N-TUPLE COMPOUND AND COMPOUND COMBINATION SYNCHRONIZATION IN DIFFERENT CHAOTIC MODELS AND THEIR CIRCUITS IMPLEMENTATION

Tarek M. Abed-Elhameed[a,b,†], Gamal M. Mahmoud[b]
Hesham Khalaf[c,b]

[a]Department of Mathematics, College of Science and Humanities in Al-Kharj
Prince Sattam bin Abdulaziz University
Al-Kharj 11942, Saudi Arabia
[b]Department of Mathematics, Faculty of Science, Assiut University
Assiut 71516, Egypt
[c]Department of Technology, College of Technology and Business
Riyadh Elm University, Riyadh 12734, Saudi Arabia

Received 22 July 2025, accepted 29 September 2025,
published online 14 October 2025

In this paper, we introduced two definitions of N-tuple compound synchronization (NCS) and N-tuple compound combination synchronization (NCCS) in $(2N+2)$ and $(3N+2)$ different chaotic models, respectively. The analytical formulas for the control functions are derived from two established theorems in order to perform these types of synchronization. These new types of synchronization are considered a generalization of various types disccussed in the literature. Numerous applications in engineering and physics may benefit from these new types of synchronization, *e.g.*, image encryption and electronic circuits. Using the active control technique for the choice $N = 5$, we study two examples of 5CS and 5CCS in 12 and 17 different chaotic models. The analytical forms of the control functions are used and good agreement is found. The Runge–Kutta method of order four is used in our numerical simulations. Other examples can be similarly studied. We designed an electronic circuit for the proposed N-tuple compound synchronization in 12 different chaotic models. Using MATLAB/Simulink, both numerical and simulation results show good agreement. For other drive models, similar circuit implementations can be created.

DOI:10.5506/APhysPolB.56.10-A3

† Corresponding author: tarekmalsbagh@aun.edu.eg

1. Introduction

Chaotic and hyperchaotic models have received increased attention over the past few decades. Recently, they have many applications in many fields such as machine learning, data analysis, and biology science [1–4]. As a basic system of atmospheric convection, Lorenz [5] introduced the first canonical chaotic attractor. By developing the complex Lorenz models as an extension of the original Lorenz equations, the authors of [6] opened a way for investigating complex chaos. The simplified Lorenz model with a single parameter was proposed by the authors in [7]. For the integer order derivative, the dynamical characteristics and synchronization of the complex simplified Lorenz model were examined [8].

The simplified Lorenz system [7] can be expressed as follows:

$$\begin{aligned}\dot{x}_1 &= 10(x_2 - x_1)\,,\\ \dot{x}_2 &= (24-4\alpha)x_1 + \alpha x_2 - x_1 x_3\,,\\ \dot{x}_3 &= -\frac{8}{3}x_3 + x_1 x_2\,,\end{aligned} \tag{1}$$

where the real state variables $x_1, x_2, x_3 \in \boldsymbol{R}$, the real bifurcation parameter $\alpha \in \boldsymbol{R}$ of model (1), and the time derivative is dot.

For $\alpha = 2$ and the initial conditions $(x_1, x_2, x_3)^{\mathrm{T}}(0) = (0.1, 0.2, 0.3)^{\mathrm{T}}$, system (1) has a chaotic attractor [7] (see Fig. 1).

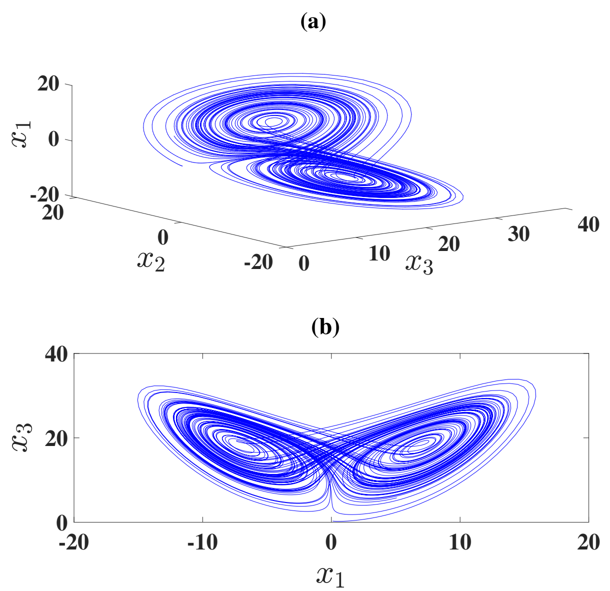

Fig. 1. Chaotic attractor of model (1) for $\alpha = 2$, (a) (x_3, x_2, x_1) space and (b) (x_1, x_3) space.

The synchronization of hyperchaotic (or chaotic) models is an essential subject of research in non-linear dynamical models. It has been thoroughly examined in numerous areas such as image encryption, secure communication, neural networks, astronomy, economics, ecology, and many more [9–14]. Two models can simultaneously generate the same signals when they are in synchronization. This indicates that the error states of synchronized models approach zero. We are motivated by the original technique presented by [15] to drive two identical models into synchronization. Several types of synchronization were investigated for integer-order models such as complete synchronization [15], combination synchronization [16, 17], combination–combination synchronization [17, 18], compound synchronization [19], double compound synchronization [20], compound combination synchronization [21], double compound combination synchronization [22], triple compound synchronization [23], and triple compound combination synchronization [24]. Jahanzaib *et al.* [25] proposed a novel synchronization scheme called quad compound combination anti-synchronization in ten fractional-order chaotic models. Recently, Khattar *et al.* [26–29] investigated many other types of synchronization.

The chaotic Rössler model is given by [30]

$$\begin{aligned} \dot{y}_1 &= -y_2 - y_3\,, \\ \dot{y}_2 &= y_1 + \frac{2}{10}y_2\,, \\ \dot{y}_3 &= \frac{2}{10} + y_3(y_1 - 5.7)\,, \end{aligned} \tag{2}$$

where $y_1, y_2, y_3 \in \boldsymbol{R}$ are real state variables of model (2).

The chaotic Chen model [31] is

$$\begin{aligned} \dot{z}_1 &= b_1(z_2 - z_1)\,, \\ \dot{z}_2 &= (b_3 - b_1)z_1 - z_1 z_3 + b_3 z_2\,, \\ \dot{z}_3 &= -b_2 z_3 + z_1 z_2\,, \end{aligned} \tag{3}$$

where the real state variables $z_1, z_2, z_3 \in \boldsymbol{R}$ and $b_1, b_2, b_3 \in \boldsymbol{R}$ are constant parameters (3).

