# BOHR HAMILTONIAN OF THE EVEN–EVEN NUCLEI WITH QUADRUPOLE AND OCTUPOLE DEFORMATIONS

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Quantization of the kinetic energy of a nucleus with deformation in curvilinear coordinates in the presence of octupole vibrations of its surface is performed. The obtained Hamiltonian differs from the previously known expression for quadrupole vibrations only in the coefficients before the differentiation operators $\partial/\partial\gamma$ and $\partial/\partial\eta$. This is due to the difference in the components of the moment of inertia tensor of the nucleus for quadrupole and octupole modes. An exact expression for the full Hamiltonian of an even–even nucleus is determined, taking into account both quadrupole and octupole deformations, including seven dynamic variables. Some fields of application of the proposed Hamiltonian are discussed.

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## 1. Introduction

The main modes of excitation of medium and heavy atomic nuclei are associated with collective forms of motion, including surface and elastic vibrations [1, 2]. In this case, the shape of the nuclear surface is represented in an expansion in five parameters corresponding to the spherical harmonics of the second order, which describes quadrupole vibrations. The corresponding Hamiltonian includes both kinetic and potential components in a certain set of variables, which allows one to describe the dynamics of vibrations of the nuclear surface.

After the discovery of the Bohr Hamiltonian, many solutions of the Schrödinger eigenvalue equation have been proposed, but in many cases, they are limited to the quadrupole degree of freedom [4–6]. The Hamiltonian describing quadrupole and octupole deformations was investigated in [7], where the core surface is represented by an expansion in seven parameters, which correspond to third-order spherical harmonics. Strong octupole correlations leading to pear-shaped shapes can occur for certain numbers of protons and neutrons $Z$ and $N$ [8]. A joint parameterization of quadrupole and octupole deformations is considered in [3], where internal variables are defined in the rest frame of the general moment of inertia tensor. In this case, the non-axial deformation parameters are not fixed at zero [9].

The collective model with quadrupole and octupole deformations was applied to deformed axially symmetric nuclei in Refs. [10–15]. The importance of taking into account deformation asymmetry parameters is shown in Refs. [7, 16]. As well as in triaxial nuclei, the eigenvalue of the third component of the angular momentum is not a good quantum number ($K$-mixing) [16]. The Schrödinger equation with the collective Bohr Hamiltonian is analyzed in detail in Ref. [17]. In addition, it is of interest to analyze solutions of the Schrödinger equation with the Bohr Hamiltonian, and to study their connection with the critical behavior of nuclei [18]. Note that in all the above works, the full form of the Bohr Hamiltonian with quadrupole and octupole deformations is not given. However, it should be taken into account that it is necessary to quantize the classical kinetic energy octupole vibrations of the surface of the nucleus in curvilinear coordinates.

In [20], using the Pauli procedure [19], the quantization of the kinetic energy of a deformed nucleus in curvilinear coordinates is considered in detail as applied to quadrupole vibrations of its surface. However, the quantization of the classical kinetic energy caused by the rotation during octupole vibrations of the nucleus surface has not been carried out.

The present work is aimed at obtaining the complete form of the Bohr Hamiltonian, taking into account quadrupole and octupole deformations.

## 2. Quadrupole and octupole vibrations of the nuclear surface

The distance from the center of the nucleus to its surface, measured in the laboratory coordinate system for small deviations from the radius of the sphere $R_0$, $R(\theta,\varphi)$, can be expanded into spherical functions

$$R(\theta,\varphi) = R_0\left[1+\sum_{\lambda\mu}\alpha_{\lambda\mu}Y^*_{\lambda\mu}(\theta,\varphi)\right], \tag{1}$$

where $\theta$ and $\varphi$ are polar angles, $\lambda$ is the rank of spherical tensors $\alpha_{\lambda\mu}$, which are dynamic variables of collective motions in the nucleus and satisfy the condition $\alpha^*_{\lambda\mu} = (-1)^\mu\alpha_{\lambda,-\mu}$, which follows from the condition that spherical functions are real

$$Y^*_{\lambda\mu}(\theta,\varphi) = (-1)^{-\mu}Y_{\lambda,-\mu}(\theta,\varphi)\,. \tag{2}$$

