BEHAVIOR OF THE LOW-LYING ALTERNATING-PARITY ENERGY LEVELS OF THE RIGID ASYMMETRIC ROTATOR DEPENDING ON THE ANGULAR VARIABLE OF THE POLAR COORDINATE

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Recently, even–even nuclei with quadrupole and octupole deformations have been studied, taking into account the triaxiality of their shape. This approach allows one to simultaneously describe the alternating-parity spectrum of different bands of these nuclei. In this case, the values of the parameters of quadrupole and octupole triaxiality go beyond the traditional characteristic values of separate consideration of these deformations: $\gamma_{\rm eff}$ ($0 \leq \gamma_{\rm eff} \leq \frac{\pi}{6}$) and $\eta_{\rm eff}$ ($0 \leq \eta_{\rm eff} \leq \frac{\pi}{2}$). To identify the reasons for these differences, we studied the behavior of low-lying energy levels of a rigid asymmetric rotator with alternating parity depending on the angular variable of the polar coordinates ε_0 for fixed values of the triaxiality parameters $\gamma_{\rm eff}$ and $\eta_{\rm eff}$. We determined the region of experimentally observed energy levels with alternating parity corresponding to the given values of the triaxial parameters $\gamma_{\rm eff}$ and $\eta_{\rm eff}$ for different values of the angular parameter ε_0.

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1. Introduction

Obtaining reliable information on the spectroscopic characteristics of heavy nuclei is one of the most important tasks of the modern theory of the structure of atomic nuclei [1]. The properties of excited collective states depend on the shape of the nucleus and its deformability [2]. Quadrupole modes are the dominant and most important type of deformation; they are conveniently described by the parameters β_2- and γ for axial deformation and deviation from axiality, respectively [1, 3]. In Ref. [4], nonadiabatic collective models providing qualitative and quantitative description of positive parity spectra of excited collective states of heavy deformed even–even nuclei are analyzed. In this paper, it is shown that the free “triaxiality“ approximation describes well the energy levels of excited states of the ground-state band, β- and γ-bands of lanthanide and actinide nuclei, and unknown energy levels of the ground-state band, β- and γ-bands of superheavy even–even nuclei are predicted. Based on this work, in Ref. [5], the reduced E2-transitions probabilities of excited states of the ground-state band, β- and γ-bands, considered in the above-mentioned nuclei are obtained. Furthermore, in Ref. [6], on the basis of the coherent state model and the triaxial rotation–vibration model are described energy spectrum of the ground-state band, β- and γ-bands of some even–even triaxial nuclei.

The study of the spectroscopic properties of triaxial even–even nuclei with quadrupole β_2 and γ as well as octupole β_3 and η deformations is an actual problem. However, the asymmetry of quadrupole and octupole deformations during nuclear rotation remains insufficiently studied.

The solution of the Schrödinger equation for the dynamic radial part of the polar coordinates of the Bohr Hamiltonian with quadrupole and octupole deformations is analyzed in [7]. The Davidson potential was chosen to describe the radial deformations. The energy spectrum and wave functions of alternating parity of the triaxial even–even nucleus are obtained [8]. In Ref. [8], these energy levels are described, using eight adjusting model parameters. They are: energy factor $\hbar\omega$ (in keV), dimensionless splitting parameters: Δ_0^+, Δ_0^-, Δ_1^+, Δ_1^-, angular parameter of polar coordinates ε_0 (in degrees), and quadrupole and octupole triaxiality parameters γ_{eff} (in degrees) and η_{eff} (in degrees), respectively [9].