The chaotic Burke–Show model [32] is

$$\begin{aligned} \dot{u}_1 &= -\rho(u_1 + u_2)\,, \\ \dot{u}_2 &= -u_2 - \rho u_1 u_3\,, \\ \dot{u}_3 &= \sigma + \rho u_1 u_2\,, \end{aligned} \tag{4}$$

where the real state variables $u_1, u_2, u_3 \in \boldsymbol{R}$ and $\rho, \sigma \in \boldsymbol{R}$ are constant parameters of (4).

The chaotic Lü model is defined by [33]

$$\begin{aligned} \dot{v}_1 &= a_1(v_2 - v_1)\,, \\ \dot{v}_2 &= a_3 v_2 - v_1 v_3\,, \\ \dot{v}_3 &= -a_2 v_3 + v_1 v_2\,, \end{aligned} \tag{5}$$

where the real state variables $v_1, v_2, v_3 \in \boldsymbol{R}$ and $a_1, a_2, a_3 \in \boldsymbol{R}$ are constant parameters of (5).

Mahmoud *et al.* [34] designed a circuit realization for the hyperchaotic complex Lü systems using MATLAB/Simulink in order to confirm the hyperchaotic behaviour. In [35], the viability of the new chaotic model with a stable fixed point is examined by demonstrating its electronic circuit implementation. Using MATLAB/Simulink, the simulation of the chaotic Chua system is presented, for more details, see [22]. The authors of [36] designed an electronic circuit implementation of a chaotic model having stationary points in the rounded square loop. Furthermore, the implementation of an electronic circuit of the novel chaotic model with a line of stationary points is presented, and a very good agreement is demonstrated between the MultiSim outcomes and the MATLAB simulations of the theoretical model in [37].

Motivated by previous research, this paper aims at proposing two definitions of N-tuple compound synchronization (NCS) and N-tuple compound combination synchronization (NCCS) in $(2N + 2)$ and $(3N + 2)$ different chaotic models using an active control technique [38]. To achieve these kinds of synchronization, we present two theorems to derive the analytical formulas for the control functions. In the literature, these synchronization types are regarded as generalizations of many other [19–25]. Using models (1)–(5), we give two examples of the 5-tuple compound and compound combination synchronization in twelve and seventeen different chaotic models, respectively. The validity of the analytical control functions is verified using numerical tests, which show a good degree of agreement between them. Also, we designed a circuit implementation as an application for the proposed synchronization using MATLAB/Simulink.

Our article is organised as follows: Section 2 contains a definition of NCS in different chaotic models. The analytical formula for the control functions is derived from verified theorem. In like manner, we stated the definition of NCCS in different chaotic models in Section 3. A scheme to achieve NCCS is given. In Section 4, two examples of N-tuple compound and compound combination synchronization are given to indicate a very good agreement between both numerical and analytical results. The electronic circuit of the proposed synchronization is given in Section 5. We used MATLAB/Simulink to construct a block diagram and we found a good agreement of numerical and simulation results of our proposed synchronization. Section 6 contains the conclusion of our observations.

2. N-tuple compound synchronization in $(2N+2)$ chaotic models

This section presents the layout of a distinct type of NCS using an active control method. This synchronization type is considered a generalization of several types described in the literature [19–25]. We give the definition and scheme of NCS in $(2N+2)$ different chaotic models. We consider the $(N+2)$ drive models and N response models which, respectively, are

$$\begin{aligned}\dot{x} &= f(x)\,, \\ \dot{y} &= g(y)\,, \\ \dot{z}_i &= h(z_i)\,, \qquad\qquad i = 1, 2, \ldots, N\,, \end{aligned} \tag{6}$$

$$\dot{u}_i = \xi_i(u_i) + m_i\,, \qquad i = 1, 2, \ldots, N\,, \tag{7}$$

where $x = \mathrm{diag}(x_1, x_2, \ldots, x_n)$, $y = \mathrm{diag}(y_1, y_2, \ldots, y_n)$, $z_i = \mathrm{diag}(z_{i1}, z_{i2}, \ldots, z_{in})$, and $u_i = \mathrm{diag}(u_{i1}, u_{i2}, \ldots, u_{in})$ represent the state matrices of models (6)–(7). Additionally, f, g, h_i, ξ_i are $(n \times n)$ diagonal continuous functions and $m_i = \mathrm{diag}(m_{i1}, m_{i2}, \ldots, m_{in})$ are the matrices of control functions, which depend on x, y, z_i, u_i, $i = 1, 2, \ldots, N$.

Definition 2.1. The NCS in $(N + 2)$ drive models (6) and another N response models (7) can be achieved if

$$\lim_{t\to\infty} \|e\| = \lim_{t\to\infty} \left\| \sum_{i=1}^{N} \left[(Ax + By)z_i - C_i u_i\right] \right\| = 0\,, \tag{8}$$

where $A = \mathrm{diag}(a_1, a_2, \ldots, a_n)$, $B = \mathrm{diag}(b_1, b_2, \ldots, b_n)$, $C_i = \mathrm{diag}(c_{i1}, c_{i2}, \ldots, c_{in})$ are nonzero constant matrices, the synchronization error is $e = \mathrm{diag}(e_1, e_2, \ldots, e_n)$, and $\|\cdot\|$ is the matrix norm.

Remark 2.1. For the choice $N = 4$, the quad compound synchronization [25] can be obtained.

Remark 2.2. We get the triple compound synchronization [23] for the choice $N = 3$.

Remark 2.3. The double compound synchronization [20] is given, if $N = 2$, while we can get the compound synchronization [19] for the choice $N = 1$.

Remark 2.4. One can obtain the combination synchronization [16, 17], if $N = 1$ and $z_1 = \text{constant}$.

Theorem 2.1. *If the control functions are built as follows, the NCS in $(N+2)$ drive models (6) and N response models (7) will be achieved*

$$U=\sum_{i=1}^{N} C_i m_i = Ke+\sum_{i=1}^{N}\left[-C_i\xi_i(u_i)+(Af(x)+Bg(y))z_i+(Ax+By)h_i(z_i)\right] , \tag{9}$$

where the control gain matrix is denoted $K = diag(k_1, k_2, \ldots, k_n)$.

