# ON INSTABILITY OF GROUND STATES IN 2D $\mathbb{CP}^{N-1}$ AND $\mathbb{O}^{N}$ MODELS AT LARGE $N$*

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We consider properties of the inhomogeneous solution found recently for the $\mathbb{CP}^{N-1}$ model. The solution was interpreted as a soliton. We reevaluate its energy in three different ways and find that it is negative contrary to the previous claims. Hence, instead of the solitonic interpretation, it calls for reconsideration of the issue of the true ground state. While complete resolution is still absent, we show that the energy density of the periodic elliptic solution is lower than the energy density of the homogeneous ground state. We also discuss similar solutions for the $\mathbb{O}(N)$ model and for SUSY extensions.

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## 1. Introduction

The two-dimensional $\mathbb{CP}^{N-1}$ sigma-model allows for the exact solution at large $N$ [1, 2] and represents such nonperturbative effects as gap generation, condensates, and nontrivial $\theta$-dependence. It is an asymptotically free theory and in many respects serves as the laboratory for investigation of complicated nonperturbative phenomena in QCD [3]. It was usually assumed that in the infinite volume, the theory is in the confinement phase. However, more recently, it was demonstrated that the phase transition from the confinement phase to the Higgs phase occurs if the model is perturbed by the twisted mass term [4, 5], considered on $S^1$ [6] or at the finite interval [7, 8].

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It was known for a while that in spite of many similar properties of 2D $\mathbb{CP}^{N-1}$ and QCD, there is one notable difference — the signs of the nonperturbative vacuum energies in the 2D $\mathbb{CP}^{N-1}$ sigma-model and QCD are opposite [3]. In QCD, the vacuum energy density is proportional to the gluon condensate

$$\epsilon_{\rm vac}^{\rm QCD} = \frac{1}{4}\langle\theta^\mu_\mu\rangle = \left\langle M\,\frac{\mathrm{d}\mathcal{L}^{\rm QCD}}{\mathrm{d}M}\right\rangle = \frac{1}{32g^4}\,M\,\frac{\mathrm{d}g(M)}{\mathrm{d}M}\,\left\langle \mathrm{Tr}\,G^{\mu\nu}G_{\mu\nu}\right\rangle\,, \tag{1}$$

while in $\mathbb{CP}^{N-1}$, it is the $\langle -D_\mu \bar{n}_a D^\mu n^a\rangle$ condensate instead

$$\epsilon_{\rm vac}^{\rm CP} = \frac{1}{2}\langle\theta^\mu_\mu\rangle = \left\langle M\,\frac{\mathrm{d}\mathcal{L}^{\rm CP}}{\mathrm{d}M}\right\rangle = \frac{1}{4g^4}\,M\,\frac{\mathrm{d}g(M)}{\mathrm{d}M}\,\langle -D_\mu \bar{n}_a D^\mu n^a\rangle\,. \tag{2}$$

Both theories are asymptotically free, *i.e.* have $M\mathrm{d}g/\mathrm{d}M<0$, and both condensates $\left\langle \mathrm{Tr}\,G^{\mu\nu}G_{\mu\nu}\right\rangle$ and $\langle -D_\mu \bar{n}_a D^\mu n^a\rangle$ are positively definite in the Euclidean signature. However, the gluon condensate is positive in its both perturbative and nonperturbative pieces, while the positivity of $\langle -D_\mu \bar{n}_a D^\mu n^a\rangle$ is due to the perturbative part only — the nonperturbative part is negative, see [3] for details.

The model can be also considered in the SUSY setting and it turns out that the observed similarity between the $\mathbb{CP}^{N-1}$ model and QCD has a very attractive explanation in the SUSY context. The SQCD allows for the non-Abelian strings [9–11] and the SUSY–$\mathbb{CP}^{N-1}$ is just the worldsheet theory on the non-Abelian string (see [12, 13] for the review). The degrees of freedom in the $\mathbb{CP}^{N-1}$ model are identified with the orientational modes on the non-Abelian string. A similar non-Abelian string solution occurs also in the non-SUSY 4D gauge model which is essentially the bosonic part of the SQCD Lagrangian [14]. In this case, the worldsheet theory on the string is the non-SUSY $\mathbb{CP}^{N-1}$ model.

There is 2D–4D correspondence [15] between SQCD and the worldsheet theory on the defect. It claims that running the coupling constant, spectrum of the stable particles, twisted superpotentials in 4D and 2D theories fit each other. The very 2D–4D correspondence reflects the property that the non-Abelian string can exist on the top of the SQCD vacuum not destroying it as the electron can propagate at the top of the Cooper condensate. It just makes quantitative that properties of any object considered from the viewpoints of 2D and 4D observers should be the same.

Recently, the new inhomogeneous solution to the $\mathbb{CP}^{N-1}$ model has been found in Ref. [16]. The key tool for the derivation of the solution was the particular mapping of the $\mathbb{CP}^{N-1}$ model to the Gross–Neveu (GN) model. The new solution of the $\mathbb{CP}^{N-1}$ model was obtained from the kink solution of the GN model interpolating between two vacua with the different values of the fermion condensate. A more general kink lattice configuration has been found as well using the elliptic solution to the GN model. This inhomogeneous solution and especially the lattice solution has some common properties with the inhomogeneous condensates in the GN and the chiral GN models [17, 18]. Note that there is also some analogy with the Peierls model of $1+1$ superconductivity. In that case, the electron propagates along some nontrivial profile of the lattice state and the integrability of the model allows to get its exact solution in some continuum [19] and discrete cases [20]. The fermions play the role of the eigenfunctions for the Lax operator for some integrable model and the spectral curve describing the finite-gap solution simultaneously plays the role of the dispersion law for the fermions. The ground state of the system strongly depends on the fermionic density and the temperature.

In this study, we focus on some aspects of this new solution. We reevaluate accurately its energy and find that it is negative contrary to the statement made in [16]. Three different approaches of derivation of the ground-state energy yield the same result. This raises the question concerning the true ground state of the model. We shall argue that the inhomogeneous solution and, in particular, the elliptic soliton lattice are the candidate ground state of the model. Recently, the model ground state was studied in [38–41]. Results of [38] yield the homogeneous ground state for the model on a torus of arbitrary size, while [40] and [41] suggest the possibility of a periodic inhomogeneous solution for some boundary conditions in the finite interval and at the finite chemical potential for one field component respectively.

However, there are some reservations due to the IR properties of the solution. The soliton solution is plagued by the presence of a zero mode, which makes the functional determinant and effective action ill-defined. We argue that this zero mode is a result of the breaking of rotational invariance by the solitonic scalar field. The zero modes describe low-energy soliton dynamics in moduli-space approximation. To establish a connection between zero mode and soliton rotation, we follow [42] and construct a coherent state with the correct solitonic formfactor. We study $\mathrm{SU}(N)$ generators action on this state and show that their behavior is consistent with rotational interpretation.

Let us recall that the conventional viewpoint implies the existence of single homogeneous ground state separated by the small gaps of the order of $1/N$ from the set of the metastable vacua. The ground state of the $\mathbb{CP}^1$ model becomes degenerate only at $\theta = \pi$ when kinks are allowed, and in the SUSY case for $\mathbb{CP}^{N-1}$, when $N$ degenerate vacua exist. At one-loop level, the kinetic term for the photon is generated which yields the linear potential between charges. It was argued in [2] that the excitations of the model are identified as the singlet $n^*n$ states. It was also noted in [2] that the $n$-particle corresponds to the kink between two vacua if the fermions are added to the Lagrangian. To some extent, the $n^*n$ pair corresponds to the interpolation between the excited metastable vacuum and the true one. In this paper, we question this standard picture.

The soliton solution in the $\mathbb{CP}^{N-1}$ model obtained in [16] is the counterpart of the elementary kink solution in the GN model or the composite kink solution in the chiral GN model. In the GN model, there are two vacua, therefore, the interpolating kink with the well-defined topological charge does exist. The topology guarantees its stability. Since it is this solution which gets mapped into the $\mathbb{CP}^{N-1}$ solution, we could wonder if there is some topological reason which yields the stability of the new solution in the $\mathbb{CP}^{N-1}$ case.

We also discuss a similar solution in the $\mathbb{O}^N$ model and in the $\mathcal{N}=1$ SUSY extensions. Although the kinks in the SUSY case are well-defined BPS particles saturating the corresponding central charges, the evaluation of their masses was a controversial issue for a while with several different answers. This puzzle has been resolved in [21, 22] where the effect of anomalies has been taken into account carefully. The finite effects of the anomalies in the mode counting have been also found in the non-SUSY $\mathbb{CP}^{N-1}$ model in [23].

The paper is organized as follows. In Section 2, we recall the main features of the nonperturbative solution to the $\mathbb{CP}^{N-1}$ model and the inhomogeneous solution is derived via the method of resolvent. Its energy is evaluated by three different approaches in Section 3. The zero-mode problem is discussed in detail in Section 4. Some remarks concerning the connection with the GN model and the SUSY generalization of the solution are presented in Section 5, while the elliptic kink crystal solution is considered in Section 6. The results and open questions are summarized in Discussion, Section 7, while some technical details are collected in Appendices.

## 2. $\mathbb{CP}^{N-1}$ model

### *2.1. Saddle-point equations*

Let us remind the standard derivation of the saddle-point approximation to the solution. The Lagrangian of the $\mathbb{CP}^{N-1}$ model in Minkowski space is

$$\mathcal{L} = D^\mu \bar{n}_a \, D_\mu n^a - \lambda \left( \bar{n}_a n^a - r \right) , \tag{3}$$

where $n^a$, $a = 1, \ldots, N$ are complex fields in the fundamental representation of $\mathrm{SU}(N)$, $r = 1/g^2$ defines the coupling constant, $\bar{n}_a = (n^a)^*$, and $\lambda$ is the Lagrange multiplier. Moreover, $D_\mu n^a = (\partial_\mu + iA_\mu)\, n^a$, where $A_\mu$ is a dummy field.

