EXPLICIT SOLUTION OF THE MEAN-VARIANCE OPTIMAL INVESTMENT MODEL FOR DEFINED-CONTRIBUTION PENSION UNDER NON-EXTENSIVE STATISTICAL MECHANICS

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With the increasingly serious population aging, people pay more attention to the operation and management of pension. The optimization of pension investment has attracted the research interest of many scholars. Firstly, an asset price model is established by using the non-extensive statistical theory, which can well describe the high-peak and fat-tail characteristics of asset returns. Then, under the mean-variance criterion, the optimal investment model of a defined-contribution pension is constructed. Moreover, the explicit solution to the optimal investment strategy of defined-contribution pension is obtained by using dynamic programming, Legendre transformation, and duality theory. This conclusion not only broadens the application of non-extensive statistics in the financial field, but also provides a new theory for the investment of pension funds.

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1. Introduction

The United Nations report shows that the global population over the age of 65 has exceeded 10% of the total population, and the number will reach 16% by 2050. Population aging has become a major factor restricting economic development. In order to cope with population aging and promote social and economic development, many countries have established pension systems. According to the different payment settings during the

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accumulation period and distribution stage of pension, it is generally divided into two types: one is a defined-benefit pension, and the other is a defined-contribution pension. The payment amount of the defined benefit pension in the payment stage is determined in advance by the pension manager at the beginning of the pension plan, and the participants do not need to bear the investment risk of the pension. However, the participants of defined-contribution pension plan often need to bear many market risks, such as inflation, interest rate change, and asset price fluctuation. Therefore, the research on investment management and risk control of defined contribution pension has important theoretical and practical significance, and it has attracted the attention of many scholars. Xiao [1] studied the optimal investment problem of defined-contribution pension by using Legendre transformation and duality theory, and obtained the explicit solution of the optimal investment strategy in two different periods before and after retirement. Gao [2] obtained the optimal investment strategy of the power utility function and exponential utility function under the assumption that the price of risky assets obeys the CEV model and the optimization goal is to maximize the expected utility of pension terminal wealth. Han [3] obtained the explicit solution of the optimal investment strategy under the CRRA-type utility function by using the stochastic dynamic programming method and the index bond hedging inflation risk. Under the mean-variance criterion, Yao [4] studied the asset allocation of defined-contribution pension, taking into account factors such as multi-period, random income, and death risk. Using stochastic control, the Lagrange duality theory and variable transformation method, the analytical expressions of optimal investment strategy and effective boundary were obtained. Liang [5] studied the optimal investment strategy of defined-contribution pension by using the stochastic programming method and considering the risk of death and random wage fluctuation, and further analyzed the influence of mortality and wage fluctuation on the optimal investment strategy through numerical simulation. Wu [6], considering inflation and wage risks, studied the optimal investment strategy of defined-contribution pension under the mean-variance criterion, gave the Hamilton–Jacobi–Bellman equation, and obtained the closed form time consistent investment strategy and the explicit solution of the equilibrium effective boundary by using the stochastic control technology. Based on the loss aversion and VaR constraints, Dong [7] gave the optimal investment strategy of the defined-contribution pension. Baltas [8] studied the optimal management of defined-contribution pension in the distribution stage under the influence of inflation and model uncertainty, and clarified the impact of robustness and inflation on the optimal investment decision. Wang [9] studied the optimal asset allocation of defined-contribution pension. Under the condition that the risk asset follows the O–U process, and considering the influence of random income and inflation risk, the Hamilton–Jacobi–Bellman equation is derived by using the stochastic control method. Tiro [10] derived the Hamilton–Jacobi–Bellman equation by using dynamic programming under the random interest rate and inflation risk, and further obtained the optimal investment strategy of the defined-contribution pension. Using state correlation function to describe investors' risk tolerance attitude and stochastic control method, Wang and Chen [11] studied the equilibrium behavior strategy of defined-contribution pension during the accumulation stage. The above research results can provide important theoretical assistance for investment decision-making of pension plans.

However, most of the above studies were conducted under the assumption that stock prices follow the geometric Brownian motion, which means that the distribution of asset returns follows a normal distribution. In fact, a large number of empirical studies have shown that the distribution of asset returns has the characteristics of high peak and fat tail, and the prices of stocks have different long-term correlations. For example, the empirical analysis results of Lo [12], Beben [13], Conrad [14, 15], McLean [16], Mantegna [17], Mandelbrot [18], and Gopikrishnan [19, 20] indicate that stock returns have obvious long-term correlation and fat-tail characteristics, in many countries' financial markets. In order to more accurately approach the actual stock price changes, many scholars have replaced the geometric Brownian motion. Xiao [21] and Gu [22] used the fractional Brownian motion with self-similarity and long-term correlation to describe the price changes of stocks. Kim [23] and Geman [24] used Lévy processes with peak and thick-tail characteristics to depict the stock prices. Duffie [25] and Duan [26] employed processes with jump diffusion. In particular, in 1988, Tsallis [27] proposed the non-extensive statistical theory. The theory can regard the financial market as a complex system, define the asset price process as an abnormal diffusion process, and then obtain a concise distribution function form, which can describe the complex system with non-linear, long-range interaction and long-term memory effect. Soon, the Tsallis statistical method was widely used in the financial field. For example, Tsallis [28], Rak [29], Kozaki [30], Queirós [31], and Biró [32] used the Tsallis statistical theory to study the fluctuation of stock price. Using the non-extensive statistical theory, respectively, Ryuji [33] studied the variation of foreign exchange rate, Borland [34] studied the pricing problem of European options, and Katz [35] studied the problem of asset default risk assessment. In addition, Liu [36] employed the non-extensive statistical mechanics to model stock prices, studied the allocation of funds in stocks, and provided the optimal investment strategy within the framework of maximizing the investors' expected utility. Trindade [37] applied the Tsallis statistics in the Cover's portfolio theory and formulated the non-extensive version of the Cover's portfolio using the q-deformed functions and the q-product as key elements. As is well known, it is difficult to obtain an explicit solution for the optimal investment strategy under the condition of non-Gaussian distribution, especially in the case of pension investment because it includes two different time stages before and after retirement, representing the accumulation and consumption of wealth, respectively.