Proof. Using (8), we get

$$\begin{aligned}\dot{e} &= \sum_{i=1}^{N}[(Af(x)+Bg(y))z_i+(Ax+By)h_i(z_i)-C_i\xi_i(u_i)-C_i m_i] \\ &= \sum_{i=1}^{N}[(Af(x)+Bg(y))z_i+(Ax+By)h_i(z_i)-C_i\xi_i(u_i)]-U\,, \end{aligned} \tag{10}$$

substituting Eq. (9) into Eq. (10), we have

$$\begin{aligned}\dot{e} &= \sum_{i=1}^{N}[(Af(x)+Bg(y))z_i+(Ax+By)h_i(z_i)-C_i\xi_i(u_i)]-Ke \\ &\quad -\sum_{i=1}^{N}[-C_i\xi_i(u_i)+(Af(x)+Bg(y))z_i+(Ax+By)h_i(z_i)]\,, \end{aligned} \tag{11}$$

then,

$$\dot{e} = -Ke\,. \tag{12}$$

A Lyapunov function is defined as follows:

$$V(t)=\frac{1}{2}e^2\,, \tag{13}$$

then, we get

$$\dot{V}(t)=e\dot{e}=-Ke^2=-2KV(t)\,. \tag{14}$$

The error $e(t)\to 0$ as $t\to\infty$, because $V(t)$ is a positive definite and $\dot{V}(t)$ is negative definite, and hence the NCS in $(N+2)$ drive models (6) and N response models (7) can be done. □

3. N-tuple compound combination synchronization in $(3N+2)$ chaotic models

In a similar way, we present the definition and scheme of NCCS in $(3N+2)$ different chaotic models. The $(N+2)$ drive models are considered as in

Eqs. (6) and the first N response models are given in Eqs. (7), while the second N response models are

$$\dot{v}_i = \eta_i(v_i) + s_i \,, \qquad i = 1, 2, \ldots, N \,, \tag{15}$$

where $v_i = \mathrm{diag}(v_{i1}, v_{i2}, \ldots, v_{in})$ represent the state matrices of models (15) and $s_i = \mathrm{diag}(s_{i1}, s_{i2}, \ldots, s_{in})$ are the matrices of control functions, which depend on $x, y, z_i, u_i, v_i, i = 1, 2, \ldots, N$.

Definition 3.1. The NCCS in $(N + 2)$ drive models (6) and $2N$ response models (7) and (15) can be performed if

$$\lim_{t\to\infty} \|e\| = \lim_{t\to\infty} \left\| \sum_{i=1}^{N} [(Ax + By)z_i - C_i u_i - D_i v_i] \right\| = 0 \,, \tag{16}$$

where $D_i = \mathrm{diag}(d_{i1}, d_{i2}, \ldots, d_{in})$ are nonzero constant matrices.

Remark 3.1. For $N = 4$, the quad compound combination synchronization [25] is obtained.

Remark 3.2. By selecting $N = 3$, the triple compound combination synchronization [24] can be obtained.

Remark 3.3. The double compound combination synchronization [22] is given, if $N = 2$, while we can get the compound combination synchronization [21] for the choice $N = 1$.

Remark 3.4. One can obtain the combination–combination synchronization [18], if $N = 1$ and $z_1 =$ constant.

Remark 3.5. In the literature, the NCCS is considered a generalization of various synchronization types, see, *e.g.*, [19–25].

Theorem 3.1. *The NCCS in (N + 2) drive models (6) and 2N response models (7) and (15) will be done, if the control functions are described as*

$$\begin{aligned} U &= \sum_{i=1}^{N} C_i m_i \\ &= Ke + \sum_{i=1}^{N} [-C_i\xi_i(u_i) - D_i\eta_i(v_i) + (Af(x) + Bg(y))z_i + (Ax + By)h_i(z_i)] \,, \end{aligned} \tag{17}$$

where the control gain matrix is denoted $K = \mathit{diag}(k_1, k_2, \ldots, k_n)$.

Proof. The proof can be written as we did in Theorem 2.1. □

4. Examples of N-tuple compound, and compound combination synchronization

Two illustrative examples are presented in this section to show the validity of our schemes which were investigated in Sections 2 and 3. We choose $N = 5$, which gives a 5-tuple compound and compound combination synchronization.

4.1. An example of 5-tuple compound synchronization

In this part, we give an example of the 5-tuple compound synchronization in twelve models. We consider the first two drive models from Eqs. (1)–(2) and the other five drive models can be written from model (3) as

$$\begin{aligned}
\dot{z}_{i1} &= b_{i1}(z_{i2} - z_{i1}),\\
\dot{z}_{i2} &= (b_{i3} - b_{i1})z_{i1} - z_{i1}z_{i3} + b_{i3}z_{i2},\\
\dot{z}_{i3} &= -b_{i2}z_{i3} + z_{i1}z_{i2}, \qquad i = 1, 2, \ldots, 5,
\end{aligned} \tag{18}$$

while five response models can be written from model (4) as follows:

$$\begin{aligned}
\dot{u}_{i1} &= -\rho_i(u_{i1} + u_{i2}) + m_{i1},\\
\dot{u}_{i2} &= -u_{i2} - \rho_i u_{i1}u_{i3} + m_{i2},\\
\dot{u}_{i3} &= \sigma_i + \rho_i u_{i1}u_{i2} + m_{i3}, \quad i = 1, 2, \ldots, 5,
\end{aligned} \tag{19}$$

where $m_i = \mathrm{diag}(m_{i1}, m_{i2}, m_{i3})$ are the matrices of control functions.

