A NOVEL GENERALISATION OF SUPERSYMMETRY: QUANTUM $\mathbb{Z}_2^2$-OSCILLATORS AND THEIR 'SUPERISATION'

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We propose a very simple toy model of a $\mathbb{Z}_2^2$-supersymmetric quantum system and show, via Klein's construction, how to understand the system as being an $N = 2$ supersymmetric system with an extra $\mathbb{Z}_2^2$-grading. That is, the commutation/anticommutation rules are defined via the standard boson/fermion rules, but the system still has an underlying $\mathbb{Z}_2^2$-grading that needs to be taken into account.

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1. Introduction

Recently, a novel generalisation of supersymmetry that is inherently $\mathbb{Z}_2^2$-graded ($\mathbb{Z}_2^2 := \mathbb{Z}_2 \times \mathbb{Z}_2$) has been proposed (see [1]), and some classical and quantum models studied (see [2–8]). The basic degrees of freedom in these models are a bosonic, exotic bosonic, and two species of fermions. The novel aspect is that the exotic bosons anticommute with the fermions and the two species of fermions mutually commute. In this sense, these systems have exotic relative statistics, to use a term borrowed from the Green–Volkov parastatistics (see [9] and references therein). Moreover, these systems have a pair of supersymmetry generators whose commutator, rather than anticommutator, is a (possibly vanishing for certain models) central term. At the time of writing, it is not clear if physical systems can exhibit this kind of generalised supersymmetry. Low-dimensional systems are certainly a candidate as the spin-statistics theorem need not hold. For sure, more models need to be constructed and studied, and their relation with conventional models made clear. There are promising results within multiparticle theories where, in principle, there are observable consequences of the $\mathbb{Z}_2^2$-grading (see [10]). Furthermore, we remark that no experimental evidence today has

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emerged that nature realises supersymmetry at the level of fundamental particles. Rather speculatively, we consider the construction of simple models as an important step in experimentally realising $\mathbb{Z}_2^n$-supersymmetry in, say, trapped ion quantum simulators (see [11] for the case of a supersymmetric quantum mechanical model and [12] for the case of para-boson oscillators). We further remark that particles with exotic statistics have been proposed as a candidate for dark matter (see, for example, [13]).

With the above comments in mind, we construct a $\mathbb{Z}_2^2$-graded version of Nicolai's supersymmetric oscillator (see [14]), examine some of its elementary properties, and then use Klein's construction (see [15]) to render the commutation/anticommutation rules to the standard ones and the system supersymmetric. The 'superised' system is not completely standard as there is still an underlying $\mathbb{Z}_2^2$-grading: the Hilbert–Fock space is $\mathbb{Z}_2^2$-graded and the supersymmetry generators still carry a $\mathbb{Z}_2^2$-grading. The extra grading is encoded in two Witten parity operators rather than a single one as found in standard supersymmetry (this was first observed in [8]). It is remarkable that a very simple mathematical system can exhibit this $\mathbb{Z}_2^2$-graded generalisation of supersymmetry.

Conventions: For notational simplicity, we work in units such that the mass of the particles $m = 1$ and Planck's constant $\hbar = 1$ (as are any possible coupling constants). As an ordered set, we define $\mathbb{Z}_2^2 := \{(0,0),(1,1),(0,1),(1,0)\}$. The $\mathbb{Z}_2^2$-commutator is defined as

$$[A,B]_{\mathbb{Z}_2^2} := AB - (-1)^{\langle \deg(A) | \deg(B) \rangle}\, BA\,,$$

where $\langle - | - \rangle$ is the standard scalar product. For the conjugation operation, we take the convention that $(ab)^\dagger = b^\dagger a^\dagger$ irrespective of the $\mathbb{Z}_2^2$-degree.