Let us go to Euclidian signature and integrate over $N-1$ fields $n^a$, $a = 1, \dots, N-1$, but not over $n^N = n$. Due to gauge invariance, the $n^N$ field can be chosen to be real. Besides the field $n$, the arising effective action depends on two more real fields: $\lambda$ and $A_\mu$. For $A_\mu = 0$, the Euclidian effective action takes the form

$$S = (N-1)\mathrm{Tr}\log\left(-\partial^2 + \lambda\right) + \int \mathrm{d}^2x \left((\partial n)^2 + \lambda\left(n^2 - r\right)\right) . \tag{4}$$

Let us write now the saddle-point equation implying that the fields $\lambda$ and $n$ are static, *i.e.*, do not depend on time, but could depend on space coordinate $x$. Variation over $n(x)$ leads to

$$\left(\partial_x^2 - \lambda\left(x\right)\right) n\left(x\right) = 0\,, \tag{5}$$

what allows to express $\lambda$ in terms of $n$

$$\lambda = \frac{\partial_x^2 n}{n}\,. \tag{6}$$

From variation over $\lambda(x)$, we get (neglecting the difference between $N-1$ and $N$)

$$\int \mathrm{d}t \left[N\Big\langle x,t\Big|\frac{1}{-\partial_t^2 - \partial_x^2 + \lambda}\Big|x,t\Big\rangle + n^2(x) - r\right] = 0\,, \tag{7}$$

what is equivalent to

$$\frac{N}{2\pi}\int \mathrm{d}\omega \left\langle x\Big|\frac{1}{-\partial_x^2 + \omega^2 + \lambda}\Big|x\right\rangle + n^2\left(x\right) - r = 0\,. \tag{8}$$

For the homogeneous solution with $\lambda = m^2$, the field $n = 0$ and

$$r = \frac{N}{(2\pi)^2}\int \mathrm{d}\omega \mathrm{d}k\,\frac{1}{k^2 + \omega^2 + \lambda} = \frac{N}{4\pi}\int \mathrm{d}\omega\,\frac{1}{\sqrt{\omega^2 + m^2}} = \frac{N}{2\pi}\log\frac{M}{m}\,, \tag{9}$$

where $M$ denotes the UV cut-off introduced via the Pauli–Villars regularization (see Section 3.2 for details).

For the inhomogeneous solution, we can then rewrite Eq. (8) as

$$n^2\left(x\right) = \frac{N}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega \left[\frac{1}{2\sqrt{\omega^2 + m^2}} - R_\omega\left(x\right)\right]\,, \tag{10}$$

where $R_\omega$ denotes the resolvent

$$R_\omega = \left\langle x\Big|\frac{1}{-\partial_x^2 + \omega^2 + \lambda}\Big|x\right\rangle. \tag{11}$$

Equation (10) can be also written as a sum over eigenfunctions of the operator $-\partial_x^2 + \lambda$

$$n^2 = r - N \sum \frac{|f_k(x)|^2}{2\omega_k}, \qquad \left(-\partial_x^2 + \lambda(x)\right) f_k(x) = \omega_k^2 f_k(x). \tag{12}$$

To construct an inhomogeneous solution, we use the well-known fact that the resolvent $R_\omega$ satisfies the Gelfand–Dikii equation

$$-2R_\omega \partial_x^2 R_\omega + (\partial_x R_\omega)^2 + 4\left(\omega^2 + \lambda(x)\right) R_\omega^2 = 1. \tag{13}$$

If we use relation (6) to substitute $\lambda$ and propose some ansatz for $R_\omega$, we obtain a differential equation for $n$ with parameter $\omega$. This equation must hold for all values of $\omega$ which is possible only for a special choice of coefficients.

Assume that the spectrum of the Schrödinger operator consists of one translational zero mode and continuum starting at eigenvalue $\omega^2 = m^2$. Hence, we suppose that

$$R_\omega = a(\omega) + b(\omega) n^2(x). \tag{14}$$

This is the simplest choice which is consistent with (10). It is also reasonable to assume that

$$a(\omega) = \frac{1}{2\sqrt{\omega^2 + m^2}}, \tag{15}$$

but for a moment, we will not use this assumption. After substitution of (14) and (5) into (13), we obtain the equation

$$4a\left(a + bn^2\right)\partial_x^2 n + 4\omega^2 n \left(a + bn^2\right)^2 - 4abn(\partial_x n)^2 = n. \tag{16}$$

If we use (15) and assume $b = Ca/\omega^2$, where $C$ is some constant, we obtain that (16) is equivalent to two equations

$$n\partial_x^2 n + Cn^4 - (\partial_x n)^2 = 0, \tag{17}$$

$$\partial_x^2 n + 2Cn^3 = m^2 n. \tag{18}$$

From these equations, we easily obtain that

$$(\partial_x n)^2 = n^2\left(m^2 - Cn^2\right). \tag{19}$$

For $C > 0$, the solution is

$$n(x) = \frac{m}{\sqrt{C}} \frac{1}{\cosh(m(x - x_0))}, \tag{20}$$

where $x_0$ is the center of the soliton. Thus, the condensate $\lambda$ is

$$\lambda(x) = \frac{\partial_x^2 n}{n} = m^2\left[1 - \frac{2}{\cosh^2(m(x - x_0))}\right]. \tag{21}$$

This is the solution found in [16]. Eigenfunctions with a given momentum at infinity may be found via supersymmetric quantum mechanics

$$\begin{aligned}
&\left(-\partial_x^2 + \lambda(x)\right) f_k(x) = \omega_k^2 f_k(x) \, , \\
&\omega_k^2 = m^2 + k^2 \, , \qquad f_k(x) = \frac{-ik + m \tanh mx}{\sqrt{m^2+k^2}} \exp(ikx) \, . 
\end{aligned} \tag{22}$$

We put $x_0 = 0$ above. These functions are normalized as

$$\int\limits_{-\infty}^{+\infty} \mathrm{d}x f_k(x) f_{k'}^*(x) = 2\pi\delta\left(k-k'\right) \, .$$

Thus, from Eq. (12), we get the same solution

$$\begin{aligned}
n^2(x) &= N \int \frac{\mathrm{d}k}{2\pi} \left[ \frac{1}{2\sqrt{k^2+m^2}} - \frac{|f_k(x)|^2}{2\sqrt{k^2+m^2}} \right] = \frac{N}{4\pi} \int \mathrm{d}k \frac{m^2\left(1-\tanh^2 mx\right)}{\left(k^2+m^2\right)^{3/2}} \\
&= \frac{N}{2\pi} \frac{1}{\cosh^2 mx} \, .
\end{aligned} \tag{23}$$

Note that we explicitly excluded the bound state with zero energy from summation. Otherwise, the sum would contain 0 in the denominator making the equation meaningless. More carefull treatment of the zero mode is presented in Section 4.

Let us comment on the topological aspect of the solution. In the GN model, the kink interpolates between two vacuum states and has the standard topological charge which is due to the difference in the field at two spatial infinities. Our soliton has no naive local topological charge since the values of the fields at two space asymptotics are the same. The solution looks like the soliton solution in the KdV equation and in the integrability context, one could say that by selecting the soliton solution which has positive energy we select the topological sector of the theory, and the topology can be read off only from the geometry of the spectral curve.

In our case, if our solution would have the conserved topological charge and positive energy, one could claim that it is just a particular sector of excitations above the ground state. However, there is no local conserved charge and its energy is negative hence we interpret it as the instability mode for the homogeneous ground state.

## 3. Energy of the soliton

In this section, we will provide three different ways of evaluation of energy for the solution obtained in the previous section. Firstly, we will use simple regularization by introducing ultraviolet cut-off and taking into account the anomaly found in [23]. Then we obtain the same result using the Pauli–Villars regularization. Finally, we calculate the average of energy-momentum tensor. A bit surprisingly in all calculations, we obtain a negative value for the soliton energy

$$E = -\frac{2Nm}{\pi} \,. \tag{24}$$

### *3.1. Regularized sum over the modes*

Firstly, we use the result of [23] for a static configuration of $\lambda(x)$ energy

$$\varepsilon\left(x\right) = \varepsilon_0 + \frac{N}{2\pi}\,\lambda\left(x\right)\,, \tag{25}$$

where $\varepsilon_0$ is some constant that does not depend on the coordinate, and we assume that $\lambda$ is a solution to the gap equation. Let us emphasize that this expression takes into account the anomalous contribution emerging from the regularization of the sum over the modes.

If we subtract the vacuum energy density $\varepsilon_{\text{vac}}$ given by the same expression with $\lambda = m^2$, we obtain

$$\varepsilon\left(x\right) - \varepsilon_{\text{vac}} = \text{const.} + \frac{N}{2\pi}\left(\lambda\left(x\right) - m^2\right)\,.$$

At spacial infinity, all fields approach their vacuum values, and the energy density is the same as in vacuum thus const. $= 0$. After the substitution of solution (21) into the energy density and integration, we find

$$E = \int\limits_{-\infty}^{+\infty} \mathrm{d}x\left(\varepsilon\left(x\right) - \varepsilon_{\text{vac}}\right) = -\frac{Nm^2}{\pi}\int\limits_{-\infty}^{+\infty} \mathrm{d}x\,\frac{1}{\cosh^2 mx} = -\frac{2Nm}{\pi}\,. \tag{26}$$

Since the energy of the soliton derived in [16] is different and positive, one could wonder what is the reason for the discrepancy. In [16], the following expression for the energy was used $E = N\sum\omega_n - r\int \mathrm{d}x\lambda + \text{b.t.}$ and the derived energy of the soliton is positive and reads as $E_{\text{sol}} - E_0 = r\int(\lambda_0 - \lambda_{\text{sol}}) = 4rm$, where the complete cancellation of the sum over the modes around the vacuum and soliton was assumed. The first point of concern is the presence of the bare coupling constant $r$ in the expression for the quantum energy. The second point which is not correct is the complete cancellation of the modes at the top of the solution which was shown to be incomplete [21, 22]. Finally, the anomaly for the energy due to the proper regularization procedure [23] has not been taken into account.

