DARBOUX POLYNOMIALS AND FIRST INTEGRALS FOR THE GENERALIZED LORENZ SYSTEM

Zdzisław A. Golda

Astronomical Observatory, Jagiellonian University
Orla 171, 30-244 Kraków, Poland

Andrzej Woszczyna

Institute of Physics, Cracow University of Technology
Podchorążych 1, 30-084 Kraków, Poland

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We study the generalized Lorenz dynamical system describing convective phenomena in magnetized fluid. We employ computer algebra to effectively support the Darboux method of solving differential equations. Darboux polynomials and various types of first integrals are determined.

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1. Introduction

The original Lorenz dynamical system [1] played a key role in the discovery of the phenomenon of deterministic chaos. Numerous articles, books, and even educational works have been devoted to this system. Although the Lorenz system was extensively studied numerically, a rigorous mathematical proof of its chaotic behavior was provided only three decades later [2].

Within a wide range of problems discussed in the literature, there are also higher-dimensional extensions of this dynamical system (see, *e.g.*, [3–7] and the references listed there). The four-dimensional extension of the Lorenz system, describing convection in a magnetic medium [5], deserves special attention. Understanding convective phenomena in magnetized liquids, and in particular demonstrating their chaotic behavior, is important for plasma physics — both in laboratory studies and in interstellar and intergalactic space research.

In our work, we apply algebraic methods to the study of the generalized Lorenz dynamical system [5], with a particular emphasis on finding first integrals. Knowledge of first integrals enables reduction of the order of the system of differential equations, finding particular solutions and invariant surfaces. This, in turn, provides insight into the structure of phase space and its partitioning into sectors with specific solution behavior. This is important for identifying appropriate numerical methods and controlling numerical errors.

2. Dynamical system of the generalized Lorenz system

The Lorenz system [1], supplemented with an additional linear equation and an extra linear term, constitutes a mathematical model of hydromagnetic convection in a magnetized fluid, as first noted in [5]. In the same work, preliminary numerical studies of this generalized dynamical system were also conducted.

Continuing to use numerical methods, in [8], the authors investigated in detail the dynamical system introduced earlier. They discovered a number of interesting properties that may shed new light on the physical nature of hydromagnetic convection observed in space plasma, planetary atmospheres, and stars. The third article in this series [9] numerically demonstrated the possibility of hyperchaotic motions. New strange attractors emerge.

The evolution of a viscous and magnetized fluid is described by the following system of partial differential equations [10]:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v} = -\frac{1}{\rho}\nabla\left(p + \frac{1}{2\mu_0}\boldsymbol{B}\cdot\boldsymbol{B}\right) + \frac{1}{\mu_0\rho}\left(\boldsymbol{B}\cdot\nabla\right)\boldsymbol{B} + \nu\,\Delta\boldsymbol{v} + \boldsymbol{F}\,, \\ \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{B} = \left(\boldsymbol{B}\cdot\nabla\right)\boldsymbol{v} + \eta\,\Delta\boldsymbol{B}\,, \\ \frac{\mathrm{d}}{\mathrm{d}t}T = \kappa\,\Delta T\,, \end{cases} \tag{1}$$

with the total derivative $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \boldsymbol{v}\cdot\nabla$, dynamical variables: $\boldsymbol{v}$ — the velocity of the flow, $\boldsymbol{B}$ — the magnetic field, $\boldsymbol{F}$ — the vector of external forces, and T — the temperature. Parameters ν, η, and κ are, respectively, kinematic viscosity, magnetic diffusive viscosity, and thermal conductivity of the fluid. ρ stands for the mass density and p for pressure. The first equation of system (1) is the vector Navier–Stokes equation, the second describes the magnetic diffusion, and the third is the scalar thermal diffusion equation. The fluid flow is considered incompressible ($\nabla\cdot\boldsymbol{v} = 0$), therefore, by close analogy to electrodynamics ($\boldsymbol{B} = \nabla\times\boldsymbol{A}$), one may introduce a vector potential $\boldsymbol{\Psi}$, such that $\boldsymbol{v} = \nabla\times\boldsymbol{\Psi}$.