In this study, to be closer to the actual financial market, the non-extensive statistical theory is employed to describe the variation of stock prices. On the other hand, to better depict the trade-off between risk and return, and avoid inconsistency between the utility function and the risk preference, the mean-variance criterion method is used to describe the investment decision-making of pension investors. Furthermore, the explicit solution of the optimal investment strategy in different periods before and after retirement is obtained. This conclusion not only expands the application of non-extensive statistical theory in the financial field, but also provides a new means for pension investment decision-making.

This article is organized as follows. In Section 2, the stock price model based on the Tsallis non-extensive statistical theory is given, which can describe the high-peak and fat-tail characteristics of asset returns, and accurately approach the actual market. In Section 3, according to the financial market model established in Section 2, the wealth equation of defined-contribution pension in two different periods before and after retirement is derived. In Section 4, under the mean-variance criterion, the optimal investment model of pension funds is constructed. That is to say, under the condition that the wealth expectation of pension investors at the end time is given in advance, the variance of the terminal moment wealth is minimized. In Section 5, by using stochastic control, the Legendre transformation and duality theory, the explicit expression of the optimal investment strategy for the defined-contribution pension plan is solved. In Section 6, the study is summarized and future research directions are proposed.

2. Market model

Suppose there are only two kinds of assets in the financial market, one is risk-free assets called bonds, and the other is risky assets called stocks. The price process $B(t)$ of the risk-free bond satisfies

$$\begin{cases} \mathrm{d}B(t) = rB(t)\mathrm{d}t\,, \\ B(0) = B_0\,, \end{cases} \tag{1}$$

where r is a positive risk-free rate. The price process $S(t)$ of the risky stock satisfies

$$\begin{cases} \mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}\Omega(t)\,, \\ S(0) = S_0\,, \end{cases} \tag{2}$$

where μ is an expected return rate of the risky asset, and satisfies $\mu > r$. σ is the volatility of the risky asset. Moreover, the random variable $\Omega(t)$ satisfies

$$\mathrm{d}\Omega(t) = P(\Omega, t)^{\frac{1-q}{2}}\mathrm{d}W(t)\,. \tag{3}$$

The process $\{W(t)\}_{t\geq 0}$ is a standard Brownian motion defined on the probability space $(\mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$. $P_q(\Omega)$ is a Tsallis distribution of index q as follows (see [34]):

$$P(\Omega, t) = \frac{1}{z(t)}\left(1 - \beta(t)(1-q)\Omega^2\right)^{\frac{1}{1-q}}\,, \tag{4}$$

where

$$z(t) = ((2-q)(3-q)ct)^{\frac{1}{3-q}}\,, \tag{5}$$

$$\beta(t) = c^{\frac{1-q}{3-q}}((2-q)(3-q)t)^{\frac{2}{q-3}}\,, \tag{6}$$

$$c = \frac{\pi}{q-1}\frac{\Gamma^2\left(\frac{1}{q-1} - \frac{1}{2}\right)}{\Gamma^2\left(\frac{1}{q-1}\right)}\,.$$

In the $q \to 1$ limit, Eq. (4) recovers a Gaussian distribution. When $q > 1$, the distribution exhibits a fat tail relative to a normal distribution. Hence, this model generalizes the geometric Brownian motion and can accurately fit the variation of asset prices.

3. Wealth model of pensions

In this paper, the optimal investment problem of defined-contribution pension is studied. It is consistent with the research of [1, 2, 4] and [9]. The optimal investment of pension is divided into two stages, namely, before retirement and after retirement. It is assumed that the payment method of pension after retirement is annuity, and the amount is predetermined by the pension fund manager. In addition, during the annuity payment period after retirement, this study does not consider the death of the policyholder. Therefore, the wealth process of pension can be divided into two stages. T is defined as the retirement time, and N is the payment cycle of the annuity after retirement.