In this paper, we will consider in detail the quadrupole ($\lambda = 2$) and octupole ($\lambda = 3$) deformations. Then we will rewrite expression (1) as

$$R(\theta,\varphi) = R_0\left[1 + \sum_{\mu=-2}^{2} \alpha_{2\mu} Y^*_{2\mu}(\theta,\varphi) + \sum_{m=-3}^{3} \alpha_{3m} Y^*_{3m}(\theta,\varphi)\right] . \tag{3}$$

In our model space, only quadrupole and octupole deformations are taken into account. We also assume that their magnitude is small enough that the introduction of a monopole or dipole term is not required to maintain constant values of the volume of the nucleus and the position of its center of mass. The hydrodynamic Bohr model is used for the description of the kinetic energy of the quadrupole and octupole deformation of the nucleus. A system of orthogonal coordinates $\xi\eta\zeta$ associated with the nucleus is introduced, and its orientation relative to the laboratory system is defined by three Euler angles $\theta_i$ ($i = 1{,}2$, and 3)

$$\begin{aligned} R(\theta,\varphi) &= R_0\left[1 + \sum_{\nu=-2}^{2} a_\nu Y^*_{2\nu}\left(\theta',\varphi'\right) + \sum_{m=-3}^{3} a_m Y^*_{3m}\left(\theta',\varphi'\right)\right] , \qquad (4)\\ a_\nu &= \sum_\mu \alpha_{\lambda\mu} D^{*\lambda}_{\lambda\mu}(\theta) , \qquad \alpha_{\lambda\mu} = \sum_\nu D^{\lambda}_{\lambda\mu}(\theta) a_\nu , \end{aligned}$$

where $D^{\lambda}_{\lambda\mu}(\theta)$ is the Wigner function.

Now we will choose the coordinate system $\xi\eta\zeta$ in such a way that the following relations are satisfied $a_1 = a_{-1} = 0$, $a_2 = a_{-2}$ and

$$a_0 = \beta_2 \cos\gamma, \qquad a_2 = a_{-2} = \frac{\beta_2 \sin\gamma}{\sqrt{2}} , \tag{5}$$

where $\beta_2 \geq 0$ is the quadrupole deformation parameter and $\gamma$ ($0 \leq \gamma \leq \frac{\pi}{3}$) is the quadrupole deformation asymmetry parameter [1, 5].

We choose the body-fixed coordinate axes, and assume that the shape of the nucleus becomes symmetrical at large octupole deformation, then the asymmetry parameters tend to zero, *i.e.* [21–23] $a_{3,\pm1} = a_{3,\pm3} = 0$, $a_{3,2} = a_{3,-2}$, and

$$a_{30} = \beta_3 \cos\eta , \qquad a_{32} = a_{3,-2} = \frac{\beta_3 \sin\eta}{\sqrt{2}} , \tag{6}$$

where $\beta_3$ is the octupole deformation parameter and the octupole deformation asymmetry parameter is $\eta$ ($0 \leq \eta \leq \pi/2$) [22, 24].

We remark that the case of $a_{3,\pm1} \neq 0$, $a_{3,\pm3} \neq 0$ was considered in Ref. [25], whereas in Ref. [26], it was shown that deformation degrees of freedom determined by $a_{3,\pm3}$ can be associated with pure two-quasiparticle states and may not need to be considered as collective variables.