For the choice $A = \mathrm{diag}(1, 2, 1)$, $B = \mathrm{diag}(1, -2, 1)$, and $C_i = \mathrm{diag}(7, 6, 1.5)$, $i = 1, 2, \ldots, 5$, and applying Theorem 2.1, the control functions are

$$\begin{aligned}
&\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix}\\
&= \begin{pmatrix}
\sum_{i=1}^{5}[-7(\rho(u_{i1} + u_{i2})) - (10(x_2 - x_1) - (y_2 + y_3))z_{i1} - b_{i1}(x_1 + y_1)(z_{i2} - z_{i1})]\\
\sum_{i=1}^{5}[6(-u_{i2} - \rho u_{i1}u_{i3}) - (2((24 - 4\alpha)x_1 + \alpha x_2 - x_1x_3) - 2(y_1 + 0.2y_2))z_{i2}]\\
\sum_{i=1}^{5}[1.5(\sigma + \rho u_{i1}u_{i2}) - (x_1x_2 - \frac{8}{3}x_3 + 0.2 + y_3(y_1 - 5.7))z_{i3}]
\end{pmatrix}\\
&- \begin{pmatrix}
-k_1e_1\\
\sum_{i=1}^{5}[(2x_2 - 2y_2)((b_{i3} - b_{i1})z_{i1} - z_{i1}z_{i3} + b_{i3}z_{i2})] - k_2e_2\\
\sum_{i=1}^{5}[(x_3 + y_3)(-b_{i2}z_{i3} + z_{i1}z_{i2})] - k_3e_3
\end{pmatrix},
\end{aligned} \tag{20}$$

where $U \equiv \mathrm{diag}(U_1, U_2, U_3) = \sum_{i=1}^{5} C_i m_i$ is the matrix of control functions, $K = \mathrm{diag}(k_1, k_2, k_3)$ is gain control matrix, and $e \equiv \mathrm{diag}(e_1, e_2, e_3) \equiv \mathrm{diag}(d_{11}, d_{12}, d_{13}) - \mathrm{diag}(r_{11}, r_{12}, r_{13}) = \mathrm{diag}(\sum_{i=1}^{5}[(x_1 + y_1)z_{i1} - 7u_{i1}], \sum_{i=1}^{5}[(2x_2 - 2y_2)z_{i2} - 6u_{i2}], \sum_{i=1}^{5}[(x_3 + y_3)z_{i3} - 1.5u_{i3}])$.

4.1.1. Numerical simulations

In numerical simulations for $\alpha = 2$, $b_{1i} = 40$, $b_{2i} = 3$, $b_{3i} = 28$, $\rho_i = 10$, $\sigma_i = 8/3$; $i = 1, 2, \ldots, 5$, $K = \mathrm{diag}(3, 5, 7)$, and the initial values for the seven drive models (1)–(2) and (18) and the five response models (19), respectively, $x_0 = \mathrm{diag}(0.1, 0.3, 1)$, $y_0 = \mathrm{diag}(-0.1, 0.5, -0.6)$, $z_{0i} = \mathrm{diag}(0.1, 0.3, 1)$, and $u_{0i} = \mathrm{diag}(0.3, -0.6, 2)$; $i = 1, 2, \ldots, 5$, the 5-tuple compound synchronization is performed as depicted in Figs. 2–3. The state variables after the 5-tuple compound synchronization among the drive models (1)–(2), and (18) and the response models (19) is shown in Fig. 2. Figure 3 describes the relationship between the response and drive models, which is evidently expressed as the line $y = x$. This means the synchronization errors approach zeros.

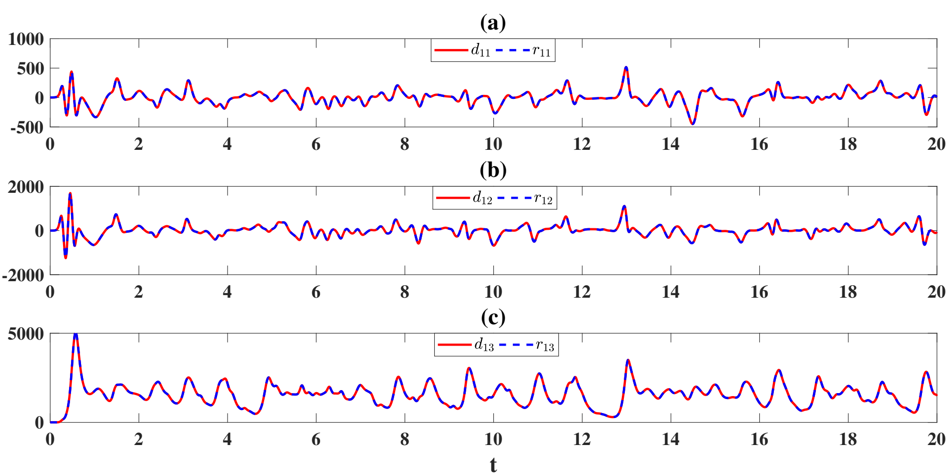

Fig. 2. The state variables for the 5-tuple compound synchronization for the seven drive models (1)–(2), and (18) (solid curves) and the five response models (19) (dashed curves): (a) d_{11} and r_{11} *versus* t, (b) d_{12} and r_{12} *versus* t, (c) d_{13} and r_{13} *versus* t.

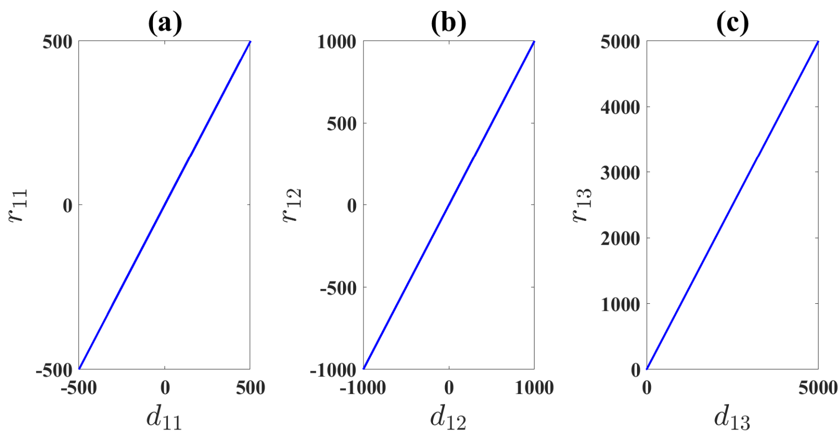

Fig. 3. The 5-tuple compound synchronization results in the seven drive models (1)–(2), and (18) and the five response models (19): (a) (d_{11}, r_{11}) space, (b) (d_{12}, r_{12}) space, (c) (d_{13}, r_{13}) space.