2. Quantum $\mathbb{Z}_2^2$-oscillators

2.1. The model via creation and annihilation operators

We construct a very simple $\mathbb{Z}_2^2$-supersymmetric system over a single point, thus we have a model of zero-dimensional field theory. This can be thought of as a spin–lattice model consisting of four independent spin degrees of freedom on a single lattice point. As the system is inherently zero-dimensional, we do not have the proper notion of spin. Nonetheless, we can consider operators that are $\mathbb{Z}_2^2$-graded and they satisfy canonical $\mathbb{Z}_2^2$-graded commutation relations. That is, we take a $*$-algebraic approach. In particular, consider a set of creation and annihilation operators acting on a Hilbert space $\mathcal{H}$:

$b^\dagger, b$	degree $(0,0)$	Standard Boson,
$e^\dagger, e$	degree $(1,1)$	Exotic Boso,
$f_1^\dagger, f_1$	degree $(0,1)$	Fermion Type 1,
$f_2^\dagger, f_2$	degree $(1,0)$	Fermion Type 2.

We will use classical notation $[-,-]$ and $\{-,-\}$ for the $\mathbb{Z}_2^2$-commutators for clarity. The commutation rules are the standard CCR and CAR

$$\left[b, b^\dagger\right] = 1\,, \qquad \left[e, e^\dagger\right] = 1\,, \qquad \left\{f_i, f_i^\dagger\right\} = 1\,, \tag{2.1}$$

with all other $\mathbb{Z}_2^2$-commutators vanishing. For example, $f_1 f_2 = f_2 f_1$ and $e f_i = -\, f_i e$. We define the Hamiltonian as

$$H_{00} := b^\dagger b + e^\dagger e + f_1^\dagger f_1 + f_2^\dagger f_2\,. \tag{2.2}$$

This means that we are considering no interaction between the individual oscillators. Note that this Hamiltonian is the sum of the number operators $N_b = b^\dagger b$, $N_e = e^\dagger e$, $N_{f_1} = f_1^\dagger f_1$, and $N_{f_2} = f_2^\dagger f_2$. Thus, this system is the natural generalisation of Nicolai's supersymmetric oscillator (see [14]). The number operators are of degree $(0,0)$ and satisfy the usual relations $[N_a, a^\dagger] = a^\dagger$ and $[N_a, a] = -a$, where $a \in \{b, e, f_1, f_2\}$. This system exhibits $\mathbb{Z}_2^2$-supersymmetry as first defined by Bruce [1] and Bruce and Duplij [8]. We naturally take observables to be degree $(0,0)$ self-adjoint operators.

Theorem 2.1. *The following self-adjoint $\mathbb{Z}_2^2$-graded operators*

$$Q_{01} := f_1^\dagger b + b^\dagger f_1 + f_2^\dagger e + e^\dagger f_2\,, \tag{2.3a}$$

$$Q_{10} := f_2^\dagger b + b^\dagger f_2 + f_1^\dagger e + e^\dagger f_1\,, \tag{2.3b}$$

satisfy the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded, $\mathcal{N} = (1,1)$ supertranslation algebra (with vanishing central term), i.e.,

$$\{Q_{01}, Q_{01}\} = \{Q_{10}, Q_{10}\} = 2H_{00}\,, \qquad [Q_{10}, Q_{01}] = 0\,. \tag{2.4}$$

Proof. This follows from a series of direct computations. First, using the commutation rule for e and $e^\dagger$

$$\begin{aligned} Q_{01}^2 \;=\; & f_1^\dagger b b^\dagger f_1 + b^\dagger f_1 f_1^\dagger b + f_2^\dagger e e^\dagger f_2 + e^\dagger f_2 f_2^\dagger e \\ & + \left(f_1^\dagger f_2^\dagger - f_2^\dagger f_1^\dagger\right) be + \left(f_2 f_1^\dagger - f_1^\dagger f_2\right) be^\dagger + \left(f_1 f_2^\dagger - f_2^\dagger f_1\right) b^\dagger e \\ & + (f_2 f_1 - f_1 f_2)\, b^\dagger e^\dagger\,, \end{aligned}$$

then, using the commutation rules for f_1, f_2 and their conjugates, the second line vanishes

$$= f_1^\dagger b b^\dagger f_1 + b^\dagger f_1 f_1^\dagger b + f_2^\dagger e e^\dagger f_2 + e^\dagger f_2 f_2^\dagger e\,,$$

next, using the non-trivial commutation relations, we obtain

$$\begin{aligned} &= b b^\dagger \left(\mathbb{1} - f_1 f_1^\dagger\right) + \left(\mathbb{1} + b b^\dagger\right) f_1 f_1^\dagger + e e^\dagger \left(\mathbb{1} - f_2 f_2^\dagger\right) + \left(\mathbb{1} + e e^\dagger\right) f_2 f_2^\dagger \\ &= b^\dagger b + e^\dagger e + f_1^\dagger f_1 + f_2^\dagger f_2 = H_{00}\,. \end{aligned}$$

