APPLICATION OF THE RIEMANN–HILBERT APPROACH AND N-SOLITON SOLUTIONS FOR THE GENERALIZED FOKAS–LENELLS EQUATION

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*Received 4 February 2025, accepted 28 April 2025,
published online 30 May 2025*

This paper investigates the generalized Fokas–Lenells equation by the Riemann–Hilbert approach. A gauge transformation is introduced to symmetrize the originally asymmetric spectral problem. A novel Riemann–Hilbert method is developed for the generalized Fokas–Lenells equation, performing spectral analysis on the temporal component of the Lax pair rather than the spatial part. N-soliton solutions are rigorously derived by solving the Riemann–Hilbert problem with this complex spectral symmetry. Additionally, the dynamics of the one and two solitons of the generalized Fokas–Lenells equation are discussed.

DOI:10.5506/APhysPolB.56.6-A1

1. Introduction

The nonlinear Schrödinger (NLS) equation [1]

$$iu_t + \nu u_{xx} + \sigma |u|^2 u = 0\,, \tag{1}$$

where ν is a real-valued parameter and $\sigma = \pm 1$, is a fundamental partial differential equation that describes a wide range of nonlinear wave phenomena. There has been found extensive applications in various fields [2, 3], including nonlinear optics, quantum mechanics, fluid dynamics, and plasma physics. Numerous studies have been conducted related to Eq. (1). For instance, Hirota's bilinear method [4] and the Darboux transformation (DT) [5] have been employed to construct a variety of soliton solutions and to analyze their dynamic behavior. In the context of optical fiber systems, detailed studies have been conducted on the interactions and collision characteristics of optical solitons, revealing important insights into the stability and interaction mechanisms of solitons during propagation [6]. Moreover, the generation

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mechanisms of higher-order soliton solutions and rogue waves have been extensively explored, further enriching the understanding of nonlinear wave dynamics [7].

The Fokas–Lenells (FL) equation

$$iu_t - \iota u_{xt} + \nu u_{xx} + \sigma |u|^2 (u + i\iota u_x) = 0\,, \tag{2}$$

as a generalization of the NLS equation, where ι and ν are real parameters and $u(x,t)$ represents a complex-valued function, was initially proposed by Fokas using the bi-Hamiltonian formulation [8]. Numerous studies on the FL equation have been conducted. For example, it has been investigated via the dressing method [9], Hirota's bilinear method [10, 11], the DT [12, 13], and other approaches [14–19].

The main purpose of this paper is to study the following generalized FL equations [20]:

$$\begin{aligned} u_{xt} &= i\left(2|u|^2 u_t + \mu u_x + 2u_t w\right) - u\,, \\ iw_t &= \mu\left(u_x u_t^* - u_t u_x^*\right)\,. \end{aligned} \tag{3}$$

by the Riemann–Hilbert (RH) method [21–32]. Here, $*$ denotes the complex conjugation, μ is a real parameter, $u(x,t)$ represents a complex-valued function, and $w(x,t)$ is a real-valued function.

The outline of this paper is as follows: In Section 2, a gauge transformation is introduced to endow the originally asymmetric spectral problem with symmetric properties. Furthermore, a novel RH method for Eq. (3) is developed. This method involves conducting spectral analysis on the temporal component of the Lax pair, while the spatial component serves as an auxiliary element. Through this approach, the complex spectral symmetry structure of Eq. (3) is thoroughly investigated. In Section 3, N-soliton solutions for Eq. (3) in the reflectionless cases will be rigorously obtained by solving the RH problem. Moreover, the one-soliton solutions of Eq. (3) will be explicitly given, and the corresponding dynamical behaviors of one- and two-soliton solutions will be shown graphically.

2. The Riemann–Hilbert problem

In this section, we shall study the scattering and inverse-scattering transformations for Eq. (3) by using the RH formulations. The Lax pair of Eq. (3) is established as follows:

$$Y_x = UY\,, \qquad Y_t = VY\,, \tag{4}$$

where $Y = Y(x,t;\lambda)$ is a matrix function and

$$U = \frac{1}{1-\mu\lambda}\begin{pmatrix} \frac{i\lambda}{2} + iuu^* + iw & -iu - \mu u_x \\ \lambda\left(iu^* - \mu u_x^*\right) & -\frac{i\lambda}{2} - iuu^* - iw \end{pmatrix},$$
$$V = \begin{pmatrix} \frac{i\lambda^{-1}}{2} & \lambda^{-1}u_t \\ u_t^* & -\frac{i\lambda^{-1}}{2} \end{pmatrix}, \tag{5}$$

where, λ is a constant spectral parameter.

