

PRESSURE INSIDE HADRONS: CRITICISM, CONJECTURES, AND ALL THAT*

CÉDRIC LORCÉ 

CPHT, CNRS, École polytechnique, Institut Polytechnique de Paris 
91120 Palaiseau, France

PETER SCHWEITZER 

Department of Physics, University of Connecticut , Storrs, CT 06269, USA

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The interpretation of the energy-momentum tensor form factor $D(t)$ of hadrons in terms of pressure and shear force distributions is discussed, concerns raised in the literature are reviewed, and ways to reconcile the concerns with the interpretation are indicated.

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1. Introduction

This contribution is dedicated to the memory of Dmitry Diakonov, Victor Petrov, and Maxim Polyakov whose lives and works strongly influenced the authors. Especially Maxim was a dear colleague, mentor, and friend for us. The purpose of this contribution is to review the pressure interpretation of the D -term form factor, one of the most influential works of Maxim.

Continuous systems have well-defined mechanical properties described by the theory of elasticity [1]. In 2002, Maxim Polyakov developed an appealing interpretation [2] of hadronic energy-momentum tensor (EMT) form factors [3] under the assumption that the nucleon and nuclei can be considered as “continuous media” [2]. Especially, the interpretation of the $D(t)$ form factor [4] in terms of shear forces and pressure inside hadrons [2] has attracted considerable interest [5–42], for a recent review see [43]. The interpretation has also raised concerns [44–52], which are valuable as they help improve our physics understanding.

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The intention of this work is to review the concerns raised in the literature, collect thoughts, and initiate a discussion. This work is organized as follows: In Section 2, we collect personal memories related to Maxim. In Section 3, we briefly review the interpretation proposed by Maxim [2]. In Section 4, we review the criticism brought up in the literature, and collect thoughts in Sections 5–12 to argue that the proposed interpretation is physically sound and how the raised issues can be resolved, before concluding in Section 13.

2. Personal memories

Some personal memories of P.S.

In the summer of 2001, I acquired in the university book shop in Bochum the German translation of Landau & Lifshitz Volume 7 on elasticity theory. I was happy to get it as a special offer (for DM 25, 50% discounted, see Fig. 1 (left); the Euro as practical currency was introduced on 1/1/2002). As I was entering the physics building in Bochum, I met Maxim who immediately noticed my new book and glanced at it with astonishment. I never asked why he was astonished, but in the retrospective it became clear. At that time, Maxim must have been working on the pressure interpretation. It was Maxim's style to think deeply about the problems he was working on. His thoughts were often months and sometimes years ahead of his publishing. In this case, we know that the seminal work of Maxim about the mechanical interpretation of $D(t)$ appeared more than a year later in October 2002 [2] (see the citation count from Inspires in Fig. 1 (right); this work was preceded by a short preview of Maxim's main ideas in an earlier unpublished preprint in July 2002 [53]).

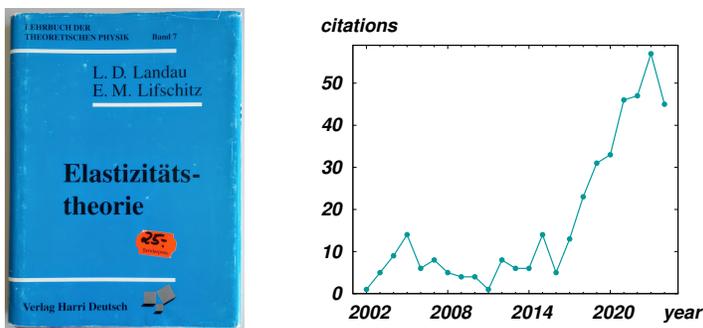


Fig. 1. Left: Copy of Landau & Lifshitz's Volume 7, Elasticity Theory (in German) [1] acquired in the summer of 2001 by one of the authors (P.S.) (special offer for DM 25). Right: Citation count of Maxim's seminal article [2] on the pressure interpretation of the EMT form factor $D(t)$ since its appearance in October 2002 until the end of 2024.

I had known Maxim since I started in 1996 my Diploma project (which later grew into a Ph.D. project) in the Institute for Theoretical Physics II in Bochum under Klaus Goeke who had a unique skill to create a vibrant and motivating environment where students and postdocs could thrive [54]. Maxim was always approachable and happy to discuss with students. He always stressed that he needed interaction with students to get new ideas. Shortly after the episode with Landau & Lifshitz Volume 7, I became a postdoc at the Università degli Studi di Pavia in Italy and interested in generalized parton distribution functions (GPDs). My first self-chosen project was the proof of polynomiality of GPDs in the chiral quark-soliton model [5] where GPDs and the D -term were previously computed by Maxim and collaborators [55–57] (along with many other properties). As it played out, I returned to Bochum two years later, while Maxim accepted a professor position in Liège which he held for two years before returning to Bochum as a professor. In the subsequent 2005–2008 period, Maxim stimulated and guided the studies in soliton models in Bochum [6–8], where the first insights were obtained on how the pressure and other EMT densities may look like in the nucleon. In this period, the Landau & Lifshitz Volume 7, Fig. 1 (left), became very helpful for me. But more important than that were the discussions and collaborations with Maxim. Maxim became for me over the years a collaborator, colleague, mentor, and friend for whom I had always the deepest admiration and respect as a scientist and person.

Some personal memories of C.L.

While I was a graduate student, I met Maxim in 2003 when he had just been appointed professor at the University of Liège, Belgium. He was very impressive and at the same time very accessible. It was an intense period where the existence of the strange pentaquark Θ^+ , predicted in 1997 by Diakonov, Petrov, and Polyakov [58], was heavily debated. In 2004, I started a Ph.D. on the chiral quark-soliton model under Maxim’s guidance. The following year, I followed him to Bochum where I met so many great and inspiring physicists, including both Victor Petrov and Dmitri Diakonov during one of their visits to Maxim.

As far as I know, I have been one of Maxim’s first Ph.D. students. He told me he considered himself “too old” (he was actually only about 40 years old at that time) and wanted a fresh, independent, and unbiased look on the task he assigned me. As a result, I focused essentially on topics closely related to my Ph.D. project. I had no idea about his works on the interpretation of the EMT form factors [2, 53], and I did not know that he was investigating further the EMT properties within soliton models [6, 7] while I was in Bochum.

After having worked on the nucleon spin puzzle and charge distributions during my postdoctoral stay in Mainz, I naturally became interested in the EMT and to my great surprise discovered Maxim’s seminal work on pressure [2]. As usual, Maxim’s paper was clear and filled with deep physical insights. However, it did not seem to attract much attention in 2012, as indicated by the relatively small number of citations around that period, see Fig. 1 (right). It took about six years to mature some ideas on the extension of Maxim’s work to the infinite-momentum frame picture. While we were writing the manuscript in 2018, the first experimental extraction of the quark contribution to the pressure inside the proton appeared in *Nature* [18], triggering e-mail exchanges between some theorists. Shortly after, Maxim and Peter uploaded on the arXiv a key review paper on the mechanical properties of hadrons [19]. These two articles came out as complete surprises to me. I reached out to Maxim to let him know about our work and we decided to meet a month later to exchange our ideas. It was such a pleasure to be back in Bochum after a long time. My collaborator, Arek Trawiński, presented our results in a seminar and we discussed them with Maxim for a couple of days. It was both strange and amusing to realize that, after more than a decade, we independently converged on similar ideas and physics interests. Our paper finally appeared on the arXiv four months later [20].

Besides being a great physicist with a strong impact on QCD, Maxim was also humble, kind, supportive, and a free-thinker. He remains a model to me and I strive to pass over his legacy to future generations. Using Maxim’s own wording, meeting him has been “my greatest pleasure”.

3. The D -term, pressure, and shear forces

The gravitational form factors (GFFs) were introduced in [59, 60]. In [3], Ji noted that the second Mellin moments of GPDs give access to GFFs. The latter are Lorentz-invariant functions that parametrize the EMT matrix elements, just like electromagnetic form factors parametrize the matrix elements of the electromagnetic current. For a nucleon, the symmetric EMT matrix elements can be written as [3, 59, 60], for a recent review see [43],

$$\langle p', \vec{s}' | T_a^{\mu\nu}(0) | p, \vec{s} \rangle = \bar{u}(p', \vec{s}') \left[\frac{P^\mu P^\nu}{M} A_a(t) + \frac{P^{\{\mu} i\sigma^{\nu\}\lambda} \Delta_\lambda}{M} J_a(t) + \frac{\Delta^\mu \Delta^\nu - g^{\mu\nu} \Delta^2}{4M} D_a(t) + M g^{\mu\nu} \bar{C}_a(t) \right] u(p, \vec{s}), \quad (1)$$

where M is the nucleon mass, $P = (p' + p)/2$ is the average four-momentum, $\Delta = p' - p$ is the four-momentum transfer, and $t = \Delta^2 < 0$. The label $a = q, g$

stands for the quark or gluon contribution. We also used for convenience the notation $x^{\{\mu}y^{\nu\}} = (x^\mu y^\nu + x^\nu y^\mu)/2$. The form factor $\bar{C}_a(t)$ accounts for non-conservation of the separate quark and gluon parts of the EMT, and vanishes identically when summing over all the constituents.

In the forward limit, the $A_a(0)$ represent the fractions of linear momentum carried by quarks or gluons [61, 62], while the $J_a(0)$ describe the corresponding fractions of total angular momentum as shown by Ji [3]. Inspired by this and in analogy to works in gravity¹, Maxim defined the static EMT of the nucleon via a Fourier transform in the Breit frame $\vec{p}' = -\vec{p} = \vec{\Delta}/2$ [2]

$$\mathcal{T}_a^{\mu\nu}(\vec{r}, \vec{s}) = \int \frac{d^3\Delta}{(2\pi)^3} e^{-i\vec{\Delta}\cdot\vec{r}} \frac{\langle p', \vec{s} | T_a^{\mu\nu}(0) | p, \vec{s} \rangle}{2E} \quad (2)$$

with $E = p^0 = p^0 = \sqrt{M^2 + \vec{\Delta}^2/4}$, similarly to the charge distribution [65]. The time independence of this EMT follows from the fact that the energy transfer Δ^0 , which is Fourier conjugate to the time coordinate, vanishes in the Breit frame.