### *3.2. Pauli–Villars regularization*

We calculate the energy of the soliton by regularizing its effective action by the Pauli–Villars method. In this calculation, we follow ideas from [3]. The regularized action is

$$S = N\sum_{i=0}^{I} C_i \mathrm{Tr}\log\left(-\partial^2 + m_i^2 + \lambda\right) + \int \mathrm{d}^2x \left[(\partial n)^2 + \lambda\left(n^2 - r\right)\right] . \quad (27)$$

Following the Pauli–Villars procedure, we introduce in addition to each original field with $m_0 = 0$ a number $I$ of regulator fields with masses $m_i,\ i = 1,\ldots, I$, and constants $C_i,\ i = 0, 1,\ldots, I$, satisfying

$$\sum_{i=0}^{I} C_i = 0\,, \qquad \sum_{i=0}^{I} C_i m_i^2 = 0\,, \qquad C_0 = 1\,, \qquad m_0 = 0\,.$$

For our purposes, it is sufficient to take $I = 2$. Then the constants $C_i$ are

$$C_1 = \frac{m_2^2}{m_1^2 - m_2^2}\,, \qquad C_2 = -\frac{m_1^2}{m_1^2 - m_2^2}\,.$$

At the end of calculation, we will take a limit when all regulator masses $m_i\,(i = 1,\ldots, I)$ go to the UV cut-off $M$. The connection between effective action and energy is $S = E \cdot T$, where $T$ is a large time cut-off.

The general scheme of calculations is as follows. First, we find coupling constant $r$ in terms of regulator fields masses and mass scale of the theory from the gap equation for the homogeneous solution $\lambda = m^2$. Next, we can show that terms with the $n$ field do not contribute to the energy because $n$ is proportional to zero mode

$$\int_{-\infty}^{+\infty} \mathrm{d}x \left[(\partial_x n)^2 + \lambda\, n\right] = \int_{-\infty}^{+\infty} \mathrm{d}x\, n\left(-\partial_x^2 n + \lambda\, n\right) = 0\,.$$

After that, we express the trace term as a sum over eigenvalues and take into account the change in the density of states for the inhomogeneous solution. Finally, we perform integration over eigenvalues and confirm the result of (24). Details of the computation are presented in Appendix A.

### *3.3. Energy of soliton, explicit evaluation*

In this section, we are going to calculate the average of energy-momentum tensor for a soliton solution. We quantize the $n$ fields canonically and introduce the Pauli–Villars regulator fields to deal with divergences and take into account the conformal anomaly. The energy-momentum tensor in Minkowski space is

$$\begin{aligned}\theta_{\mu\nu} &= \sum C_i \theta^i_{\mu\nu}\,,\\ \theta^i_{\mu\nu} &= \partial_\mu n_i \partial_\nu n_i^* + \partial_\mu n_i^* \partial_\nu n_i - g_{\mu\nu}\left(|\partial n_i|^2 - \lambda\left(|n_i|^2 - r\right) - m_i^2\,|n_i|^2\right).\end{aligned}$$

The components $\theta_{00}$ , $\theta_{11}$, and $\theta_{01}$ are

$$\begin{aligned}\theta_{00} &= \sum C_i\left(|\partial_t n_i|^2 + |\partial_x n_i|^2 + \lambda\,|n_i|^2 + m_i^2\,|n_i|^2\right) - \lambda r\,,\\ \theta_{11} &= \sum C_i\left(|\partial_t n_i|^2 + |\partial_x n_i|^2 - \lambda\,|n_i|^2 - m_i^2\,|n_i|^2\right) + \lambda r\,,\\ \theta_{01} &= \sum C_i\left(\partial_t n_i\,\partial_x n_i^* + \partial_t n_i^*\,\partial_x n_i\right).\end{aligned} \tag{28}$$

We consider field $\lambda$ as classical and suppose that the $n$ field has a classical component

$$\lambda = m^2\left(1 - \frac{2}{\cosh^2 mx}\right), \qquad n_{\rm cl} = \sqrt{\frac{N}{2\pi}}\,\frac{1}{\cosh mx}\,.$$

The modes on the $n$ field in the continuum spectrum are given by Eq. (22). Also, there is a zero mode

$$\psi_0 = \sqrt{\frac{m}{2}}\,\frac{1}{\cosh mx}\,.$$

Quantization of field $n = n^N$ and regulator fields $n_i$ , $(i = 1, 2)$, are slightly different. The $n$ field has a classical component, proportional to zero mode, while the regulator field has an additional component with frequency $m_i$. The masses of auxiliary fields and coefficients $C_i$ are the same as in the calculation of the determinant via the Pauli–Villars regularization. The frequencies for regulator fields are $\omega_{k,\,i} = \sqrt{\omega_k^2 + m_i^2}$. In terms of creation and annihilation operators, we have

$$n^a\,(x,\,t) = \delta^a_N n_{\rm cl}\,(x) + \int \frac{{\rm d}k}{2\pi}\,\frac{1}{\sqrt{2\omega_k}}\left(a^a_k f_k\,(x)\,{\rm e}^{-i\omega_k t} + b^{a\dagger}_k f^*_k\,(x)\,{\rm e}^{+i\omega_k t}\right) \tag{29}$$

for the $n^a$ field. For the regulator fields $n^a_i$ , $i = 1, ..., I$, we have

$$\begin{aligned}n^a_i &= \frac{1}{\sqrt{2m_i}}\left(A^a_i {\rm e}^{-im_i t} + B^{a\dagger}_i {\rm e}^{+im_i t}\right)\psi_0\,(x)\\ &\quad + \int \frac{{\rm d}k}{2\pi}\,\frac{1}{\sqrt{2\omega_{k,\,i}}}\left(a^a_{k,\,i} f_k\,(x)\,{\rm e}^{-i\omega_{k,i} t} + b^{a\dagger}_{k,\,i} f^*_k\,(x)\,{\rm e}^{+i\omega_{k,i} t}\right).\end{aligned} \tag{30}$$

The canonical commutation relations for the $n$ field are modified by the presence of zero mode

$$[n^a(x,\,t)\,,\partial_t\bar{n}_b(y,\,t)] = i\delta^a_b\left(\delta(x-y) - i\delta^a_N\delta^N_b\psi_0(x)\,\psi_0(y)\right)\,.$$

However, for regulator fields, the commutation relation is unchanged

$$[n^a_i(x,\,t)\,,\partial_t\bar{n}_{kb}(y,\,t)] = i\delta_{ik}\delta^a_b\delta(x-y)\,.$$

We take average over the state, which is annihilated by all operators $a_k$, $a_{k,i}$, $A_i$ and $b_k$, $b_{k,i}$, $B_i$. For the product of two $n = n^N$ fields, we get

$$\begin{aligned}\left\langle n(x_1,t_1)\,n^\dagger(x_2,t_2)\right\rangle &= n_{\mathrm{cl}}(x_1)\,n_{\mathrm{cl}}(x_2)\\ &\quad + N\int\frac{\mathrm{d}k}{2\pi}\,\frac{1}{2\sqrt{k^2+m^2}}\,\mathrm{e}^{i\omega_k(t_1-t_2)}f^*_k(x_1)\,f_k(x_2)\,.\end{aligned}$$

For the corresponding regulators, it gives

$$\begin{aligned}\left\langle n_i(x_1,t_1)\,n^\dagger_i(x_2,t_2)\right\rangle &= N\,\frac{\psi_0(x)\,\psi_0(y)}{2m_i}\,\mathrm{e}^{im_i(t_1-t_2)}\\ &\quad + N\int\frac{\mathrm{d}k}{2\pi}\,\frac{\mathrm{e}^{i\omega_{k,i}(t_1-t_2)}}{2\sqrt{k^2+m^2+m_i^2}}\,f^*_k(x_1)\,f_k(x_2)\,.\end{aligned}$$

The expression for the regularized square of the field is then

$$\begin{aligned}\sum_{i=0}^{i=2}C_i\left\langle|n_i(x)|^2\right\rangle &= n^2_{\mathrm{cl}}(x) + N\int\frac{\mathrm{d}k}{2\pi}\sum_i\frac{C_i\,|f_k(x)|^2}{2\sqrt{k^2+m^2+m_i^2}}\\ &\quad + N\psi_0(x)^2\sum_i\frac{C_i}{2m_i} = r\,.\end{aligned}$$

This equality is equivalent to the gap equation, therefore, the $r$ term in the energy-momentum tensor cancels by the $n^2$ term.