4.2. An example of 5-tuple compound combination synchronization

In this part, we give an example of the 5-tuple compound combination synchronization in seventeen models. We consider the seven drive models (1)–(2) and (18), and the first five response models (19), while the second five response models can be written from model (5) as

$$\begin{aligned} \dot{v}_{i1} &= a_{i1}(v_{i2} - v_{i1}) + s_{i1} \,, \\ \dot{v}_{i2} &= a_{i3}v_{i2} - v_{i1}v_{i3} + s_{i2} \,, \\ \dot{v}_{i3} &= -a_{i2}v_{i3} + v_{i1}v_{i2} + s_{i3} \,, \end{aligned} \tag{21}$$

where $s_i = \operatorname{diag}(s_{i1}, s_{i2}, s_{i3})$ are the matrices of control functions.

For the same choice of A, B, C_i as in Subsection 4.1 and $D_i = \operatorname{diag}(7, 6, 1.5)$, $i = 1, 2, \ldots, 5$, and applying Theorem 3.1, we can write the control functions (17) as

$$\begin{aligned} &\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \\ &= \begin{pmatrix} \sum\limits_{i=1}^{5} [7(-\rho(u_{i1} + u_{i2}) + a_{i1}(v_{i2} - v_{i1})) - (10(x_2 - x_1) - (y_2 + y_3))z_{i1} - b_{i1}(x_1 + y_1)(z_{i2} - z_{i1})] \\ \sum\limits_{i=1}^{5} [6(-u_{i2} - \rho u_{i1}u_{i3} + a_{i3}v_{i2} - v_{i1}v_{i3}) - (2((24 - 4\alpha)x_1 + \alpha x_2 - x_1x_3) - 2(y_1 + 0.2y_2))z_{i2}] \\ \sum\limits_{i=1}^{5} [1.5(\sigma + \rho u_{i1}u_{i2} - a_{i2}v_{i3} + v_{i1}v_{i2}) - \left(x_1x_2 - \frac{8}{3}x_3 + 0.2 + y_3(y_1 - 5.7)\right) z_{i3}] \end{pmatrix} \\ &- \begin{pmatrix} -k_1e_1 \\ \sum\limits_{i=1}^{5} [(2x_2 - 2y_2)((b_{i3} - b_{i1})z_{i1} - z_{i1}z_{i3} + b_{i3}z_{i2})] - k_2e_2 \\ \sum\limits_{i=1}^{5} [(x_3 + y_3)(-b_{i2}z_{i3} + z_{i1}z_{i2})] - k_3e_3 \end{pmatrix} , \end{aligned} \tag{22}$$

where $U \equiv \mathrm{diag}(U_1, U_2, U_3) = \sum_{i=1}^{5}(C_i m_i + D_i s_i)$ is the matrix of control functions, $K = \mathrm{diag}(k_1, k_2, k_3)$ is gain control matrix, and $e \equiv \mathrm{diag}(e_1, e_2, e_3) \equiv \mathrm{diag}(d_{21}, d_{22}, d_{23}) - \mathrm{diag}(r_{21}, r_{22}, r_{23}) = \mathrm{diag}(\sum_{i=1}^{5}[(x_1 + y_1)z_{i1} - 7(u_{i1} + v_{i1})],$ $\sum_{i=1}^{5}[(2x_2 - 2y_2)z_{i2} - 6(u_{i2} + v_{i2})], \sum_{i=1}^{5}[(x_3 + y_3)z_{i3} - 1.5(u_{i3} + v_{i3})])$.

4.2.1. Numerical simulations

In numerical simulations, for the same initial values and the same parameters for the seven drive models (1)–(2) and (18) which were used in Subsection 4.1 except $K = \mathrm{diag}(1, 2, 3)$ and the initial values of the ten response models (19) and (21) $u_{0i} = v_{0i} = \mathrm{diag}(-0.14, -0.43, 6.7)$; $i = 1, 2, \ldots, 5$, the 5-tuple compound combination synchronization is done as seen in Figs. 4–5. Figure 4 indicates the 3D space of state variables after the 5-tuple compound combination synchronization in the drive models (1)–(2) and (18), and the response models (19) and (21). The synchronization errors approach zeros as illustrated in Fig. 5.

Remark 4.1. Further examples of N-tuple compound and compound combination synchronization can be similarly studied.

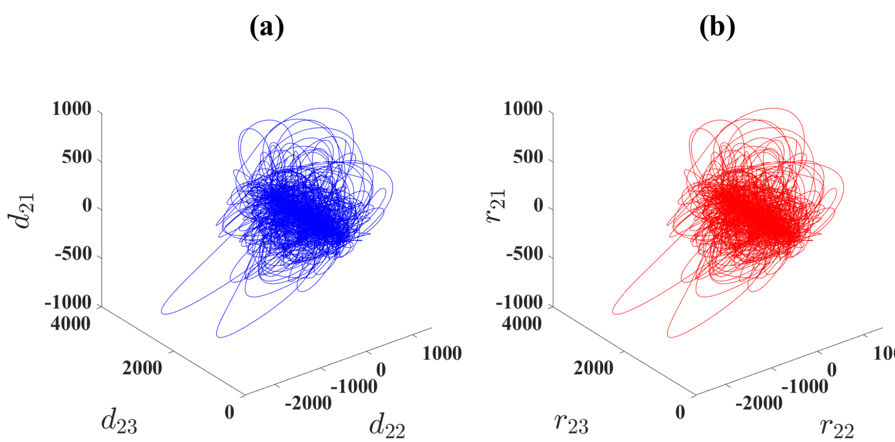

Fig. 4. The 5-tuple compound combination synchronization of: (a) the seven drive models (1)–(2) and (18) in (d_{22}, d_{23}, d_{21}) space, and (b) the ten response models (19) and (21) in (r_{22}, r_{23}, r_{21}) space.

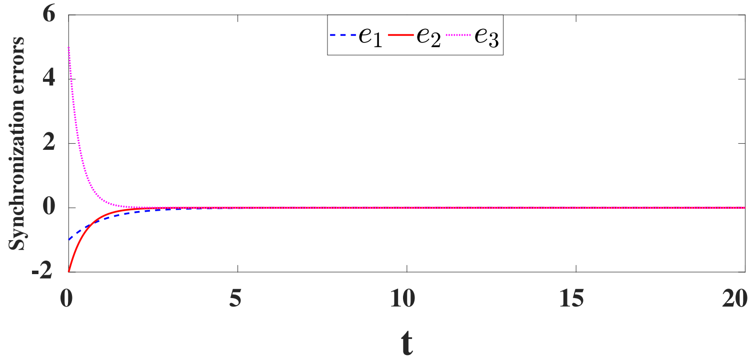

Fig. 5. The 5-tuple compound combination synchronization errors for the drive models (1)–(2), and (18) and the response models (19) and (21).