The components $\mathcal{T}_a^{00}(\vec{r}, \vec{s})$ and $\mathcal{T}_a^{0k}(\vec{r}, \vec{s})$ are naturally interpreted as spatial distributions of energy and momentum carried by quarks or gluons. The purely spatial components $\mathcal{T}_a^{ij}(\vec{r}, \vec{s})$ are then interpreted as the stress tensor², in the sense that if the nucleon were to be considered as a continuous medium, then $\mathcal{T}_a^{ij}(\vec{r}, \vec{s})$ would characterize the forces exerted by quarks or gluons in an infinitesimal volume at a position \vec{r} relative to the center of the nucleon.

In the following, we will focus on the force interpretation for the total EMT, $\mathcal{T}^{ij}(\vec{r}, \vec{s}) = \sum_a \mathcal{T}_a^{ij}(\vec{r}, \vec{s})$. For discussions of forces for the separate quark and gluon subsystems, we refer to [68, 69].

It turns out that for spin-0 as well as spin-1/2 particles, the stress tensor is spin-independent and can be expressed as

$$\mathcal{T}^{ij}(\vec{r}) = \delta^{ij} p(r) + \left(\frac{r^i r^j}{r^2} - \frac{1}{3} \delta^{ij} \right) s(r). \quad (3)$$

The functions $p(r)$ and $s(r)$ are related to each other by the conservation of the total EMT $\nabla^i \mathcal{T}^{ij}(\vec{r}) = 0$ which implies

$$\frac{d}{dr} \left(p(r) + \frac{2}{3} s(r) \right) = -\frac{2s(r)}{r}. \quad (4)$$

¹ A reference to works [63, 64] can be found above Eq. (15) of [53] in which a not yet fully matured preview of Maxim's EMT interpretation ideas is included.

² The stress tensor is usually defined as $\sigma^{ij} = -T^{ij}$ and represents the *external* forces acting on a volume element [66]. Sometimes T^{ij} itself is called the stress tensor [67] and represents the *internal* forces inside a volume element.

The structure of the stress tensor (3) is the same as that of an anisotropic fluid [70–72], where the monopole function $p(r)$ is interpreted as the *isotropic pressure* and the quadrupole function $s(r)$ as the *pressure anisotropy*. Equation (4) can then be understood as the differential form of the Young–Laplace equation, where $s(r)$ can be regarded as the radial distribution of surface tension [73, 74].

The function $s(r)$ describes also the off-diagonal part of the symmetric stress tensor, and so is related to the distribution of shear forces inside the nucleon. A measure of these shear forces is provided by the so-called D -term which is given by [2, 19]

$$D = -\frac{2}{5} M \int d^3r \mathcal{T}^{ij}(\vec{r}) \left(\frac{r^i r^j}{r^2} - \frac{1}{3} \delta^{ij} \right) = -\frac{4}{15} M \int d^3r r^2 s(r), \quad (5)$$

$$= \frac{1}{3} M \int d^3r \mathcal{T}^{ij}(\vec{r}) \delta^{ij} = M \int d^3r r^2 p(r) \quad (6)$$

provided that the integrals converge. The second representation for D in Eq. (6) follows from Eq. (4).

4. Concerns in the literature regarding the pressure interpretation

In the following, we list the concerns raised in [44–50, 52], which are centered around the following points (the sections where we discuss these points and indicate potential resolutions are mentioned in brackets):

- The notion of EMT distributions in the Breit frame is known to be plagued by relativistic recoil corrections, see *e.g.* Refs. [44, 45] for recent criticism (Section 5).
- In “standard thermodynamics” the pressure is always positive, while $p(r)$ from the interpretation of $D(t)$ can be negative [46] (Sections 6–8).
- In “fluid dynamics”, an “effective description in terms of pressure and internal energy [requires] that the particle’s mean free path must be much smaller” than the size of the system which is the case in QCD “only at high-temperature and density” [47] (Section 9).
- It was argued that the D -term is negative based on mechanical stability arguments [15] in agreement with results from hadronic models, lattice QCD, and dispersion relation studies [43], but no quantum field theoretical proof of this conjecture exists in QCD [48]. (Section 10)

- More precisely, $D < 0$ appears in systems governed by short-range forces which include hadrons when QED is neglected, a common approximation in hadronic physics. When the long-range QED effects are present, $D(t) \propto \frac{1}{\sqrt{-t}}$ for $t \rightarrow 0$ and the D -term becomes undefined³ (Section 11).
- While hadronic D -terms are negative (when QED effects are neglected), it was shown that the D -term of the hydrogen atom is positive [46, 49, 50, 52]. Does this invalidate the pressure interpretation of hadrons? (Section 12).

In the following we explain and review the points of concern in detail, and present ideas and indicate paths along which one may seek their resolutions.

5. Is the use of the Breit frame problematic?

The short answer is: not necessarily. Before we go into more detail, it is worth stressing that this issue is not specific to the pressure interpretation. Criticism of the interpretation of the nucleon electromagnetic form factors in terms of 3D spatial distributions of electric charge or magnetization [65] is as old as the interpretation itself [75, 76]. However, this did not prevent Hofstadter from getting the 1961 Nobel Prize for investigating how the electric charge is distributed in nuclei and measuring the proton radius⁴.

The discussions about the interpretation of form factors has continued to this day [20, 44, 82–95]. A modern view is that the Breit frame definition can be fully justified by adopting a phase-space perspective, and contenting ourselves with a quasi-probabilistic interpretation [20, 91]. Alternatively, a pragmatic solution is to consider that the target is so heavy that recoil corrections are negligible. This step is well-justified for (not too light) nuclei. A theoretically rigorous justification of a 3D density interpretation in the Breit frame in the case of the nucleon consists in working in the limit of a large number of colors N_c [6, 19, 34]. When the target mass is kept finite, some recoil corrections can be included via a multipole expansion [82, 92, 96].

³ This point is not a criticism of the pressure interpretation. But it is a very interesting point whose resolution may shed light on the next point related to D -terms of atoms.

⁴ Today, many think first in terms of form factors which need to be interpreted. At Hofstadter's time [77], the natural approach was the opposite: electric form factors were *derived* for (infinitely heavy) nuclei with finite-size electric charge distributions $\rho(r)$ in non-relativistic quantum mechanics in the Born approximation [78–80]. With electron beam energies of up to 150 MeV in the early 1950s [80], the assumption of infinitely heavy nuclei and neglect of recoil and spin effects were well justified. Sachs was, to the best of our knowledge, one of the first to pursue the nowadays familiar route starting from a fully relativistic setting with electric and magnetic nucleon form factors and introducing their interpretation in some definite frame [65, 81].

To avoid recoil corrections while preserving at the same time a probabilistic interpretation, others have proposed to define instead 2D densities in the infinite-momentum frame or using the light-front formalism [20, 31, 82–86]. Averaging over all the directions, one can then define 3D densities in the so-called zero average momentum frame [89, 94]. An interpolation between the Breit frame and the infinite-momentum frame pictures is possible within the phase-space approach, and demonstrates the key role played by Wigner rotations [20, 91, 93, 97].

In what follows, we will use the 3D interpretation in the Breit frame.

6. Poincaré stresses and von Laue condition

In this section, we show that the function $p(r)$ obtained from the interpretation of $D(t)$ not only can, but even must be negative in some region of r . To understand this, it is instructive to review two (not unrelated) aspects of the early history of particle physics and special relativity.

First, the discovery of the electron by J.J. Thompson in 1897 triggered efforts to develop classical models of the electron, which was an endeavour at the center of attention of the founders of special relativity [98], with works by Thomson, Abraham, Lorentz, Poincaré, Einstein, Wien, Planck, Sommerfeld, Langevin, Ehrenfest, Born, Pauli, von Laue, and others [99]. In these early pre-quantum days, the electron was thought of to be a little sphere carrying some classical electric charge distribution, and conflicting results were obtained like $E_e = \frac{4}{3} m_e c^2$ for the rest energy of the electron. Poincaré recognized the necessity to introduce forces of non-electromagnetic origin⁵, known as “Poincaré stresses”, to compensate for the Coulomb repulsion of the electron’s own charge distribution and ensure a bound system [101]⁶.

Second, it is well known that Einstein’s original proof of $E = mc^2$ [107] and subsequent variants thereof were based on simplifying assumptions like low speeds $v \ll c$, and cannot be considered valid “fully relativistic” proofs, as Einstein did not yet have the concept of a conserved EMT $T^{\mu\nu}$. The generally accepted proof was given by Max von Laue in 1911 for closed systems with a time-independent EMT [108], and generalized by Felix Klein in 1918 to closed systems with an arbitrary time dependence [109], for reviews of the interesting history see [110]. Maxim’s static EMT in Eq. (3) corresponds to a closed, time-independent system in von Laue’s proof of $E = mc^2$, for which

⁵ Strictly speaking, this resolves some but not all inconsistencies in classical calculations of the electromagnetic mass of a particle. For a pedagogical exposition, see [100].

⁶ Even after the pointlike nature of the electron became clear, with its classical divergent electrostatic self-energy put routinely under the carpet in dimensional regularization [50], the formulation of consistent classical models of finite-size charged particles remained an intellectually challenging problem with interest until the present [102–106].

an essential ingredient is the so-called von Laue condition [108]

$$\int d^3r T^{ii}(r) = 0 \quad \Leftrightarrow \quad \int_0^\infty dr r^2 p(r) = 0 \quad (7)$$

that follows from EMT conservation. It is a necessary (but not sufficient⁷) condition for stability of a system, and can be understood as a field-theoretic version of the virial theorem [29].

After these preparations, it is clear why $p(r)$ in Eq. (3) cannot be positive everywhere: it must have at least one node and change sign somewhere to comply with the von Laue condition (7).