3.1. Wealth process of pension before retirement

According to the financial market mentioned above in the article, before retirement, pension is allowed to invest in a risky asset stock and the other risk-free asset bond. Let $V(t)$ be the wealth value of the pension at time t, $\pi(t)$ and $1-\pi(t)$ the proportion of the investor's wealth invested in the risky asset stock and the risk-free asset bond, respectively. Without losing generality, it is assumed that the pension contribution rate is a positive constant c, and the salary is unit 1. Then the wealth process of pre-retirement pension satisfies the following stochastic differential equation:

$$\begin{cases} \mathrm{d}V(t) = \pi(t)V(t)\frac{\mathrm{d}S(t)}{S(t)} + (1-\pi(t))V(t)\frac{\mathrm{d}B(t)}{B(t)} + c\mathrm{d}t\,, \\ V(0) = V_0\,, \end{cases} \tag{7}$$

where V_0 is the initial wealth of the pension. By substituting (1) and (2) into (7), the wealth process of the pre-retirement pension is

$$\begin{cases} \mathrm{d}V(t) = [\pi(t)V(t)\mu + (1-\pi(t))V(t)r + c]\,\mathrm{d}t + \pi(t)V(t)\sigma P(\Omega,t)^{\frac{1-q}{2}}\mathrm{d}W(t)\,, \\ V(0) = V_0\,. \end{cases} \tag{8}$$

3.2. Wealth process of pension after retirement

Suppose that the pension accumulated by T at the time of retirement is all used to purchase annuities. Let D be the fund paid when purchasing the N-term annuity. It obviously satisfies $D \le V(T)$. Define $\bar{D}$ as the payment amount at time t after retirement, then $\bar{D} = D/\bar{a}_{\bar{N}|}$, where $\bar{a}_{\bar{N}|} = (1-\mathrm{e}^{-\delta N}/\delta)$ and δ is a continuous technical rate.

It is assumed that after retirement, the pension must be used to pay a certain annuity, and it is allowed to invest in a risk-free asset bond and other risky asset stock. As described, before retirement, $V(t)$ is defined as the wealth value of the pension at time t. Let $\pi(t), 1-\pi(t)$ be the proportion of the investor's wealth invested in the risky asset stock and the risk-free asset bond, respectively. Then the wealth process of pension after retirement satisfies the following stochastic differential equation:

$$\begin{cases} \mathrm{d}V(t) = \left[\pi(t)V(t)\mu + (1-\pi(t))V(t)r - \bar{D}\right]\mathrm{d}t + \pi(t)V(t)\sigma P(\Omega,t)^{\frac{1-q}{2}}\mathrm{d}W(t)\,, \\ V(0) = V_0\,. \end{cases} \tag{9}$$

4. Mean-variance optimal investment problem

In this section, the optimal investment of defined-contribution pension under the mean-variance criterion is considered. That is to say, minimize the risk (variance) of investors' terminal wealth under the condition of maximizing the return (expectation) of investors' terminal wealth. Therefore, we can consider minimizing the variance of pension investors' terminal wealth under the condition that the expected value of pension investors' terminal wealth is given in advance.

4.1. Optimal investment model of pension before retirement

Definition 1. If the investment strategy $\pi(t)$ is the solution of the stochastic differential equation (8), then the investment strategy $\pi(t)$ is said to be feasible. Note that the set of all feasible solutions is $L^2_{\mathrm{F}}(0, T; R)$, then $\pi(t) \in L^2_{\mathrm{F}}(0, T; R)$.

Therefore, under the mean-variance criterion, we can write the optimal investment problem of pre-retirement pension as

$$\begin{cases} \min & \mathrm{Var} V(T) = E[V(T) - C]^2\,, \\ \text{s.t.} & E[V(T)] = C\,, \\ & \pi(t) \in L^2_{\mathrm{F}}(0, T; R)\,, \end{cases} \tag{10}$$

where the expected return C is a positive constant.

According to the Sharpe theory in the financial field, the return of assets is in direct proportion to the risks they face. It can be obtained that investors' income from investing pensions in high-risk stocks is higher than that from investing all their funds in risk-free bonds. Moreover, when $\pi(t) = 0$, the corresponding wealth process equation (8) becomes

$$\begin{cases} \mathrm{d}V(t) = [V(t)r + c]\,\mathrm{d}t\,, \\ V(0) = V_0\,. \end{cases} \tag{11}$$

The solution is $V(T) = \frac{(c+rV_0)\,\mathrm{e}^{rT} - c}{r}$. Thus, naturally, $E[V(T)] = C \geq ((c + rV_0)\,\mathrm{e}^{rT} - c)/r$.

4.2. Optimal investment model of pension after retirement

According to the above analysis on the optimal investment of pre-retirement pension, similarly, we can easily obtain the mathematical description of the optimal investment of pension after retirement under the mean-variance criterion as

$$\begin{cases} \min & \mathrm{Var} V(T) = E\left[V(T+N) - \widetilde{C}\right]^2\,, \\ \text{s.t.} & E[V(T)] = \widetilde{C}\,, \\ & \pi(t) \in L^2_{\mathrm{F}}(T, T+N; R)\,, \end{cases} \tag{12}$$

where the expected return $\widetilde{C}$ is a positive constant.