Now, we introduce a gauge transformation as follows:

$$\widehat{Y} = TY, \qquad T = \begin{pmatrix} 1 & 0 \\ 0 & k^{-1} \end{pmatrix}, \quad (k \neq 0) \tag{6}$$

where $\lambda = k^2$. Then Eq. (4) turns into the following equivalent form:

$$\widehat{Y}_x = \widehat{U}\widehat{Y}, \qquad \widehat{Y}_t = \widehat{V}\widehat{Y}, \tag{7}$$

with

$$\widehat{U} = \frac{1}{1-\mu k^2}\begin{pmatrix} \frac{i}{2}k^2 + iuu^* + iw & k\left(-iu - \mu u_x\right) \\ k\left(iu^* - \mu u_x^*\right) & -\frac{i}{2}k^2 - iuu^* - iw \end{pmatrix},$$
$$\widehat{V} = \begin{pmatrix} \frac{i}{2}k^{-2} & k^{-1}u_t \\ k^{-1}u_t^* & -\frac{i}{2}k^{-2} \end{pmatrix}. \tag{8}$$

To facilitate our research, we introduce a new matrix spectral function $J = J(x,t;k)$ defined by

$$\widehat{Y} = JE, \tag{9}$$

where $E = \mathrm{e}^{\frac{ik^2}{2(1-\mu k^2)}\sigma_3 x + \frac{i}{2}k^{-2}\sigma_3 t}$. Under the transformation (9), the Lax pair (7) can be rewritten in the following form:

$$J_x = \frac{ik^2}{2\left(1-\mu k^2\right)}\left[\sigma_3, J\right] + \widetilde{U}J,$$
$$J_t = \frac{i}{2}k^{-2}\left[\sigma_3, J\right] + \widetilde{V}J, \tag{10}$$

where σ_3 is a specific Pauli matrix, and $[\sigma_3, J] = \sigma_3 J - J\sigma_3$ represents the commutators

$$\widetilde{U} = \frac{1}{1-\mu k^2}\begin{pmatrix} iuu^* + iw & k\left(-iu - \mu u_x\right) \\ k\left(iu^* - \mu u_x^*\right) & -iuu^* - iw \end{pmatrix},$$
$$\widetilde{V} = \begin{pmatrix} 0 & k^{-1}u_t \\ k^{-1}u_t^* & 0 \end{pmatrix}. \tag{11}$$

In the following direct scattering process, we focus solely on the t-part of the Lax pair (10), where x is treated as a dummy variable and is omitted. Now, we introduce the matrix Jost solutions $J_\pm = J_\pm(t,k)$ for the t-part of the Lax pair (10)

$$J_- = ([J_-]_1, [J_-]_2)\ , \qquad J_+ = ([J_+]_1, [J_+]_2)\ , \tag{12}$$

with the asymptotic conditions

$$J_\pm \to I\,, \qquad t \to \pm\infty\,, \tag{13}$$

where each $[J_\pm]_l$ $(l = 1, 2)$ denotes the l^{th} column of $J_\pm$, respectively, and the symbol I is the 2×2 identity matrix.

Using the large-t asymptotic condition (13), we can transform the t-part of (10) into the Volterra integral equations

$$J_\pm(t,k) = I + \int_{\pm\infty}^{t} \mathrm{e}^{-\frac{i}{2}k^{-2}\hat{\sigma}_3(\tau-t)} \left(\widetilde{V}(\tau)J_\pm(\tau,k)\right) \mathrm{d}\tau\,, \tag{14}$$

where $\hat{\sigma}_3$ acts on a 2×2 matrix X by $\hat{\sigma}_3 X = [\sigma_3, X]$. Additionally, $\mathrm{e}^{\hat{\sigma}_3}X$ is defined as $\mathrm{e}^{\sigma_3}X\mathrm{e}^{-\sigma_3}$.