7. Gases, liquids, positive and negative pressures

The above discussion of Poincaré stresses indicates that we cannot simply rely on the intuition about pressure from “standard thermodynamics” [46] familiar, *e.g.*, from the ideal gas equation of state $PV = NkT$, where P is always positive. The point is that an ideal gas is an inherently unstable system: without a container, *i.e.* a fixed volume V , the ideal gas would immediately expand and disperse. Providing a container corresponds to introducing cohesive forces, the Poincaré stresses, necessary to form a stable, closed system. (Notice that in thermodynamic equilibrium and when external forces are absent, both the density $\rho = N/V$ and pressure P are constant throughout the volume occupied by the gas. In the presence of external forces like gravity, the density and pressure are not constant anymore and containers are not always necessary, as illustrated *e.g.* by the earth atmosphere, stars, and confined gas clouds in galaxies.)

An actual gas is more realistically described by the van der Waals equation of state $(P + \frac{aN^2}{V^2})(V - Nb) = NkT$, where $a > 0$ accounts for the attractive forces between the particles and $b > 0$ corrects for their finite volume. Rewriting this equation as $P = \frac{NkT}{V - Nb} - \frac{aN^2}{V^2}$, we see that attractive forces between the particles reduce the pressure, but of course not enough to reverse the sign: the attractive forces in a gas are relatively small compared to the inertial forces and so the total pressure remains positive. A van der Waals gas still needs a container to form a closed system.

Loosely speaking, the region with $p(r) > 0$ corresponds to where the matter in the system sits. This matter would disperse without the Poincaré stresses in the region with $p(r) < 0$ which bind the system. Although this is

⁷ EMT conservation implies that momentum current densities $T^{ij}(\vec{r})$ are divergenceless. This means only that the system is at equilibrium and says nothing about its stability, contrary to what is suggested in [49].

a gross oversimplification in general, in one ideal system this is literally the case, namely in the liquid drop which was, in fact, explored by Maxim as a pedagogical illustration for his newly introduced concepts in Ref. [2].

The liquid drop is bound by surface tension (one may think of astronauts and cosmonauts playing with water drops in spacecraft in Earth's orbit). The surface tension γ enters the shear force as $s(r) = \gamma \delta(r - R)$, where R is the radius of the drop. Since $s(r) = 0$ for $r \neq R$, Eq. (4) dictates that $p(r)$ is constant for $r \neq R$. More precisely, $p(r)$ is equal to the constant hydrostatic pressure p_0 for $r < R$ where the liquid is, and zero in the region $r > R$ where no liquid is present. Solving the Young–Laplace equation with these boundary conditions for given $s(r) = \gamma \delta(r - R)$ yields for the pressure $p(r) = p_0 \Theta(R - r) - \frac{2}{3} \gamma \delta(R - r)$, and the von Laue condition (7) yields the Kelvin relation $p_0 = 2\gamma/R$ for the pressure inside a drop in terms of γ and R [111]. We see that, for an ideal drop, one has $p(r) < 0$ only at one single point, namely at $r = R$ where the surface tension in $s(r) = \gamma \delta(r - R)$ acts, providing in this way the Poincaré stresses necessary to bind the system.

Since many bulk properties of nuclei can be approximately described in the liquid-drop model, Maxim made predictions for the dependence of nuclear D -terms on the mass number A as $D \propto -A^{7/3}$ [2] which were confirmed in more elaborate nuclear models, like Walecka [112] or Skyrme model [113], and await experimental tests.

Although this is not always discussed in standard thermodynamics textbooks, negative pressures do exist in thermodynamics. From a microscopic perspective, the “standard thermodynamic” pressure is understood as the macroscopic manifestation of a large collection of particles zipping around and bouncing off walls. As discussed above, in classical gases, the kinetic energy dominates and pressure is naturally positive.

In liquids, where the cohesive forces are stronger than in gases, it is usually taught that the hydrostatic pressure is isotropic and positive. This is true in most practical cases since the density usually does not change much in the bulk of a liquid. But the situation is different at the interfaces, where the density gradients become large and surface tension effects (*i.e.* anisotropic stresses) have to be invoked to describe the observations. Bulk pressure can even become negative when the liquid is in a metastable state, *e.g.* when it is trapped in a closed system and cannot expand or contract freely [114, 115]. In particular, negative pressure resulting from evaporation provides a simple explanation for why water can rise well above 10 m in the trees [114, 116], a phenomenon that cannot be quantitatively explained with capillarity arguments.

For inhomogeneous media at rest, the pressure, defined as the magnitude of the force per unit area normal to the surface element, is not isotropic and receives a contribution from both $p(r)$ and $s(r)$. In the context of neutron stars, the combinations $p_r(r) = p(r) + \frac{2}{3}s(r)$ and $p_t(r) = p(r) - \frac{1}{3}s(r)$ are respectively called *radial pressure* and *tangential pressure* [72], and correspond to the normal stresses along the radial and tangential directions. For static fluid spheres, it is usually expected that $p_r(r) \geq 0$ [117, 118]. In the familiar non-relativistic situations, we typically observe that $p(r) \gg s(r)$ (except at the interfaces) and so the positivity of the radial pressure effectively reduces to the positivity of the isotropic pressure.

In summary, pressure is not always isotropic and can even become negative. Therefore, the fact that the function $p(r)$ does necessarily become negative in some region does not jeopardize its interpretation as isotropic pressure, but simply indicates that there are situations where the isotropic part of the attractive or confining forces dominates.

8. Poincaré stresses in bag model

It is instructive to discuss the Poincaré stresses in the bag model [119]. For pedagogical purposes, let us start with its predecessor, the Bogoliubov model [120] (the history of the two models is reviewed in [121], Section 8.4). The Bogoliubov model does not describe a stable nucleon. It corresponds basically to the bag model with N_c non-interacting massless quarks in anti-symmetric color state, inserted in a cavity of radius R with a boundary condition such that there is no energy-momentum flow across the surface (this is how quarks are confined). The only difference with the bag model is that there is no bag contribution in the Bogoliubov model. The EMT in the Bogoliubov model is given solely in terms of $T_{\text{Dirac}}^{\mu\nu}$ of the quarks. In particular, the nucleon mass is given by $M_{\text{Bogob}} = N_c \omega_0 / R$, where $\omega_0 \simeq 2.04$ is the (ground state) solution of a transcendental equation dictated by the boundary condition. The free parameter R must be fixed by hand to reproduce the experimental value of the nucleon mass. The pressure in the Bogoliubov model is $p(r)_{\text{Bogob}} > 0$ [24], *i.e.* according to the von Laue condition (7) we deal with an unstable system: the nucleon explodes. This can be seen equivalently from $M_{\text{Bogob}} = N_c \omega_0 / R$, namely: if one does not fix the bag radius by hand, then the nucleon mass is minimized for $R \rightarrow \infty$. The reason for this is obvious: the model lacks the Poincaré stresses and the quarks disperse due to the positive Fermi pressure. (At this point, it is helpful to consider the large- N_c limit, so that we deal with a “macroscopic number” of quarks and the notion of thermodynamic pressure can be defined.)

This problem is remedied in the bag model by supplementing $T_{\text{Dirac}}^{\mu\nu}$ in the EMT with the bag contribution⁸ $T_{\text{bag}}^{\mu\nu} = B \Theta(R - r) g^{\mu\nu}$, where $B > 0$ is the bag constant. This has two effects:

- (i) the mass in the bag model is given by $M_{\text{bag}} = N_c \omega_0 / R + \frac{4}{3} \pi R^3 B$ and
- (ii) the pressure receives a negative constant contribution $(-B)$.

One of the two model parameters, R and B , can be fixed by minimizing M_{bag} with respect to R which yields⁹ the condition $N_c \omega_0 = 4\pi R^4 B$ sometimes referred to as virial theorem in the bag model. If this condition is satisfied, then the von Laue condition (7) is also satisfied. The equivalence of the von Laue condition and the virial theorem has been observed in many models, see *e.g.* [6–13] and discussed in the general field-theoretic context in [29]. Thus, the Poincaré stresses are explicitly incorporated in $T_{\text{bag}}^{\mu\nu} = B \Theta(R - r) g^{\mu\nu}$ and ensure in this way a consistent model of a stable nucleon.

The bag model provides another instructive lesson, namely regarding the form factor $\bar{C}^a(t)$ and nicely illustrating the usefulness and theoretical consistency of the EMT form factor interpretation. One of the earliest model studies of EMT form factors was presented in the bag model [127] where, starting from the well-known quark wave functions, the EMT form factors of quarks were computed and found non-zero. Interestingly, the EMT form factors $A(t) = \sum_q A^q(t)$, $J(t) = \sum_q J^q(t)$, and $D(t) = \sum_q D^q(t)$ (summed over quark flavors) are given entirely in terms of quarks only. This can be seen, *e.g.*, from the fact that $\sum_q A^q(0) = 1$ and $\sum_q J^q(0) = \frac{1}{2}$ [127].

However, the situation is different for the fourth EMT form factor $\bar{C}^a(t)$ since one finds that $\sum_q \bar{C}^q(t) \neq 0$ [127]. Clearly, something is missing: we should have $\sum_a \bar{C}^a(t) = 0$ when summing over *all* degrees of freedom because the EMT is certainly conserved in this model [119] and the von Laue condition (7) is satisfied [24]. What we are missing is the contribution from the bag, but we encounter here an unpleasant problem: it is not possible to treat the bag as a degree of freedom, determine its wave function, and compute in this way $\bar{C}^{\text{bag}}(t)$ [127].

Fortunately, at this point, the 3D interpretation of EMT densities comes to the rescue. The bag contribution $T_{\text{bag}}^{\mu\nu} = B \Theta(R - r) g^{\mu\nu}$ to the EMT can be Fourier transformed and yields exactly $\bar{C}^{\text{bag}}(t) = -\sum_q \bar{C}^q(t)$ [24]. This

⁸ In cosmology, dark energy is often modeled as a bag-like contribution with R equal to the size of the Universe and B proportional to the cosmological constant [122, 123]. This dark energy provides a negative pressure contribution which drives the observed accelerating expansion of the Universe. The analogy between the cosmological constant and pressure contributions in hadrons has been discussed in [124–126].