The statement that $Q_{10}^2 = H_{00}$ follows from the above proof upon interchanging $1 \leftrightarrow 2$ for the fermions.

Moving on to the mixed commutator, writing out only the terms that do not obviously commute, we have

$$\begin{aligned} [Q_{01}, Q_{10}] \;=\; & f_2^\dagger b b^\dagger f_1 - b^\dagger f_1 f_2^\dagger b + f_2^\dagger b e^\dagger f_2 - e^\dagger f_2 f_2^\dagger b \\ & + b^\dagger f_2 f_1^\dagger b - f_1^\dagger b b^\dagger f_2 + b^\dagger f_2 f_2^\dagger e - f_2^\dagger e b^\dagger f_2 \\ & + f_1^\dagger e b^\dagger f_1 - b^\dagger f_1 f_1^\dagger e + f_1^\dagger e e^\dagger f_2 - e^\dagger f_2 f_1^\dagger e \\ & + e^\dagger f_1 f_1^\dagger b - f_1^\dagger b e^\dagger f_1 + e^\dagger f_1 f_2^\dagger e - f_2^\dagger e e^\dagger f_1\,, \end{aligned}$$

using the (anti)commutation rules, we obtain

$$= f_1 f_2^\dagger - b e^\dagger - f_2 f_1^\dagger + e b^\dagger - e b^\dagger + f_2 f_1^\dagger + b e^\dagger - f_1 f_2^\dagger = 0\,.$$

The fact that H_{00} is central, so $[H_{00}, Q_{01}] = [H_{00}, Q_{10}] = 0$, follows from the Jacobi identity and the antisymmetry of the graded commutators. □

Remark 2.2. In general, we have a central term $[Q_{10}, Q_{01}] = 2\mathrm{i}\, Z_{11}$, where Z_{11} is of $\mathbb{Z}_2^2$-degree $(1,1)$. Note that we have a commutator here and not an anticommutator as would be the case for standard supersymmetry. The vanishing of Z_{11} is completely expected as all the oscillators are independent of each other. Adding interactions requires the presence of coupling constants that carry a non-zero $\mathbb{Z}_2^2$-degree (see [5–7]).

Observations:

1. The Hamiltonian (2.2) is the sum of two bosonic and two fermionic harmonic oscillators ($\hbar = \omega = 1$). Thus, in the basis $|n_b, n_e, n_{f_1}, n_{f_2}\rangle$, the energy is given by $E = n_b + n_e + n_{f_1} + n_{f_2} = n$. See Proposition 2.3 for the degeneracy of these states. The first four excited states are given in Table 1.

Table 1. The first few energy levels of the $\mathbb{Z}_2^2$-oscillator (2.2) in the 'particle number' basis.