We write the total energy of the quadrupole and octupole deformations of the shape of the nucleus in the following form [5]:

$$E = \frac{B_2}{2}\left(\dot{\beta}_2^2 + \beta_2^2\dot{\gamma}^2\right) + \frac{B_3}{2}\left(\dot{\beta}_3^2 + \beta_3^2\dot{\eta}^2\right) + V\left(\beta_2, \beta_3, \gamma, \eta\right) + \sum_{\kappa=1}^{3}\frac{\omega_\kappa^2 J_\kappa}{2}, \quad (7)$$

where quantities $B_2$ and $B_3$ are the quadrupole and octupole mass parameters, respectively, $V(\beta_2, \beta_3, \gamma, \eta)$ are the potential energy of $\beta_2$-, $\beta_3$-, $\gamma$-, and $\eta$-vibrations, and the last term in (7) is the rotational energy. The angular velocity components of the nucleus in the body-fixed system $\omega_\kappa$ ($\kappa = 1,2,3$) can be expressed through time derivatives of three Euler angles., $J_\kappa$ are components of the total moment of inertia and have the following form:

$$J_1 = B_2\left(3a_{20}^2 + 2\sqrt{6}a_{20}a_{22} + 2a_{22}^2\right) + 4B_3\left(3a_{30}^2 + 4a_{32}^2 + \sqrt{30}a_{30}a_{32}\right), \quad (8)$$

$$J_2 = B_2\left(33a_{20}^2 - 2\sqrt{6}a_{20}a_{22} + 2a_{22}^2\right) + 4B_3\left(3a_{30}^2 + 4a_{32}^2 - \sqrt{30}a_{30}a_{32}\right), \quad (9)$$

$$J_3 = 8B_2a_{22}^2 + 8B_3a_{32}^2. \quad (10)$$

To quantize the total energy (7), the following conditions must be met. The kinetic energy is expressed explicitly in terms of the time derivatives of collective variables and the potential energy must be written in the form of scalar variables. The total moment of inertia associated with the collective rotations was diagonal because we consider the quadrupole and octupole separately.

## 3. Quantization of the kinetic energy of an octupole deformed nucleus in curvilinear coordinates

In order to quantize the classical kinetic energy arising from rotational motion, as well as quadrupole and octupole vibrations of the nuclear surface, it is necessary to introduce the Euler angles $\theta_1$, $\theta_2$, $\theta_3$, and the internal variables $a_{20}$, $a_{22}$, $a_{30}$, and $a_{33}$. In this paper, curvilinear coordinates are understood as coordinates that cannot be represented as linear combinations of Cartesian coordinates with constant coefficients [27]. A detailed analysis of the quantization of the kinetic energy of a deformed nucleus in a curvilinear coordinate space, as applied to the internal variables $a_{20}$ and $a_{22}$, was presented in [20].

Here, we illustrate a very similar procedure for the variables $a_{30}$ and $a_{32}$. In this paper, we consider the simultaneous occurrence of quadrupole and octupole deformation of the shape of an even–even nucleus. However, we do not take into account their interaction because the problem becomes mathematically very complicated and we start with the same approximation that we can neglect their interaction. Therefore, we can write the energy of octupole vibrations by analogy with quadrupole vibrations as in [1, 3, 5]

$$E_{\beta_3} = \frac{B_3}{2}\left(\dot{\beta}_3^2 + \beta_3^2\dot{\eta}^2\right) + V(\beta_3, \eta) + \sum_{\kappa=1}^{3} \frac{{\omega'}_\kappa^2 J_\kappa^{(3)}}{2} \,. \tag{11}$$