By performing the standard procedures on the Volterra integral equations (14), one can prove the existence and uniqueness of the Jost solutions $J_\pm$. Moreover, it is crucial to note that $[J_-]_1$, $[J_+]_2$ can be analytically extended into D_+, while $[J_+]_1$, $[J_-]_2$ can be analytically extended into D_-, where the regions $D_\pm$ are defined by

$$\begin{aligned} D_+ &= \{k \in \mathbb{C} |\arg k \in (0, \pi/2) \cup (\pi, 3\pi/2)\}\,, \\ D_- &= \{k \in \mathbb{C} |\arg k \in (\pi/2, \pi) \cup (3\pi/2, 2\pi)\}\,, \end{aligned}$$

and $\partial D = \{\mathbb{R} \cup i\mathbb{R}\}$.

Now we investigate the properties of $J_\pm$. Indeed, the fact that the potential matrix $\widetilde{V}$ is zero-trace implies that $\det J_\pm$ are independent of the variable t. Particularly, by evaluating $\det J_+$ as $t \to +\infty$ and $\det J_-$ as $t \to -\infty$, respectively, we have

$$\det J_\pm = 1\,, \qquad k \in \partial D\,. \tag{15}$$

Since J_-E_1 and J_+E_1 are both fundamental solutions of the t-part of Eq. (7) for $k \in \partial D$, with $E_1 = \mathrm{e}^{\frac{i}{2}k^{-2}\sigma_3 t}$, they are linearly related. That is, there exists a scattering matrix $S(k)$ such that

$$J_-E_1 = J_+E_1 S(k)\,, \qquad k \in \partial D\,, \tag{16}$$

with $\det S(k) = 1$, $k \in \partial D$, and $S(k) = \begin{pmatrix} a(k) & -\widetilde{b}(k) \\ b(k) & \widetilde{a}(k) \end{pmatrix}$.

Furthermore, we find from the scattering relation (16) that

$$\begin{aligned} a(k) &= W([J_-]_1, [J_+]_2), \qquad b(k) = \mathrm{e}^{ik^{-2}t} W([J_+]_1, [J_-]_1), \\ \widetilde{a}(k) &= W([J_+]_1, [J_-]_2), \qquad \widetilde{b}(k) = \mathrm{e}^{-ik^{-2}t} W([J_+]_2, [J_-]_2), \end{aligned} \tag{17}$$

where $W(\cdot,\cdot)$ denotes the Wronski determinant. Then it follows from the analytic property of $J_\pm$ that $a(k)$ can be analytically extended to D_+, and $\widetilde{a}(k)$ allows for analytic extensions to D_-.

To construct the RH problem on ∂D by using the analytic properties of the Jost solutions $J_\pm$, it is important to introduce a matrix function $P_1 = P_1(t,k)$ which is analytic in D_+. Specifically,

$$P_1 = ([J_-]_1, [J_+]_2), \tag{18}$$

which solves the linear spectral problem (10). Furthermore, by considering the large-k asymptotic behavior of P_1, we find that

$$P_1 \to I, \qquad k \in D_+ \to \infty. \tag{19}$$

On the other hand, in order to construct an analytic matrix function P_2 in D_-, we write the inverse of $J_\pm$ as

$$J_-^{-1} = \begin{pmatrix} [J_-^{-1}]^1 \\ [J_-^{-1}]^2 \end{pmatrix}, \qquad J_+^{-1} = \begin{pmatrix} [J_+^{-1}]^1 \\ [J_+^{-1}]^2 \end{pmatrix}, \tag{20}$$

where each $[J_\pm^{-1}]^l$ $(l = 1,2)$ denotes the l^{th} row of $J_\pm^{-1}$, respectively. It is easy to verify that $J_\pm^{-1}$ satisfy the equation of K

$$\begin{aligned} K_x &= \frac{ik^2}{2(1-\mu k^2)}[\sigma_3, K] - K\widetilde{U}, \\ K_t &= \frac{i}{2}k^{-2}[\sigma_3, K] - K\widetilde{V}. \end{aligned} \tag{21}$$