Energy	States			
	Boson $(0,0)$	Exotic $(1,1)$	Fermion 1 $(0,1)$	Fermion 2 $(1,0)$
0	$\lvert 0,0,0,0\rangle$	—	—	—
1	$\lvert 1,0,0,0\rangle$	$\lvert 0,1,0,0\rangle$	$\lvert 0,0,1,0\rangle$	$\lvert 0,0,0,1\rangle$
2	$\lvert 2,0,0,0\rangle$	$\lvert 1,1,0,0\rangle$	$\lvert 1,0,1,0\rangle$	$\lvert 1,0,0,1\rangle$
	$\lvert 0,2,0,0\rangle$	$\lvert 0,0,1,1\rangle$	$\lvert 0,1,0,1\rangle$	$\lvert 0,1,1,0\rangle$
3	$\lvert 3,0,0,0\rangle$	$\lvert 2,1,0,0\rangle$	$\lvert 2,0,1,0\rangle$	$\lvert 2,0,0,1\rangle$
	$\lvert 1,2,0,0\rangle$	$\lvert 1,0,1,1\rangle$	$\lvert 1,1,0,1\rangle$	$\lvert 1,1,1,0\rangle$
	$\lvert 0,1,1,1\rangle$	$\lvert 0,3,0,0\rangle$	$\lvert 0,2,1,0\rangle$	$\lvert 0,2,0,1\rangle$
4	$\lvert 4,0,0,0\rangle$	$\lvert 3,1,0,0\rangle$	$\lvert 3,0,1,0\rangle$	$\lvert 3,0,0,1\rangle$
	$\lvert 2,2,0,0\rangle$	$\lvert 2,0,1,1\rangle$	$\lvert 2,1,0,1\rangle$	$\lvert 2,1,1,0\rangle$
	$\lvert 1,1,1,1\rangle$	$\lvert 1,3,0,0\rangle$	$\lvert 1,2,1,0\rangle$	$\lvert 1,2,0,1\rangle$
	$\lvert 0,4,0,0\rangle$	$\lvert 0,2,1,1\rangle$	$\lvert 0,3,0,1\rangle$	$\lvert 0,3,1,0\rangle$

2. Clearly, $H_{00}|0,0,0,0\rangle = 0$ and the zero-energy ground state is a singlet. We use the shorthand $|0\rangle := |0,0,0,0\rangle$ for this ground state. The ground state being a zero energy state implies, as standard, $Q_{01}|0\rangle = 0$ and $Q_{10}|0\rangle = 0$, meaning that $\mathbb{Z}_2^2$-supersymmetry is unbroken. This is exactly the same situation as of the standard supersymmetric oscillator. The ground state is bosonic.

3. There is a version of R-symmetry which shifts the $\mathbb{Z}_2^2$-degree and is given by

$$\begin{aligned} b &\mapsto \exp(\mathrm{i}\lambda)\, e\,, & e &\mapsto \exp(-\mathrm{i}\lambda)\, b\,,\\ f_1 &\mapsto \exp(\mathrm{i}\lambda)\, f_2\,, & f_2 &\mapsto \exp(-\mathrm{i}\lambda)\, f_1\,, \end{aligned}$$

together with the conjugates, and here $\lambda \in \mathbb{R}$. Note that the Hamiltonian H_{00} is invariant and that

$$Q_{01} \longleftrightarrow Q_{10}\,.$$

Proposition 2.3. *The n^{th} energy level for $n \geq 1$ of the Hamiltonian* (2.2) *is $4n$-fold degenerate.*

Proof. For any fixed $n \geq 1$, a state is labelled $|n_b, n_e, n_{f_1}, n_{f_2}\rangle$ where $n_b, n_e \in \mathbb{N}$ and $n_{f_1}, n_{f_2} \in \{0, 1\}$, subject to the constraint $n_b + n_e + n_{f_1} + n_{f_2} = n$. The fermionic labels are 00, 01, 10, and 11 and thus the proof reduces to arranging pairs of integers that sum to n, $n-1$ (counted twice), and $n-2$. This problem is equivalent to the number of ways one can place a single stick between n balls, and then $n-1$ balls (counted twice), and finally $n-2$ balls. For the n-case, we have

$$\begin{aligned} |OO\cdots OO &= (0, n)\,, \\ O|O\cdots OO &= (1, n-1)\,, \\ &\vdots \\ OO\cdots O|O &= (n-1, 1)\,, \\ OO\cdots OO| &= (n, 0)\,. \end{aligned}$$

Thus, the number of pairs of numbers that sum to n is $n+1$. Then it is clear that the number of states for a given n is just $n+1+2n+n-1 = 4n$. $\square$

We have a pair of the Witten parity operators[1] defined as

$$K_1 = \cos\left(\pi\left(N_e + N_{f_1}\right)\right)\,, \qquad K_2 = \cos\left(\pi\left(N_e + N_{f_2}\right)\right)\,, \tag{2.5}$$

which are both clearly $\mathbb{Z}_2^2$-degree $(0,0)$ and self-adjoint, thus they correspond to observables. By construction, we have

$$K_i\,|n_b, n_e, n_{f_1}, n_{f_2}\rangle = (-1)^{n_e + n_{f_i}}\,|n_b, n_e, n_{f_1}, n_{f_2}\rangle\,. \tag{2.6}$$