5. Model solution

To obtain the optimal investment strategy of pre-retirement pension under the mean-variance criterion, that is, the solution of Eq. (10), we introduce a Lagrange multiplier $2\lambda \in R$ (the coefficient 2 is introduced to simplify the calculation). Then, Eq. (10) becomes

$$\begin{cases} \min & J(\pi(t), \lambda) = E\left[(V(T) - C)^2 + 2\lambda(V(T) - C)\right] \,, \\ \text{s.t.} & E[V(T)] = C \,, \\ & \pi(t) \in L^2_{\mathrm{F}}(0, T; R) \,. \end{cases} \tag{13}$$

After calculation, it can be written as

$$\begin{cases} \min & J(\pi(t), \lambda) = E[V(T) - (C - \lambda)]^2 - \lambda^2 \,, \\ \text{s.t.} & E[V(T)] = C \,, \\ & \pi(t) \in L^2_{\mathrm{F}}(0, T; R) \,. \end{cases} \tag{14}$$

Let $\eta = C - \lambda$. Substituting it into Eq. (14), we have

$$\begin{cases} \min & J(\pi(t), \eta) = E[V(T) - \eta]^2 - (C - \eta)^2 \,, \\ \text{s.t.} & E[V(T)] = C \,, \\ & \pi(t) \in L^2_{\mathrm{F}}(0, T; R) \,. \end{cases} \tag{15}$$

According to the Lagrange duality theorem (see [13]), there is an equivalent relationship between model (10) and model (15) as follows:

$$\min \mathrm{Var} V(T) = \max\min J(\pi(t), \lambda) = \max\min J(\pi(t), \eta) \,. \tag{16}$$

Obviously, when $\eta = C - \lambda$ is a constant, Eq. (15) is equivalent to the following equation:

$$\begin{cases} \min & J(\pi(t), \eta) = E[V(T) - \eta]^2 \,, \\ \text{s.t.} & E[V(T)] = C \,, \\ & \pi(t) \in L^2_{\mathrm{F}}(0, T; R) \,. \end{cases} \tag{17}$$

Similarly, under the mean-variance criterion, the optimal investment problem of pension after retirement, that is, model (12) can be equivalently transformed into the following equation:

$$\begin{cases} \min & J\left(\pi(t), \widetilde{\eta}\right) = E\left[V(T + N) - \widetilde{\eta}\right]^2 \,, \\ \text{s.t.} & E[V(T) + N] = \widetilde{C} \,, \\ & \pi(t) \in L^2_{\mathrm{F}}(T, T + N; R) \,, \end{cases} \tag{18}$$

where $\widetilde{\eta} = \widetilde{C} - \lambda$. It is not difficult to see that the optimal investment problems (17) and (18) of the defined-contribution pension under the mean-variance criterion before and after retirement have the same structure. Therefore, we can generalize the objective function in Eqs. (17) and (18) as follows:

$$U(v) = (v - \eta^{\tau})^2 \,. \tag{19}$$

When $\eta^\tau = \eta$, it means before retirement. When $\eta^\tau = \widetilde{\eta}$, it means after retirement. Thus, Eqs. (17) and (18) are equivalent to Eqs. (20) and (21) below

$$\begin{cases} \min & J(\pi(t), \eta) = E[U(V(T))]\,, \\ \text{s.t.} & \pi(t) \in L^2_{\mathrm{F}}(0, T; R)\,, \end{cases} \tag{20}$$

$$\begin{cases} \min & J\,(\pi(t), \widetilde{\eta}\,) = E[U(V(T+N))]\,, \\ \text{s.t.} & \pi(t) \in L^2_{\mathrm{F}}(T, T+N; R)\,. \end{cases} \tag{21}$$

In the following, we will use stochastic control theory to transform equations (20) and (21) of the optimization problem into the corresponding Hamilton–Jacobi–Bellman equation to obtain a non-linear quadratic partial differential equation. Then, the non-linear quadratic partial differential equation is changed into a linear quadratic partial differential equation by using Legendre transformation and duality theory to obtain the explicit expression of its solution.

5.1. Solve before retirement

Theorem 1. Under the mean-variance criterion, the optimal investment strategy of the defined-contribution pension before retirement is

$$\pi_t^* = \frac{\mu - r}{\sigma^2 P^{(1-q)}}\left[-1 + \frac{a(t)}{v}\right], \tag{22}$$

where $a(t) = (\eta + c/r)\,\mathrm{e}^{-r(T-t)} - c/r$.

Proof. First, we define the value function (23) of the optimization problem before retirement (20)

$$H(t, s, v) = \inf E\{U(V(T))|S(t) = s, V(t) = v\}\,, \qquad 0 < t < T\,, \tag{23}$$

where $H(T, s, v) = U(v)$. Therefore, the corresponding Hamilton–Jacobi–Bellman equation is

$$\begin{aligned} &H_t + \mu s H_s + (rv + c)H_v + \frac{1}{2}\sigma^2 P^{(1-q)} s^2 H_{ss} \\ &+\min\left[\pi_t(\mu - r)vH_v + \pi_t\sigma^2 P^{(1-q)} svH_{sv} + \frac{1}{2}\pi_t^2\sigma^2 P^{(1-q)} v^2 H_{vv}\right] = 0\,, \end{aligned} \tag{24}$$

where $H_t, H_s, H_v, H_{sv}, H_{ss}$ and H_{vv} are the first-order and second-order partial derivatives of time, stock price and pension wealth, respectively. Solve the partial derivative of the above equation with respect to the investment strategy π_t and make it equal to zero, then the optimal investment strategy can be obtained

$$\pi_t^* = -\frac{(\mu - r)H_v + \sigma^2 P^{(1-q)} s H_{sv}}{\sigma^2 P^{(1-q)} v H_{vv}} . \tag{25}$$