Here, $V(\beta_3, \eta)$ is the potential energy of $\beta_3$- and $\eta$-vibrations, the last term is the rotational energy operator in the case of octupole vibrations, where $\omega'_\kappa$ are components of angular velocities of the nucleus with octupole deformation in the body-fixed system, and components of the total moment of inertia for octupole vibrations are equal to

$$\begin{aligned} J_1^{(3)} &= 4B_3\left(3a_{30}^2 + 4a_{32}^2 + \sqrt{30}a_{30}a_{32}\right) \\ &= 8B_3\beta_3^2\left(\frac{3}{2}\cos^2\eta + \sin^2\eta + \frac{\sqrt{15}}{2}\sin\eta\cos\eta\right) \,, \end{aligned} \tag{12}$$

$$\begin{aligned} J_2^{(3)} &= 4B_3\left(3a_{30}^2 + 4a_{32}^2 - \sqrt{30}a_{30}a_{32}\right) \\ &= 8B_3\beta_3^2\left(\frac{3}{2}\cos^2\eta + \sin^2\eta - \frac{\sqrt{15}}{2}\sin\eta\cos\eta\right) \,, \end{aligned} \tag{13}$$

$$J_3^{(3)} = 8B_3a_{32}^2 = 8B_3\beta_3^2\sin^2\eta \,. \tag{14}$$

From (11), we extract the kinetic energy

$$T = \frac{B_3}{2}\left(\dot{\beta}_3^2 + \beta_3^2\dot{\eta}^2\right) \,. \tag{15}$$

We quantize the classical kinetic energy associated with the rotation and octupole oscillations of the surface of the nucleus by choosing as independent variables the Euler angles $\theta_1$, $\theta_2$, $\theta_3$, and the internal variables $a_{30}$ and $a_{33}$.

Hence, we can write the kinetic energy operator [20]

$$\hat{T} = -\frac{1}{2}\hbar^2 \sum_\mu \frac{1}{m_\mu}\frac{\partial^2}{\partial x_\mu^2}, \tag{16}$$

which we use for quantization of the classical kinetic energy (15).

Now, the kinetic energy operator corresponding to the classical expression (11) can be written as [20]

$$\hat{T} = -\frac{\hbar^2}{2}\frac{1}{\sqrt{G}}\sum_{ij}\frac{\partial}{\partial q_i}\left(\sqrt{G}g^{ij}\frac{\partial}{\partial q_j}\right), \tag{17}$$

where $G$ is the determinant of the metric tensor $g_{ij}$, which is determined by the following expression:

$$g_{\mu\nu} = \sum_k J_k V_{k,\mu}(\theta_i) V_{k,\nu}(\theta_i)\,, \qquad \mu,\nu \geq 3\,, \tag{18}$$

where $V_{k,\nu}(\theta_i)$ is as in [20]

$$V_{k,l}(\theta_1,\theta_2,\theta_3) = \begin{pmatrix} -\sin\theta_2\cos\theta_3 & \sin\theta_3 & 0 \\ \sin\theta_2\cos\theta_3 & \cos\theta_3 & 0 \\ \cos\theta_2 & 0 & 1 \end{pmatrix}. \tag{19}$$

We find non-zero matrix elements of $g_{\mu\nu}$

$$g_{11} = B_3\,, \tag{20}$$
$$g_{22} = 2B_3\,, \tag{21}$$
$$g_{33} = \left(J_1\cos^2\theta_3 + J_2\sin^2\theta_3\right)\sin^2\theta_2 + J_3\cos^2\theta_2\,, \tag{22}$$
$$g_{34} = (J_2 - J_1)\sin\theta_2\cos\theta_3\sin\theta_3 = g_{43}\,, \tag{23}$$
$$g_{44} = J_1\sin^2\theta_3 + J_2\cos^2\theta_3\,, \tag{24}$$
$$g_{53} = J_3\cos\theta_2 = g_{35}\,, \tag{25}$$
$$g_{55} = J_3\,. \tag{26}$$

Using expressions (20)–(26), it is easy to find the determinant of this matrix

$$\begin{aligned} G &= \det(g_{\mu\nu}) = 2B_3^2\sin^2\theta_2 J_1 J_2 J_3 \\ &= 32B_3^5\sin^2\theta_2 a_{32}^2\left[\left(3a_{30}^2 + 4a_{32}^2\right)^2 - 30a_{30}^2 a_{32}^2\right]. \end{aligned} \tag{27}$$