In Ref. [7], it was shown that the theoretical alternating-parity spectra of the yrast band are well described for deformed even–even nuclei: ^{154}Sm, ^{154,160}Gd, ^{156}Dy, ^{162,164}Er, ^{230,232}Th, ^{230,232,234,236,238}U, and ^{238,240}Pu by the simultaneous contribution of the triaxial K-mixing effect (K is the component of the angular momentum $\hat{I}$ on the third axis of the intrinsic frames) and the angular quadrupole–octupole vibration modes [8]. In this paper, we considered in detail the alternating-parity spectra of the yrast and first non-yrast bands of even–even nuclei ^{150}Nd and ^{172}Yb [8]. It is shown that in their description, the values of the triaxiality parameters of the quadrupole and octupole deformations go beyond the traditional values for the purely quadrupole γ_{eff} ($0 \leq \gamma_{\text{eff}} \leq \frac{\pi}{6}$) [2] and octupole η_{eff} ($0 \leq \eta_{\text{eff}} \leq \frac{\pi}{2}$) [10] deformations. It is worth noting that these parameters are used in combination with the parameter ε_0 as given in [8]. By these components, the alternating-parity energy levels of a rigid asymmetric rotator are determined [11]. They are also included in the list of parameters of the expansion coefficients $\mathrm{A}_{IK}^{\tau}(\gamma_{\text{eff}}, \eta_{\text{eff}}, \varepsilon_0)$ of the wave function of a rigid asymmetric rotator [11], here $\tau = 1, 2, 3, \ldots$ labels the triaxial-rotator states.

Thus, the behavior of the low-lying alternating-parity energy levels of a rigid asymmetric rotator on the dependence parameter ε_0 for the fixed values of the triaxiality parameters γ_{eff}, η_{eff} is an urgent problem. Therefore, the purpose of this paper is to investigate the behavior of the low-lying alternating-parity energy spectra of the yrast band (where $\tau = 1$) depending on the angular parameter of the polar coordinates at fixed values of the quadrupole and octupole triaxiality deformations parameters: γ_{eff} and η_{eff}.

2. Model formalism

The general solution of the Schrödinger equation with the Hamiltonian [7] is complicated, so various simplifications are used. Here, we will briefly dwell on the approximations used by other authors. By applying approximations of an axially-symmetric nucleus, where $\gamma = 0°$ and $\eta = 0°$, and also $K = 0$ (K is a good quantum number), the Hamiltonian operator contains five dynamic variables (β_2, β_3, Euler angles) and is developed in [12–17]. However, the asymmetry of quadrupole and octupole deformations during nuclear rotation remains insufficiently studied. The approximation of a triaxial asymmetric rotator is developed in [10, 11, 18, 19], where the deformation variables are replaced by their effective values; in addition, K is a bad quantum number (K-mixing). Then the dynamic variables remain the Euler angles.

Obviously, the next step is to combine the axially-symmetric nucleus approximation with the rigid asymmetric rotator approximation. The solution of the Schrödinger equation for the total Bohr Hamiltonian is very complicated [7, 8]. Therefore, following the simplification suggested in Ref. [21], we freeze the γ and η degrees of freedom in the vibration part, while keeping the axial variables β_2 and β_3 dynamic. We replace in the moment of inertia the variables γ and η in the rotation part with their effective values γ_{eff} and η_{eff} [2].

Going to polar coordinate σ ($0 \leq \sigma \leq \infty$) and ε ($-\frac{\pi}{2} \leq \varepsilon \leq \frac{\pi}{2}$) [16], we get

$$\beta_2 = \sqrt{\frac{B}{B_2}}\sigma\cos\varepsilon\,, \qquad \beta_3 = \sqrt{\frac{B}{B_3}}\sigma\sin\varepsilon\,, \qquad B = \frac{B_2 + B_3}{2}\,. \tag{1}$$

Here, B_2 and B_2 are quadrupole and octupole mass parameters, respectively, $\hat{T}_{\text{rot}}$ is the rotational energy operator, and $W(\sigma,\varepsilon)$ is the potential energy of σ and ε vibrations [7].

We will write the Schrödinger equation as

$$\left\{-\frac{\hbar^2}{2B}\left[\frac{\partial^2}{\partial\sigma^2} + \frac{1}{\sigma\partial\sigma} + \frac{\partial^2}{\sigma^2\partial\varepsilon^2}\right] + \hat{T}_{\text{rot}} + W(\sigma,\varepsilon) - E_I^{\pm}\right\}\Phi_I^{\pm}(\sigma,\varepsilon,\theta) = 0\,. \tag{2}$$

We search for a solution of this equation in the following form:

$$\Phi^{\pm}_{I\tau}(\sigma,\varepsilon,\theta) = F^{\pm}_{I}(\sigma,\varepsilon)\varphi^{\pm}_{I\tau}(\theta)\,. \tag{3}$$