Considering the geometrically simplified problem of convection between two horizontal layers separated by h in a uniform gravitational field, we set $\boldsymbol{\Psi} = (0, \psi(x,z,t), 0)$ and $\boldsymbol{A} = (0, \alpha(x,z,t), 0)$. Following [11], we introduce the function $\tau(x,z,t)$, which characterizes the deviation of the temperature from the linear profile in a fluid layer of thickness h with temperature T_0 at its lower boundary

$$\tau(x,z,t) = T(x,z,t) - T_0 - \frac{z}{h}\delta T_0 \,. \tag{2}$$

Using Fourier analysis, we reduce the problem of partial differential equations to a system of ordinary differential equations. Accordingly, we assume that the functions α and ψ can be represented as Fourier series satisfying certain boundary conditions. We further restrict the solutions to their first terms only

$$\psi(x,z,t) = \sin\left(\frac{\pi a}{h}x\right)\sin\left(\frac{\pi}{h}z\right)\bar{\psi}(t)\,, \tag{3}$$

$$\alpha(x,z,t) = \sin\left(\frac{\pi a}{h}x\right)\sin\left(\frac{\pi}{h}z\right)\bar{\alpha}(t)\,, \tag{4}$$

$$\tau(x,z,t) = \cos\left(\frac{\pi a}{h}x\right)\sin\left(\frac{\pi}{h}z\right)T_1(t) - \sin\left(\frac{2\pi}{h}z\right)T_2(t)\,. \tag{5}$$

We introduce four harmonic amplitudes

$$X = \frac{a}{\sqrt{2}(1+a^2)\kappa}\bar{\psi}(t)\,, \tag{6}$$

$$Y = \frac{a^2\beta g}{\sqrt{2}(1+a^2)^3\left(\frac{\pi}{h}\right)^3\nu\,\kappa}T_1(t)\,, \tag{7}$$

$$Z = \frac{a^2\beta g}{\sqrt{2}(1+a^2)^3\left(\frac{\pi}{h}\right)^3\nu\,\kappa}T_2(t)\,, \tag{8}$$

$$W = \frac{a}{\sqrt{2}(1+a^2)\kappa}\bar{\alpha}(t)\,, \tag{9}$$

to finally express the dynamical system in the form

$$\begin{cases} X' = \sigma(-X+Y) - \omega_0 W\,, \\ Y' = rX - Y - XZ\,, \\ Z' = -bZ + XY\,, \\ W' = \omega_0 X - \sigma_{\mathrm{m}} W\,. \end{cases} \tag{10}$$

Parameter $\sigma = \nu/\kappa$ is the Prandtl number, r is the Rayleigh number, $b = \frac{4}{1+a^2}$ is a geometrical factor, ω_0 characterizes the fundamental dimensionless magnetic frequency associated with Alfvén waves, and $\sigma_{\mathrm{m}} = \eta/\kappa$ is the magnetic Prandtl number. To avoid the explicit inclusion of physical quantities that themselves are not parameters of the dynamical system, the following scaling was used: $x \to x/h$, $z \to z/h$, and $t \to t\,D_T/h^2$ (where

D_T stands for the fluid diffusion coefficient). The five parameters explicitly present in the equations have precise physical meanings, and all are positive. The meaning of the dynamical variables is as follows:

— $X \to$ the intensity of the convective motion,

— $Y \to$ the temperature difference between the ascending and descending currents,

— $Z \to$ the deviation of the vertical temperature profile from linearity,

— $W \to$ the magnetic field induced in the convected, magnetized fluid.

For the case of a vanishing magnetic field, system (10) reduces to the Lorenz dynamical system.

