \tag{75}$$

where

$$f_1 = \varepsilon^{-1}L^{-1}\left[a\left(1-2z_1^{-1}\right)^{-1/2}(1+L)^{1/2} - 1\right], \tag{76}$$

$$f_2 = 3z_4^2 z_1^{-1}\left(1+L^{-1}\right)^{1/2}, \tag{77}$$

$$f_3 = -3z_3z_4z_1^{-1}\left[\left(1-2z_1^{-1}\right)z_1^2z_4^2 + z_3^2\right]^{-1/2}, \tag{78}$$

$$L \equiv \left(1-2z_1^{-1}\right)z_3^2 + z_1^2z_4^2\,. \tag{79}$$

The values z_5 and z_6 in (74) and (75) are equal to $a_{(2)}$ and $a_{(3)}$ in the dimensionless notation, respectively. The parameter a on the right-hand side of (76) is determined in (54).

Integration of Eqs. (74) and (75) together with (51)–(53) shows that on the time interval from 0 to 0.1, the value of $a_{(2)}$ is constant and close to 0 (this interval is close to that on which the graphs in Figs. 1–6 are shown). While $a_{(3)}$ on this interval is also practically unchanged, but has a large numerical value of the order of -10^4 (the sign "$-$" means that the acceleration is directed towards the Schwarzschild source).

The numerical values for $a_{(2)}$ and $a_{(3)}$ correspond to the solutions of exact Eqs. (16) and (17). These values differ from those that take place in the linear spin approximation. In particular, for the circular orbits by (13), (27), (40), and (41), we have $a_{(2)} = 0$ (in addition to $a_{(1)} = 0$) and

$$a_{(3)} = \frac{M}{r^2}\frac{3S}{mr}|u_\perp|u^4\left(1-2Mr^{-1}\right)^{-1/2}. \tag{80}$$

It follows from (80) with (27) that for low velocities of a spinning particle, when $|u_\perp| \ll 1$ and u^4 is close to 1,

$$|a_{(3)}| \ll a_{\rm class}\,, \tag{81}$$

where $a_{\rm class} = M/r^2$ is the known classical acceleration in the Newtonian theory of gravitation. In the case of the highly relativity motions, when $|u_\perp| \gg 1$ and for the circular orbits $u^4 \approx |u_\perp|(1 - 2Mr^{-1})^{-1/2}$, according to (80), we have

$$|a_{(3)}| = \frac{3S}{mr} u_\perp^2 a_{\rm class}\,. \tag{82}$$

That is, in contrast to (81), in (82), the small value $|S|/mr$ is multiplied by the large multiplier $|u_\perp|$. Let us use Eq. (82) for the above-considered case of the circular orbit with $z_1 = 20$ and $\varepsilon = 10^{-4}$. In notation (50), expression (82) is

$$|a_{(3)}| = 3\varepsilon z_1 z_4^2\, a_{\rm class}\,, \tag{83}$$

where the numerical value of z_4 is written in (68). Then by (83), we obtain

$$|a_{(3)}| \approx 3.35 a_{\rm class}\,. \tag{84}$$

That is, in (84) $|a_{(3)}| > a_{\rm class}$, in contrast to (81). Let us compare expressions (80) and (84) with the corresponding expression for $a_{(3)}$ which follows from the exact Eq. (17) for the circular orbits. According to (17), we write

$$a_{(3)} = -\frac{s_{(1)}}{m} R_{(3)(4)(2)(3)} \left(1 - \frac{s_{(1)}}{m}\gamma_{(2)(3)(4)}\right)^{-1}\,. \tag{85}$$

The contribution of the nonlinear spin terms is determined by the value

$$\frac{s_{(1)}}{m}\gamma_{(2)(3)(4)} \tag{86}$$

on the right-hand side of Eq. (85). Using (86) and (42), we write

$$\left|\frac{s_{(1)}}{m}\gamma_{(2)(3)(4)}\right| = \frac{S}{mr}\left(1 - 3Mr^{-1}\right)|u_\perp| u^4. \tag{87}$$

According to (87) and (27), the contribution of the nonlinear spin terms in the expression for the acceleration component (85) is very small for the low velocities, when $|u_\perp| \ll 1$, and become much greater for the high velocities, when $|u_\perp| \gg 1$. This property is similar to those described for the linear spin approximation. In our case of the circular orbit with $z_1 = 20$, $\varepsilon = 10^{-4}$, and z_4 from (68), it follows from (85) that

$$|a_{(3)}| \gg a_{\rm class}\,, \tag{88}$$

in contrast to (84). Note that the MP equations under condition (3) in Schwarzschild's field for $r = 20M$ (as well as for other r in the region $r > 3M$) have the two circular solutions with different physical meaning. The first of them describes the above-considered highly relativistic circular orbit when condition (34) for a spinning test particle is taken into account. The second is the direct generalization of the known solution of Wyessenhoff's type [31] in the Minkowski spacetime to the case of Schwarzschild's spacetime. According to the known interpretation of the physical meaning of the last solution, it describes the motion of some nonproper center of mass inside the rotation body with the corresponding dimension greater than r, and then the test condition is not satisfied.