In the rotation part, the equation for $\varphi^{\pm}_{I\tau}(\theta)$ is

$$\left(\frac{1}{2}\sum_{i=1}^{3}\frac{\hat{I}_i^2}{\mathcal{J}_i^2} - \epsilon^{\pm}_{I\tau}\right)\varphi^{\pm}_{I\tau}(\theta) = 0\,. \tag{4}$$

Here, $\mathcal{J}_i^2 = 8B\sigma^2 J_i$ is the component of the moment of inertia; the present approximation J_i of the reduced component of the moment of inertia has the following form as in Refs. [7, 8]:

$$\begin{aligned}
J_1 &= \cos^2\varepsilon_0 \sin^2\left(\gamma_{\text{eff}} - \frac{2\pi}{3}\right) \\
&\quad + \sin^2\varepsilon_0\left(\frac{3}{2}\cos^2\eta_{\text{eff}} + \sin^2\eta_{\text{eff}} + \frac{\sqrt{15}}{2}\sin\eta_{\text{eff}}\cos\eta_{\text{eff}}\right), \\
J_2 &= \cos^2\varepsilon_0 \sin\left(\gamma_{\text{eff}} - \frac{4\pi}{3}\right) \\
&\quad + \sin^2\varepsilon_0\left(\frac{3}{2}\cos^2\eta_{\text{eff}} + \sin^2\eta_{\text{eff}} - \frac{\sqrt{15}}{2}\sin\eta_{\text{eff}}\cos\eta_{\text{eff}}\right), \\
J_3 &= \cos^2\varepsilon_0\sin^2\gamma_{\text{eff}} + \sin^2\varepsilon_0\sin^2\eta_{\text{eff}}\,,
\end{aligned}$$

where $\epsilon_{I\tau}$ is the energy of the triaxial rotator [11]. In order to separate the rotational variables θ in equation (2), we can assume that the angular variable ε in the expressions for components of the moment of inertia $\mathcal{J}_i$ is replaced by its average value $\varepsilon_0 = \langle\varepsilon\rangle$, which in the adiabatic approximation can be considered as a general constant with respect to the rotational motion [8].

The eigenfunctions $\varphi^{\pm}_{I\tau}(\theta)$ of $\hat{T}_{\text{rot}}$ are obtained as follows:

$$\varphi^{\pm}_{I\tau}(\theta) = \sum_{K\geq 0}^{I} A^{\tau}_{IK}|IMK\pm\rangle\,, \tag{5}$$

$$|IMK\pm\rangle = \frac{1}{\sqrt{2(1+\delta_{K,0})}}\Big(|IMK\rangle \pm (-1)^{I-K}|IM-K\rangle\Big)\,, \tag{6}$$

with $|IMK\rangle = \sqrt{\frac{2I+1}{8\pi^2}}D^I_{MK}(\theta)$, where $D^I_{MK}(\theta)$ is the Wigner function. Here, M are the projections of the angular momentum $\hat{I}$ on the third axis of the laboratory frames [8, 9].

3. Behavior of the alternating-parity energy levels of a rigid asymmetric rotator depending on parameter ε_0

In this section, we will study the behavior of the low-lying alternating-parity energy levels $\epsilon_{I\tau}$ of a rigid asymmetric rotator depending on the angular parameter ε_0 for fixed values of the triaxiality parameters γ_{eff} and η_{eff}. The contributions of the triaxiality parameters to full shape deformations of the nucleus are determined by the contribution of the parameter ε_0 [8, 9].

It is known that the first non-yrast band of the alternating-parity spectra is more rotational–vibrational, while the spectra of the yrast bands are more rotational. The main factor that forms the spectrum of alternating parity, the yrast and first non-yrast bands in triaxial even–even nuclei, is the rotational energy spectrum of the rigid asymmetric rotator [8, 9, 20]. The energy spectrum of the rigid asymmetric rotator, in turn, is determined by the triaxiality parameters of the quadrupole and octupole deformations γ_{eff} and η_{eff}, as well as the angular parameter ε_0. Therefore, in this paper, it is sufficient to analyze the behavior of the energy spectrum of the rigid asymmetric rotator depending on the parameter ε_0.