For the convenience of further analysis and calculations, uniform notations are introduced for the dynamical variables, $\{X, Y, Z, W\} \to \{x_1, x_2, x_3, x_4\}$, and for the parameters occurring in the dynamical system, $\{\sigma, r, b, \omega_0, \sigma_{\mathrm{m}}\} \to \{c_1, c_2, c_3, c_4, c_5\}$ (this is consistent with the notation introduced in [12] for the Lorenz model). Then, the dynamical system in the new notation takes the following uniform form:

$$\begin{cases} x_1' = F_1 := -c_1 x_1 + c_1 x_2 - c_4 x_4 \,, \\ x_2' = F_2 := c_2 x_1 - x_2 - x_1 x_3 \,, \\ x_3' = F_3 := -c_3 x_3 + x_1 x_2 \,, \\ x_4' = F_4 := c_4 x_1 - c_5 x_4 \,. \end{cases} \tag{11}$$

From the physical interpretation, it follows that all components of $\boldsymbol{c} = \{c_1, c_2, c_3, c_4, c_5\}$ are positive. The components of the vector field $\boldsymbol{F} := \{F_1, F_2, F_3, F_4\}$ are second-degree polynomials of the dynamical variables $\boldsymbol{x} := \{x_1, x_2, x_3, x_4\}$, and the field components F_2 and F_3 contain two quadratic nonlinearities, which also appear in the Lorenz system. In the case $c_4 = 0$ and $x_4 = 0$, system (11) reduces to the classical Lorenz system [1].

The dynamical system (11) has several important properties:

— is an autonomous dynamical system,

— the divergence of the vector field $\nabla \cdot \boldsymbol{F} = -(1 + c_1 + c_3 + c_5)$ is constant; this means that the dynamical system for $1 + c_1 + c_3 + c_5 = 0$ is conservative, while for $1 + c_1 + c_3 + c_5 > 0$, it is dissipative (non-Hamiltonian system),

— the system also has the symmetry $\Sigma = \{t \to t, x_1 \to -x_1, x_2 \to -x_2, x_3 \to x_3, x_4 \to -x_4\}$,

— the symmetry Σ implies that if $\{x_1(t), x_2(t), x_3(t), x_4(t)\}$ is a solution, then its symmetric counterpart is also a solution $\{-x_1(t), -x_2(t), x_3(t), -x_4(t)\}$,

— x_3-axis is an invariant manifold and

$$\{x_1(t) = 0, x_2(t) = 0, x_3(t) = C\mathrm{e}^{c_3 t}, x_4(t) = 0\}$$

is a solution of the dynamical system (11).

The use of these symmetries will significantly increase the computational efficiency of the Darboux method.

3. Darboux polynomials

Strictly speaking, there are no uniform standard techniques for finding the first integrals of ordinary differential equations. However, a number of specialized methods for computing them have been widely discussed in the literature. These include methods based on the Noether symmetry [13], Lie symmetry [14–16], Lax pairs [17], Painlevé analysis [18], Carleman immersion [19–22], the differential Galois theory [23, 24], and the Darboux method [25–29]. In particular, the Darboux polynomial method is discussed in detail in several textbooks [30–32].

In this work, we focus on the Darboux method, since it is closely related to dynamical systems with polynomial vector fields. Moreover, this method can be readily implemented algorithmically, and the calculations can be performed using symbolic computation software. A significant portion of the computations in this study was carried out using Wolfram Mathematica [33].

Although the Darboux method for integrating differential equations is largely algorithmic, the computational complexity is high and increases rapidly with the degree of the dynamical system. For a fourth-degree system, 126 quadratic equations in 75 variables must be solved, but for a fifth-degree system, these equations number is 210, and the number of variables increases to 131. While successfully solving these equations, one obtains a number of interdependent polynomial expressions, which have to be decomposed in a basis of independent polynomials. The algebraic complexity of these expressions is significant, and the basis of independent polynomials is not given *a priori* but must be identified by analyzing regular expressions in the originally obtained solutions. We omit discussion of the algorithm and code in this paper, as it falls outside the scope of physics.

The starting point for calculating the first integrals is the determination of the Darboux polynomials. Notably, these polynomials also exhibit the Σ symmetry. Exploiting the Σ symmetry in the computation of the Darboux polynomials significantly reduces the computational time.