The calculation of other contributions to the energy-momentum tensor is straightforward. Details are provided in Appendix B. The final answer is consistent with other methods

$$\langle\theta_{00}\rangle = \frac{Nm^2}{4\pi} - \frac{N}{\pi}\,\frac{m^2}{\cosh^2 mx} = \frac{Nm^2}{4\pi} + \frac{N}{2\pi}\left(\lambda - m^2\right)\,. \tag{31}$$

The other components of the energy-momentum tensor are the same as ones of the homogeneous phase

$$\langle\theta_{11}\rangle = -\frac{Nm^2}{4\pi}\,, \qquad \langle\theta_{01}\rangle = 0\,. \tag{32}$$

This can be compared with the evaluation of the energy density of the homogeneous ground state via the conformal anomaly [3]. Since there is no scale at the classical level, the trace of the energy stress tensor gets contribution from the running of the coupling constant only and, therefore, is proportional to the $\beta$-function, $\theta_\mu^\mu = N\lambda/2\pi$. Hence, the vacuum energy density $\epsilon_{\rm vac} = (1/2)\langle {\rm vac}\,|\,\theta_\mu^\mu\,|{\rm vac}\rangle = Nm^2/4\pi$. Similarly, the mass of the particle can be evaluated from the matrix element of the $\theta_\mu^\mu$ over the corresponding state [3]. For instance, we can use the relation for the $\sigma$-particle mass, $2m^2 = \langle\sigma|\,\theta_\mu^\mu\,|\sigma\rangle$ and express it via the propagator of the $\lambda$-field $D(p^2)$ at zero momentum $D(0)$ and simple $\sigma\sigma\lambda$ vertex proportional to $2m^2/N$.

To complete this section, let us make a comment concerning the spectrum of excitations. First, note that the photon acquires finite inhomogeneous mass in the inhomogeneous vacuum. This implies that there is no linear confinement of charged degrees of freedom. According to the emerging picture, the homogeneous state is metastable and the kink–antikink pair in the homogeneous state now yields the bounce configuration in the Euclidean space. We shall discuss the spectrum and the $\theta$-dependence in the inhomogeneous ground state in more detail elsewhere.

## 4. Zero-modes treatment

When we considered the construction of the inhomogeneous solution, we stated that zero-mode contribution should be omitted from the summation over eigenfrequencies. We are going to justify this exclusion. To that end, we argue that these modes describe rotation excitation of the soliton. Therefore, the integration over zero modes is non-Gaussian. The action does not depend on the zero-mode excitation amplitude and this amplitude becomes large, making quadratic approximation incorrect. Instead, we should integrate over all field configurations with the same energy. The space of such configurations is nothing but the moduli space of the soliton. Thus, to obtain the correct partition function of the theory, we should multiply contribution from all $\omega_n^2 > 0$ by the contribution of the moduli space part. If we consider a small fluctuation of the $\lambda$ field, which replaces the zero mode with some small frequency $\omega_0$ mode, it induces a potential on moduli space. Such potential changes the partition function, but it remains regular and no terms like $\log\omega_0$ arise. Therefore, zero mode does not disrupt the solution.

To make the connection between the zero modes and soliton rotation more robust, we construct a quantum mechanical state corresponding to the solitonic field configuration. The construction is very similar to the kink analysis in scalar $\phi^4$ and Sine–Gordon models, performed in the series of papers [42]. Then we study the quantized field fluctuations around this state. We find out that zero modes of the fluctuation are related to the SU($N$) symmetry generators. Therefore, these modes could be treated as soliton rotations in the internal state.

Let the ket $|0\rangle$ be the homogeneous state of the sigma model and $f_0(x)$ zero-mode profile. In the homogeneous state, scalar field expectation values are zero

$$\langle 0|\, n^a \,|0\rangle = \langle 0|\, n^{a\dagger} \,|0\rangle = 0\,.$$

Consider the operator

$$D = \exp\left(-iA\, l^a \int \mathrm{d}x\, \pi^a(x) f_0(x) - iA\, l^{a*} \int \mathrm{d}x\, \pi^{a\,\dagger}(x) f_0(x)^*\right).$$

Here, $\pi^a$ is a canonical momentum conjugated to $n^a$ and $l^a$ is a unit complex vector. The constant $A$ relates the normalized zero mode and classical value of the $n$ field, $n_{\mathrm{cl}} = Af_0$. The operator $D$ acts as a shift operator in the field space. One can show using commutation relations that

$$\langle 0|\, D^\dagger n^a D \,|0\rangle = Af_0(x) l^a\,, \qquad \langle 0|\, D^\dagger n^{a\dagger} D \,|0\rangle = Af_0 l^{a*}\,.$$

Therefore, the state $|K\rangle = D\,|0\rangle$ has the correct field expectation value. Note that we cannot construct the translation operator for the $\lambda$ field because it does not have a canonical momentum. Therefore, we set this field by hand.

We consider the state $|K\rangle$ a first approximation to the solitonic state. The operator $D$ is a unitary transformation which can be applied to all the theory observables. This transformation shifts all scalar fields

$$D^\dagger n^a D = n^a + f l^a\,, \qquad D^\dagger n^{a\dagger} D = n^{a\dagger} + f l^{a*}\,.$$

Let us turn to the internal symmetry of the theory. Let us denote Hermitian generators ($N \times N$ matrices) of $\mathrm{SU}(N)$ as $T_i^{ab}$, $i = 1, \ldots, N^2 - 1$. The corresponding conserved charges are

$$Q_i = \int \mathrm{d}x \left(\pi^a(x) T_i^{ab} n^b(x) + n^{b\dagger}(x) T_i^{ab} \pi^{a\dagger}(x)\right).$$

Let us consider how these operators transform after the shift

$$D^\dagger Q_i D = Q_i + \int \mathrm{d}x f(x) \left(\pi^a(x) T_i^{ab} l^b(x) + l^{b*}(x) T_i^{ab} \pi^{a\dagger}(x)\right).$$

We rewrite the mode expansion of the fields in slightly a different from the previous section form, taking into account the quantization of the zero-mode component

$$n^a(x) = f_0(x) n_0^a + \int \frac{\mathrm{d}k}{2\pi} \frac{1}{\sqrt{2\omega_k}} \left(a_k^a f_k(x)\, \mathrm{e}^{-i\omega_k t} + b_k^{a\dagger} f_k^*(x)\, \mathrm{e}^{+i\omega_k t}\right), \tag{33}$$

$$\pi^a(x) = f_0(x) \pi_0^a + i \int \frac{\mathrm{d}k}{2\pi} \sqrt{\frac{\omega_k}{2}} \left(-a_k^a f_k(x)\, \mathrm{e}^{-i\omega_k t} + b_k^{a\dagger} f_k^*(x)\, \mathrm{e}^{+i\omega_k t}\right). \tag{34}$$

The commutation relation between the $a_k^a$ and $b_k^a$ operators are standard, $n_0^a$ and $\pi_0^a$ behave as usual coordinate and momentum operators

$$\left[n_0^a,\, \pi_0^b\right] = i\delta^{ab}\,.$$

The integral of the $\pi^a$ field multiplied by the zero-mode $f$ yields the $\pi_0^a$-mode operator with some normalization constant $A$

$$D^\dagger Q_i D = Q_i + A\left(\pi_0^a T_i^{ab} l^b(x) + l^{b*}(x) T_i^{ab} \pi_0^{a\dagger}\right) = Q_i'\,.$$

The homogeneous state is rotational invariant: $Q_i\left|0\right\rangle = 0$, therefore we obtain

$$\begin{aligned}\left\langle K\right| Q_i \left|K\right\rangle &= \left\langle 0\right| D^\dagger Q_i D \left|0\right\rangle = \left\langle 0\right| Q_i + A\left(\pi_0^a T_i^{ab} l^b(x) + l^{b*}(x) T_i^{ab}\pi_0^{a\dagger}\right)\left|0\right\rangle \\ &= A\left\langle 0\right|\left(\pi_0^a T_i^{ab} l^b(x) + l^{b*}(x) T_i^{ab}\pi_0^{a\dagger}\right)\left|0\right\rangle\,. \end{aligned} \tag{35}$$

Therefore, for the solitonic state, we can express the charges corresponding to internal rotations through the zero-mode components of the canonical momentum. This connection supports the idea that the zero modes correspond to rotations in internal space.

The moduli space effective theory can be read off the action. We substitute the expression for the $n$ fields in the form of $n^a(x,t) = Af_0(x)l^a(t)$ and obtain

$$S = \int \mathrm{d}t\,\mathrm{d}x A^2 f_0^2(x) \dot{l}^a \dot{l}_a^{\,2} = \frac{M}{2}\int \mathrm{d}t \dot{l}^a \dot{l}_a^{\,*}\,.$$

Here, we take into account the identity $\int \mathrm{d}x\left((\partial_x f_0)^2 + \lambda f_0^2\right) = 0$. Thus, we obtain the quantum mechanical problem on the $\mathbb{CP}^{N-1}$ moduli space.

## 5. Connection with the Gross–Neveu model

In this section, we study the connection between our soliton and the kink of the GN model. To this end, we use the fact that the GN model is nothing but the fermionic part of the $\mathcal{N}=1$ SUSY $\mathbb{O}^{N-1}$ sigma-model. The $\mathbb{O}^{N-1}$ is very similar to the $\mathbb{CP}^{N-1}$ model considered in the previous section. The main difference is that the theory is written in terms of real scalar fields and there is no gauge field. For our solution $A_\mu = 0$, thus its derivation is not affected.

First, let us argue that the $\mathbb{O}(N)$ model admits the similar inhomogeneous solution and then consider its minimal SUSY extension. The Lagrangian of the model reads as

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu n_a\right)^2 - \frac{\lambda}{2}\left(\left(n_a\right)^2 - r\right)\,. \tag{36}$$

There are $N$ real fields $n_a$ and the Lagrange multiplier $\lambda$ leads to constraint $n_a n_a = r = 1/g^2$. Similar to the case of the $\mathbb{CP}^{N-1}$ model, this model demonstrates dynamical mass generation, so in vacuum, $\lambda = m^2$. It is a simple issue to show that in the large-$N$ limit, model (36) possesses a soliton solution similar to the one being discussed in the case of the $\mathbb{CP}^{N-1}$ model. The difference is only in a number of degrees of freedom.