5. Circuit implementation of the 5-tuple compound synchronization (5CS)

We designed a circuit implementation for the 5-tuple compound synchronization of twelve models. The circuit for one model is presented in detail, while the circuits for the remaining eleven are omitted to avoid repetition, as they are functionally identical. An electronic circuit for the chaotic simplified Lorenz model (1) is designed, see Fig. 6, which is one of five drive models of 5CS (see Subsection 4.1). The simulation results of this circuit are compared with the numerical ones of Section 1 (Fig. 1). The circuit equations with their parameters take the form

$$\begin{aligned}
\dot{x}_1 &= \frac{1}{C_1 R_1}(x_2 - x_1)\,,\\
\dot{x}_2 &= \frac{1}{C_2 R_2}x_1 + \frac{1}{C_2 R_3}x_2 - \frac{1}{C_2 R_4}x_1 x_3\,,\\
\dot{x}_3 &= \frac{-1}{C_3 R_5}x_3 + \frac{1}{C_3 R_6}x_1 x_2\,.
\end{aligned} \tag{23}$$

The circuit has three channels that are used to add, subtract, and integrate the state variables x_1, x_2, and x_3. The state voltages of the three channels are represented by these variables. The values of circuit elements which correspond to model (1) with the same parameters values as in Fig. 1, are $R_1 = 0.005\ k\Omega$, $R_2 = 0.00416\ k\Omega$, $R_3 = 0.03333\ k\Omega$, $R_4 = 0.06667\ k\Omega$, $R_5 = 0.075\ k\Omega$, $R_6 = 0.2\ k\Omega$, $R = 10\ k\Omega$, and the capacitors are $C_1 = 20\ \mu$F, $C_2 = 15\ \mu$F, and $C_3 = 5\ \mu$F. Figure 6 shows the circuit implementation of model (23), the output of this circuit is the solution of model (23). The block diagram simulation is illustrated in Fig. 7. The numerical solution in Fig. 1 agrees with the equivalent simulation observations of the chaotic simplified Lorenz model (23), as seen in Fig. 8. Similarly, we can design circuit implementation for the other five drive models.

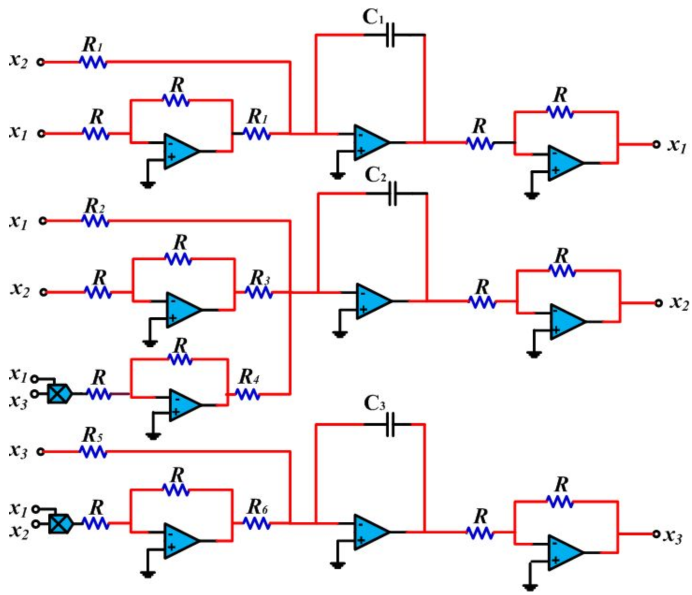

Fig. 6. Circuit diagram of the chaotic simplified Lorenz model (23).

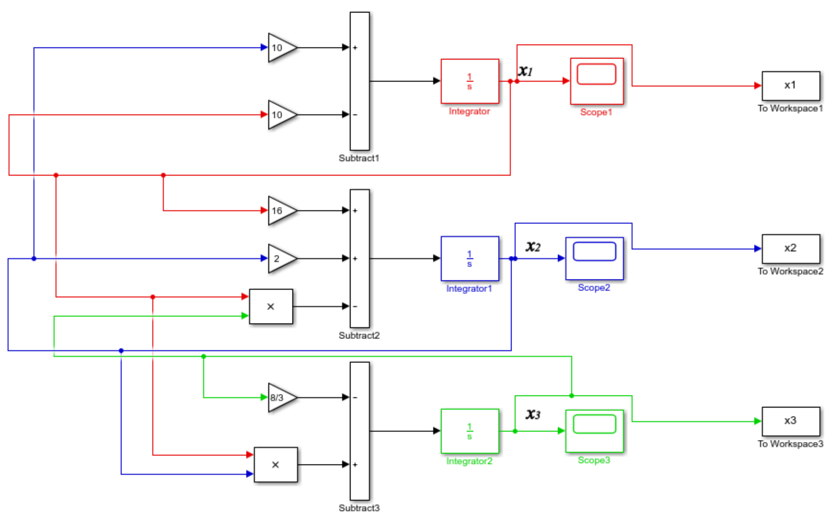

Fig. 7. Block diagram of the chaotic simplified Lorenz model (23) using MATLAB/Simulink.

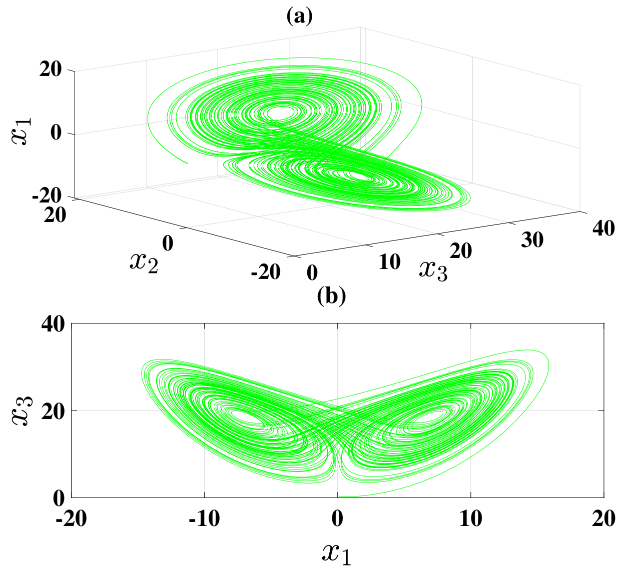

Fig. 8. Simulation observations of model (23) for the same parameters as in Fig. 1: (a) (x_3, x_2, x_1) space and (b) (x_1, x_3) space.