⁹ Evaluating the nucleon mass at the minimum gives $M_{\text{bag}} = \frac{4}{3} N_c \omega_0 / R$. It is amusing to note that $M_{\text{bag}} = \frac{4}{3} M_{\text{Bogogo}}$, which is reminiscent of the famous $\frac{4}{3}$ paradox in the incomplete classical electron models, see Section 6. Notice that here the Bogoliubov model is incomplete, lacks Poincaré stresses, and *misses* $\frac{1}{4}$ of the nucleon mass.

corresponds to *inverting* the EMT form factor interpretation and is, to the best of our knowledge, the only way to compute $\bar{C}^{\text{bag}}(t)$ in this model. This provides an independent proof of EMT conservation in the bag model and illustrates the theoretical consistency of the EMT distribution formalism. Notice that the calculation in Ref. [24] was carried out in the large- N_c limit, where the 3D interpretation of EMT form factors is justified [6, 19].

9. The mean free path and continuum description

In classical physics, matter is described at the microscopic level as a large collection of particles that interact with each other. At the macroscopic level, one can use instead an effective continuum description. This is typically justified when the size of the volume over which one averages the microscopic details is much larger than the typical mean free path of the particles. In quantum theory, the particle-like description makes sense provided that the typical sizes of the problem are much larger than the Compton wavelength¹⁰. Below the Compton wavelength, one should rather switch to a wave-like picture.

We naturally tend to think of hadrons as bound states of quarks and gluons. However, it is not clear at all that a particle-like picture is well adapted to describe the structure of light hadrons, since the latter are typically states with an *indefinite* number of quarks and gluons. We should keep in mind that *fields* are the fundamental degrees of freedom in quantum field theory¹¹. Particles appear only as effective degrees of freedom arising from the complicated collective dynamics and excitations of the fields.

In the context of hadronic physics, the notion of pressure is defined directly in terms of the fundamental microscopic EMT, and not in terms of an effective macroscopic one. There is therefore no need to justify it by mean free path arguments. This is to be contrasted with studies of the quark–gluon plasma, where a macroscopic effective description of the system, motivated by the large number of quasifree excitations of the quark and gluon fields, is usually adopted [130].

10. The negative D -term sign conjecture

Maxim, in collaboration with Christian Weiss, proved a soft pion theorem stating that the quark contribution to the pion D -term is $D^q = -A^q(0)$ in chiral limit [4]. This holds also for the gluon part implying the total D -term of the pion $D_{\text{pion}} = -1$. This is true also for other Goldstone bosons

¹⁰ When the particle is moving, the characteristic wavelength is given in general by the inverse of the energy and not the inverse of the mass [82].

¹¹ Note, *e.g.*, that the Casimir effect, which consists of a modification of the vacuum expectation value of the electromagnetic field energy, results in a force per unit area acting on two parallel conducting plates, sometimes referred to as the Casimir pressure [128, 129].

associated with chiral symmetry breaking, namely kaons and η -meson, and was known from early works on hadronic decays of quarkonia [131–134]. The D -term of the nucleon was found negative in the bag model [127] and chiral quark-soliton model [55, 57]. Maxim also predicted that nuclear D -terms are negative [2].

The first theoretical study of spatial EMT distribution was, to the best of our knowledge, carried out in the chiral quark-soliton model in Ref. [6] of which Maxim is a co-author. Stressing the relevance of the von Laue condition [108] and its connection to the virial theorem, this work among others also raised the suspicion that the negative sign of D might be a natural consequence of stability [6], as illustrated in Fig. 2.

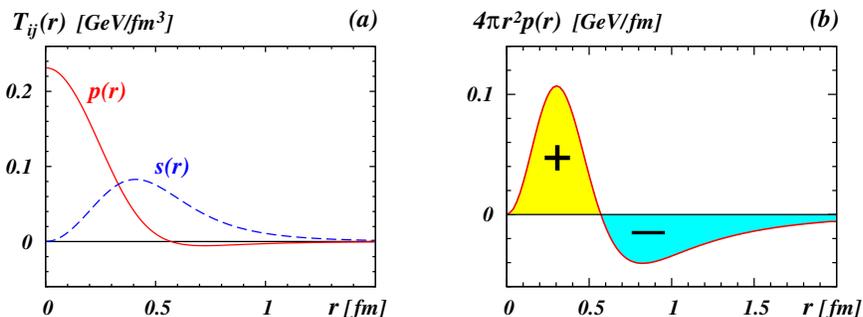


Fig. 2. (a) Shear and pressure distributions, $s(r)$ and $p(r)$, describing the nucleon stress tensor $T_{ij}(r)$ in Eq. (3) as obtained from the chiral quark-soliton model [6]. (b) $4\pi r^2 p(r)$ illustrating how the von Laue condition is satisfied in the said model: the positive and negative contributions compensate each other according to Eq. (7).

The shear and pressure distributions $s(r)$ and $p(r)$ from the chiral quark-soliton model exhibit a remote similarity to the liquid-drop picture, see Fig. 2 (a): throughout $s(r) \geq 0$, while $p(r) > 0$ in the inner region and $p(r) < 0$ in the outer region, whereby the δ - and Θ -functions of the ideal liquid drop, see Section 7, are strongly smeared out in the nucleon. This is not surprising after all, as the nucleon is considerably more diffuse than a nucleus.

Figure 2 (b) shows how the internal forces balance each other in the nucleon: $p(r) > 0$ in the center region corresponding to repulsive forces are balanced by the attractive forces, Poincaré stresses, in the outer region where $p(r) < 0$. The sign change occurs at the node $r_0 = 0.57$ fm (in this model). Multiplying $p(r)$ by r^2 yields the integrand of the von Laue condition and is shown in Fig. 2 (b). The areas above and below the r -axis compensate each other exactly according to Eq. (7). (A factor 4π is included in Fig. 2 (b) to compare the force = (pressure) \times (area $4\pi r^2$) to the string tension in the confining QCD potential known, *e.g.*, from quarkonium studies $V_{\text{conf}}(r) = kr$ with $k \simeq 1$ GeV/fm [135]. The magnitude of forces in the nucleon is understandably nowhere close to the string tension.)

Notice that the von Laue condition (7) would be equally well satisfied if the picture in Fig. 2 (b) had reversed signs, with a negative pressure in the inner region and a positive pressure in the outer region. From the point of view of mechanical stability, however, this would correspond to a seemingly unphysical situation because

- (i) nothing would prevent the matter within the node r_0 from collapsing into the center, and
- (ii) nothing would prevent the matter beyond r_0 from dispersing into infinity (which reflects the fact that the von Laue condition is necessary but not sufficient for stability).

Thus, the sign pattern in Fig. 2 (b) is a consequence of mechanical stability (we will elaborate on this point below). Now, compliance with the von Laue condition (7) in conjunction with the physical sign pattern of $p(r)$ dictates that the D -term $D = 4\pi M \int_0^\infty dr r^4 p(r)$ is negative, since the additional weight of r^2 under the r -integration as compared to Eq. (7) and Fig. 2 (b) suppresses the contribution of the small- r region and enhances that of the large- r region [6].

This physically appealing insight was supported by subsequent works where negative D -terms were obtained in other hadron models, lattice QCD, dispersion relations, Q -ball systems, and phenomenological hints [7–40]. A physical pressure sign pattern and negative D -term are necessary but not sufficient mechanical stability conditions and apply equally to unstable systems (*e.g.* Roper resonance, ρ -meson, Q -clouds) which correspond to local minima of the action in the pertinent quantum number sector.

The above observations were solidified by Irina Perevalova, Maxim *et al.* in [15], where the conclusion that $D < 0$ was drawn based on the assumptions that (i) the pressure interpretation is exact, (ii) mechanical stability criteria *can* be applied to hadrons, (iii) the densities $p(r)$ and $s(r)$ decay at long distances faster than $\frac{1}{r^4}$. The first point is justified for the nucleon in the large- N_c limit. The second is an assumption which, if one accepts its validity, can be explored as shown below. The third is a technical assumption which is satisfied for hadrons, provided one neglects electromagnetic effects (we will come back to this point below).

The argument is that at any point inside a mechanically stable system the radial pressure $p_r(r)$ must satisfy the local stability criterion

$$p_r(r) = p(r) + \frac{2}{3}s(r) \geq 0, \quad (8)$$

or else one would encounter mechanical instability [15]. (This criterion has already been mentioned in Section 7 in the context of neutron stars.) The conclusion follows immediately from the fact that if (8) holds, then

$4\pi \int_0^\infty dr r^4 p_r(r) > 0$ which can be expressed with the help of Eqs. (5) and (6) as $(-3D)/(2M)$, q.e.d. [15].

To the best of our knowledge, this is presently the most compelling explanation of why $D < 0$ is found in all physically sound (stable, metastable, unstable) systems governed by short-range forces. In the next section, we will discuss what happens in the presence of long-range forces.

Before going into that, it is worth discussing where the Poincaré stresses come from in the chiral quark-soliton model. They are provided by the strong chiral forces in the soliton (mean) field. Such a mean field picture is known to arise in QCD in the large- N_c limit [136]. At long distances, the distributions behave like $s(r) = 3 \frac{a}{r^6}$ and $p(r) = -\frac{a}{r^6}$ in the chiral limit with $a = (\frac{3g_A}{8\pi f_\pi})^2$ for large N_c , where $g_A = 1.26$ and $f_\pi = 91 \text{ MeV}$ denote, respectively, the nucleon axial coupling and pion decay constant (off the chiral limit, when the pion mass $m_\pi \neq 0$, the EMT densities are exponentially suppressed with tails proportional to $e^{-2m_\pi r}$) [6].

It is well known that the exchange of spin-0 particles yields attractive forces, and the chiral Goldstone bosons are spin zero. In this sense, the Poincaré stresses in the nucleon emerge from the pion cloud (see Ref. [137] for a careful review of the history and folklore of this term).

The results in Fig. 2 are from a specific chiral model, but the long-distance behavior $s(r) = 3 \frac{a}{r^6}$ and $p(r) = -\frac{a}{r^6}$ is model-independent and dictated by chiral symmetry. This fact was in turn explored by Maxim and Jambul Gegelia to determine a conservative upper bound for the nucleon D -term of $D \leq -(0.20 \pm 0.02)$ [26]. Noteworthy, the upper bound indicates that the D -term is negative. In the derivation of this bound, long-range forces were neglected whose impact we shall discuss in the next section.