The polynomial approach to a special class of differential equations was developed by Darboux in the late 19$^{\mathrm{th}}$ century and published in 1878 [25, 26]. Let $\mathbb{R}[\boldsymbol{x}]$ be a ring of polynomials in $\boldsymbol{x} \in \mathbb{R}^n$ with real coefficients. Consider a polynomial vector field operator of the form

$$\mathcal{X} = \sum_{i=1}^{n} F_i(\boldsymbol{x}) \frac{\partial}{\partial x_i}\,, \tag{12}$$

where $F_i(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$. The quantity $d = \max\{\deg F_i; i = 1, \dots, n\}$ is called the degree of the vector field $\mathcal{X}$. In the case of the field (11), $d = 2$. The polynomial $\mathcal{P}(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$ is called the Darboux polynomial of the dynamical system if the following equation holds:

$$\mathcal{X}(\mathcal{P}) = \sum_{i=1}^{n} F_i(\boldsymbol{x}) \frac{\partial \mathcal{P}}{\partial x_i} = K_{\mathcal{P}} \mathcal{P}\,. \tag{13}$$

The quantity $K_{\mathcal{P}}(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$ is called a cofactor of the Darboux polynomial $\mathcal{P}$. The cofactor $K_{\mathcal{P}}(\boldsymbol{x})$ is of degree at most $d - 1$. The form of the above equation defines the Darboux polynomials with accuracy to the multiplicative constant.

If $\mathcal{P}$ is a set of the Darboux polynomials, then the algebraic surface $\mathcal{P} = 0$ in the space $\mathbb{R}^n$ is called an invariant algebraic surface (see [34, 36]). Some authors employ the term *secondary integral* [30] interchangeably.

Table 1 presents the Darboux polynomials of the first to the fifth degree $\mathcal{P}_j^{(i)}$ (where i is the degree of the polynomial and j denotes the ordinal number of the polynomial) and their corresponding cofactors. All cofactors are constant quantities that depend only on the parameters $\boldsymbol{c}$.

The second-degree Darboux polynomials $\mathcal{P}_1^{(2)}$, $\mathcal{P}_2^{(2)}$, and $\mathcal{P}_3^{(2)}$ in Table 1 are the original Lorenz polynomials [34–36]. Two passes to the limit give us, respectively: $\lim_{x_4 \to 0} \mathcal{P}_5^{(2)} = \mathcal{P}_1^{(2)}$ and $\lim_{x_4 \to 0} \mathcal{P}_6^{(2)} = \mathcal{P}_3^{(2)}$. A similar situation exists for the fourth-degree Darboux polynomials. The Darboux polynomials $\mathcal{P}_1^{(4)}$, $\mathcal{P}_2^{(4)}$, and $\mathcal{P}_3^{(4)}$ are identical to the Darboux polynomials of the original Lorenz system [34–36].

Table 1. Darboux polynomials.