Additionally, MATLAB/Simulink provided the 5-tuple compound synchronization results, which are displayed in Figs. 9–10. Figure 9 shows the same results as Fig. 2, while Fig. 10 indicates the synchronization errors

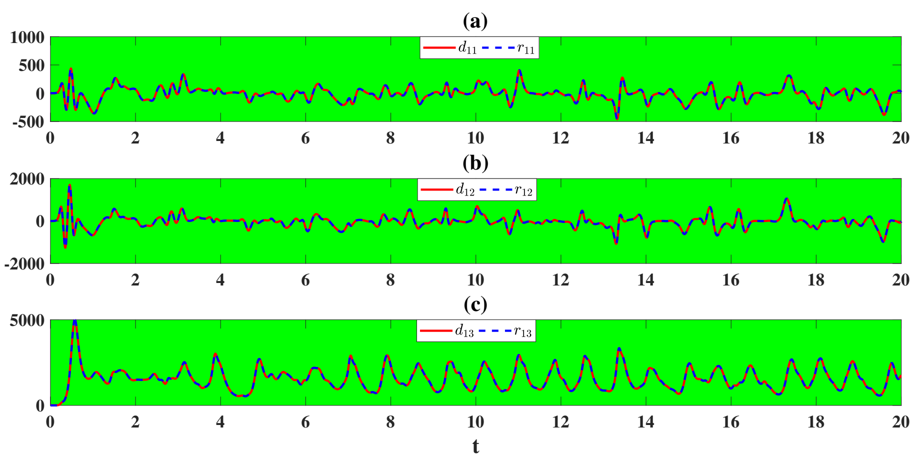

Fig. 9. Simulation observations of the 5-tuple compound synchronization for the seven drive models (1)–(2) and (18) (solid curves), and the five response models (19) (dashed curves): (a) d_{11} and r_{11} *versus* t, (b) d_{12} and r_{12} *versus* t, (c) d_{13} and r_{13} *versus* t.

which go to zeros as in Fig. 3. This means that there is a good agreement between the results of Figs. 9–10 and 2–3. Similar results for 5CCS can be calculated in the similar way as for 5CS.

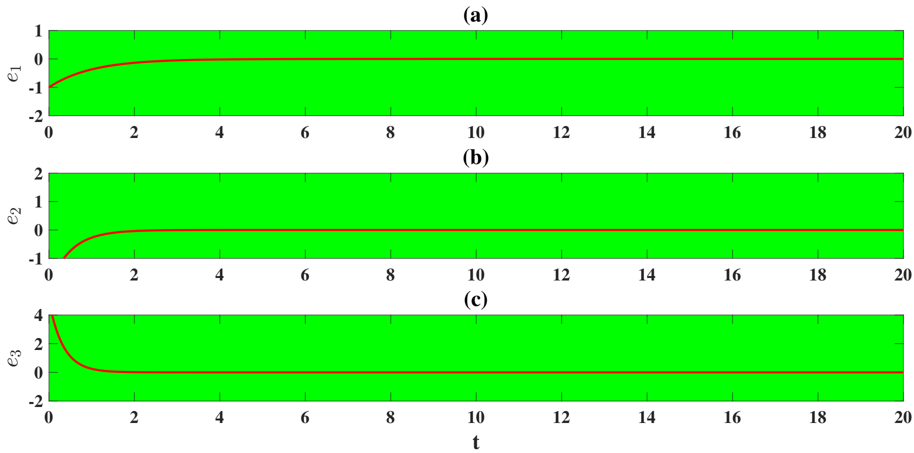

Fig. 10. Simulation observations of the 5-tuple compound synchronization errors for the seven drive models (1)–(2) and (18), and the five response models (19): (a) (t, e_1), (b) (t, e_2), (c) (t, e_3).

6. Conclusions

The definitions 2.1 and 3.1 of the N-tuple compound synchronization (NCS) and N-tuple compound combination synchronization (NCCS) are introduced. Theorems 2.1 and 3.1 are presented to give us the analytical expressions of control functions (9) and (17) to achieve NCS and NCCS, respectively. The comparisons between the two proposed kinds of synchronization and the previous work are shown in Remarks 2.1–2.4 and 3.1–3.5. These kinds of synchronization may be useful in a wide range of engineering and physics applications, including electronic circuits and image encryption. As special cases, we present two examples of the 5-tuple compound and 5-tuple compound combination synchronization in twelve and seventeen models depending on models (1)–(5). Very good agreement between numerical and analytical results is demonstrated, see Figs. 2–5. The electronic circuit for the chaotic simplified Lorenz model (1) is designed as shown in Fig. 6. Good agreement between the simulation observations of this circuit and the numerical solutions is found.

This study is supported via funding from Prince Sattam bin Abdulaziz University, Saudi Arabia project number (PSAU/2025/R1447).

REFERENCES

[1] M. Nonaka, M. Agüero, M. Kovalsky, «Machine learning algorithms predict experimental output of chaotic lasers», *Optics Lett.* **48**, 1060 (2023).

[2] D.G. Nair, K.S. Kumar, S.A.H. Nair, «Chaos Game Optimization with Machine Learning Enabled Social Distance Detection and Classification Model», *Int. J. Intell. Syst. Appl. Eng.* **12**, 675 (2024).

[3] G. Margazoglou, L. Magri, «Stability analysis of chaotic systems from data», *Nonlinear Dyn.* **111**, 8799 (2023).

[4] S. Raczynski, «Chaotic and Oscillatory Behavior of an Epidemic: Agent-Based Model», *Nonlinear Dynamics Psychol. Life Sci.* **28**, 1 (2024).

[5] E.N. Lorenz, «Deterministic Nonperiodic Flow», *J. Atmos. Sci.* **20**, 130 (1963).

[6] A. Fowler, J. Gibbon, M. McGuinness, «The complex Lorenz equations», *Physica D* **4**, 139 (1982).

[7] K. Sun, J.C. Sprott, «Dynamics of a Simplified Lorenz System», *Int. J. Bifur. Chaos* **19**, 1357 (2009).