11. Long-range forces and D -terms of charged particles

The D -term is arguably the particle property most sensitive to changes in the parameters or dynamics of a theory [16]. For instance, masses of nuclei grow linearly with mass number A while $D \propto -A^{7/3}$ [2]. Similarly, masses of large Q -balls grow approximately linearly with their charge $|Q|$, while their D -terms grow like $D \propto -|Q|^{7/3}$ [10]¹². Or, if one goes to high-lying excitations in the spectra of Q -balls and baryons (in the bag model), then $D \propto -M^{8/3}$ as the masses of the excited states increase [11, 24]¹³. However, the strongest sensitivity of the D -term observed so far is when electromagnetic effects are included [23], which were neglected in prior hadronic studies (with few exceptions [64, 139]).

¹² The appearance of the same power $7/3$ in both systems is not accidental: the ground states of large nuclei and those of Q -balls share basic similarities with liquid drops (radii growing like $A^{1/3}$ or $|Q|^{1/3}$, one can define a surface tension, *etc.*) [138].

¹³ Whether the appearance of the same power $8/3$ in the case of excitations in these very different systems has deeper underlying reasons or is accidental is unknown to us.

In Ref. [23], the *classical* model of the proton by Białynicki-Birula [105] was explored. The latter is remarkable (*cf.* footnote 6) in that it provided the first classical framework with Poincaré stresses introduced in a dynamical way (as opposed to an *ad hoc* way in prior works). This is a classical relativistic field theory which describes the proton as a static pressureless dust bound within a radius $R_{\text{dust}} \simeq 1$ fm (which emerges from the model dynamics) by the interplay of strong massive scalar and vector as well as electrostatic forces. The strong forces are taken from nuclear models, *i.e.* the dust is bound by residual strong forces [105].

An inspection of the contributions to the von Laue integral (7) from the scalar (S), vector (V) meson, and electrostatic (em) fields leads to

$$\begin{aligned} \int_0^\infty dr r^2 p(r)_S &= -10.916 \text{ MeV}, & \frac{g_S^2}{4\pi\hbar c} &= 7.29, \\ \int_0^\infty dr r^2 p(r)_V &= +10.891 \text{ MeV}, & \frac{g_V^2}{4\pi\hbar c} &= 10.8, \\ \int_0^\infty dr r^2 p(r)_{\text{em}} &= +0.025 \text{ MeV}, & \frac{e^2}{4\pi\hbar c} &= \frac{1}{137}. \end{aligned} \quad (9)$$

The coupling constants included in Eq. (9) show that strong forces are three orders of magnitude stronger than electric ones, and the latter make an accordingly minuscule contribution to the balance of forces.

The negative sign signals attractive (scalar) forces, while the positive signs indicate repulsive (vector, electrostatic) forces. The contributions to $p(r)$ and other EMT distributions from the massive strong fields exhibit Yukawa-type tails proportional to $e^{-m_i r}$ with $i = S, V$. The stronger repulsive vector fields are more massive with $m_V = 783$ MeV (ω -meson mass), have a shorter range, and make $p(r)$ positive in the inner region. The somewhat weaker and lighter scalar fields with $m_S = 550$ MeV (“ σ -meson” mass) have a longer range, dominate the outer region, and provide there the Poincaré stresses. The electromagnetic effects in Eq. (9) seem to play a naturally subordinate role for the structure of the proton. The results of the classical proton model look qualitatively very similar to those in Fig. 2, except that $s(r)$ and $p(r)$ are about one order of magnitude smaller (since the residual nuclear forces of the classical model are significantly weaker than the strong mean field forces in the chiral quark-soliton model). The electromagnetic effects make an impact only at long distances, beyond the range displayed in Fig. 2.

The strong fields govern the structure of the inner region in the classical proton model and somewhat beyond until about $r \simeq 2$ fm. But they are exponentially suppressed, and around $r \gtrsim (2-3)$ fm, the electrostatic contribution becomes comparably strong, and makes $s(r)$ and $p(r)$ change the sign again. The Maxwell stress tensor $T^{ij} = -E^i E^j + \frac{1}{2} \delta^{ij} \vec{E}^2$ in the elec-

trostatic case, see footnote 2 on sign conventions, explains the sign pattern of $s(r)$ and $p(r)$ at long distances, where the electric field in the classical proton model corresponds to that of a point charge $E^i = \frac{e}{4\pi} \frac{r^i}{r^3}$ and implies asymptotically $s(r) = -\frac{\alpha}{4\pi} \frac{\hbar c}{r^4}$ and $p(r) = \frac{\alpha}{24\pi} \frac{\hbar c}{r^4}$.

This has several consequences. First, the radial pressure $p_r(r)$ in the classical proton model does not comply with the condition (8) for $r > 2$ fm, but this is irrelevant as argued in [23]. The point is that the mechanical system is confined within the radius $R_{\text{dust}} \simeq 1$ fm, where the matter (dust) sits in the classical model. Here, the mechanical stability criterion (8) should hold and does so. At distances $r > R_{\text{dust}}$, we have only fields. The change of sign of $p_r(r)$ in (8) at $r \simeq 2$ fm *would* imply mechanical instability and expel matter to infinity *if* there was any at that point. But there is none. Hence the system is mechanically stable. In fact, the stability of the proton in the classical model was proven in [105]. Within a classical field theoretical framework, such an argumentation is physically sound [23]¹⁴.

The second consequence is that the D -term of the proton is undefined. With $s(r)$ and $p(r)$ exhibiting $\frac{1}{r^4}$ -tails at long distances, the integrals in Eqs. (5) and (6) diverge. The form factor $D(t)$, however, is well-defined for $t < 0$ and given by $D(t) = \frac{\alpha\pi}{4} \frac{m}{\sqrt{-t}} + \text{subleading terms}$ for $(-t) \ll m^2$ [23].

The impact of QED effects on $D(t)$ was actually known in literature from chiral perturbation theory studies of charged pion EMT properties [139] and other contexts [64], and further investigated subsequently in QED [25, 30]. It is not surprising that a classical model can correctly reproduce QED results [23], because classical electrodynamics is the long-distance limit of QED [64]. Can this QED effect be seen experimentally? The answer is no.

The QED asymptotics $D(t) = \frac{\alpha\pi}{4} \frac{m}{\sqrt{-t}}$, where the D -term form factor is positive, becomes apparent in the classical model for $(-t) \lesssim 10^{-5} \text{ GeV}^2$, which is beyond the reach of realistic experiments [18]. In the $10^{-2} \text{ GeV}^2 \lesssim (-t) \lesssim 1 \text{ GeV}^2$ region, the proton $D(t)$ is negative and well described in the classical model by a multipole form $D(t) = D_{\text{reg}}/(1 - t/m_D^2)^n$, where $n \simeq (2-3)$ and D_{reg} is a “regularized value” of the D -term which can be derived within the classical model in a “regularization procedure” that effectively removes the contribution from the classical $\frac{1}{r^4}$ -tail [23]. This is the behavior of the proton D -term one will be able to infer experimentally.

The introduction of massless degrees of freedom like photons¹⁵ in theory affects also other EMT form factors. But the values $A(0) = 1$ and $J(0) = \frac{1}{2}$,

¹⁴ How one could transpose such reasoning to a quantum field theoretical setting is an interesting question which remains to be elaborated.

¹⁵ The classical model can be used again for pedagogical purposes: the contributions of the massive carriers of the strong scalar and vector forces to, e.g., $s(r)$ are proportional to $\frac{(1+m_i r)^2}{r^4} e^{-2m_i r}$ for $r > R_{\text{dust}}$. In the $m_i \rightarrow 0$ limit, we recover the $\frac{1}{r^4}$ -behavior of the massless Coulomb field, which makes the D -term in Eq. (5) undefined.

in the proton case, are fixed by Poincaré invariance and the QED effects manifest themselves only in the slopes of these form factors [64]. Once again, the form factor $D(t)$ appears as the proton property most sensitive to changes in the dynamics. Julia Panteleeva has shown that these issues are not specific to QED and that D -terms are undefined also in other theories with long-range forces [35].

12. Thoughts on D -terms in atomic systems

Atoms, and especially the exactly solvable hydrogen atom, provide an appealing theoretical framework for investigating the EMT interpretation. This has been pursued with the interesting finding that the D -term of the hydrogen atom is positive [46, 49, 50, 52], at variance with the findings of negative D -terms obtained in systems governed by short-range forces, see Section 10. Can one reconcile these different behaviors? The cautious answer at this point is: possibly yes, though more work may be needed to settle the issue.

First, it is important to stress that a different behavior of atomic D -terms may not be entirely surprising because atoms are bound states of the electromagnetic interaction. In fact, we learned in Section 11 that the presence of long-range forces has drastic effects and generates, *e.g.*, due to long-distance behavior of the electric field $\propto \frac{1}{r^2}$ for the proton, a behavior like $D(t) \rightarrow \frac{\alpha\pi}{4} \frac{m}{\sqrt{-t}} \rightarrow +\infty$ as $|t| \rightarrow 0$, see Section 11.

This universal behavior is, of course, the same for the proton and electron. In the hydrogen atom, the proton charge is screened by the electron cloud and the system is overall neutral. The QED singularities due to the proton and electron charges compensate, and $D(t)$ is finite. Although the hydrogen atom is exactly solvable, the calculation of its D -term is too technical to reproduce in a few lines, but the positive sign can easily be explained. The ground-state wave function of the electron in the hydrogen atom is $\Psi(r) = e^{-r/a}/\sqrt{\pi a^3}$, where a is the Bohr radius. The “charge density” of the electron cloud is $\rho_e(r) = -e|\Psi(r)|^2$. At distances r much larger than the proton radius R_p , the electric field of the proton–electron–cloud system is $E^i = \frac{e}{4\pi}(\frac{1}{r^2} + \frac{2}{ar} + \frac{2}{a^2})\frac{r^i}{r}e^{-2r/a}$. From $T^{ij} = -E^i E^j + \frac{1}{2}\delta^{ij}\vec{E}^2$, we then see that $p(r)$ is positive and $s(r)$ is negative at asymptotic distances, *i.e.* the same sign pattern emerges as in Section 11, except that now both functions are suppressed as $e^{-4r/a}$ at long distances. This makes the integrals (5) and (6) convergent, and we obtain $0 < D < \infty$. This back-of-the-envelope estimate is in line with the more involved calculations of Refs. [46, 49, 50, 52].