By substituting the optimal investment strategy (25) into (24), a partial differential equation of the value function can be obtained

$$\begin{aligned} &H_t + \mu s H_s + (rv + c)H_v + \frac{1}{2}\sigma^2 P^{(1-q)} s^2 H_{ss} \\ &-\frac{\left((\mu - r)H_v + \sigma^2 P^{(1-q)} s H_{sv}\right)^2}{2\sigma^2 P^{(1-q)} H_{vv}} = 0 . \end{aligned} \tag{26}$$

Then, the value function can be obtained by solving the partial differential equation (26). Finally, the specific expression of the optimal investment strategy can be solved by substituting the value function into Eq. (25). In the following, we will focus on solving the partial differential Eq. (26).

Definition 2. Let $R^n \to R$ be a convex function, and $z > 0$. Then, define Legendre transform as

$$L(z) = \max(f(x) - zx) . \tag{27}$$

$L(z)$ is called the Legendre dual function (see [1]). According to Definition 2, by applying Legendre transformation to the value function $H(t, s, v)$, we can get

$$\widetilde{H}(t, s, z) = \sup \{H(t, s, v) - zv | 0 < v\} , \qquad 0 < z , \qquad 0 < t < T , \tag{28}$$

where z and v are dual variables (see [1]). Equation (28) above is equivalent to the following equation:

$$g(t, s, z) = \inf \left\{ v | H(t, s, v) \geq zv + \widetilde{H}(t, s, z) \right\} , \qquad 0 < t < T . \tag{29}$$

Both $g(t, s, z)$ and $\widetilde{H}(t, s, z)$ are dual functions of $H(t, s, v)$ and have the following relation:

$$\widetilde{H}(t, s, z) = H(t, s, g) - zg , \qquad g(t, s, z) = v , \qquad H_v = z . \tag{30}$$

At the terminal time, let $\widetilde{U}(z) = \sup\{U(v) - zv | 0 < v\}$, $G(z) = \sup\{v | U(v) \geq zv + \widetilde{U}(z)\}$. So obviously, there is $G(z) = (U^{'})^{-1}(z)$. Since $H(T,s,v) = U(v)$, at the terminal time T, we can obtain $g(T,s,z) = \inf\{v | U(v) \geq zv + \widetilde{H}(T,s,z)\}$, $\widetilde{H}(T,s,z) = \sup\{U(v) - zv\}$, and $g(T,s,z) = (U^{'})^{-1}(z)$. By solving the derivatives of variables t, s, and z for (30), the derivative relationship between the value function H and its dual function $\widetilde{H}$ can be obtained as follows:

$$
\begin{aligned}
H_v &= z\,, & H_t &= \widetilde{H}_t\,, & H_s &= \widetilde{H}_s\,,\\
H_{ss} &= \widetilde{H}_{ss} - \frac{\widetilde{H}_{sz}^2}{\widetilde{H}_{zz}}\,, & H_{sv} &= -\frac{\widetilde{H}_{sz}}{\widetilde{H}_{zz}}\,, & H_{vv} &= -\frac{1}{\widetilde{H}_{zz}}\,.
\end{aligned} \tag{31}
$$

Substituting (31) into (26) and solving the derivative of variable z combined with $v = g = -\widetilde{H}_z$, we can get the partial differential equation about g

$$
\begin{aligned}
&g_t + rsg_s - rg - c + \frac{1}{2}\sigma^2 P^{(1-q)} s^2 g_{ss} + \left(\frac{(\mu - r)^2}{\sigma^2 P^{(1-q)}} - r\right) z g_z\\
&+ \frac{(\mu - r)^2}{2\sigma^2 P^{(1-q)}} z^2 g_{zz} - (\mu - r) s z g_{sz} = 0\,.
\end{aligned} \tag{32}
$$

Therefore, the dual function g is used to express the optimal investment strategy (24), which is

$$
\pi_t^* = \frac{-(\mu - r) z g_z + \sigma^2 P^{(1-q)} s g_s}{\sigma^2 P^{(1-q)} g}\,. \tag{33}
$$

Using $g(T,s,z) = (U^{'})^{-1}(z)$ and (19), we have

$$
g(T,s,z) = \frac{1}{2} z + \eta\,. \tag{34}
$$

Suppose that the form of a solution of the partial differential equation (32) is

$$
g(t,s,z) = z h(t,s) + a(t)\,. \tag{35}
$$

Moreover, the boundary condition $a(T) = \eta$, $h(T,s) = \frac{1}{2}$ is satisfied. Substituting Eq. (35) into Eq. (32), we can get

$$
\left[h_t + (2r - \mu) s h_s + \frac{1}{2}\sigma^2 P^{(1-q)} s^2 h_{ss} + \frac{(\mu - r)^2}{\sigma^2 P^{(1-q)}} h - 2rh\right] z + a^{'}(t) - r a(t) - c = 0\,. \tag{36}
$$