Dependence of the alternating-parity energy levels of the ground-state band with spins $I = 1^-; 2^+; 3^-; 4^+$ on a parameter ε_0 at fixed values of the triaxiality parameters $\gamma_{\text{eff}} = 10^\circ$ and $\eta_{\text{eff}} = 10^\circ$ is presented in Fig. 1. The energy levels of alternating parity are arranged sequentially and gradually

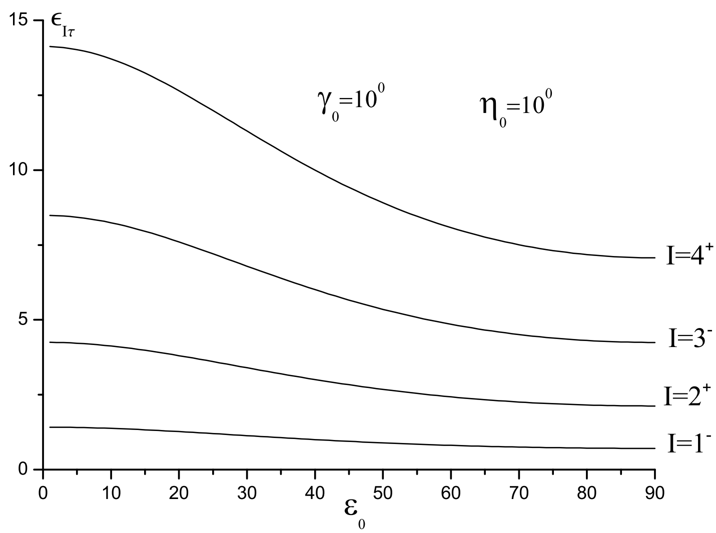

Fig. 1. Dependence of alternating-parity energy levels of the yrast band with spins $I = 1^-; 2^+; 3^-; 4^+$ on a parameter ε_0 at fixed values of the triaxiality parameters $\gamma_{\text{eff}} = 10^\circ$ and $\eta_{\text{eff}} = 10^\circ$.

decrease as the parameter ε_0 increases. Such a sequence of energy levels in experiments, where the energies of positive parity levels are located higher than the energies of negative parity levels, has not yet been observed [22].

Dependence of the alternating-parity energy levels of the ground-state band with spins $I = 1^-; 2^+; 3^-; 4^+; 5^-; 6^+; 7^-; 8^+; 9^-; 10^+$ on a parameter ε_0 at fixed values of the triaxiality parameters $\gamma_{\text{eff}} = 50°$ and $\eta_{\text{eff}} = 100°$ is presented in Fig. 2. The energy levels of alternating parity are arranged sequentially and gradually decrease as the parameter ε_0 increases. To date, no sequence of energy levels has been observed in experimental studies in which levels with positive parity are located above levels with negative parity [22].

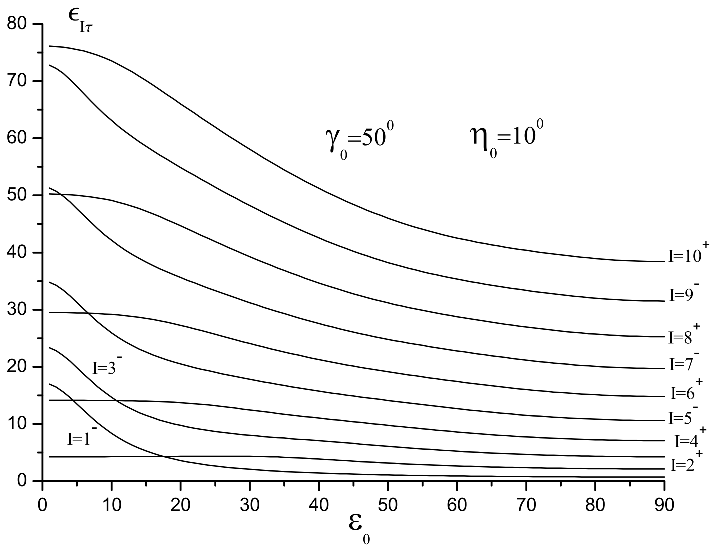

Fig. 2. Dependence of alternating-parity energy levels of the yrast band with spins $I = 1^-; 2^+; 3^-; 4^+; 5^-; 6^+; 7^-; 8^+; 9^-; 10^+$ on a parameter ε_0 at fixed values of the triaxiality parameters $\gamma_{\text{eff}} = 30°$ and $\eta_{\text{eff}} = 20°$.