However, we should keep in mind that the main point of Maxim’s pressure interpretation is not that the D -term is negative, but that a hadron can be treated as a mechanical continuum system (and only if it can, then the mechanical stability criteria can be applied and imply that $D < 0$ [15]). In order to keep the discussion simple, let us have in mind for the following

a comparison of the D -terms of a neutron and hydrogen atom. Both are electrically neutral systems and have well-defined finite D -terms, but one has a negative and the other a positive D -term.

One possible approach to put the positive D -terms of atoms into context is to rephrase our question as: can the interiors of atoms and hadrons both be reasonably treated as mechanical continuum systems?

Arguably, atoms and hadrons are very different systems and are not equally amenable to a mechanical continuum interpretation [43]. Hadrons are extremely dense systems. In the case of the nucleon, the density is of the order of $1 \text{ GeV}/\text{fm}^3$. The dynamics of this dense medium inside hadrons is governed by the extremely strong QCD forces. It is an assumption that mechanical continuum concepts can be applied to hadrons, but it is an assumption which is physically not unreasonable.

We need to contrast this with the “medium” inside the hydrogen atom. The “medium” here is basically the electron cloud. Hence, we deal with an extremely dilute system with a density of one electron mass distributed over a volume of 1 \AA^3 , *i.e.* the medium in the interior of an atom is 18 orders of magnitude more dilute compared to the medium inside a hadron.

Let us also consider “what kind of medium” we deal with inside a hydrogen atom. The hydrogen atom is a two-body system with a pointlike, positive charge in the center which binds the electron. Thus, the “medium” is actually one single body, the electron, which swirls around. The quantum mechanical solution determines the probability where the electron can be found. Does this constitute a physically viable “mechanical continuum system”? Notice, that even if we consider atoms in the large- Z limit, then there still remain fundamental differences to, *e.g.*, baryons in the large- N_c limit [140].

To conclude this section, one possible explanation why we cannot compare neutral hadrons and atoms is because we might be comparing apples and oranges. This is an admittedly intuitive approach. More work may be needed to fully understand why, in the context of the D -term, atoms may not provide useful toy-model systems to get insights about the dynamics inside hadrons¹⁶.

¹⁶ For completeness, let us mention that positive D -terms have systematically been found in the covariant quark–diquark model calculations of Refs. [33, 37, 38, 40]. A positive D -term has also been found in the scalar diquark model calculation of Ref. [36], but only in the unrealistic case where the parent hadron and the active quark have the same mass. Depending on the modeling of the effective nucleon–quark–diquark vertex, diquark approaches do not always describe the nucleon as a dynamical quark–diquark bound state compliant with the virial theorem. Often additional model assumptions are needed to obtain a negative $D(t)$ [141]. Notice that the compliance of the pressure obtained from $D(t)$ with the von Laue condition does not demonstrate the stability of the system because *any* $D(t)$ satisfying $\lim_{t \rightarrow 0} tD(t) = 0$ (and such that the integral in Eq. (2) exists) yields a pressure satisfying the von Laue condition [16].

13. Conclusions and outlook

Interpreting hadronic matrix elements of the stress tensor in terms of the concepts of “pressure” and “shear forces” borrowed from continuum mechanics is far from a trivial step. In this contribution, we reviewed the interpretation proposed by Maxim Polyakov in the seminal paper [2] and addressed the criticism raised in the recent literature [44–50].

We showed that the concept of pressure is not limited to the familiar situation found in homogeneous gases. Mechanical *equilibrium* requires that the spatial integral of pressure must vanish for a bound system, which is basically the physical content of the virial theorem. This implies that pressure is not always positive and isotropic. A bound system results from the balance between repulsive and attractive forces.

Mechanical *stability* requires, however, stronger constraints. Based on the analogy with continuum mechanics, it has been argued that a stable system should be characterized by a negative D -term [15], in agreement with results from hadronic models, lattice QCD, and dispersion relation studies [43].

The fact that the D -term is found positive in the hydrogen atom has, however, led some authors to conclude that the above mechanical interpretation does not make sense [46, 47, 49, 50]. However, we share the opinion expressed in Refs. [43, 52] and believe that this conclusion may be premature. We stressed, in particular, that a negative D -term is a natural consequence of mechanically stable systems determined by short-range forces, whereas the structure of atoms is governed by the long-range electromagnetic forces.

While clarifying the domain of validity of the stability criterion requires more work, we conclude that none of the raised criticism to date has really invalidated Maxim’s mechanical interpretation of the hadronic stress tensor.

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REFERENCES

- [1] L.D. Landau, E.M. Lifshitz, «Theory of Elasticity», Vol. 7, *Pergamon Press*, 1970.
- [2] M.V. Polyakov, «Generalized parton distributions and strong forces inside nucleons and nuclei», *Phys. Lett. B* **555**, 57 (2003).
- [3] X.D. Ji, «Gauge-Invariant Decomposition of Nucleon Spin», *Phys. Rev. Lett.* **78**, 610 (1997).

- [4] M.V. Polyakov, C. Weiss, «Skewed and double distributions in pion and nucleon», *Phys. Rev. D* **60**, 114017 (1999).
- [5] P. Schweitzer, S. Boffi, M. Radici, «Polynomiality of unpolarized off forward distribution functions and the D term in the chiral quark soliton model», *Phys. Rev. D* **66**, 114004 (2002).
- [6] K. Goeke *et al.*, «Nucleon form-factors of the energy momentum tensor in the chiral quark-soliton model», *Phys. Rev. D* **75**, 094021 (2007).
- [7] K. Goeke *et al.*, «Pion mass dependence of the nucleon form factors of the energy momentum tensor in the chiral quark-soliton model», *Phys. Rev. C* **75**, 055207 (2007).
- [8] C. Cebulla, K. Goeke, J. Ossmann, P. Schweitzer, «The nucleon form-factors of the energy momentum tensor in the Skyrme model», *Nucl. Phys. A* **794**, 87 (2007).
- [9] H.C. Kim, P. Schweitzer, U. Yakhshiev, «Energy-momentum tensor form factors of the nucleon in nuclear matter», *Phys. Lett. B* **718**, 625 (2012).
- [10] M. Mai, P. Schweitzer, «Energy momentum tensor, stability, and the D -term of Q -balls», *Phys. Rev. D* **86**, 076001 (2012).
- [11] M. Mai, P. Schweitzer, «Radial excitations of Q -balls, and their D -term», *Phys. Rev. D* **86**, 096002 (2012).
- [12] J.H. Jung, U. Yakhshiev, H.C. Kim, «Energy-momentum tensor form factors of the nucleon within a π - ρ - ω soliton model», *J. Phys. G: Nucl. Part. Phys.* **41**, 055107 (2014).
- [13] J.H. Jung, U. Yakhshiev, H.C. Kim, P. Schweitzer, «In-medium modified energy-momentum tensor form factors of the nucleon within the framework of a π - ρ - ω soliton model», *Phys. Rev. D* **89**, 114021 (2014).
- [14] B. Pasquini, M.V. Polyakov, M. Vanderhaeghen, «Dispersive evaluation of the D -term form factor in deeply virtual Compton scattering», *Phys. Lett. B* **739**, 133 (2014).
- [15] I.A. Perevalova, M.V. Polyakov, P. Schweitzer, «On LHCb pentaquarks as a baryon- $\psi(2S)$ bound state: prediction of isospin- $\frac{3}{2}$ pentaquarks with hidden charm», *Phys. Rev. D* **94**, 054024 (2016).
- [16] J. Hudson, P. Schweitzer, « D term and the structure of pointlike and composed spin-0 particles», *Phys. Rev. D* **96**, 114013 (2017).
- [17] J. Hudson, P. Schweitzer, «Dynamic origins of fermionic D -terms», *Phys. Rev. D* **97**, 056003 (2018).
- [18] V.D. Burkert, L. Elouadrhiri, F.X. Girod, «The pressure distribution inside the proton», *Nature* **557**, 396 (2018).
- [19] M.V. Polyakov, P. Schweitzer, «Forces inside hadrons: Pressure, surface tension, mechanical radius, and all that», *Int. J. Mod. Phys. A* **33**, 1830025 (2018).
- [20] C. Lorcé, H. Moutarde, A.P. Trawiński, «Revisiting the mechanical properties of the nucleon», *Eur. Phys. J. C* **79**, 89 (2019).