It is immediately clear that

$$[K_1, K_2] = 0\,, \qquad [K_1, H_{00}] = [K_2, H_{00}] = 0\,, \qquad K_1^2 = K_2^2 = \mathbb{1}\,. \tag{2.7}$$

In particular, the above implies that we can have simultaneous eigenfunctions of the Hamiltonian and the two Witten parity operators (2.5). We can then pick a basis for the states $|n, \varepsilon_1, \varepsilon_2\rangle$, where $\varepsilon_i \in \{+1, -1\}$. That is, the space of states $\mathcal{H}$ has a decomposition into four sectors depending on the sign of the Witten parity operators, *i.e.*, $\mathcal{H} = \mathcal{H}_{++} \oplus \mathcal{H}_{--} \oplus \mathcal{H}_{+-} \oplus \mathcal{H}_{-+}$. These sectors correspond to bosons, exotic bosons, fermions of type 1, and fermions of type 2. It is then convenient to relabel these sectors via the corresponding $\mathbb{Z}_2^2$-degree, *i.e.*, $\mathcal{H} = \mathcal{H}_{00} \oplus \mathcal{H}_{11} \oplus \mathcal{H}_{01} \oplus \mathcal{H}_{10}$. Moreover, the Witten parity operators imply the superselection rule that only states that

[1] Also known as Klein operators, chirality operators or fermion number operators, though this last name is not appropriate in the current situation.

are homogeneous in $\mathbb{Z}_2^2$-degree are physically realisable (see, for example, [16]). It is straightforward to observe that

$$\begin{aligned} Q_{01}K_1 &= -K_1Q_{01}\,, \qquad Q_{10}K_1 = +K_1Q_{10}\,, \\ Q_{01}K_2 &= +K_1Q_{01}\,, \qquad Q_{10}K_2 = -K_2Q_{10}\,, \end{aligned} \tag{2.8}$$

and thus

$$\begin{aligned} Q_{01}\mathcal{H}_{00} &\subset \mathcal{H}_{01}\,, \qquad Q_{10}\mathcal{H}_{00} \subset \mathcal{H}_{10}\,, \\ Q_{01}\mathcal{H}_{11} &\subset \mathcal{H}_{10}\,, \qquad Q_{10}\mathcal{H}_{11} \subset \mathcal{H}_{01}\,, \\ Q_{01}\mathcal{H}_{01} &\subset \mathcal{H}_{00}\,, \qquad Q_{10}\mathcal{H}_{01} \subset \mathcal{H}_{10}\,, \\ Q_{01}\mathcal{H}_{10} &\subset \mathcal{H}_{11}\,, \qquad Q_{10}\mathcal{H}_{01} \subset \mathcal{H}_{11}\,. \end{aligned}$$

The system really is $\mathbb{Z}_2^2$-supersymmetric, *i.e.*, states from one sector are mapped to other sectors using Q_{01} and Q_{10}. It is important to note that applying $Q_{10}Q_{01}$ (or equivalent in this case $Q_{01}Q_{10}$) does *not* return one to the starting sector as it would in the standard supersymmetry. For example, $Q_{10}Q_{01}\mathcal{H}_{00} \subset \mathcal{H}_{11}$.

2.2. Klein operators and 'superisation'

We now proceed to apply Klein's construction (see [15]) to redefine the operators we work with to render the system super, *i.e.*, with the standard commutation/anticommutation rules for the creation and annihilation operators defined by a $\mathbb{Z}_2$-grading. The natural choice here is to use the total degree of the assigned $\mathbb{Z}_2^2$-degree. Moreover, we want the construction to lead to two standard supersymmetries.

Remark 2.4. Quesne showed that the algebra of $\mathbb{Z}_2^2$-graded supersymmetric quantum mechanics is realisable in terms of a single bosonic degree of freedom using the Calogero–Vasiliev algebras *i.e.*, there is a minimal bosonisation of the theory (see [17]). We will content ourselves with a 'superisation' in this note.