No.	Darboux polynomial	Cofactor	Parameters
1.1	$\mathcal{P}_1^{(1)} = c_4 x_1 - \alpha\, x_4\,,$ $\alpha = \frac{1}{2}\left[c_5 - \sqrt{c_5^2 - 4c_4^2}\right]$ and $\lvert c_5\rvert \geq 2\lvert c_4\rvert$	$-\alpha$	$\{0, c_2, c_3, c_4, c_5\}$
1.2	$\mathcal{P}_2^{(1)} = c_4 x_1 - \beta\, x_4\,,$ $\beta = \frac{1}{2}\left[c_5 + \sqrt{c_5^2 - 4c_4^2}\right]$ and $\lvert c_5\rvert \geq 2\lvert c_4\rvert$	$-\beta$	$\{0, c_2, c_3, c_4, c_5\}$
1.3	$\mathcal{P}_3^{(1)} = x_4$	$-c_5$	$\{c_1, c_2, c_3, 0, c_5\}$
2.1	$\mathcal{P}_1^{(2)} = x_1^2 - 2c_1 x_3$	$-2c_1$	$\{c_1, c_2, 2c_1, 0, c_5\}$
2.2	$\mathcal{P}_2^{(2)} = x_2^2 + x_3^2$	-2	$\{c_1, 0, 1, c_4, c_5\}$
2.3	$\mathcal{P}_3^{(2)} = -c_2 x_1^2 + x_2^2 + x_3^2$	-2	$\{1, c_2, 1, 0, c_5\}$
2.4	$\mathcal{P}_4^{(2)} = c_4 x_1^2 - c_5 x_1 x_4 + c_4 x_4^2$	$-c_5$	$\{0, c_2, c_3, c_4, c_5\}$
2.5	$\mathcal{P}_5^{(2)} = x_1^2 - 2c_1 x_3 + x_4^2$	$-2c_1$	$\{c_1, c_2, 2c_1, c_4, c_1\}$
2.6	$\mathcal{P}_6^{(2)} = -c_2 x_1^2 + x_2^2 + x_3^2 - c_2 x_4^2$	-2	$\{1, c_2, 1, c_4, 1\}$
3.1	$\mathcal{P}_1^{(3)} = x_1\left[x_2^2 + (x_3 - c_2)^2\right] + x_2\left[x_3 - c_2\right]$	-1	$\{0, c_2, 0, 0, c_5\}$
3.2	$\mathcal{P}_2^{(3)} = \left[1 - c_3\right] x_2 x_3 + x_1\left[x_2^2 + x_3^2\right]$	$c_3 - 1$	$\{0, 0, c_3, 0, c_5\}$
4.1	$\mathcal{P}_1^{(4)} = 9x_1^4 - 8x_1 x_2 - 4x_2^2 + 12x_1^2\left[c_2 - x_3\right]$	$-\frac{4}{3}$	$\left\{\frac{1}{3}, c_2, 0, 0, c_5\right\}$
4.2	$\mathcal{P}_2^{(4)} = x_1^4 + 8x_1 x_2 - 4x_2^2 + 16\left[c_2 - 1\right] x_3 - 4x_1^2\left[x_3 + c_2\right]$	-4	$\{1, c_2, 4, 0, c_5\}$
4.3	$\mathcal{P}_3^{(4)} = x_1^4 - 4c_1 x_1^2 x_3 - 4\left[(2c_1 - 1)x_1 - c_1 x_2\right]^2$	$-4c_1$	$\{c_1, 2c_1 - 1, 6c_1 - 2, 0, c_5\}$
4.4	$\mathcal{P}_4^{(4)} = \left[x_1^2 + x_4^2\right]\left[x_1^2 - 2x_3 + x_4^2\right] - x_2^2$	-2	$\left\{\frac{1}{2}, 0, 1, c_4, \frac{1}{2}\right\}$
5.1	$\mathcal{P}_1^{(5)} = x_1^3\left(x_2^2 + (c_2 - x_3)^2\right) - (c_3 - 1)c_3 x_2 x_3 - x_1^2 x_2 (c_2(c_3 + 1) + (c_3 - 1)x_3) + c_3 x_1\left(x_2^2 + x_3(c_2(c_3 - 1) + x_3)\right)$	$-1 - c_3$	$\{0, c_2, c_3, 0, c_5\}$

4. The generator of exponential factors and Darbouxian first integrals

For coprime polynomials G and $H \in \mathbb{R}[\boldsymbol{x}]$, the quantity

$$\exp\left[\frac{G}{H}\right] \tag{14}$$

is an exponential factor of the vector field $\mathcal{X}$ if there exists a polynomial $L_{\mathrm{e}}(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$ of degree $d-1$ (cofactor) such that

$$\mathcal{X}\left[\exp\left[\frac{G}{H}\right]\right] = L_{\mathrm{e}} \exp\left[\frac{G}{H}\right] . \tag{15}$$

For the dynamical system (11), there are two generators of the exponential factor of the form:

— $\mathcal{G}_1 = \exp[x_1/c_1]$ with cofactor $L_{\mathrm{e}} = -x_1 + x_2 - \frac{c_4}{c_1} x_4$ for parameter $c_1 \neq 0$,

— $\mathcal{G}_2 = \exp[x_4]$ with cofactor $L_{\mathrm{e}} = -c_4 x_1 - c_5 x_4$.