Resorting to the spectral analysis of (21), we can define another matrix function $P_2 = P_2(t,k)$, which is analytic for k in D_-

$$P_2 = \begin{pmatrix} [J_-^{-1}]^1 \\ [J_+^{-1}]^2 \end{pmatrix}. \tag{22}$$

Moreover, the large-k asymptotic behavior of P_2 can be shown to be

$$P_2 \to I, \qquad k \in D_- \to \infty. \tag{23}$$

To establish the RH problem for Eq. (3), we notice the symmetry relations for the matrices $\widehat{U}$ and $\widehat{V}$ such that

$$\begin{aligned} \sigma_3\widehat{U}^{\dagger}(k^*)\sigma_3 &= -\widehat{U}(k)\,, \qquad \sigma_3\widehat{V}^{\dagger}(k^*)\sigma_3 = -\widehat{V}(k)\,,\\ \sigma_3\widehat{U}(-k)\sigma_3 &= \widehat{U}(k)\,, \qquad \sigma_3\widehat{V}(-k)\sigma_3 = \widehat{V}(k)\,, \end{aligned} \tag{24}$$

where $\dagger$ means the Hermitian conjugate. Therefore, we get the following relations:

$$\begin{aligned} a(-k) &= a(k)\,, \qquad \widetilde{a}(-k) = \widetilde{a}(k)\,,\\ b(-k) &= -b(k)\,, \qquad \widetilde{b}(-k) = -\widetilde{b}(k)\,,\\ \widetilde{a}(k) &= a^*(k^*)\,, \qquad \widetilde{b}(k) = -b^*(k^*)\,, \qquad k \in \partial D\,. \end{aligned} \tag{25}$$

In addition, the following equality also holds:

$$\begin{aligned} \sigma_3 P_1^{\dagger}(k^*)\sigma_3 &= P_2(k)\,,\\ \sigma_3 P_j(-k)\sigma_3 &= P_j(k)\,, \qquad j=1,2\,, \qquad k \in \partial D\,. \end{aligned} \tag{26}$$

Summarizing the above results, we have constructed two matrix functions P_1 and P_2, which are analytic in D_+ and D_-, respectively. Now, we denote the limit of P_1 as k approaches ∂D for $k \in D_+$ by P^+, and the limit of P_2 as k approaches ∂D for $k \in D_-$ by P^-.

Consequently, we can formulate the RH problem, that is, two matrix functions P^+ and P^- satisfy the jump condition on the curve ∂D

$$P^-(t,k)P^+(t,k) = G(t,k)\,, \qquad k \in \partial D\,, \tag{27}$$

where $G(t,k) = \begin{pmatrix} 1 & \mathrm{e}^{ik^{-2}t}\widetilde{b}(k) \\ \mathrm{e}^{-ik^{-2}t}b(k) & 1 \end{pmatrix}$.

Let us consider the asymptotic expansion of P_1,

$$P_1(k) = I + k^{-1}P_1^{(1)} + k^{-2}P_1^{(2)} + \ldots\,, \qquad k \to \infty\,, \tag{28}$$

and substitute this expansion into (10). Then we find that the potential functions u and w can be reconstructed by

$$\begin{aligned} u &= \left(P_1^{(1)}\right)_{12}\,,\\ w &= i\mu\left[\left(P_1^{(2)}\right)_{11,x} - u_x u^*\right]\,, \end{aligned} \tag{29}$$

where $(P_1^{(1)})_{12}$ is the $(1,2)$-entry of $P_1^{(1)}$ and $(P_1^{(2)})_{11}$ is the $(1,1)$-entry of $P_1^{(2)}$.

3. Soliton solutions

From the definitions of P_1 and P_2 as well as the scattering relations between J_+ and J_-, we obtain

$$\det P_1(k) = a(k)\,, \qquad k \in D_+\,, \tag{30}$$

$$\det P_2(k) = \widetilde{a}(k)\,, \qquad k \in D_-\,, \tag{31}$$

which means that the zeros of $\det P_1$ and $\det P_2$ are the same as $a(k)$ and $\widetilde{a}(k)$, respectively.

From (26), we find that if k_j is a zero of $\det P_1$, then $-k_j$ is also a zero of $\det P_1$, and $\hat{k}_j = k_j^*$ is a zero of $\det P_2$. We first assume that $\det P_1$ has $2N$ simple zeros $\{k_j\}_1^{2N}$ satisfying $k_{N+j} = -k_j$, $1 \le j \le N$, all of which lie in D_+. Hence, $\det P_2$ possesses $2N$ simple zeros $\{\hat{k}_j\}_1^{2N}$ satisfying $\hat{k}_j = k_j^*$ for $1 \le j \le 2N$, all of which lie in D_-.