The large-$N$ effective action is obtained similarly to the case of the $\mathbb{CP}^{N-1}$ model by integration over fields $n_a$, $a = 1, 2, \dots, N-1$, but not over $n_N = n$. In the Euclidean signature, the effective action is

$$S_{\text{eff}} = \frac{N-1}{2} \operatorname{Tr} \log\left(-\partial^2 + \lambda\right) + \frac{1}{2} \int \mathrm{d}^2 x \left((\partial n)^2 - \lambda(n^2 - r)\right) . \tag{37}$$

The actions (4) and (37) differ only by a numerical factor of 1/2. Thus, their stationary points are the same and (21) is a solution in the $\mathbb{O}^N$ model with the energy

$$E = -\frac{Nm}{\pi} .$$

Let us turn now to the case of the $\mathcal{N} = 1$ supersymmetric $\mathbb{O}^N$ model. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu n_a)^2 + \bar{\psi}_a i \partial\!\!\!/ \, \psi_a + \frac{1}{4r}\left(\bar{\psi}_a \psi_a\right)^2\right] .$$

Here, $\psi_a$ are Majorana fermions, $\partial\!\!\!/ = \gamma^\mu \partial_\mu$, $\gamma^0 = \sigma_2$, $\gamma^1 = i\sigma_3$, $\gamma^5 = -\gamma^0\gamma^1 = \sigma_1$. The constraints $n_a n_a = r$ and $n_a \psi_a = 0$ are taken into account by the Lagrange multipliers $\lambda$ and $\chi$. Also, we introduce auxiliary field $\sigma \sim \bar{\psi}\psi$

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu n_a)^2 + \bar{\psi}_a \left(i\partial\!\!\!/ - \sigma\right)\psi_a - r\sigma^2 - \lambda\left((n_a)^2 - r\right) - \bar{\chi}\psi_a n_a - \bar{\psi}_a \chi n_a\right] .$$

In order to obtain the effective action, we have to integrate over all fermionic fields and all fields $n_a$ but $n_N = n$. To integrate over $\psi_a$, we make a shift of variables

$$\psi_a \to \psi_a + \phi_a , \qquad \phi_a = \left(i\partial\!\!\!/ - \sigma\right)^{-1} \chi n_a .$$

Then the terms in action linear in $\psi_a$ are canceled, but we have additional term
$n_a n_a \, \bar{\chi}\left(i\partial\!\!\!/ - \sigma\right)^{-1}\chi = r\,\bar{\chi}(i\partial\!\!\!/ - \sigma)^{-1}\chi$. Then integration over $\chi$ can also be performed. Integration over $\psi_a$ and $\chi$ yields determinant contributions to the effective action

$$-\frac{iN}{2} \operatorname{Tr} \log\left(i\partial\!\!\!/ - \sigma\right) + \frac{i}{2} \operatorname{Tr} \log\left(i\partial\!\!\!/ - \sigma\right) ,$$

hence, the field $\chi$ integration reduces the number of degrees of freedom by 1.

The effective action is

$$S_{\rm eff} = \frac{i}{2}(N-1)\left[\mathrm{Tr}\log\left(-\partial^2-\lambda\right)-\mathrm{Tr}\log\left(i\partial\!\!\!/-\sigma\right)\right] + \frac{1}{2}\int \mathrm{d}^2x\left[(\partial n)^2-\lambda(n^2-r)-\sigma^2 r\right]. \tag{38}$$

Note that this action can be rewritten in a slightly different way, making the situation more clear. Before integration over $n_a$, we can use constraint $n_a n_a = r$ to put a factor $n_a n_a$ before the $\sigma$ term in Lagrangian

$$\mathcal{L}=\frac{1}{2}\left[(\partial_\mu n^a)^2+\bar\psi_a\left(i\partial\!\!\!/-\sigma\right)\psi_a-\sigma^2(n_a)^2-D\left((n_a)^2-r\right)-\bar\chi\psi_a n_a-\bar\psi_a\chi n_a\right].$$

In this equation, we rename the Lagrange multiplier $\lambda$ and call it $D$. Thus, mass of both bosons and fermions is given by v.e.v. of the same field $\sigma$ and in homogeneous vacuum state $D=0$ corresponds to unbroken supersymmetry. The effective action is

$$S_{\rm eff} = \frac{i(N-1)}{2}\mathrm{Tr}\log\left(-\partial^2-D-\sigma^2\right)-\frac{i(N-1)}{2}\mathrm{Tr}\log\left(i\,\partial\!\!\!/-\sigma\right) + \frac{1}{2}\int \mathrm{d}^2x\left[(\partial n)^2-\left(\sigma^2+D\right)n^2+rD\right]. \tag{39}$$

The first form of effective action (38) shows that the fermionic part of the model is nothing but the Gross–Neveu model (with the number of degrees of freedom reduced by factor 2 because Majorana fermions are used instead of Dirac ones).

From identity $\gamma^5\left(i\partial\!\!\!/-\sigma\right)\gamma^5=-\left(i\partial\!\!\!/+\sigma\right)$, we can obtain

$$\mathrm{Tr}\log\left(i\partial\!\!\!/-\sigma\right)=\frac{1}{2}\mathrm{Tr}\log\left(-\left(i\partial\!\!\!/-\sigma\right)\left(i\partial\!\!\!/+\sigma\right)\right)=\frac{1}{2}\mathrm{Tr}\log\left(\partial^2+\sigma^2-i\gamma^\mu\partial_\mu\sigma\right).$$

If $\sigma$ does not depend on time, we have

$$\mathrm{Tr}\log\left(i\partial\!\!\!/-\sigma\right)=\frac{1}{2}\mathrm{Tr}\log\left(\partial^2+\sigma^2+\partial_x\sigma\right)+\frac{1}{2}\mathrm{Tr}\log\left(\partial^2+\sigma^2-\partial_x\sigma\right). \tag{40}$$

If $\sigma$ is a topologically nontrivial solution for the GN model, then $\lambda=\sigma^2\pm\partial_x\sigma$ is a solution to the $\mathbb{CP}^{N-1}$ model and, thus, to the $\mathbb{O}^N$ model. In terms of $D$, it means $D=\pm\partial_x\sigma$. For definiteness, we set $\lambda=\sigma^2-\partial_x\sigma$. Thus,

$$S_{\rm eff}=\frac{i(N-1)}{4}\mathrm{Tr}\log\left(-\partial^2-\sigma^2+\partial_x\sigma\right)-\frac{i(N-1)}{4}\mathrm{Tr}\log\left(-\partial^2-\sigma^2-\partial_x\sigma\right) + \frac{r}{2}\int \mathrm{d}^2x D\,.$$

Here, we used the fact that $n$ is zero mode and that the overall sign of expression under the logarithm is unimportant because leads only to pure imaginary constant contribution. The simplest inhomogeneous solution

$$\sigma = m \tanh mx \tag{41}$$

leads to $\lambda$ in the form of (21). For this solution, $\sigma^2 + \partial_x \sigma = m^2$, so we can see that one of two terms in (40) is just a vacuum determinant and does not change the energy. It is consistent with the fact that the GN energy ($E = Nm/2\pi$ instead of $E = Nm/\pi$ as in Ref. [24] because we consider Majorana fermions) kink is minus half of the energy of the $\mathbb{O}^N$ soliton. The difference in signs of energies can be formally explained by the different signs of logarithms of bosonic and fermionic determinants.

## 6. Periodic inhomogeneous solution

In this section, we analyze the periodic solution, which corresponds to the kink crystal in the Gross–Neveu model. We explicitly check that the gap equation is true for this solution. However, the amplitude of the $n^2$ condensate has an infrared divergence. Therefore, the condensate amplitude in infinite space turns out to be also infinite. We formally calculate the energy of this solution and find that it is well-defined and lower than for the homogeneous solution. Therefore, such a solution might be relevant for a system on the finite interval with some choice of boundary conditions.

### *6.1. Gap equation*

In this section, we check the self-consistency of the periodic solution. In this calculation, we follow the ideas from [30] and use results from [31]. For this purpose, we consider a possible solution $\lambda = \sigma^2 - \partial_x \sigma$, where

$$\sigma = \nu m \, \frac{\operatorname{sn}(mx;\, \nu)\operatorname{cn}(mx;\, \nu)}{\operatorname{dn}(mx;\, \nu)} \tag{42}$$

is proportional to $\bar{\psi}\psi$ condensate in the GN model. It is also possible to write this condensate in the form

$$\sigma = m\, \frac{2\sqrt{\nu_1}}{1+\sqrt{\nu_1}} \operatorname{sn}\left(\frac{2mx}{1+\sqrt{\nu_1}};\, \nu_1\right), \tag{43}$$

where parameters are connected as

$$\nu = \frac{4\sqrt{\nu_1}}{\left(1+\sqrt{\nu_1}\right)^2}\,. \tag{44}$$

Note that solutions of $\lambda = \sigma^2 \pm \partial_x \sigma$ are different only by the shift on a half of period, so we do not need to consider the solution with a plus sign. For simplicity, we will use only form (42) and omit the second argument of elliptic functions. The standard calculation yields

$$\lambda = m^2 \nu \left(2\,\mathrm{sn}^2\,(mx) - 1\right)\,.$$

We need to find eigenfunctions of the operator $-\partial_x^2 + \lambda$. For the operator $-\partial_y^2 + 2\nu\,\mathrm{sn}^2 y$ (where $y = mx$), eigenfunctions are found in [31]

$$\begin{aligned} &\left(-\partial_y^2 + 2\nu\,\mathrm{sn}^2 y\right) f = \mathcal{E} f\,, \\ &f\,(y) = \frac{\theta_1\left(\frac{\pi(y+\alpha)}{2K}\,,\,q\right)}{\theta_4\left(\frac{\pi y}{2K}\,,\,q\right)} \exp\left(-yZ\,(\alpha)\right)\,, \qquad q = \exp\left(-\pi K'/K\right)\,. \end{aligned} \tag{45}$$