We have to choose one of the Witten operators (2.5) to be our Klein operator. We pick K_1 for no particular reason other than our choice of ordering with the elements of $\mathbb{Z}_2^2$. We then define a new set of fermionic creation and annihilation operators as

$$a_1 = f_1K_1\,, \qquad a_1^\dagger = K_1f_1^\dagger\,, \tag{2.9a}$$

$$a_2 = f_2K_1\,, \qquad a_2^\dagger = K_1f_2^\dagger\,. \tag{2.9b}$$

Proposition 2.5. *The set of operators* $\{b, b^\dagger, e, e^\dagger, a_1, a_1^\dagger, a_2, a_2^\dagger\}$ *satisfy the standard commutation/anticommutation rules for a pair of bosonic and a pair of fermionic creation and annihilation operators where the supercommutation rules are defined by the total degree of the operators.*

Proof. This follows from the properties of the Witten parity operators. We do not need to check all expressions, just those that involve a_1 and a_2 (and their conjugates). Moreover, we need not consider any expression involving b (and its conjugate). For example, $\{e, f_1\}K_1 = ef_1K_1 + f_1eK_1 = ef_1K_1 - f_1K_1e = [e, a_1] = 0$. Similarly, $K_1[f_2, f_1]K_1 = K_1f_2f_1K_1 - K_1f_1f_2K_1 = f_2f_1K_1 + f_1K_1f_2K_1 = \{a_2, a_1\} = 0$. Finally, just to further illustrate the point, $K_1\{f_1, f_1^\dagger\}K_1 = K_1f_1f_1^\dagger K_1 + K_1f_1^\dagger f_1K_1 = f_1K_1K_1f_1^\dagger + K_1f_1^\dagger f_1K_1 = \{a_1, a_1^\dagger\} = 1$. All other commutators and anticommutators can similarly be deduced. The claim that these are now all supercommutators using the total degree follows directly. □

Furthermore, we define the following self-adjoint operators:

$$H := b^\dagger b + e^\dagger e + a_1^\dagger a_1 + a_2^\dagger a_2 = H_{00}\,, \tag{2.10a}$$
$$Q_1 := \mathrm{i}K_1Q_{01} = \mathrm{i}a_1^\dagger b - \mathrm{i}b^\dagger a_1 + \mathrm{i}a_2^\dagger e - \mathrm{i}e^\dagger a_2\,, \tag{2.10b}$$
$$Q_2 := K_1Q_{10} = a_2^\dagger b + b^\dagger a_2 + a_1^\dagger e + e^\dagger a_1\,. \tag{2.10c}$$

Note that the Hamiltonian is unchanged, but now has the interpretation of the sum of the Hamiltonians for a pair of distinguishable bosons and a pair of distinguishable fermions.

Theorem 2.6. *The above operators* (2.10a), (2.10b), *and* (2.10c) *satisfy the* $\mathcal{N} = 2$ *supertranslation algebra (with vanishing central charge)*

$$\{Q_1, Q_1\} = \{Q_2, Q_2\} = 2H\,, \qquad \{Q_2, Q_1\} = 0\,,$$

and all other commutators vanishing.

Proof. Note that we are now dealing with commutators/anticommutators defined by the total degree of the operators. Direct computation using the properties of the Witten operators and Theorem 2.1 gives

$$\begin{aligned}
\{Q_1, Q_1\} &= 2\mathrm{i}K_1Q_{01}\mathrm{i}K_1Q_{01} = 2Q_{01}Q_{01} = 2H_{00} = 2H\,,\\
\{Q_2, Q_2\} &= 2K_1Q_{10}K_1Q_{10} = 2Q_{10}Q_{10} = 2H_{00} = 2H\,,\\
\{Q_2, Q_1\} &= \mathrm{i}(K_1Q_{10}K_1Q_{01} + K_1Q_{01}K_1Q_{10}) = \mathrm{i}(Q_{10}Q_{01} - Q_{01}Q_{10}) = 0\,.
\end{aligned}$$

Checking that the Hamiltonian is central is similarly straightforward

$$[H, Q_1] = \mathrm{i}K_1[H_{00}, Q_{01}] = 0\,, \qquad [H, Q_2] = K_1[H_{00}, Q_{10}] = 0\,.$$

□

Remark 2.7. This construction is not completely canonical, there is the other obvious choice of using K_2 and the obvious amendments to the above constructions. This would be no more than exchanging the labelling of fermions of type 1 and type 2.