The matrix $g^{\mu\nu}$, the inverse of the matrix $g_{\mu\nu}$ (18), has been found from the relation

$$g^{\mu\nu} = g_{\mu\nu}^{-1} = \frac{G_{\mu\nu}}{G}\,, \tag{28}$$

where $G_{\mu\nu}$ is the algebraic complement of the element $g_{\mu\nu}$ in the determinant of $G$. Now, we find non-zero matrix elements of $g_{\mu\nu}^{-1}$

$$g_{11}^{-1} = \frac{1}{B_3}\,, \qquad g_{22}^{-1} = \frac{1}{2B_3}\,, \tag{29}$$

$$
\begin{aligned}
g_{33}^{-1} &= \frac{2B_3^2 J_3}{G}\left(J_1 \sin^2\theta_3 + J_2\cos^2\theta_3\right) , && (30)\\
g_{34}^{-1} &= g_{43}^{-1} = \frac{-2B_3^2 J_3}{G}\sin\theta_2 \sin\theta_3\cos\theta_3\left(J_1 - J_2\right) , && (31)\\
g_{35}^{-1} &= g_{53}^{-1} = \frac{-2B_3^2 J_3}{G}\cos\theta_2\left(J_1 \sin^2\theta_3 + J_2\cos^2\theta_3\right) , && (32)\\
g_{44}^{-1} &= \frac{2B_3^2 J_3}{G}\sin^2\theta_2\left(J_1 \cos^2\theta_3 + J_2\sin^2\theta_3\right) , && (33)\\
g_{45}^{-1} &= g_{54}^{-1} = \frac{2B_3^2 J_3}{G}\cos\theta_2 \\
&\quad\times\left(-J_1\sin\theta_2\cos\theta_3\sin\theta_3 + J_2\sin\theta_2\sin\theta_3\cos\theta_3\right) , && (34)\\
g_{55}^{-1} &= \frac{2B_3}{G}\sin^2\theta_2\left[J_1J_2\cos^4\theta_3 + J_2\cos^2\theta_3\left(J_3\cot\theta_2 + 2J_1\sin^2\theta_3\right)\right.\\
&\quad\left.+J_1\sin^2\theta_3\left(J_3\cot^2\theta_2 + J_2\sin^2\theta_3\right)\right] . && (35)
\end{aligned}
$$

Note that the expression for $g_{55}^{-1}$ (35) differs from that in work [20].

Further, we find the volume element in the form

$$
\mathrm{d}\tau = |G|\mathrm{d}q_1 \ldots \mathrm{d}q_N , \tag{36}
$$

where $q_i$ are generalized coordinates.

Using formula

$$
\omega_k' = \frac{\mathrm{d}\beta_k}{\mathrm{d}t} = \sum_i V_{ki}\frac{\partial\theta_i}{\mathrm{d}t} , \tag{37}
$$

we rewrite formula (11) in the following form:

$$
E_{\beta_3} = \frac{1}{2}\sum_k J_k(a_{30}, a_{32})\left(\sum_i V_{kl}(\theta_1,\theta_2,\theta_3)\frac{\mathrm{d}\theta_i}{\mathrm{d}t}\right)^2 + \frac{1}{2}B_3\left(\dot{a}_{30}^2 + \dot{a}_{33}^2\right) . \tag{38}
$$

We choose the generalized coordinates $q_i$ (36)

$$
q_1 = a_{30} , \qquad q_2 = a_{32} , \qquad q_3 = \theta_1 , \qquad q_4 = \theta_2 , \qquad q_5 = \theta_3 . \tag{39}
$$

We determine some matrix elements $g_{\mu\nu}$ of matrix (19)

$$
\begin{aligned}
g_{11} &= B_3 , \qquad g_{22} = 2B_3 ,\\
g_{1k} &= g_{k1} \quad \text{for} \quad k \neq 1 ,\\
g_{2k} &= g_{k2} \quad \text{for} \quad k \neq 2 .
\end{aligned}
$$