In this case, each of $\ker[P_1(k_j)]$ and $\ker[P_2(\hat{k}_j)]$ contains only a single column vector v_j and a single row vector $\hat{v}_j$, respectively,

$$P_1(k_j)v_j = 0\,, \qquad \hat{v}_j P_2\left(\hat{k}_j\right) = 0\,, \qquad 1 \le j \le 2N\,. \tag{32}$$

It is easy to see that these vectors satisfy the following relations:

$$v_{N+j} = \sigma_3 v_j\,, \qquad \hat{v}_{N+j} = \hat{v}_j \sigma_3\,, \qquad 1 \le j \le N\,, \tag{33}$$

$$\hat{v}_j = v_j^\dagger \sigma_3\,, \qquad 1 \le j \le 2N\,. \tag{34}$$

Now, we shall deduce the spatial evolutions of the vectors v_j, for $1 \le j \le 2N$. For this purpose, we take the x-derivative of $P_1(k_j)v_j = 0$. Then, utilizing the x-part of (10), we obtain

$$v_{j,x} = \left(\frac{ik_j^2}{2\left(1 - \mu k_j^2\right)} \sigma_3 + \alpha_j I \right) v_j\,, \tag{35}$$

where α_j are arbitrary constants. Noticing the t-part of (10), we similarly have

$$v_{j,t} = \left(\frac{i}{2} k_j^{-2} \sigma_3 + \beta_j I \right) v_j\,, \tag{36}$$

where β_j are arbitrary constants. By solving (36) and (37) explicitly, we get

$$v_j = \mathrm{e}^{\left(\frac{ik_j^2}{2\left(1-\mu k_j^2\right)} \sigma_3 + \alpha_j I \right) x + \left(\frac{i}{2} k_j^{-2} \sigma_3 + \beta_j I \right) t} v_{j,0}\,, \qquad 1 \le j \le 2N\,, \tag{37}$$

where each $v_{j,0}$, for $1 \le j \le 2N$ is a nonzero complex constant vector.

If we set $k_j = \xi_j + i\eta_j$ and $v_{j,0} = (\mathrm{e}^{\alpha_{j,0}+i\beta_{j,0}}, 1)^T$, then v_j takes the form

$$v_j = \mathrm{e}^{\epsilon_j}\left(\mathrm{e}^{(z_j+i\varphi_j)/2}, \mathrm{e}^{-(z_j+i\varphi_j)/2}\right)^T, \qquad 1 \le j \le 2N\,, \tag{38}$$

with

$$\begin{aligned}
\epsilon_j &= \alpha_j x + \beta_j t + \frac{\alpha_{j,0}+i\beta_{j,0}}{2}\,,\\
z_j &= \frac{-2\xi_j\eta_j}{\left[1-\mu\left(\xi_j^2-\eta_j^2\right)\right]^2 + 4\mu^2\xi_j^2\eta_j^2}x + \frac{2\xi_j\eta_j}{\left(\xi_j^2+\eta_j^2\right)^2}t + \alpha_{j,0}\,,\\
\varphi_j &= \frac{\xi_j^2-\eta_j^2-\mu\left(\xi_j^2+\eta_j^2\right)^2}{\left[1-\mu\left(\xi_j^2-\eta_j^2\right)\right]^2 + 4\mu^2\xi_j^2\eta_j^2}x + \frac{\xi_j^2-\eta_j^2}{\left(\xi_j^2+\eta_j^2\right)^2}t + \beta_{j,0}\,.
\end{aligned} \tag{39}$$

As is well known, the RH problem (27) with the canonical normalization condition can be transformed into the RH problem without zeros by utilizing the methods from Ref. [28]. To obtain soliton solutions for Eq. (3), we choose the jump matrix G to be the 2×2 identity matrix, which corresponds to the reflectionless case. Consequently, the unique solution for this particular RH problem reads

$$\begin{aligned}
P_1(k) &= I - \sum_{m=1}^{2N}\sum_{j=1}^{2N}\frac{v_m\hat{v}_j\left(M^{-1}\right)_{mj}}{k-\hat{k}_j}\,,\\
P_2(k) &= I + \sum_{m=1}^{2N}\sum_{j=1}^{2N}\frac{v_m\hat{v}_j\left(M^{-1}\right)_{mj}}{k-k_m}\,,
\end{aligned} \tag{40}$$

where $M = (M_{mj})_{2N\times 2N}$ is a matrix whose entries are

$$M_{mj} = \frac{\hat{v}_m v_j}{k_j - \hat{k}_m}\,, \qquad 1 \le m, j \le 2N\,. \tag{41}$$