The first exponential factor, $\mathcal{G}_1$, in the limit $x_4 \to 0$ reduces to the exponential factor of the Lorenz model [34, 36], whereas the second, $\mathcal{G}_2$, vanishes in this limit.

For a small number of dynamical systems reported in the literature, there exists a special class of first integrals known as the Darbouxian first integrals. When such integrals exist, they take the form of a product of the Darboux polynomials and exponential factors.

5. First integrals

Knowledge of the Darboux polynomials of the dynamical system (11) allows us to determine four different classes of first integrals (see, *e.g.*, [32]):

— polynomial first integrals;

— first integrals with exponential time dependence;

— rational first integrals;

— Darbouxian first integrals (as already discussed in Section 4).

In the order given above, we will discuss the first three classes of first integrals below.

5.1. Polynomial first integrals

Table 2 presents polynomial first integrals of system (11) $\mathcal{I}_j^{(i)}$, where i is the degree of the polynomial, and j denotes the number of the first integral in the table. These integrals depend on the parameters $\boldsymbol{c}$. Some of these integrals involve up to three free parameters c_i (out of five). It should be noted that for the parameter value $c_4 = 0$, the dynamical equation for x_4 decouples from the Lorenz model. In this case, we do not provide the corresponding first integrals in Table 2.

Table 2. Polynomial first integrals.

No.	First integrals	Values of parameters c_i
1	$\mathcal{I}_1^{(2)} = x_1^2 + x_4^2$	$\{0, c_2, c_3, c_4, 0\}$
2	$\mathcal{I}_2^{(3)} = \left[x_2^2 + x_3^2\right]\left[c_4 x_1 + 2x_4\right]$	$\left\{0, 0, 1, c_4, -\frac{1}{2}\left(4 + c_4^2\right)\right\}$
3	$\mathcal{I}_3^{(4)} = \left[x_2^2 + x_3^2\right]\left[c_4 x_1 + x_4\right]^2$	$\{0, 0, 1, c_4, -1 - c_4^2\}$
4	$\mathcal{I}_4^{(4)} = \left[x_2^2 + x_3^2\right]\left[c_4 x_1^2 + 2x_1 x_4 + c_4 x_4^2\right]$	$\{0, 0, 1, c_4, -2\}$
5	$\mathcal{I}_5^{(4)} = \left[(1 - c_3)x_2 x_3 + x_1\left(x_2^2 + x_3^2\right)\right] x_4$	$\{0, 0, c_3, 0, -1 - c_3\}$
6	$\mathcal{I}_6^{(5)} = \left[x_2^2 + x_3^2\right]\left[3c_4 x_1 + 2x_4\right]^3$	$\left\{0, 0, 1, c_4, -\frac{1}{6}\left(4 + 9c_4^2\right)\right\}$
7	$\mathcal{I}_7^{(5)} = \left[x_2^2 + x_3^2\right]^2\left[c_4 x_1 + 4x_4\right]$	$\left\{0, 0, 1, c_4, -\frac{1}{4}\left(16 + c_4^2\right)\right\}$
8	$\mathcal{I}_8^{(5)} = \left[x_2^2 + x_3^2\right]\left[c_4 x_1 + \left[1 + \sqrt{1 - 2c_4^2}\right] x_4\right] \times \left[2c_4 x_1 + \left[1 - \sqrt{1 - 2c_4^2}\right] x_4\right]^2$	$\left\{0, 0, 1, \lvert c_4 \rvert \leq \frac{\sqrt{2}}{2}, -\frac{1}{2}\left[\sqrt{1 - 2c_4^2} + 3\right]\right\}$
9	$\mathcal{I}_9^{(5)} = \left[x_2^2 + x_3^2\right]\left[c_4 x_1 + \left[1 - \sqrt{1 - 2c_4^2}\right] x_4\right] \times \left[2c_4 x_1 + \left[1 + \sqrt{1 - 2c_4^2}\right] x_4\right]^2$	$\left\{0, 0, 1, \lvert c_4 \rvert \leq \frac{\sqrt{2}}{2}, \frac{1}{2}\left[\sqrt{1 - 2c_4^2} - 3\right]\right\}$