Now, we rewrite the determinant $G$ (27) as

$$G^{1/2} = 16B_3^{5/2} \sin\theta_2 \left[9a_{30}^4 a_{32}^2 - 6a_{30}^2 a_{32}^4 + 16a_{32}^6\right]^{1/2}.$$

We calculate the kinetic energy operator for $g_{11}^{-1}$ and

$$-\frac{\hbar^2}{2}\frac{1}{G^{1/2}}\frac{\partial}{\partial q_1}G^{1/2}g_{11}^{-1}\frac{\partial}{\partial q_1}$$
$$= -\frac{\hbar^2}{2B_3}\left[\frac{6a_{30}\left(3a_{30}^2 - a_{32}^2\right)}{9a_{30}^4 - 6a_{30}^2a_{32}^2 + 16a_{32}^4}\frac{\partial}{\partial a_{30}} + \frac{\partial^2}{\partial a_{30}^2}\right].$$

Now we will calculate the kinetic energy operator for $g_{22}^{-1}$ and

$$-\frac{\hbar^2}{2}\frac{1}{G^{1/2}}\frac{\partial}{\partial q_2}G^{1/2}g_{22}^{-1}\frac{\partial}{\partial q_2}$$
$$= \left[\frac{3\left(3a_{30}^4 - 4a_{30}^2a_{32}^2 + 16a_{32}^4\right)}{2a_{32}\left(9a_{30}^4 - 6a_{30}^2a_{32}^2 + 16a_{32}^4\right)}\frac{\partial}{\partial a_{32}} + \frac{\partial^2}{2\partial a_{32}^2}\right].$$

We obtain the expression for the kinetic energy operator

$$\hat{T}_{a_{30},a_{32}} = -\frac{\hbar^2}{2B_3}\left[\frac{6a_{30}\left(3a_{30}^2 - a_{32}^2\right)}{9a_{30}^4 - 6a_{30}^2a_{32}^2 + 16a_{32}^4}\frac{\partial}{\partial a_{30}} + \right.$$
$$\left. + \frac{\partial^2}{\partial a_{30}^2} + \frac{3\left(3a_{30}^4 - 4a_{30}^2a_{32}^2 + 16a_{32}^4\right)}{2a_{32}\left(9a_{30}^4 - 6a_{30}^2a_{32}^2 + 16a_{32}^4\right)}\frac{\partial}{\partial a_{32}} + \frac{\partial^2}{2\partial a_{32}^2}\right].$$

We move on from $a_{30}$ and $a_{32}$ to variables $\beta_3$ and $\eta$ and consider

$$\beta_3 = \sqrt{a_{30}^2 + 2a_{32}^2}\,, \qquad \eta = \arctan\sqrt{2}\frac{a_{32}}{a_{30}}.$$

We have the final form of the expression for the classical kinetic energy (11) of the octupole $\beta_3$- and $\eta$-vibrations of the nuclear surface in curvilinear coordinates

$$\hat{T}_{\beta_3} = -\frac{\hbar^2}{2B_3}\frac{1}{\beta_3^4}\frac{\partial}{\partial\beta_3}\left(\beta_3^4\frac{\partial}{\partial\beta_3}\right), \tag{40}$$

the operator of the kinetic energy of $\beta_3$-vibration, and

$$\hat{T}_\eta = -\frac{\hbar^2}{2B_3\beta_3^2}\left[\frac{\partial^2}{\partial\eta^2} + \frac{24\cos^2 2\eta - 6\cos 2\eta}{5 + 5\cos 2\eta + 8\cos^2 2\eta}\frac{\cos\eta}{\sin\eta}\frac{\partial}{\partial\eta}\right], \tag{41}$$

the operator of the kinetic energy of $\eta$-vibration. Operators of the kinetic energy given by formulas (40) and (41) are determined in the space of variables $\beta_3$, $\eta$, $\theta_1$, $\theta_2$, $\theta_3$ with the volume element (36)

$$\mathrm{d}\tau = \beta_3^4 |\sin 3\eta| |\sin\theta_2| \mathrm{d}\beta_3 \, \mathrm{d}\eta \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2 \, \mathrm{d}\theta_3 \,. \tag{42}$$