Therefore, we obtain from (40) that

$$\begin{aligned}
P_1^{(1)} &= -\sum_{m=1}^{2N}\sum_{j=1}^{2N} v_m\hat{v}_j\left(M^{-1}\right)_{mj}\,,\\
P_1^{(2)} &= -\sum_{m=1}^{2N}\sum_{j=1}^{2N} \hat{k}_j v_m\hat{v}_j\left(M^{-1}\right)_{mj}\,.
\end{aligned} \tag{42}$$

Then, we deduce N-soliton solutions formula of Eq. (3)

$$\begin{aligned} u &= -\sum_{m=1}^{2N}\sum_{j=1}^{2N}\left(v_m\hat{v}_j\right)_{12}\left(M^{-1}\right)_{mj}\,, \\ w &= -i\mu\left[\partial_x\left(\sum_{m=1}^{2N}\sum_{j=1}^{2N}\hat{k}_j\left(v_m\hat{v}_j\right)_{11}\left(M^{-1}\right)_{mj}\right)\right. \\ &\quad \left.+\sum_{m=1}^{2N}\sum_{j=1}^{2N}\left(v_m\hat{v}_j\right)^*_{12}\left(M^{-1}\right)^*_{mj}\partial_x\left(\sum_{m=1}^{2N}\sum_{j=1}^{2N}\left(v_m\hat{v}_j\right)_{12}\left(M^{-1}\right)_{mj}\right)\right]. \end{aligned} \tag{43}$$

As a special example, we choose $N = 1$ in formula (44). Consequently, one-soliton solution of Eq. (3) takes the form

$$\begin{aligned} u &= \frac{2i\xi_1\eta_1\,\mathrm{e}^{i\varphi_1}}{\xi_1\sinh z_1 - i\eta_1\cosh z_1}\,, \\ w &= \frac{-8\xi_1^2\eta_1^2\mu\left[2\left(\eta_1^2-\xi_1^2\right)+\mu\left(\xi_1^2+\eta_1^2\right)\right]}{\left[1-\mu\left(\xi_1-i\eta_1\right)^2\right]\left[1-\mu\left(\xi_1+i\eta_1\right)^2\right]\left(\xi_1^2\sinh^2 z_1+\eta_1^2\cosh^2 z_1\right)}\,, \end{aligned} \tag{44}$$

with $k_1 = \xi_1 + i\eta_1 \in D_+$, and

$$\begin{aligned} z_1 &= \frac{-2\xi_1\eta_1}{\left[1-\mu\left(\xi_1^2-\eta_1^2\right)\right]^2+4\mu^2\xi_1^2\eta_1^2}x + \frac{2\xi_1\eta_1}{\left(\xi_1^2+\eta_1^2\right)^2}t + \alpha_{1,0}\,, \\ \varphi_1 &= \frac{\xi_1^2-\eta_1^2-\mu\left(\xi_1^2+\eta_1^2\right)^2}{\left[1-\mu\left(\xi_1^2-\eta_1^2\right)\right]^2+4\mu^2\xi_1^2\eta_1^2}x + \frac{\xi_1^2-\eta_1^2}{\left(\xi_1^2+\eta_1^2\right)^2}t + \beta_{1,0}\,. \end{aligned} \tag{45}$$

where $v_{1,0} = (\mathrm{e}^{\alpha_{1,0}+i\beta_{1,0}}, 1)^T$ is chosen.

Considering the complexity and length of the two-soliton solution, we refrain from presenting its explicit form in this paper to maintain conciseness. However, to provide an intuitive understanding of the dynamical behavior of the two-soliton solution, graphical representations of its evolution are presented in Figs. 1, 2, and 3.