Thus, two equations (37) and (38) are obtained

$$h_t + (2r-\mu)sh_s + \frac{1}{2}\sigma^2 P^{(1-q)} s^2 h_{ss} + \frac{(\mu-r)^2}{\sigma^2 P^{(1-q)}} h - 2rh = 0\,, \tag{37}$$

$$a^{'}(t) - ra(t) - c = 0\,. \tag{38}$$

Using the boundary condition $a(T) = \eta$, we can obtain the solution of Eq. (38) as follows:

$$a(t) = (\eta + c/r)\,\mathrm{e}^{-r(T-t)} - c/r\,. \tag{39}$$

Let $x = \ln s$, $\tau = T - t$, then $V(x,\tau) = h(s,t) = h(t,s)$ and $h_t = \frac{\partial V}{\partial \tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = -V_\tau$, $h_s = \frac{\partial V}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{1}{s}V_x$, $h_{ss} = -\frac{1}{s^2}V_x + \frac{1}{s^2}V_{xx}$. Substituting into (37), we have

$$-\frac{\partial V}{\partial \tau} + \frac{1}{2}\sigma^2 P^{(1-q)}\frac{1}{s^2}(V_{xx} - V_x) + (2r-\mu)V_x + \left[\frac{(\mu-r)^2}{\sigma^2 P^{(1-q)}} - 2r\right]V = 0 \tag{40}$$

with the boundary condition $V(x,0) = \frac{1}{2}$. Suppose that the form of function $V(x,\tau)$ is

$$V(x,\tau) = W(x,\tau)\,\mathrm{e}^{\alpha\tau+\beta x}\,. \tag{41}$$

By taking the derivative of Eq. (41) and making it equal to zero, we can get

$$\begin{aligned} &-\mathrm{e}^{\alpha\tau+\beta x}[W_\tau + \alpha W] + \frac{1}{2}\sigma^2 P^{(1-q)}\,\mathrm{e}^{\alpha\tau+\beta x}\left[W_{xx} + 2\beta W_x + \beta^2 W\right] \\ &+ \left[2r - \mu - \frac{1}{2}\sigma^2 P^{(1-q)}\right]\mathrm{e}^{\alpha\tau+\beta x}[W_x + \beta W] \\ &+ \left[\frac{(\mu-r)^2}{\sigma^2 P^{(1-q)}} - 2r\right] W\mathrm{e}^{\alpha\tau+\beta x} = 0\,. \end{aligned} \tag{42}$$

Further, we obtain

$$\begin{cases} \alpha = -2r + \frac{(\mu-r)^2}{\sigma^2 P^{(1-q)}}\,, \\ \beta = 0\,. \end{cases} \tag{43}$$

Using Poisson formula, we have

$$W(x,\tau) = \int_{-\infty}^{+\infty} \frac{1}{2}\frac{\mathrm{e}^{2\sigma^2 P^{(1-q)}\tau}}{\sqrt{2\pi}\sigma P^{\frac{(1-q)}{2}}}\,\mathrm{d}\xi = \frac{1}{2}\,. \tag{44}$$

Then, we get

$$h(s,t) = V(x,\tau) = \frac{1}{2}\,\mathrm{e}^{\left[-2r + \frac{(\mu-r)^2}{\sigma^2 P^{(1-q)}}\right](T-t)}\,. \tag{45}$$

Substituting Eq. (45) into Eq. (35), we have

$$g(t,s,z)=\frac{1}{2}z\,\mathrm{e}^{\left[-2r+\frac{(\mu-r)^2}{\sigma^2P^{(1-q)}}\right](T-t)}+a(t)\,. \tag{46}$$

Finally, substituting (46) and (30) into (33), we can obtain

$$\begin{aligned}\pi_t^* &= \frac{-(\mu-r)zh(t,s)+0}{v\sigma^2P^{(1-q)}}\\ &= \frac{-(\mu-r)[g-a(t)]}{v\sigma^2P^{(1-q)}}\\ &= \frac{-(\mu-r)[v-a(t)]}{v\sigma^2P^{(1-q)}}\\ &= \frac{\mu-r}{\sigma^2P^{(1-q)}}\left[-1+\frac{a(t)}{v}\right]. \end{aligned}\tag{47}$$

□

5.2. Solve after retirement

Theorem 2. Under the mean-variance criterion, the optimal investment strategy of the defined-contribution pension after retirement is

$$\pi_t^*=\frac{\mu-r}{\sigma^2P^{(1-q)}}\left[-1+\frac{a(t)}{v}\right], \tag{48}$$

where $a(t)=(\widetilde{\eta}-\widetilde{D}/r)\,\mathrm{e}^{-r(T+N-t)}-\widetilde{D}/r$.