The Lie superalgebra formed by H, Q_1, and Q_2 should be considered as a $\mathbb{Z}_2^2$-graded Lie superalgebra, *i.e.*, a Lie superalgebra with an additional compatible $\mathbb{Z}_2^2$-grading. Specifically, H is even and carries $\mathbb{Z}_2^2$-degree $(0,0)$, Q_1 is odd and carries $\mathbb{Z}_2^2$-degree $(0,1)$, and Q_2 is odd and carries $\mathbb{Z}_2^2$-degree $(1,0)$. Of course, these operators still act on the Hilbert–Fock space $\mathcal{H} = \mathcal{H}_{00} \oplus \mathcal{H}_{11} \oplus \mathcal{H}_{01} \oplus \mathcal{H}_{10}$, and the $\mathbb{Z}_2^2$-grading still needs to be taken into account. The Witten parity operators encode the $\mathbb{Z}_2^2$-grading and one can easily deduce the following:

$$
\begin{aligned}
\{K_1, Q_1\} &= 0\,, \qquad [K_1, Q_2] = 0\,,\\
[K_2, Q_1] &= 0\,, \qquad \{K_2, Q_2\} = 0\,.
\end{aligned}
\tag{2.11}
$$

Just as before, we have

$$
\begin{aligned}
Q_1\mathcal{H}_{00} &\subset \mathcal{H}_{01}\,, \qquad Q_2\mathcal{H}_{00} \subset \mathcal{H}_{10}\,,\\
Q_1\mathcal{H}_{11} &\subset \mathcal{H}_{10}\,, \qquad Q_2\mathcal{H}_{11} \subset \mathcal{H}_{01}\,,\\
Q_1\mathcal{H}_{01} &\subset \mathcal{H}_{00}\,, \qquad Q_2\mathcal{H}_{01} \subset \mathcal{H}_{10}\,,\\
Q_1\mathcal{H}_{10} &\subset \mathcal{H}_{11}\,, \qquad Q_2\mathcal{H}_{01} \subset \mathcal{H}_{11}\,.
\end{aligned}
$$

Via this construction, the relative statistics of the creation/annihilation operators are now standard. Moreover, the system exhibits supersymmetry, but now with an extra internal quantum number — the $\mathbb{Z}_2^2$-grading that is encoded in the two Witten parity operators. These observations sit comfortably with the results of [9]. In particular, systems with para-fermions can, under some technical conditions, be reformulated to have standard statistics, but now the observable algebra is selected by a non-Abelian gauge group. This supersymmetric system is not entirely standard. Supersymmetry generators usually anticommute with the Witten parity operator, but in the current situation, we have both commutators and anticommutators, see (2.11). Generalising supersymmetry to include internal degrees of freedom — and we view the extra $\mathbb{Z}_2^2$-grading in this light — has a long history dating back to the late 1970s (see [18] and references therein).

3. Concluding remarks

In this short note, we have constructed a simple $\mathbb{Z}_2^2$-supersymmetric model based on creation and annihilation operators. We have shown that via Klein's construction the system can be rendered supersymmetric.

One issue here is that the model does not have a central charge and it is desirable to amend this. The lack of central charge is due to the four oscillators not interacting with each other. It has already been noticed that interacting models, classical at least, seem to require coupling constants that carry non-zero $\mathbb{Z}_2^2$-grading. The physical interpretation of such constants is not exactly clear, nor is the role of exotic bosons in nature.

Moreover, building simple models with $\mathbb{Z}_2^n$-supersymmetry for $n > 2$ is a challenge as the number of elements of $\mathbb{Z}_2^n$ grows exponentially as n increases. This increase in the number of degrees of freedom has hindered model-building (for work in this direction, see [2]).

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