The oscillatory noncircular solutions can be obtained from Eqs. (51)–(53) and (74), (75) when the parameters a and b do not satisfy Eqs. (65) and (66). An example is presented in Fig. 6 where the graph is shown for the absolute value of the acceleration

$$a(M) = \sqrt{a^2_{(2)}(M) + a^2_{(3)}(M)}\,. \tag{89}$$

This graph corresponds to the initial values of z_i which are used in Sections 6 and 7, with the numerical values of a that are equal to a from (71) multiplied by 1.0001, and b from (72). The letter "k" near the vertical axis of Fig. 6 means multiplier 1000.

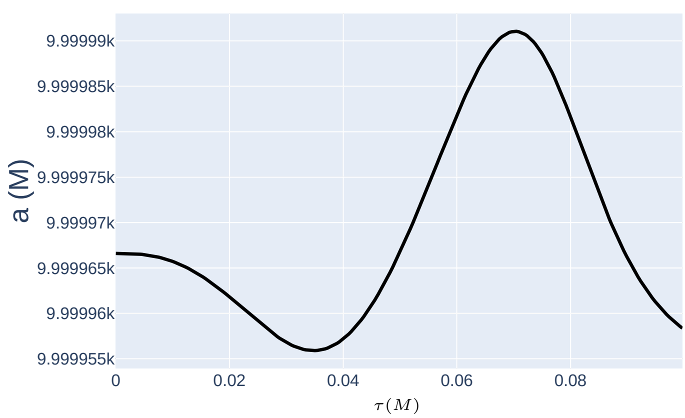

Fig. 6. Absolute value of acceleration *vs.* proper time for an oscillatory solution.

8. A case of the highly relativistic spinning particle motions close to the radial one

In Section 6, we considered the highly relativistic motions of a spinning particle with its initial tangential and radial velocities which satisfy the relations $u_\perp^2 \gg 1$ and $u_\parallel^2 \ll 1$ (in notation (50) $u_\perp \equiv z_1 z_4$, $u_\parallel \equiv z_3$). Let us consider an example of the highly relativistic motions with the initial value of $u_\parallel^2 \gg 1$, for different initial values of $u_\perp$. First of all, we recall that when a spinning particle starts with zero tangential velocity in Schwarzschild's background then its motion is exactly radial geodesic. That is, for the radial motion, there is not any influence of the spin–gravity coupling. When the initial value of the tangential velocity is nonzero, then the spinning particle motion deviates from the geodesic. To estimate the deviation depending on the tangential velocity, we compare corresponding solutions of Eqs. (51)–(53) and geodesic Eqs. (55). For the cases when the deviation from the geodesic is not very large, to describe the non-oscillatory motions of Eqs. (51)–(53), it is possible to use the values of the parameters a and b which follow directly from (48) and (49) (with notation (50)) in the corresponding approximation. Namely, we take into account the expressions for a and b which contain linear and quadratic spin corrections to the corresponding geodesic terms

$$a = R - z_1^{-1} z_4 \varepsilon - 3 z_1^{-1} z_4^2 \varepsilon^2 \,, \tag{90}$$

$$b = z_1^2 z_4 - R\varepsilon - 3 z_1^{-1} z_4 \left(1 + z_1^2 z_4^2\right) \varepsilon^2 \,, \tag{91}$$

where

$$R = \left[z_3^2 + \left(1 - 2z_1^{-1}\right)\left(1 + z_1^2 z_4^2\right)\right]^{1/2} \,. \tag{92}$$

For illustration, let us consider the motions when a particle with $\varepsilon = 2\times 10^{-2}$ starts from the position $z_1(0) = 20$, $z_2(0) = 0$, and $z_3(0) = -10^3$. Note that for $z_4(0) = 0$, *i.e.* for radial motion, all graphs $z_i(\tau)$ coincide with the corresponding geodesic graphs. When $z_4(0)$ is different from zero but relatively small, for example, equal to 1 or 2, the deviation of the graphs for the spinning particle from the geodesic graphs is insignificant and imperceptible on all graphs. As the value $z_4(0)$ increases, the deviation increases and becomes clearly visible. This is illustrated by the example for $z_4(0) = 30$ in Figs. 7–10.

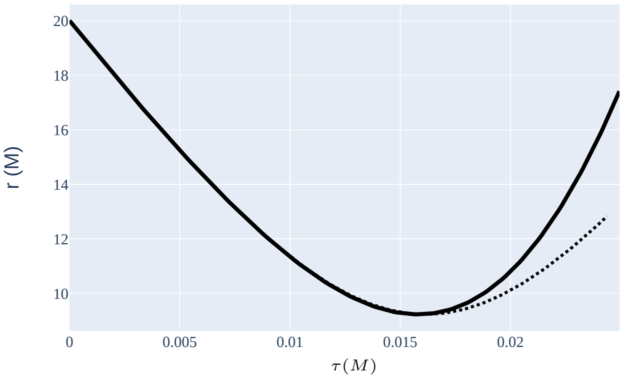

Fig. 7. Radial coordinate *vs.* proper time for the spinning (solid line) and the spinless particle (dotted line) with the same initial values of the coordinates and velocity.