Proof. Similar to the solution method before retirement, we define the value function (49) of the optimization problem (21) after retirement as follows:

$$H(t,s,v)=\inf E\{U(V(T+N))|S(t)=s,V(t)=v\}\,,\qquad T\leqslant t\leqslant T+N\,. \tag{49}$$

Then, its corresponding Hamilton–Jacobi–Bellman formula is

$$\begin{aligned}&H_t+\mu sH_s+\left(rv-\widetilde{D}\right)H_v+\frac{1}{2}\sigma^2P^{(1-q)}s^2H_{ss}\\ &+\min\left[\pi_t(\mu-r)vH_v+\pi_t\sigma^2P^{(1-q)}svH_{sv}+\frac{1}{2}\pi_t^2\sigma^2P^{(1-q)}v^2H_{vv}\right]=0\,,\end{aligned}\tag{50}$$

where $H_t, H_s, H_v, H_{sv}, H_{ss}$, and H_{vv} are the first-order and second-order partial derivatives of time, stock price and pension wealth, respectively. Taking the derivative of investment strategy π_t and making it equal to zero, we can obtain the optimal investment strategy

$$\pi_t^*=-\frac{(\mu-r)H_v+\sigma^2P^{(1-q)}sH_{sv}}{\sigma^2P^{(1-q)}vH_{vv}}\,. \tag{51}$$

Substituting the optimal investment strategy (51) into (50), a partial differential equation of the value function is derived

$$H_t + \mu s H_s + \left(rv - \widetilde{D}\right) H_v + \frac{1}{2}\sigma^2 P^{(1-q)} s^2 H_{ss} + \frac{\left((\mu - r)H_v + \sigma^2 P^{(1-q)} s H_{sv}\right)^2}{2\sigma^2 P^{(1-q)} H_{vv}} = 0\,. \tag{52}$$

Then substituting (31) into (52) and combining $v = g = -\widetilde{H}_z$, we can obtain the partial differential equation (53) of the function g

$$g_t + rsg_s - rg + \widetilde{D} + \frac{1}{2}\sigma^2 P^{(1-q)} s^2 g_{ss} + \left(\frac{(\mu - r)^2}{\sigma^2 P^{(1-q)}} - r\right) z g_z + \frac{(\mu - r)^2}{2\sigma^2 P^{(1-q)}} z^2 g_{zz} - (\mu - r) s z g_{sz} = 0\,. \tag{53}$$

Therefore, the dual function g is used to express the optimal investment strategy (51), which is

$$\pi_t^* = \frac{-(\mu - r) z g_z + \sigma^2 P^{(1-q)} s g_s}{\sigma^2 P^{(1-q)} g}\,. \tag{54}$$

In the following, it is only necessary to solve the function g through (53) and substitute it into (54) to obtain the optimal investment strategy for the defined-contribution pension. Similar to the solution before retirement, we suppose that the solution of Eq. (53) is

$$g(t, s, z) = z h(t, s) + a(t) \tag{55}$$

with the boundary condition $a(T + N) = \widetilde{\eta}$, $h(T + N, s) = \frac{1}{2}$. Substituting Eq. (55) into Eq. (53), we can get

$$\left[h_t + (2r - \mu) s h_s + \frac{1}{2}\sigma^2 P^{(1-q)} s^2 h_{ss} \frac{(\mu - r)^2}{\sigma^2 P^{(1-q)}} h - 2rh\right] z + a'(t) - r a(t) + \widetilde{D} = 0\,. \tag{56}$$

Thus, two equations (57) and (58) are obtained

$$h_t + (2r - \mu) s h_s + \frac{1}{2}\sigma^2 P^{(1-q)} s^2 h_{ss} + \frac{(\mu - r)^2}{\sigma^2 P^{(1-q)}} h - 2rh = 0\,, \tag{57}$$

$$a'(t) - r a(t) + \widetilde{D} = 0\,. \tag{58}$$

Using the boundary condition $a(T + N) = \widetilde{\eta}$, the solution of Eq. (58) can be derived

$$a(t) = \left(\widetilde{\eta} - \widetilde{D}/r\right) \mathrm{e}^{-r(T+N-t)} + \widetilde{D}/r\,. \tag{59}$$

The solution of Eq. (57) is the same as that before retirement, and it is easy to get

$$h(t,s)=\frac{1}{2}\,\mathrm{e}^{\left[-2r+\frac{(\mu-r)^2}{\sigma^2 P^{(1-q)}}\right](T+N-t)}\,. \tag{60}$$

Substituting Eq. (60) into Eq. (55), we have

$$g(t,s,z)=\frac{1}{2}z\,\mathrm{e}^{\left[-2r+\frac{(\mu-r)^2}{\sigma^2 P^{(1-q)}}\right](T+N-t)}+a(t)\,. \tag{61}$$

Then substituting (61) into (54) and combining $v=g$, we can get

$$\begin{aligned}\pi_t^* &= \frac{-(\mu-r)zh(t,s)+0}{v\sigma^2 P^{(1-q)}}\\ &= \frac{-(\mu-r)[g-a(t)]}{v\sigma^2 P^{(1-q)}}\\ &= \frac{-(\mu-r)[v-a(t)]}{v\sigma^2 P^{(1-q)}}\\ &= \frac{\mu-r}{\sigma^2 P^{(1-q)}}\left[-1+\frac{a(t)}{v}\right]\,,\end{aligned} \tag{62}$$

where $a(t)=(\widetilde{\eta}-\widetilde{D}/r)\,\mathrm{e}^{-r(T+N-t)}-\widetilde{D}/r$. □

6. Numerical results

To test the model, we select the daily closing prices of the Shanghai Composite Index as experimental datasets. The time period is from 01/04/2022 to 02/08/2024.