Here and later, $K$ and $E$ denote full elliptic integrals of the first and the second kinds with argument $\nu$, if it is not stated otherwise, and $K'\,(\nu) = K\,(1-\nu)\,.$ The parameter $\alpha = K + i\eta$ for the lower band with eigenvalues $\nu < \mathcal{E} < 1$ and $\alpha = i\eta$ for the band $\mathcal{E} > 1 + \nu$. The eigenvalue can be expressed via parameter $\alpha$ as

$$\mathcal{E} = \nu + \omega^2/m^2 = \mathrm{dn}^2\alpha + \nu\,.$$

For the states of the spectrum, $Z\,(\alpha)$ is purely imaginary and does not change the absolute value of $f$. Using the identities for the product of two theta functions, we can obtain

$$|f\,(x)|^2 = A^2\left(1 - \frac{\mathrm{cn}^2(mx)}{\mathrm{cn}^2\alpha}\right)\,.$$

We need to fix the normalization factor $A$. The normalization condition is that the average of the square of the eigenfunction is equal to 1

$$A^2 \int\limits_0^{2K/m} \left(1 - \frac{\mathrm{cn}^2(mx)}{\mathrm{cn}^2\alpha}\right) \mathrm{d}x = \frac{2K}{m}\,.$$

The integral can be readily computed and we find normalized eigenfunctions

$$|f_k|^2 = \frac{\omega^2/m^2 - \mathrm{dn}^2(mx)}{\omega^2/m^2 - E\,(\nu)\,/K\,(\nu)}\,. \tag{46}$$

Note that for the upper band, both numerator and denominator are negative.

It is convenient to integrate over the eigenvalue $\omega$ instead of the momentum $k$. To change the variable of integration, we use the formula from [31]

$$\frac{1}{m}\frac{\mathrm{d}k}{\mathrm{d}\mathcal{E}} = \frac{\nu + E/K - \mathcal{E}}{\sqrt{(1-\mathcal{E})\,(\mathcal{E}-\nu)\,(1+\nu-\mathcal{E})}}\,.$$

Therefore,

$$\frac{\mathrm{d}k}{\mathrm{d}\omega} = \frac{E/K - z^2}{\sqrt{(1-\nu-z^2)\,(1-z^2)}}\,, \qquad z = \omega/m\,.$$

The gap equation can be rewritten as

$$n^2 = r - \frac{N}{2\pi}\int \frac{\mathrm{d}z}{z}\left|\frac{\mathrm{d}k}{\mathrm{d}\omega}\right| |f_k|^2\,.$$

Integration over $z$ is over both bands. The bare coupling constant can be expressed as

$$r = \frac{N}{4\pi}\int \mathrm{d}k \left\{\frac{1}{\sqrt{k^2+\Lambda^2}} - \frac{1}{\sqrt{k^2+M^2}}\right\} = \frac{N}{2\pi}\log\frac{M}{\Lambda}\,,$$

where $\Lambda$ is the mass scale of the theory and $M$ is the Pauli–Villars UV cut-off. The explicit form of the gap equation is

$$\begin{aligned} n^2 &= \frac{N}{2\pi}\log\frac{m}{\Lambda} + \frac{N}{2\pi}\int\limits_1^\infty \mathrm{d}z \left\{\frac{1}{\sqrt{z^2-1}} - \frac{1}{z}\frac{z^2 - \mathrm{dn}^2(mx)}{\sqrt{(z^2-1+\nu)\,(z^2-1)}}\right\} \\ &\quad - \frac{N}{2\pi}\int\limits_0^{\sqrt{1-\nu}} \frac{\mathrm{d}z}{z}\frac{\mathrm{dn}^2(mx) - z^2}{\sqrt{(1-\nu-z^2)\,(1-z^2)}} = \frac{N}{2\pi}\left(a + b\cdot\mathrm{dn}^2(mx)\right)\,. \end{aligned}$$

Here, we extracted the term, proportional to the square of the zero mode of potential $\lambda$

$$\psi_0 \sim \mathrm{dn}\,(mx)\,.$$

The second gap equation is

$$\left(-\partial_x^2 + \lambda\right) n = 0\,,$$

thus $n$ must be proportional to zero mode. It means that $a = 0$ and this condition determines the parameter $m$.

From the expressions above, we obtain

$$a = \log\frac{m}{\Lambda} + \int_1^\infty \mathrm{d}z \left\{ \frac{1}{\sqrt{z^2-1}} - \frac{z}{\sqrt{(z^2-1+\nu)\,(z^2-1)}} \right\} + \int_0^{\sqrt{1-\nu}} \mathrm{d}z \,\frac{z}{\sqrt{(1-\nu-z^2)\,(1-z^2)}}\,, \tag{47}$$

$$b = \int_1^\infty \frac{\mathrm{d}z}{z} \frac{1}{\sqrt{(z^2-1+\nu)\,(z^2-1)}} - \int_0^{\sqrt{1-\nu}} \frac{\mathrm{d}z}{z} \frac{1}{\sqrt{(1-\nu-z^2)\,(1-z^2)}}\,. \tag{48}$$

All the integrals are elementary functions and their calculation is straightforward. However, the last integral in expression for $b$ is divergent in infrared. Thus, we introduce a very small cut-off $\epsilon = \omega_{\mathrm{min}}/m$. Physically it corresponds to placing the system in a box of large but finite size $L$ and dropping out zero mode from the gap equation. Then,

$$k_{\mathrm{min}} = \frac{2\pi}{L}\,, \qquad \omega_{\mathrm{min}} = k_{\mathrm{min}} \frac{\mathrm{d}\omega}{\mathrm{d}k}\,(\omega = 0) = \frac{2\pi}{L}\sqrt{1-\nu}\,\frac{K}{E}\,.$$

The calculation yields

$$a = \log\frac{m}{\Lambda} + \log\left(1+\sqrt{1-\nu}\,\right) = 0\,, \qquad m = \frac{\Lambda}{1+\sqrt{1-\nu}}\,. \tag{49}$$

Here, we recall the transformation of the elliptic parameter (44) and return to the original parameter $\nu$

$$\Lambda = \frac{2m}{1+\sqrt{\nu}}\,.$$

Thus, the fermionic condensate can be written in the form of (43)

$$\sigma = \sqrt{\nu_1}\Lambda\,\mathrm{sn}\,(\Lambda x;\, \nu_1)\,.$$

In terms of the mass of particle in the homogeneous phase this expression takes an especially simple form. However, the physical reason for this simplification is unclear.

The second coefficient

$$b = \frac{1}{\sqrt{1-\nu}} \log\left(\frac{1+\sqrt{1-\nu}}{Lm}\,\frac{\pi K}{E}\right)\,.$$

Note that this coefficient has logarithmic divergence and is negative at sufficiently large length. It implies the inequality $n^2 < 0$.

### *6.2. Energy density*

If we ignore the infrared divergence, the average energy density can be calculated in much similar way to the calculation of the energy of soliton. Omitting rather tricky technical details, we give here the final result

$$\epsilon = \frac{N\Lambda^2}{4\pi} - \frac{E(\nu)}{K(\nu)}\frac{Nm^2}{\pi}\,. \tag{50}$$

Now we discuss some arguments connected with the calculation of energy-momentum tensor (28). Due to the conservation of momentum $\partial_\mu \theta^\mu_\nu = 0$, we have $\partial_x\langle\theta_{11}\rangle = 0$. The $r$ term and $n^2$ term cancel each other similarly to the case of soliton. The mass-term contribution is

$$\sum_i C_i m_i^2 |n_i|^2 = N\int \frac{\mathrm{d}k}{2\pi}\sum_i \frac{C_i m_i^2}{2\sqrt{\omega_k^2+m_i^2}}|f_k|^2 = \frac{N}{2\pi}\left(\alpha + \beta\,\mathrm{dn}^2(mx)\right)\,,$$

where the square of the mode is given by (46). We are going to calculate only the coefficient $\beta$

$$\begin{aligned}\beta \;=\; & -\int_1^\infty \mathrm{d}z \sum_i \frac{C_i m_i^2}{\sqrt{\left(z^2+a_i^2\right)\left(z^2-1\right)\left(z^2-1+\nu\right)}} \\ & + \int_0^{\sqrt{1-\nu}} \mathrm{d}z \sum_i \frac{C_i m_i^2}{\sqrt{\left(z^2+a_i^2\right)\left(z^2-1\right)\left(z^2-1+\nu\right)}} = -m^2\,.\end{aligned}$$

We are not able to calculate derivative terms in the energy-momentum tensor but the fact that $\langle\theta_{11}\rangle = \mathrm{const.}$ suggests that

$$\sum_i C_i\left(|\partial_t n_i|^2 + |\partial_x n_i|^2\right) = \frac{N}{2\pi}\left(\alpha_1 + \beta\,\mathrm{dn}^2(mx)\right)$$

with the same coefficient $\beta$ but a different coefficient $\alpha_1$. Therefore, the energy density is

$$\epsilon(x) = \langle\theta_{00}\rangle = -\frac{Nm^2}{\pi}\mathrm{dn}^2(mx) + \mathrm{const}\,.$$

This result is consistent with the formula

$$\epsilon(x) = \frac{N}{2\pi}\lambda(x) + \mathrm{const}\,.$$

The value of the constant can be determined from the average energy density

$$\epsilon(x) = \frac{N}{2\pi}\lambda(x) - \frac{N\Lambda^2}{4\pi}\left(\frac{1-\sqrt{1-\nu}}{1+\sqrt{1-\nu}}\right)\,.$$

The obtained energy is lower than the one of homogeneous solution. However, due to infrared divergence, this solution can possibly be considered on a finite part of a plane only.