[8] M. Jin, K. Sun, H. Wang, «Dynamics and synchronization of the complex simplified Lorenz system», *Nonlinear Dyn.* **106**, 2667 (2021).

[9] G.M. Mahmoud, H. Khalaf, M.M. Darwish, T.M. Abed-Elhameed, «On the fractional-order simplified Lorenz models: Dynamics, synchronization, and medical image encryption», *Math. Methods Appl. Sci.* **46**, 15706 (2023).

[10] G.M. Mahmoud, H. Khalaf, M.M. Darwish, T.M. Abed-Elhameed, «Synchronization and desynchronization of chaotic models with integer, fractional and distributed-orders and a color image encryption application», *Phys. Scr.* **98**, 095211 (2023).

[11] G.M. Mahmoud, T. Aboelenen, T.M. Abed-Elhameed, A.A. Farghaly, «On boundedness and projective synchronization of distributed order neural networks», *Appl. Math. Comput.* **404**, 126198 (2021).

[12] R.K. Upadhyay, V. Rai, «Complex dynamics and synchronization in two non-identical chaotic ecological systems», *Chaos, Solitons Fractals* **40**, 2233 (2009).

[13] M. Lecar, «Chaos in the solar system», *Celestial Mech. Dyn. Astr.* **64**, 163 (1996).

[14] T.M. Abed-Elhameed, G.M. Mahmoud, M.M. Elbadry, M.E. Ahmed, «Nonlinear distributed-order models: Adaptive synchronization, image encryption and circuit implementation», *Chaos, Solitons Fractals* **175**, 114039 (2023).

[15] L.M. Pecora, T.L. Carroll, «Synchronization in chaotic systems», *Phys. Rev. Lett.* **64**, 821 (1990).

[16] L. Runzi, W. Yinglan, D. Shucheng, «Combination synchronization of three classic chaotic systems using active backstepping design», *Chaos* **21**, 043114 (2011).

[17] G.M. Mahmoud, T.M. Abed-Elhameed, H. Khalaf, «Synchronization of hyperchaotic dynamical systems with different dimensions», *Phys. Scr.* **96**, 125244 (2021).

[18] J. Sun *et al.*, «Combination–combination synchronization among four identical or different chaotic systems», *Nonlinear Dyn.* **73**, 1211 (2013).

[19] J. Sun, Y. Shen, Q. Yin, C. Xu, «Compound synchronization of four memristor chaotic oscillator systems and secure communication», *Chaos* **23**, 013140 (2013).

[20] B. Zhang, F. Deng, «Double-compound synchronization of six memristor-based Lorenz systems», *Nonlinear Dyn.* **77**, 1519 (2014).

[21] J. Sun *et al.*, «Compound-combination synchronization of five chaotic systems via nonlinear control», *Optik* **127**, 4136 (2016).

[22] G.M. Mahmoud, T.M. Abed-Elhameed, A.A. Farghaly, «Double compound combination synchronization among eight n-dimensional chaotic systems», *Chinese Phys. B* **27**, 080502 (2018).

[23] V.K. Yadav, G. Prasad, M. Srivastava, S. Das, «Triple Compound Synchronization Among Eight Chaotic Systems with External Disturbances via Nonlinear Approach», *Differ. Equ. Dyn. Syst.* **30**, 549 (2022).

[24] D. Khattar, N. Agrawal, G. Singh, «Triple compound combination synchronization of eleven n-dimensional chaotic systems», *Int. J. Dynam. Control* **11**, 2499 (2023).

[25] L.S. Jahanzaib, P. Trikha, D. Baleanu, «Analysis and Application Using Quad Compound Combination Anti-synchronization on Novel Fractional-Order Chaotic System», *Arab. J. Sci. Eng.* **46**, 1729 (2021).

[26] D. Khattar, N. Agrawal, G. Singh, «Chaotic Analysis of a New 3D System with Exponential Nonlinearity and its Dual Compound Combination Multiswitching Synchronization using Nonlinear Control», *Int. J. Appl. Comput. Math.* **11**, 98 (2025).

[27] D. Khattar, N. Agrawal, G. Singh, «Chaos synchronization of a new chaotic system having exponential term via adaptive and sliding mode control», *Differ. Equ. Dyn. Syst.* **33**, 475 (2025).

[28] G. Singh, D. Khattar, N. Agrawal, «Dual quadratic compound multiswitching anti-synchronization of Lorenz, Rössler, Lü and Chen chaotic systems», *Eur. Phys. J. B* **98**, 11 (2025).

[29] D. Khattar, N. Agrawal, G. Singh, «Anti-difference quadratic compound synchronization of Lorenz, Rössler, modified finance, and Shimizu–Morioka chaotic systems», *Indian J. Phys.* **98**, 1415 (2024).

[30] O.E. Rössler, «An equation for continuous chaos», *Phys. Lett. A* **57**, 397 (1976).

[31] G. Chen, T. Ueta, «Yet Another Chaotic Attractor», *Int. J. Bifurcation Chaos* **09**, 1465 (1999).

[32] R. Shaw, «Strange Attractors, Chaotic Behavior, and Information Flow», *Z. Naturforsch. A* **36**, 80 (1981).

[33] J. Lü, G. Chen, «A New Chaotic Attractor Coined», *Int. J. Bifurcation Chaos* **12**, 659 (2002).

[34] G.M. Mahmoud, M.E. Ahmed, N. Sabor, «On Autonomous and Nonautonomous Modified Hyperchaotic Complex Lü Systems», *Int. J. Bifurcation Chaos* **21**, 1913 (2011).

[35] X. Wang *et al.*, «A New Chaotic System With Stable Equilibrium: From Theoretical Model To Circuit Implementation», *IEEE Access* **5**, 8851 (2017).

[36] V.-T. Pham *et al.*, «A Chaotic System With Equilibria Located on the Rounded Square Loop and Its Circuit Implementation», *IEEE Trans. Circuits. Syst. II: Express Briefs* **63**, 878 (2016).

[37] A. Sambas *et al.*, «A Novel 3D Chaotic System With Line Equilibrium: Multistability, Integral Sliding Mode Control, Electronic Circuit, FPGA Implementation and Its Image Encryption», *IEEE Access* **10**, 68057 (2022).

[38] G.M. Mahmoud, T. Bountis, E.E. Mahmoud, «Active Control and Global Synchronization of the Complex Chen and Lü Systems», *Int. J. Bifurcation Chaos* **17**, 4295 (2007).