We compare the resulting expressions (40) and (41) with a similar expression for quadrupole $\beta_2$- and $\gamma$-vibrations of the surface of the nucleus [1, 5]

$$\hat{T}_{\beta_2} = -\frac{\hbar^2}{2B_2} \frac{1}{\beta_2^4} \frac{\partial}{\partial \beta_2} \left( \beta_2^4 \frac{\partial}{\partial \beta_2} \right) , \tag{43}$$

the operator of the kinetic energies of $\beta_2$-vibration

$$\hat{T}_{\gamma} = -\frac{\hbar^2}{2B_2} \frac{1}{\sin 3\gamma} \frac{\partial}{\partial \gamma} \left( \sin 3\gamma \frac{\partial}{\partial \gamma} \right) , \tag{44}$$

the operator of the kinetic energies of $\gamma$-vibration.

It is evident that the final formulas (41) and (44) do not coincide with each other. Although the dynamic variables $a_{20}$, $a_{22}$ and $a_{30}$, $a_{32}$ have the same expressions via (5) and (6). However, the expressions for the components of their moments of inertia differ, *i.e.* the first and second terms in formulas (8) and (9).

The volume element (36) in the space of variables $\beta_2$, $\gamma$, $\beta_3$, $\eta$, $\theta_1$, $\theta_2$, $\theta_3$ is determined as

$$\mathrm{d}\tau = \beta_2^4 \beta_3^4 |\sin 3\gamma| |\sin 3\eta| |\sin\theta_2| \mathrm{d}\beta_2 \, \mathrm{d}\beta_3 \, \mathrm{d}\gamma \, \mathrm{d}\eta \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2 \, \mathrm{d}\theta_3 \,. \tag{45}$$

As noted above, quantization of the kinetic energy of a deformed nucleus over curvilinear coordinates in quadrupole space was presented in [20]. We applied an analogy of this method for the quantization of the kinetic energy of octupole deformations on curvilinear coordinates. We agree that the proposed quantization of the kinetic energy of octupole deformations is rough since the octupole collective motion is more complicated than the quadrupole one, and the theory of spherical octupole tensors is even more complicated. The principal axes of the quadrupole tensor can be determined. However, for octupole tensors, this is impossible [28]. Taking into account these and other properties of the octupole collective motion, the Hamiltonian of the kinetic energy of the octupole motion in curvilinear coordinates is presented in [28]. In addition, such a Hamiltonian is also presented in [3]. However, the Hamiltonians obtained in these works are very complex and difficult to use.

It should be noted that the Hamiltonian we proposed, despite the above-mentioned shortcomings, is very simple and easy to use. Below are some examples of its application.

The Hamiltonian of an even–even nucleus with quadrupole and octupole deformations in the curvilinear coordinates has the following form:

$$\hat{H} = \hat{T}_{\beta_2} + \hat{T}_{\beta_3} + \hat{T}_{\gamma} + \hat{T}_{\eta} + \hat{T}_{\mathrm{rot}} + V(\beta_2, \beta_3, \gamma, \eta)\,, \tag{46}$$

where V($\beta_2$), V($\beta_3$), V($\gamma$), and V($\eta$) are potential energies of $\beta_2$-, $\beta_3$-, $\gamma$-, and $\eta$-vibrations, separately.

Usually, this problem can be solved by choosing the phenomenological form of this potential energy. Often, the general form of potential energy is expressed in the following [29]:

$$V(\beta_2, \beta_3, \gamma, \eta) = V(\beta_2) + V(\beta_3) + \frac{V(\gamma)}{\beta_2^2} + \frac{V(\eta)}{\beta_3^2}\,. \tag{47}$$

A general solution to the Schrödinger equation with the operator (46) has not yet been found. Therefore, the study of collective excitations of the deformed even–even nuclei is carried out by introducing simplifying assumptions.