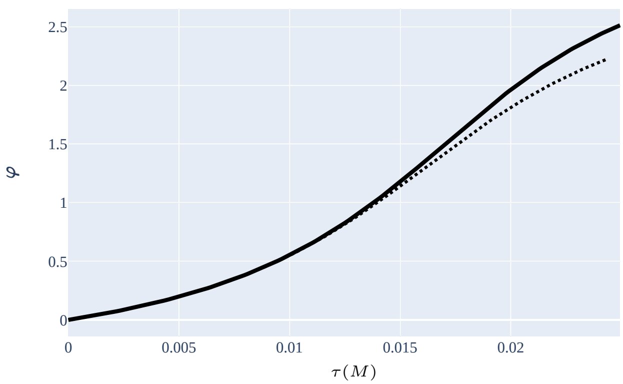

Fig. 8. Angle φ *vs.* proper time for the spinning (solid line) and the spinless particle (dotted line).

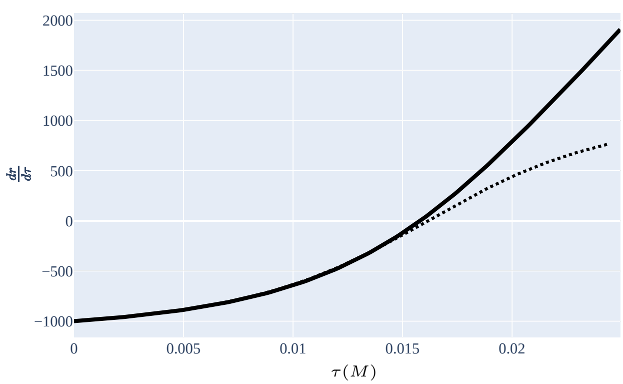

Fig. 9. Radial velocity *vs.* proper time for the spinning (solid line) and the spinless particle (dotted line).

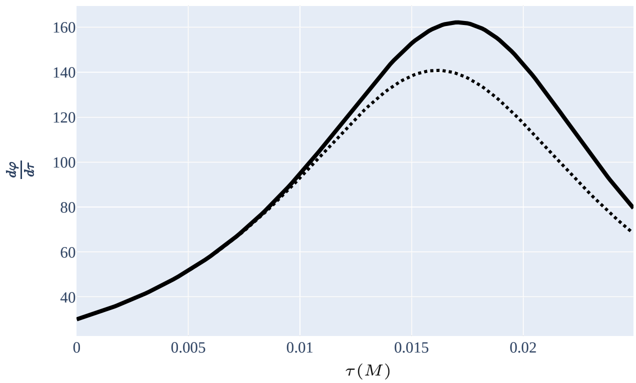

Fig. 10. Angular velocity *vs.* proper time for the spinning (solid line) and the spinless particle (dotted line).

According to Fig. 7, the spinning particle first approaches Schwarzschild's source to a value of r approximately equal to $9M$, and then moves away from it. In the process of motion, the components of its radial and tangential velocity are noticeably different from the corresponding components of the velocity of geodesic motion, which are shown in Figs. 9 and 10. That is, both Figs. 1–6 and 7–10 clearly show the determining influence of the high tangential velocity on the deviation of the motion of the spinning particle from the geodesic one.

9. Conclusions

The Mathisson–Papapetrou Eqs. (1)–(3) give possibilities to study the influence of the spin–gravity coupling on the spinning particle that moves in the gravitational field with any high velocity, up to its values which are very close to the speed of light. There is a significant difference in the scale of this influence for particles that are highly or weak relativistic by their velocities relative to the source of the gravitational field. In this paper, we used both Eqs. (1)–(3) and their consequences in terms of the local tetrads values (15)–(17) which describe the frame of reference accompanying the particle. The solutions of these equations in the case of equatorial motions of a particle in Schwarzschild's background, when its spin is orthogonal to the motion plane are considered. An important point of the study was the selection of solutions that describe the motions of the particle's proper center of mass.

Figures 1–5 show some typical cases of the world lines and trajectories of the spinning particle when its initial absolute value of the radial 4-velocity is much less than the corresponding value of the tangential velocity. The latter is equal to the value in the circular orbit with $r = 20M$. It is significant that the absolute value of the acceleration, associated with a certain force characteristic of the spin–gravity coupling, is high. This force acts on the particle as an additional attraction.

Regarding the planned experiments to record spin–gravity effects, it is important to consider that these effects are very small if the velocity of the spinning particle is small compared to the speed of light, whereas for the highly relativistic particles, the effects can be much stronger and manifest in the corresponding observations. In this context, it is appropriate to carry out an analysis of the propagation of the extremely energetic cosmic ray, which is discussed in recent article [42].

In further study of the highly relativistic spin–gravity coupling, it is appropriate to analyze solutions of Eqs. (1)–(3) under the Schwarzschild horizon surface.

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