5.2. First integrals with exponential time dependence

The Darboux polynomials $\mathcal{P}_j^{(i)}$ associated with the dynamical system (11), listed in Table 1, possess constant cofactors $K_{\mathcal{P}}$ depending on the parameters $\boldsymbol{c}$. This property makes it possible to construct first integrals with the exponential time dependence. Their general form is given by

$$\mathcal{I}_j^{(i)}(t, \boldsymbol{x}) = \mathcal{P}_j^{(i)}(\boldsymbol{x}) \exp\left(-K_{\mathcal{P}}\, t\right) , \tag{16}$$

where the parameter $K_{\mathcal{P}}$ is a cofactor for the given Darboux polynomial $\mathcal{P}_j^{(i)}(\boldsymbol{x})$. Differentiating equation (16) by sides, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_j^{(i)}(t,\boldsymbol{x}) = \exp\left(-K_{\mathcal{P}}\,t\right)\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}_j^{(i)}(\boldsymbol{x}) - K_{\mathcal{P}}\mathcal{P}_j^{(i)}(\boldsymbol{x})\right]. \tag{17}$$

By virtue of equation (13), the expression in square brackets is equal to zero, so

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_j^{(i)}(t,\boldsymbol{x}) = 0\,. \tag{18}$$

This implies that the function $\mathcal{I}_j^{(i)}(t,\boldsymbol{x})$ is a first integral. Owing to the straightforward nature of their construction, we do not provide a separate table for these integrals. First integrals with the exponential time dependence may also be used to obtain exact solutions.

5.3. Rational first integrals

Rational integrals are defined as the quotients of two Darboux polynomials. Below, the only rational first integral for $c_4 \neq 0$ is of the form

$$\mathcal{I} = \frac{x_2^2 + x_3^2}{\left[x_1^2 - x_3 + x_4^2\right]^2} \tag{19}$$

for $\boldsymbol{c} = \left\{\frac{1}{2}, 0, 1, c_4, \frac{1}{2}\right\}$. Rational first integral (19) becomes the Lorenz integral [36] in the limit $x_4 \to 0$.

6. Summary

Highly complex physical phenomena are described by non-Hamiltonian systems. The first integrals of these systems do not have a universal character, such as momentum or energy — they appear only in the context of a related problem. The Lorenz system integrals, in general, are not integrals of the generalized Lorenz system, nor is the reverse. Integrals of non-Hamiltonian systems do not appeal to particular physical quantities. Due to their contextual nature, they do not constitute a basis for introducing new physical quantities, either. Their physical significance lies in the fact that they determine invariant and critical surfaces that — like separatrices in a two-dimensional systems — cut the phase space into sectors with qualitatively different solution behavior.

For convection in the presence of a magnetic field, described by system (11), the critical surfaces determined by the integrals in Table 2 are elementary functions of the flow intensity X, temperature variables Y, Z, and the magnetic field W (for a detailed interpretation of the variables $\{x_1, x_2, x_3, x_4\} = \{X, Y, Z, W\}$, see Section 2). These critical surfaces identify regions of particularly strong instability (high sensitivity to initial conditions), and therefore they cannot be computed numerically. In contrast, knowledge of the invariant surfaces expressed by the integrals $\mathcal{I}_j^{(i)}$ provides a basis for selecting a numerical method appropriate for the sector of phase space containing the solution under investigation.

Knowledge of invariant surfaces is particularly important for systems exhibiting deterministic chaos. It provides a powerful tool for distinguishing genuine deterministic chaos — which has clear physical significance — from artifacts arising from numerical integration of differential equations.

For the generalized Lorenz system, the Darboux polynomials have been computed and are summarized in Table 1. All cofactors associated with these polynomials are constants, some of which can be expressed in terms of the parameters c_i. Knowledge of the Darboux polynomials enabled us to identify several classes of first integrals: polynomial first integrals (Table 2), first integrals with exponential time dependence (see Section 5.3), a rational first integral (equation (19)), and Darbouxian first integrals (see Section 4). We hope that the first integrals presented here will serve as a useful tool for further numerical studies of convection in the presence of a magnetic field.

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