### *6.3. Comment on the GN model at zero density*

For comparison, let us briefly comment on the periodic solution in the Gross–Neveu model with the Minkowski Lagrangian

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - \sigma\right)\psi - r\,\sigma^2\,.$$

A similar problem was considered in [30]. For more similarity, in this section, we consider the theory with Dirac fermions. Generically, the period of the elliptic solution to the GN model is fixed by the chemical potential however, for the zero density case, we do not have the Fermi momentum parameter, the period of the solution remains a free parameter.

The effective action is

$$S_{\text{eff}} = -iN\text{Tr}\log\left(i\partial\!\!\!/ - \sigma\right) - r\int \mathrm{d}^2x\sigma^2\,.$$

We look for the solution in the form of (42). The mass parameter $m$ of this solution is connected to the mass scale $\Lambda$ of the theory through the gap equation that reads as

$$\sigma\left(x\right) = \frac{N}{2r}\int\frac{\mathrm{d}k}{2\pi}\,\bar{\psi}_k\psi_k\,,$$

where eigenfunctions

$$\bar{\psi}_k\psi_k = \frac{\omega}{\omega^2 - m^2E/K}\,\sigma\left(x\right)\,.$$

Therefore, the gap equation reduces to

$$1 = \frac{N}{2r}\int\frac{\mathrm{d}k}{2\pi}\,\frac{\omega\left(k\right)}{\omega^2\left(k\right) - m^2E/K}\,.$$

The fermionic gap equation leads to the same formula (49) for mass as the bosonic one. Note that there is no infrared divergence. The energy of this solution can be calculated from relation (40) between bosonic and fermionic determinants. Using the fact that the potentials $\sigma^2 \pm \partial_x\sigma$, we find that the energy density for the fermionic case is different from the bosonic only by a sign

$$\epsilon_{\text{GN}} = -\epsilon = -\frac{N\Lambda^2}{4\pi} + \frac{E\left(\nu\right)}{K\left(\nu\right)}\frac{Nm^2}{\pi}\,.$$

Thus, the energy is minimal for the homogeneous solution which is the correct ground state. However, the nonvanishing chemical potential modifies the ground state which becomes inhomogeneous.

## 7. Discussion

In this paper, we considered the properties of the inhomogeneous solutions [16] found recently for the $\mathbb{CP}^{N-1}$ sigma-model at large $N$. We focused on the soliton-like solution and the elliptic solution to the quantum gap equation. The careful analysis shows that the energy of the soliton is lower than the energy of the homogeneous ground state. This clearly makes questionable the common viewpoint that the ground state of the $\mathbb{CP}^{N-1}$ sigma-model at large $N$ is homogeneous.

The answer to the question about the true ground state of the model does not look simple. The naïve conjecture would be that the periodic elliptic kink crystal solution yields the true ground state and vacuum is in the FFLO-like phase as in the GN model with nonvanishing chemical potential. The energy for the kink crystal solution can be evaluated and indeed it is lower than the energy of the homogeneous state. However, there are two points of concern which provide difficulties with such immediate identification. First, the kink crystal solution suffers from the IR divergence at the infinite plane and deserves some IR regularization, for instance by introducing a box. Secondly, the kink crystal solution has the free massive parameter which fixes the period whose interpretation is not completely clear in the non-SUSY case. It is the counterpart of the chemical potential in the GN model.

It is instructive to look at the massive deformations of the large-$N$ sigma-models. It has been discussed in [4] for $\mathbb{O}^N$ and in [5] for $\mathbb{CP}^{N-1}$. The mass provides the IR regularization of the models, at large masses the theory can be treated perturbatively and is proven to be in the Higgs-like phase. In both models, there is a clear-cut phase transition at the value of the mass of the order of nonperturbatively generated scale $\Lambda$. Moreover, it is demonstrated in [4] that at the phase transition point two states become massless: the bound state of two $n$-particles and the soliton.

For masses below $\Lambda$, these light states could hint at the existence of a dual, more suitable, description. This is similar to the Sine–Gordon model transition from the bosonic description at the weak coupling to the fermionic one at the strong coupling. We did not explore this opportunity. Instead, in our analysis, we suggest that the ground state of these models is a small mass deformation of the FFLO-like kink crystal solution. The (twisted) mass parameter fixes the period of the elliptic solution to the gap equation and provides the IR regularization hence everything is well-defined in this case. We hope to investigate this issue elsewhere.

The massive deformations of the 2D theories have the clear-cut 4D counterparts — these are the gauge theories with flavor and masses of fundamental matter playing a similar role. Instead of the kinks in 2D, the domain walls in 4D are considered and the nontrivial mass dependence of their tensions are of interest. We would like to mention two examples: QCD at $\theta = \pi$ and softly broken $\mathcal{N} = 2$ SQCD. In both cases, there are domain walls with mass-dependent tensions. In the QCD case, it was proved in [32] that the 3D theory on the domain wall is deconfined. However, the approach of [32] does not give exactly the critical value of the quark mass when the domain wall tension vanishes. On the other hand, in softly broken $\mathcal{N} = 2$ SQCD at $N_f = 1$, the critical value of the mass at the Argyres–Douglas point when the domain wall tension vanishes has been found exactly [33]. At the critical mass, the whole 4D theory turns out in the deconfinement phase [33] and this fits with the deconfinement in 3D theory on the domain wall observed in [32]. Indeed, when the domain wall tension is small, it becomes wide and finally the deconfined phase occupies the whole space-time at the Argyres–Douglas point.

One more comment is in order. Recently, it was recognized that the discrete anomaly matching provides a powerful tool for the analysis of the phase diagram of the strongly coupled theories. In particular, this approach has been applied to the discussion of the ground state in the spin systems with the $\mathrm{SU}(N)$ structure group in some representation [35]. As was known for a while [36], the low-energy action for the $\mathrm{SU}(2)$ group case gets identified with the $\mathbb{CP}^1$ model with the $\theta$ term which depends on the spin representation. If $\theta = \pi(2k+1)$, the ground state turns out to be gapless and can be thought of as the condensate of dimers. More recent analysis [35] suggests that similar gapless phases for higher spin chains could occur at $\theta = 2\pi/N$. For instance, in the $\mathrm{SU}(3)$ case at the proper value of $\theta$, the ground state is gapless and presumably is a kind of condensate of trimers. We could speculate that the gapless ground state we have found could be some analogue of Haldane's gapless phase and our periodic kink crystal is the generalization of the dimer and trimer condensates ground states for the low-rank spin systems. Indeed, our soliton-like solution from the chiral GN viewpoint can be considered as the superposition of $N$ elementary kinks in the hedgehog shape. In our case, we have $\theta = 0$ but presumably, it can be a reasonable approximation of $\theta = 2\pi/N$ at large $N$.

We have touched a bit the SUSY generalization of the new solution postponing the detailed analysis for a separate study. The immediate question concerns the BPS property of the solution. The SUSY picture implies also several questions concerning its brane interpretation. Let us make a few remarks:

— The nontrivial profile of the $n$-field corresponds to the pulling of D2 brane in a particular direction by D2–D4 string. Hence to some extent, the soliton is represented by the profile of the F1 D2–D4 string. It would be interesting to get the interpretation of the soliton solution from the F1 worldsheet viewpoint.

— The brane picture for the GN model [29] tells that the kink corresponds to the interpolation between two possible intersections of D4 and D6 branes. This resembles the appearance of the second vacuum in the $\mathbb{CP}^{N-1}$ model coupled to 4D degrees of freedom [28]. Hence, it is natural to expect that the brane configuration responsible for the soliton and soliton lattice configurations involves D6 branes.

— The local negative energy contribution is typical for boojoums [12] when the magnetic non-Abelian string is attached to the domain wall. The negative energy is localized on the domain wall near the intersection point. One could conjecture that the soliton solution corresponds to the region of the intersection of the D6 domain wall and D2 brane representing non-Abelian string in the 4D gauge theory.

— Recently, the so-called negative branes with the negative tensions have been found [37]. These objects are identified both for extended branes and for point-like particles with negative mass. For some of them, the supergravity solutions have been found and it was argued that they obey the fermion statistics. It is unclear if our finding is related to this issue.

Several questions concern the IR properties of the periodic solution:

— Connection between infrared divergences in the solution and Coleman's theorem deserves careful study. There are some examples of models in which 2D continuous symmetry can be broken (chiral GN and $\mathbb{CP}^{N-1}$ on a circle at large $N$, SUSY $\mathbb{CP}^{N-1}$ due to mixing of $\pi$ and $A_\mu$ propagators). Could something similar happen in our case?

— Our study implies that the homogeneous solution for the $\mathbb{CP}^{N-1}$ model certainly is not the true ground state contrary to the standard viewpoint. Therefore, it is necessary to clarify if it is the metastable minimum of just a local extremum. If it is the metastable state, the kink–antikink configuration usually considered as the excitation could be treated as the bounce responsible for the decay of the metastable vacuum.

— Even if periodic solutions do not exist on a plane, they can change phase structure on a circle. There are possible phase transitions when $n^2 = 0$.