- [21] P.E. Shanahan, W. Detmold, «Pressure Distribution and Shear Forces inside the Proton», *Phys. Rev. Lett.* **122**, 072003 (2019).
- [22] M.V. Polyakov, B.D. Sun, «Gravitational form factors of a spin one particle», *Phys. Rev. D* **100**, 036003 (2019).
- [23] M. Varma, P. Schweitzer, «Effects of long-range forces on the D -term and the energy-momentum structure», *Phys. Rev. D* **102**, 014047 (2020).
- [24] M.J. Neubelt *et al.*, «Energy momentum tensor and the D term in the bag model», *Phys. Rev. D* **101**, 034013 (2020).
- [25] A. Metz, B. Pasquini, S. Rodini, «The gravitational form factor $D(t)$ of the electron», *Phys. Lett. B* **820**, 136501 (2021).
- [26] J. Gegelia, M.V. Polyakov, «A bound on the nucleon Druck-term from chiral EFT in curved space-time and mechanical stability conditions», *Phys. Lett. B* **820**, 136572 (2021).
- [27] S. Owa, A.W. Thomas, X.G. Wang, «Effect of the pion field on the distributions of pressure and shear in the proton», *Phys. Lett. B* **829**, 137136 (2022).
- [28] D.A. Pefkou, D.C. Hackett, P.E. Shanahan, «Gluon gravitational structure of hadrons of different spin», *Phys. Rev. D* **105**, 054509 (2022).
- [29] C. Lorcé, A. Metz, B. Pasquini, S. Rodini, «Energy-momentum tensor in QCD: nucleon mass decomposition and mechanical equilibrium», *J. High Energy Phys.* **2021**, 121 (2021).
- [30] A. Freese, A. Metz, B. Pasquini, S. Rodini, «The gravitational form factors of the electron in quantum electrodynamics», *Phys. Lett. B* **839**, 137768 (2023).
- [31] A. Freese, W. Cosyn, «Spatial densities of momentum and forces in spin-one hadrons», *Phys. Rev. D* **106**, 114013 (2022).
- [32] A. Freese, W. Cosyn, «Spatial densities of the photon on the light front», *Phys. Rev. D* **106**, 114014 (2022).
- [33] D. Fu, B.D. Sun, Y. Dong, «Electromagnetic and gravitational form factors of Δ resonance in a covariant quark–diquark approach», *Phys. Rev. D* **105**, 096002 (2022).
- [34] C. Lorcé, P. Schweitzer, K. Tezgin, «2D energy-momentum tensor distributions of nucleon in a large- N_c quark model from ultrarelativistic to nonrelativistic limit», *Phys. Rev. D* **106**, 014012 (2022).
- [35] J.Y. Panteleeva, «Internal force distributions in the 't Hooft–Polyakov monopole and Julia–Zee dyon», *Phys. Rev. D* **107**, 055015 (2023).
- [36] A. Amor-Quiroz, W. Focillon, C. Lorcé, S. Rodini, «Energy-momentum tensor in the scalar diquark model», *Eur. Phys. J. C* **83**, 1012 (2023).
- [37] D. Fu, J. Wang, Y. Dong, «Form factors of Ω^- in a covariant quark–diquark approach», *Phys. Rev. D* **108**, 076023 (2023).
- [38] J. Wang, D. Fu, Y. Dong, «Form factors of decuplet baryons in a covariant quark–diquark approach», *Eur. Phys. J. C* **84**, 79 (2024).

- [39] B. Maynard, «Energy-momentum tensor in Φ^4 theory at one loop», *Phys. Rev. D* **111** (2025) in press, [arXiv:2406.08857 \[hep-ph\]](#).
- [40] J. Wang, D. Fu, Y. Dong, «A systematical study of the nucleon form factors with the pion cloud effect», [arXiv:2410.14953 \[hep-ph\]](#).
- [41] C. Adam, A. Garcia Martin-Caro, C. Naya, A. Wereszczynski, «Integral identities and universal relations for solitons», *Phys. Rev. D* **110**, 116014 (2024).
- [42] W. Broniowski, E. Ruiz Arriola, «Gravitational form factors and mechanical properties of the nucleon in a meson dominance approach», [arXiv:2503.09297 \[hep-ph\]](#).
- [43] V.D. Burkert *et al.*, «Colloquium: Gravitational form factors of the proton», *Rev. Mod. Phys.* **95**, 041002 (2023).
- [44] R.L. Jaffe, «Ambiguities in the definition of local spatial densities in light hadrons», *Phys. Rev. D* **103**, 016017 (2021).
- [45] A. Freese, G.A. Miller, «Forces within hadrons on the light front», *Phys. Rev. D* **103**, 094023 (2021).
- [46] X. Ji, Y. Liu, «Gravitational Tensor-Monopole Moment of Hydrogen Atom To Order $\mathcal{O}(\alpha)$ », [arXiv:2208.05029 \[hep-ph\]](#).
- [47] X. Ji, «Proton mass decomposition: Naturalness and interpretations», *Front. Phys.* **16**, 64601 (2021).
- [48] M. Fujita, Y. Hatta, S. Sugimoto, T. Ueda, «Nucleon D -term in holographic quantum chromodynamics», *Prog. Theor. Exp. Phys.* **2022**, 093B06 (2022).
- [49] X. Ji, Y. Liu, «Momentum-Current Gravitational Multipoles of Hadrons», *Phys. Rev. D* **106**, 034028 (2022).
- [50] A. Czarnecki, Y. Liu, S.N. Reza, «Energy-momentum Tensor of a Hydrogen Atom: Stability, D -term, and the Lamb Shift», *Acta Phys. Pol. B Proc. Suppl.* **16**, 7-A19 (2023).
- [51] X. Ji, C. Yang, «Momentum Flow and Forces on Quarks in the Nucleon», [arXiv:2503.01991 \[hep-ph\]](#).
- [52] A. Freese, «Quantum stresses in the hydrogen atom», [arXiv:2412.09664 \[hep-ph\]](#).
- [53] M.V. Polyakov, A.G. Shuvaev, «On “dual” parametrizations of generalized parton distributions», [arXiv:hep-ph/0207153](#).
- [54] M. Polyakov, P. Schweitzer, C. Weiss, «Foreword», *Ann. Phys.* **516**, 627 (2004).
- [55] V.Y. Petrov *et al.*, «Off-forward quark distributions of the nucleon in the large- N_c limit», *Phys. Rev. D* **57**, 4325 (1998).
- [56] M. Penttinen, M.V. Polyakov, K. Goeke, «Helicity skewed quark distributions of the nucleon and chiral symmetry», *Phys. Rev. D* **62**, 014024 (2000).
- [57] N. Kivel, M.V. Polyakov, M. Vanderhaeghen, «Deeply virtual Compton scattering on the nucleon: Study of the twist-3 effects», *Phys. Rev. D* **63**, 114014 (2001).

- [58] D. Diakonov, V. Petrov, M.V. Polyakov, «Exotic anti-decuplet of baryons: Prediction from chiral solitons», *Z. Phys. A* **359**, 305 (1997).
- [59] I.Y. Kobzarev, L.B. Okun, «Gravitational Interaction of Fermions», *Zh. Eksp. Teor. Fiz.* **43**, 1904 (1962),
<http://jetp.ras.ru/cgi-bin/e/index/e/16/5/p1343?a=list>
- [60] H. Pagels, «Energy-Momentum Structure Form Factors of Particles», *Phys. Rev.* **144**, 1250 (1966).
- [61] D.J. Gross, F. Wilczek, «Asymptotically Free Gauge Theories. II.», *Phys. Rev. D* **9**, 980 (1974).
- [62] H.D. Politzer, «Setting the scale for predictions of asymptotic freedom», *Phys. Rev. D* **9**, 2174 (1974).
- [63] A.A. Pomeransky, R.A. Senkov, I.B. Khriplovich, «Spinning relativistic particles in external fields», *Phys. Usp.* **43**, 1055 (2000).
- [64] J.F. Donoghue, B.R. Holstein, B. Garbrecht, T. Konstandin, «Quantum corrections to the Reissner–Nordström and Kerr–Newman metrics», *Phys. Lett. B* **529**, 132 (2002); *Erratum ibid.* **612**, 311 (2005).
- [65] R.G. Sachs, «High-Energy Behavior of Nucleon Electromagnetic Form Factors», *Phys. Rev.* **126**, 2256 (1962).
- [66] L.D. Landau, E.M. Lifschits, «The Classical Theory of Fields», Vol. 2, Pergamon Press, 1975.
- [67] D.E. Soper, «Classical Field Theory», Wiley, New York 1976.
- [68] M.V. Polyakov, H.D. Son, «Nucleon gravitational form factors from instantons: forces between quark and gluon subsystems», *J. High Energy Phys.* **2018**, 156 (2018).
- [69] C. Lorcé, «On the hadron mass decomposition», *Eur. Phys. J. C* **78**, 120 (2018).
- [70] M. Ruderman, «Pulsars: Structure and Dynamics», *Annu. Rev. Astron. Astrophys.* **10**, 427 (1972).
- [71] V. Canuto, «Equation of State at Ultrahigh Densities», *Annu. Rev. Astron. Astrophys.* **12**, 167 (1974).
- [72] S.S. Bayin, «Anisotropic fluid spheres in general relativity», *Phys. Rev. D* **26**, 1262 (1982).
- [73] J.G. Kirkwood, F.P. Buff, «The Statistical Mechanical Theory of Surface Tension», *J. Chem. Phys.* **17**, 338 (1949).
- [74] A. Marchand, J.H. Weijts, J.H. Snoeijer, B. Andreotti, «Why is surface tension a force parallel to the interface?», *Am. J. Phys.* **79**, 999 (2011).
- [75] D.R. Yennie, M.M. Lévy, D.G. Ravenhall, «Electromagnetic Structure of Nucleons», *Rev. Mod. Phys.* **29**, 144 (1957).
- [76] G. Breit, «Limitations on the Interpretation of Electromagnetic Form-Factors of Nucleons», in: «Proceedings of the XII International Conference on High Energy Physics (ICHEP 1964)», *Atomizdat*, Moscow 1966, pp. 985–987.