Figure 3 is the same as Fig. 2, but for $\gamma_{\text{eff}} = 70°$ and $\eta_{\text{eff}} = 100°$ values. The energy levels of positive parity with spins $I = 2^+; 4^+; 6^+; 8^+; 10^+$ are arranged sequentially and gradually decrease as the parameter ε_0 increases. The energy levels of negative parity with spins $I = 1^-; 3^-; 5^-; 7^-; 9^-$ at relatively small values of the parameter ε_0 intersect the energy levels with spins $I = 2^+; 4^+; 6^+; 8^+; 10^+$. They sharply decrease for small values of the parameter ($\varepsilon_0 = 1° \div 20°$) and subsequently smoothly decrease with the growth of the parameter ε_0. Experimental observations of a sequence of states alternating-parity energy levels correspond precisely to this region of the parameter ε_0, *i.e.* $\varepsilon_0 = 1° \div 20°$. Note that for small values of the angular parameter ε_0, the contribution of the quadrupole deformation will be large.

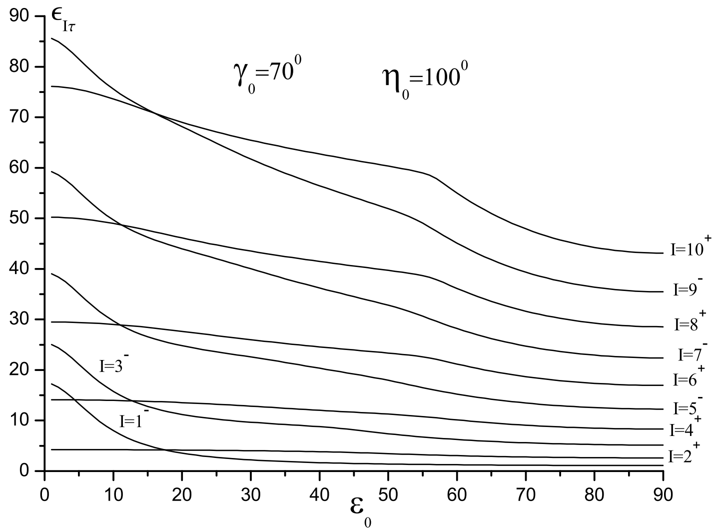

Fig. 3. The same as in Fig. 2, but at fixed values of the triaxiality parameters $\gamma_{\rm eff} = 50^{\circ}$ and $\eta_{\rm eff} = 20^{\circ}$.

The energy levels of negative parity with spins $I = 1^-$ arise as a vector sum of the states $I = 2^+$ and $I = 3^-$. Therefore, this state can satisfy the following inequalities: $1^- > 2^+$ or $1^- < 2^+$. We obtained the fulfillment of these inequalities depending on energy levels of low-lying states from the angular parameter of the polar coordinates at the different fixed values of the parameters $\gamma_{\rm eff}$ and $\eta_{\rm eff}$ in Figs. 1–2.

Energy levels of alternating parity with parity $1^- > 2^+$ have been observed in the energy spectra of most even–even nuclei [22], and within different models have been studied in Refs. [7, 11, 16, 21]. However, in experiments [22] it is possible to observe states with $1^- < 2^+$, then we can discuss the description of these states within the proposed model.

4. Conclusion

In this paper, we study the behavior of low-lying alternating-parity energy levels of the rigid asymmetric rotator as a function of the angular parameter of polar coordinates ε_0. The contribution of the triaxiality parameters to the overall deformation of the nuclear shape is determined as a function of the value of ε_0 [9]. For small values of the quadrupole and octupole triaxiality parameters $\gamma_{\rm eff} \leq 10^{\circ}$ and $\eta_{\rm eff} \leq 10^{\circ}$, with an increase in the parameter ε_0, the alternating-parity energy levels are ordered sequentially and their values gradually decrease. For small values of the parameter ϵ_0, the quadrupole deformation dominates and decreases with increasing values of this parameter. The experimentally observed region of the sequence of energy levels of excited states of alternating parity is determined as a function of the parameter ε_0 for fixed values of the triaxiality parameters γ_{eff} and η_{eff}.

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