Let us remark that the lattice studies of the $\mathbb{CP}^{N-1}$ model also show an unexpected structure of the ground state [34] which has in the Euclidean space the crystal-like double-layer structure. The distribution of the topological charge density has a dipole-like structure and vacuum was interpreted as a kind of condensate of the Wilson loops. It is unclear if the kink crystal solution we considered in this study with minimal energy has something to do with these lattice observations.

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## Appendix A

*Effective action calculation for soliton*

Here, we provide the technical details of computation soliton energy calculation via Eq. (27). Firstly, we calculate the $r$ coupling constant in terms of the Pauli–Villars regulators. For this purpose, we write the homogeneous gap equation in space of large volume $V$

$$\begin{aligned} r\,V &= \sum_{i=0}^{I} C_i \,\mathrm{Tr}\,\frac{1}{-\partial^2+m_i^2+m^2} = V\int \frac{\mathrm{d}^2k}{4\pi^2}\sum_{i=0}^{I} C_i\,\frac{1}{k^2+m_i^2+m^2}\,, \\ r &= -\frac{N}{4\pi}\sum_{i=0}^{I} C_i \log\left(m^2+m_i^2\right)\,. \end{aligned} \tag{A.1}$$

Now, the trace of the operator can be written as a sum over the eigenvalues

$$\mathrm{Tr}\log\left(-\partial^2+m_i^2+\lambda\right) = T\int \frac{\mathrm{d}\omega}{2\pi}\sum_n \log\left(\omega^2+\omega_n^2+m_i^2\right)\,.$$

Here, $T$ stands for a large time cut-off and summation is over all eigenvalues $\omega_n^2$ of the operator $-\partial_x^2+\lambda$. Therefore, we obtain the following expression for the energy:

$$E_1 = N\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\sum_n\sum_{i=0}^{I} C_i\log\left(\omega^2+\omega_n^2+m_i^2\right) - r\int_{-\infty}^{+\infty}\mathrm{d}x\lambda\,. \tag{A.2}$$

The same expression can be written for the energy of vacuum $E_{\mathrm{vac}}$ when $\lambda = m^2$ and eigenvalues are $\omega_{0n}^2$.

We use expression (A.2) for the energy and subtract vacuum contribution

$$
\begin{aligned}
E \;=\; E_1 - E_{\mathrm{vac}} &= N \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi} \sum_{i=0}^{I} C_i \log\left(\omega^2 + m_i^2\right) \\
&+N \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi} \sum_n \sum_{i=0}^{I} C_i \log \frac{\omega^2 + \omega_n^2 + m_i^2}{\omega^2 + \omega_{0n}^2 + m_i^2} - \int\limits_{-\infty}^{+\infty} \mathrm{d}x \left(\lambda - m^2\right) r\,.
\end{aligned}
$$

Here, the first term is a contribution from the zero mode and the second is contribution from the continuum. We use integral

$$
\int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi} \log\left(1 + \frac{a^2}{\omega^2}\right) = a
$$

and integrate over $\omega$ in the first and second term, and over coordinate in the third we get

$$
E = N \sum_{i=0}^{I} C_i m_i + N \sum_n \sum_{i=0}^{I} C_i \left(\sqrt{\omega_n^2 + m_i^2} - \sqrt{\omega_{0n}^2 + m_i^2}\right) + 4mr\,.
$$

Summation over all eigenvalues can be replaced with integration over all momenta

$$
\sum_n \to \int \mathrm{d}k \rho\left(k\right)\,, \qquad \omega_n^2 \to k^2 + m^2\,,
$$

where the difference in densities of states for homogeneous and inhomogeneous states is

$$
\rho\left(k\right) = \frac{1}{\pi} \frac{\mathrm{d}\delta\left(k\right)}{\mathrm{d}k} = -\frac{2m}{\pi\left(k^2 + m^2\right)}\,.
$$

Here, $\delta\left(k\right) = \pi - 2\arctan(k/m)$ is the phase shift for eigenfunctions (22). Therefore, the energy is

$$
E = N \sum_{i=0}^{I} C_i m_i - \frac{2Nm}{\pi} \int\limits_{0}^{+\infty} \mathrm{d}k \sum_{i=0}^{I} C_i \frac{\sqrt{k^2 + m^2 + m_i^2}}{k^2 + m^2} + 4mr\,.
$$

We use the integral

$$
\begin{aligned}
\int \mathrm{d}k \frac{\sqrt{k^2 + m^2 + M^2}}{k^2 + m^2} \;&=\; \frac{M}{m} \arctan \frac{Mk}{m\sqrt{k^2 + m^2 + M^2}} \\
&\quad + \log\left(k + \sqrt{k^2 + m^2 + M^2}\right)
\end{aligned}
$$

and obtain

$$E = N\sum_{i=0}^{I} C_i m_i - \frac{2Nm}{\pi}\left[\sum_{i=1}^{I} C_i \frac{m_i}{m}\arctan\frac{m_i}{m} - \frac{1}{2}\sum_{i=0}^{I} C_i \log\left(m^2+m_i^2\right)\right] + 4mr\,.$$

If we apply expression (A.1) for $r$ and assume that $m_i \gg m$ and thus $\arctan\left(m_i/m\right) = \pi/2 - m/m_i$, we obtain

$$\begin{aligned} E &= N\sum_{i=0}^{I} C_i m_i - \frac{2Nm}{\pi}\sum_{i=1}^{I} C_i \frac{m_i}{m}\frac{\pi}{2} + \frac{2Nm}{\pi}\sum_{i=1}^{I} C_i \\ &\quad + \frac{Nm}{\pi}\sum_{i=0}^{I} C_i \log\left(m^2+m_i^2\right) - \frac{Nm}{\pi}\sum_{i=0}^{I} C_i \log\left(m^2+m_i^2\right)\,. \end{aligned}$$

We see that all terms except the third cancel. The sum in the third term is $\sum_{i=1}^{I} C_i = -C_0 = -1$ and we find expression (24).

## Appendix B

*Energy-momentum tensor of the soliton*

To calculate the average of energy-momentum tensor components (28), we need the following field average values:

$$\sum_i C_i m_i^2 \left\langle \left|n_i(x)\right|^2 \right\rangle = N\int \frac{\mathrm{d}k}{2\pi}\sum_i \frac{m_i^2\left|f_k(x)\right|^2}{2\sqrt{k^2+m^2+m_i^2}} + N\psi_0(x)^2 \sum_i \frac{C_i m_i}{2}\,, \tag{B.1}$$

$$\begin{aligned} \sum_i C_i \left\langle \left|\partial_x n_i(x)\right|^2 \right\rangle &= \left(\partial_x n_{cl}(x)\right)^2 + N\int \frac{\mathrm{d}k}{2\pi}\sum_i \frac{\left|\partial_x f_k(x)\right|^2}{2\sqrt{k^2+m^2+m_i^2}} \\ &\quad + N\psi_0(x)^2 \sum_i \frac{C_i}{2m_i}\,, \end{aligned} \tag{B.2}$$

$$\begin{aligned} \sum_i C_i \left\langle \left|\partial_t n_i(x)\right|^2 \right\rangle &= N\int \frac{\mathrm{d}k}{4\pi}\sum_i C_i\sqrt{k^2+m^2+m_i^2}\left|f_k(x)\right|^2 \\ &\quad + N\psi_0(x)^2 \sum_i \frac{C_i m_i}{2}\,. \end{aligned} \tag{B.3}$$

The expressions for modes and their derivatives are

$$\begin{aligned}
|f_k(x)|^2 &= \frac{k^2 + m^2 \tanh^2 mx}{k^2 + m^2} = 1 - \frac{m^2}{k^2+m^2}\frac{1}{\cosh^2 mx}\,, \\
|\partial_x f_k(x)|^2 &= k^2 + \frac{m^2}{\cosh^2 mx} + \frac{m^4}{k^2+m^2}\left(\frac{1}{\cosh^4 mx} - \frac{1}{\cosh^2 mx}\right)\,.
\end{aligned} \tag{B.4}$$

We consider mass term (B.1) and terms with derivatives (B.2) and (B.3) separately

$$\begin{aligned}
&\sum_i C_i m_i^2 \left\langle |n_i(x)|^2 \right\rangle = N \int \frac{\mathrm{d}k}{2\pi} \sum_i \frac{C_i m_i^2}{2\sqrt{k^2+m^2+m_i^2}} \\
&+N\frac{m^2}{\cosh^2 mx}\left(-\int \frac{\mathrm{d}k}{4\pi}\sum_i \frac{C_i m_i^2}{(k^2+m^2)\sqrt{k^2+m^2+m_i^2}} + \sum_i \frac{C_i m_i}{4m}\right)\,.
\end{aligned}$$

The first term yields the energy density of the homogeneous state. Note that in expression (B.2) for the spacial derivative, the term with derivative of the classical component cancels with the convergent part of the integral, which is a contribution from the third term in (B.4). Thus, we can write down the remaining contributions

$$\begin{aligned}
&\sum_i C_i \left\langle |\partial_x n_i(x)|^2 + |\partial_t n_i(x)|^2 \right\rangle = N\frac{m^2}{\cosh^2 mx} \\
&\times\left(\int \frac{\mathrm{d}k}{4\pi}\sum C_i\left(\frac{1}{\sqrt{k^2+m^2+m_i^2}} - \frac{\sqrt{k^2+m^2+m_i^2}}{k^2+m^2}\right) + \sum_i \frac{C_i m_i}{4m}\right)\,.
\end{aligned}$$

The integral calculation is straightforward. Thus, we find that the contribution to the inhomogeneous part of energy density from derivative terms (B.3) and (B.2) and term (B.1) are equal. Therefore, corresponding contributions in the momentum flow $\theta_{11}$ in (28) cancel, and $\theta_{11}$ is constant. Combining the results, we obtain (31) and (32).

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