- [77] R. Hofstadter, «Electron Scattering and Nuclear Structure», *Rev. Mod. Phys.* **28**, 214 (1956).
- [78] E. Guth, «On the Interaction Between Fast Electrons and Nuclei», *Anzeiger der Akademie der Wissenschaften in Wien, Mathematisch-naturwissenschaftliche Klasse* **24**, 299 (1934).
- [79] M.E. Rose, «The Charge Distribution in Nuclei and the Scattering of High Energy Electrons», *Phys. Rev.* **73**, 279 (1948).
- [80] L.I. Schiff, «Interpretation of Electron Scattering Experiments», *Phys. Rev.* **92**, 988 (1953); *Erratum ibid.* **93**, 1434 (1954).
- [81] F.J. Ernst, R.G. Sachs, K.C. Wali, «Electromagnetic Form Factors of the Nucleon», *Phys. Rev.* **119**, 1105 (1960).
- [82] M. Burkardt, «Impact parameter dependent parton distributions and off-forward parton distributions for $\zeta \rightarrow 0$ », *Phys. Rev. D* **62**, 071503 (2000); *Erratum ibid.* **66**, 119903 (2002).
- [83] G.N. Fleming, «Charge Distributions from Relativistic Form-Factors», in: «Physical Reality and Mathematical Description», *Springer, Dordrecht* 1974, p. 357.
- [84] D.E. Soper, «The Parton Model, the Bethe–Salpeter Wave Function», *Phys. Rev. D* **15**, 1141 (1977).
- [85] G.A. Miller, «Transverse Charge Densities», *Annu. Rev. Nucl. Part. Sci.* **60**, 1 (2010).
- [86] A. Freese, G.A. Miller, «Unified formalism for electromagnetic and gravitational probes: Densities», *Phys. Rev. D* **105**, 014003 (2022).
- [87] J.J. Kelly, «Nucleon charge and magnetization densities from Sachs form-factors», *Phys. Rev. C* **66**, 065203 (2002).
- [88] A.V. Belitsky, X.d. Ji, F. Yuan, «Quark imaging in the proton via quantum phase-space distributions», *Phys. Rev. D* **69**, 074014 (2004).
- [89] E. Epelbaum *et al.*, «Definition of Local Spatial Densities in Hadrons», *Phys. Rev. Lett.* **129**, 012001 (2022).
- [90] J.Y. Panteleeva, E. Epelbaum, J. Gegelia, U.G. Meißner, «Definition of electromagnetic local spatial densities for composite spin-1/2 systems», *Phys. Rev. D* **106**, 056019 (2022).
- [91] C. Lorcé, «Charge Distributions of Moving Nucleons», *Phys. Rev. Lett.* **125**, 232002 (2020).
- [92] Y. Li *et al.*, «Minkowski’s lost legacy and hadron electromagnetism», *Phys. Lett. B* **838**, 137676 (2023).
- [93] Y. Chen, C. Lorcé, «Pion and nucleon relativistic electromagnetic four-current distributions», *Phys. Rev. D* **106**, 116024 (2022).
- [94] J.Y. Panteleeva, E. Epelbaum, J. Gegelia, U.G. Meißner, «Definition of gravitational local spatial densities for spin-0 and spin-1/2 systems», *Eur. Phys. J. C* **83**, 617 (2023).

- [95] J.Y. Panteleeva, E. Epelbaum, J. Gegelia, U.G. Meißner, «Electromagnetic and gravitational local spatial densities for spin-1 systems», *J. High Energy Phys.* **2023**, 237 (2023).
- [96] Y. Li, Q. Wang, J.P. Vary, «Hadronic energy-momentum tensor revisited», [arXiv:2405.06892 \[hep-ph\]](#).
- [97] H.Y. Won, C. Lorcé, «Relativistic energy-momentum tensor distributions in a polarized nucleon», [arXiv:2503.07382 \[hep-ph\]](#).
- [98] I. Bialynicki-Birula, «Classical model of the electron. Exactly soluble example», *Phys. Rev. D* **28**, 2114 (1983).
- [99] For a review of these efforts see, *e.g.*, A.I. Miller, «Albert Einstein's Special Theory of Relativity», Addison-Wesley, Reading 1981.
- [100] D.J. Griffiths, R.E. Owen, «Mass renormalization in classical electrodynamics», *Am. J. Phys.* **51**, 1120 (1983).
- [101] M.H. Poincaré, «Sur la dynamique de l'électron», *Comptes Rendus.* **140**, 1504 (1905); *Rend. Circ. Matem. Palermo* **21**, 129 (1906).
- [102] P.A.M. Dirac, «An extensible model of the electron», *Proc. Roy. Soc. Lond. A* **268**, 57 (1962).
- [103] T.H. Boyer, «Classical model of the electron and the definition of electromagnetic field momentum», *Phys. Rev. D* **25**, 3246 (1982).
- [104] J.S. Schwinger, «Electromagnetic Mass Revisited», *Found. Phys.* **13**, 373 (1983).
- [105] I. Bialynicki-Birula, «Simple relativistic model of a finite size particle», *Phys. Lett. A* **182**, 346 (1993).
- [106] F. Rohrlich, «The dynamics of a charged sphere and the electron», *Am. J. Phys.* **65**, 1051 (1997).
- [107] A. Einstein, «Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig?», *Ann. Phys.* **18**, 639 (1905).
- [108] M. Laue, «Zur Dynamik der Relativitätstheorie», *Ann. Phys.* **340**, 524 (1911).
- [109] F. Klein, «Über die Integralform der Erhaltungssätze und die Theorie der räumlich-geschlossenen Welt», Nach. Gesells. Wissensch. Göttingen, Math.-Physik. Klasse, 1918, pp. 394–423, [arXiv:2006.14743 \[physics.hist-ph\]](#) (English translation).
- [110] H.C. Ohanian, «Einstein's $E = mc^2$ mistakes», [arXiv:0805.1400 \[physics.hist-ph\]](#); «Did Einstein Prove $E = mc^2$?», *Stud. Hist. Philos. Mod. Phys.* **40**, 167 (2009); «Klein's theorem and the proof of $E = mc^2$ », *Am. J. Phys.* **80**, 1067 (2012).
- [111] W. Thomson (Lord Kelvin), «On the thermal effect of drawing out a film of liquid», *Proc. Roy. Soc. Lond.* **9**, 255 (1858).
- [112] V. Guzey, M. Siddikov, «On the A -dependence of nuclear generalized parton distributions», *J. Phys. G: Nucl. Part. Phys.* **32**, 251 (2006).
- [113] A. García Martín-Caro, M. Huidobro, Y. Hatta, «Nuclear mass radius and pressure in the Skyrme model», *Phys. Rev. D* **110**, 034002 (2024).

- [114] P.G. Debenedetti, «Metastable Liquids: Concepts and Principles. Vol. 1», *Princeton University Press*, 1996.
- [115] L. Mercury, Y. Tardy, «Negative pressure of stretched liquid water. Geochemistry of soil capillaries», *Geochim. Cosmochim. Acta* **65**, 3391 (2001).
- [116] T. Wheeler, A. Stroock, «The transpiration of water at negative pressures in a synthetic tree», *Nature* **455**, 208 (2008).
- [117] L. Herrera, N.O. Santos, «Local anisotropy in self-gravitating systems», *Phys. Rep.* **286**, 53 (1997).
- [118] M.K. Mak, T. Harko, «Anisotropic stars in general relativity», *Proc. Roy. Soc. Lond. A* **459**, 393 (2003).
- [119] A. Chodos *et al.*, «A new extended model of hadrons», *Phys. Rev. D* **9**, 3471 (1974).
- [120] P.N. Bogoliubov, «Sur un modèle à quarks quasi-indépendants», *Ann. Inst. Henri Poincaré* **8**, 163 (1968),
http://www.numdam.org/item/AIHPA_1968__8_2_163_0/
- [121] A.W. Thomas, W. Weise, «The Structure of the Nucleon», *Wiley-VCH, Berlin, Germany* 2001.
- [122] J. Frieman, M. Turner, D. Huterer, «Dark Energy and the Accelerating Universe», *Ann. Rev. Astron. Astrophys.* **46**, 385 (2008).
- [123] J. Martin, «Everything you always wanted to know about the cosmological constant problem (but were afraid to ask)», *Comptes Rendus Phys.* **13**, 566 (2012).
- [124] O. Teryaev, «Analyticity and higher twists», *Nucl. Phys. B Proc. Suppl.* **245**, 195 (2013).
- [125] O.V. Teryaev, «Gravitational form factors and nucleon spin structure», *Front. Phys.* **11**, 111207 (2016).
- [126] K.F. Liu, «Hadrons, superconductor vortices, and cosmological constant», *Phys. Lett. B* **849**, 138418 (2024).
- [127] X.D. Ji, W. Melnitchouk, X. Song, «Study of off-forward parton distributions», *Phys. Rev. D* **56**, 5511 (1997).
- [128] V.M. Mostepanenko, N.N. Trunov, «The Casimir effect and its applications», *Sov. Phys. Usp.* **31**, 965 (1988).
- [129] G.L. Klimchitskaya, U. Mohideen, V.M. Mostepanenko, «The Casimir force between real materials: Experiment and theory», *Rev. Mod. Phys.* **81**, 1827 (2009).
- [130] E. Shuryak, «Strongly coupled quark–gluon plasma in heavy ion collisions», *Rev. Mod. Phys.* **89**, 035001 (2017).
- [131] M.B. Voloshin, V.I. Zakharov, «Measuring QCD Anomalies in Hadronic Transitions between Onium States», *Phys. Rev. Lett.* **45**, 688 (1980).
- [132] V.A. Novikov, M.A. Shifman, «Comment on the $\psi' \rightarrow J/\psi\pi\pi$ decay», *Z. Phys. C* **8**, 43 (1981).

- [133] M.B. Voloshin, A.D. Dolgov, «On Gravitational Interaction of the Goldstone Bosons», *Sov. J. Nucl. Phys.* **35**, 120 (1982).
- [134] H. Leutwyler, M.A. Shifman, «Goldstone bosons generate peculiar conformal anomalies», *Phys. Lett. B* **221**, 384 (1989).
- [135] T. Barnes, S. Godfrey, E.S. Swanson, «Higher charmonia», *Phys. Rev. D* **72**, 054026 (2005).
- [136] E. Witten, «Baryons in the $1/N$ expansion», *Nucl. Phys. B* **160**, 57 (1979).
- [137] U.G. Meißner, «The Pion cloud of the Nucleon: Facts and Popular Fantasies», *AIP Conf. Proc.* **904**, 142 (2007).
- [138] S.R. Coleman, «Q-balls», *Nucl. Phys. B* **262**, 263 (1985).
- [139] B. Kubis, U.G. Meißner, «Virtual photons in the pion form factors and the energy-momentum tensor», *Nucl. Phys. A* **671**, 332 (2000); *Erratum ibid.* **692**, 647 (2001).
- [140] D. Diakonov *et al.*, «Nucleon parton distributions at low normalization point in the large N_c limit», *Nucl. Phys. B* **480**, 341 (1996).
- [141] D. Müller, D.S. Hwang, «The concept of phenomenological light-front wave functions — Regge improved diquark model predictions», [arXiv:1407.1655](https://arxiv.org/abs/1407.1655) [hep-ph].