Hamiltonian (46) can be applied to even–even nuclei with free triaxiality.

Further, if we assume $\gamma = \gamma_{\mathrm{eff}}$, $\eta = \eta_{\mathrm{eff}}$, then the Bohr Hamiltonian can be applied to even–even nuclei with effective triaxiality which has the following form:

$$\begin{aligned}\hat{H} \;=\; & -\frac{\hbar^2}{2B_2}\frac{1}{\beta_2^3}\frac{\partial}{\partial\beta_2}\left(\beta_2^3\frac{\partial}{\partial\beta_2}\right) - \frac{\hbar^2}{2B_3}\frac{1}{\beta_3^3}\frac{\partial}{\partial\beta_2}\left(\beta_2^3\frac{\partial}{\partial\beta_3}\right) \\ & + \sum_{i=1}^{3}\frac{\hbar^2\epsilon_{I\tau}}{4J_i} + V(\beta_2, \beta_3)\,,\end{aligned} \tag{48}$$

where $\epsilon_{I\tau}$ is the energy spectrum of the triaxial quadrupole–octupole rigid rotator [16], $\tau$ labels the triaxial rigid rotator states. This Hamiltonian was applied for the description of the excited collective states of the even–even nuclei of the lanthanide and actinide region [7, 30].

If we assume that $\gamma = 0$, $\eta = 0$, and $K = 0$, then the Bohr Hamiltonian can be applied for axially-symmetric even–even nuclei which has the following form:

$$\begin{aligned}\hat{H} \;=\; & -\frac{\hbar^2}{2B_2}\frac{1}{\beta_2^2}\frac{\partial}{\partial\beta_2}\left(\beta_2^2\frac{\partial}{\partial\beta_2}\right) + -\frac{\hbar^2}{2B_3}\frac{1}{\beta_3^2}\frac{\partial}{\partial\beta_2}\left(\beta_2^2\frac{\partial}{\partial\beta_3}\right) \\ & + \frac{\hbar^2 I(I+1)}{6B_2\beta_2^2 + 6B_3\beta_3^2} + V(\beta_2, \beta_3)\,.\end{aligned} \tag{49}$$

This Hamiltonian was applied for the description of the excited collective states of the even–even nuclei of the lanthanide and actinide region [10–15].

If we assume that $\beta_2 = \beta_{2\text{eff}}$, $\beta_3 = \beta_{3\text{eff}}$, $\gamma = \gamma_{\text{eff}}$, $\eta = \eta_{\text{eff}}$, we get the Hamiltonian adiabatic approximation for the rigid rotator [16]

$$\hat{H} = \sum_{\kappa=1}^{3} \frac{{\omega'}_{\kappa}^{2} J_{\kappa}^{(3)}}{2} \,. \tag{50}$$

This Hamiltonian was applied for the description of the excited collective states of the even–even nuclei of the actinide region [16].

## 4. Conclusion

To describe quadrupole and octupole excitations in even–even nuclei, a Hamiltonian is used. For ease of analysis, this operator is expressed in a curvilinear coordinate system. A formulation of the Hamiltonian in curvilinear coordinates that takes into account quadrupole deformations has already been developed previously. In this paper, the kinetic energy of a deformed nucleus with octupole vibrations of the surface presented in curvilinear coordinates is quantized for the first time.

As a result, the exact form of the Hamiltonian for an even–even nucleus with quadrupole and octupole deformations is obtained. This Hamiltonian is determined by seven dynamic variables: $\beta_2$, $\gamma$, $\beta_3$, $\eta$, $\theta_1$, $\theta_2$, $\theta_3$.

Some fields of application of the Hamiltonian we proposed are discussed.

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