In Table 1, the kurtosis coefficient of the daily returns of the Shanghai Composite Index is 5.1204 which is very different from that of the Gaussian distribution. Moreover, the value of the J–B test is 95.8563 and the test probability is 0.0010 (the significance level is set as 0.05), which means the J–B test rejects the null hypothesis that the distribution of the daily returns of the Shanghai Composite Index is normal.

Table 1. The statistical characteristics of daily returns of the Shanghai Composite Index.

Mean	Standard deviation	Kurtosis	J–B	P
0.00016	0.0099	5.1204	95.8563	0.0010

In Fig. 1, the histogram of the daily returns shows that the return distribution has the characteristics of excess kurtosis. Furthermore, the Tsallis distribution ($q = 1.45$) can more accurately fit the empirical density distribution of the daily returns than the Gaussian distribution.

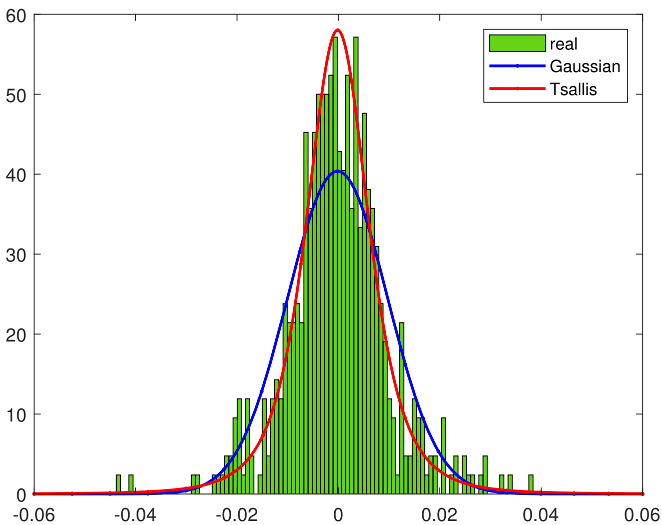

Fig. 1. Comparison of fitting of the empirical distribution of the daily returns for the empirical distribution, Gaussian distribution, and Tsallis distribution ($q = 1.45$).

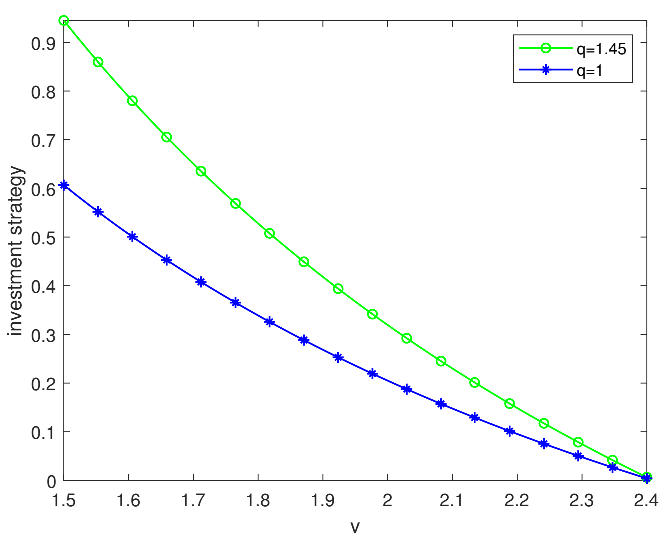

Fig. 2. The optimal investment strategy *versus* the wealth ($r = 0.04$, $\mu = 0.08$, $\sigma = 0.2$, $v_0 = 1$, $c = 0.1$, $q = 1.45$, $T = 20$, $t = 1$, $\lambda = 1$).

In Fig. 2, the optimal investment strategy is shown as a function of wealth. In order to control risks, the investor usually reduces the proportion of investment in risky assets as his wealth increases. However, at the same level of wealth, a higher value of parameter q corresponds to a higher proportion of investment in risky assets, which means that the investor needs to take greater risks to achieve the expected return when the market is more volatile.

7. Summary

As a powerful tool for dealing with long-range interactions and nonlinear complex systems, the Tsallis non-extensive statistical theory has been applied to physics, finance, and management since its birth. To accurately describe the movement of asset prices, this paper employs the Tsallis non-extensive statistical theory. Moreover, on this basis, the optimal investment problem of defined-contribution pension is considered. Under the mean-variance criterion, the explicit solution of the optimal investment strategy is obtained. This provides a reference for the further application of the Tsallis non-extensive statistical theory in the field of finance. In the future, the price model can be extended to study the investment problem under the utility maximization criterion. In addition, replacing the constant stock return and constant stock volatility in the model with variable stock return and variable stock volatility is also worth further study.

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