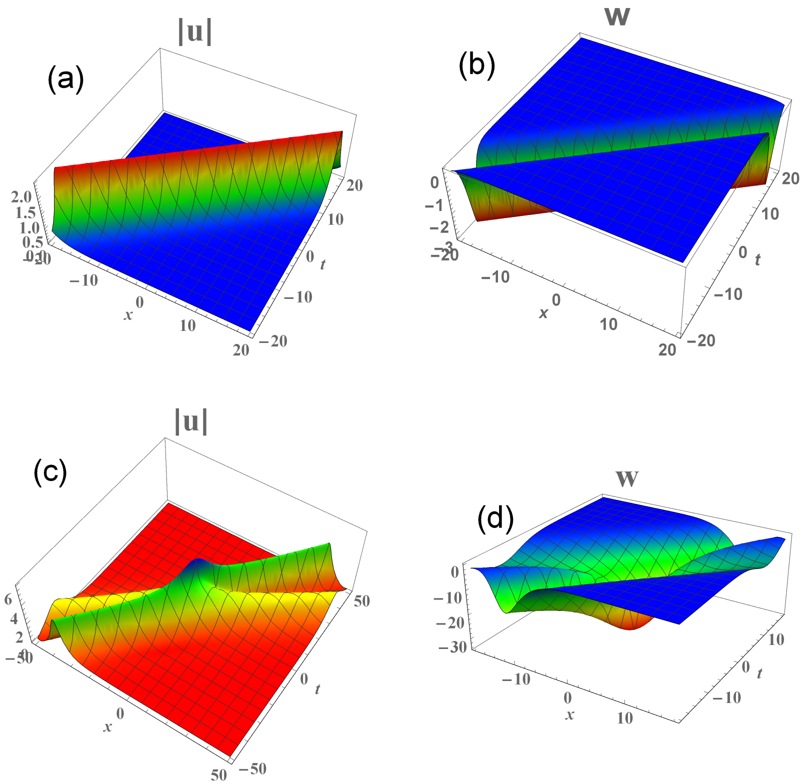

Fig. 1. (a), (b) Evolution plots of the one-soliton solution (44) with $k_1 = 1 + i$, $\mu = 1$, $\alpha_{1,0} = \beta_{1,0} = 0$; (c), (d) Evolution plots of the two-soliton solution with $k_1 = 2 + i$, $k_2 = 1 + 2i$, $\mu = 1$, $\alpha_{1,0} = \beta_{1,0} = \alpha_{2,0} = \beta_{2,0} = 0$.

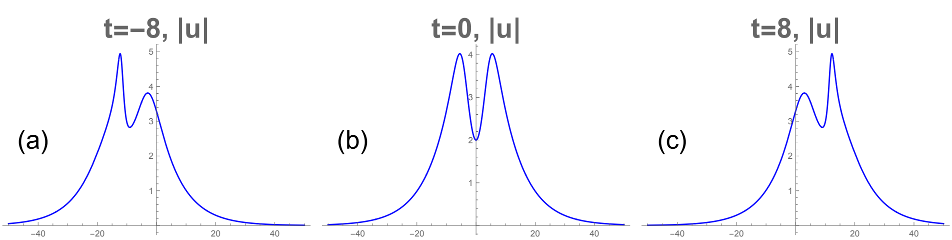

Fig. 2. (a), (b), (c) The two-soliton solution for the u component at $t = -8$, $t = 0$, and $t = 8$, respectively, with $k_1 = 1 + i$, $k_2 = -2 - 2i$, $\mu = 1$, $\alpha_{1,0} = \beta_{1,0} = \alpha_{2,0} = \beta_{2,0} = 0$.

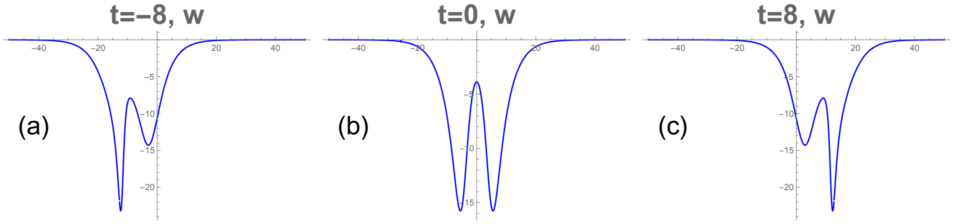

Fig. 3. (a), (b), (c) The two-soliton solution for the w component at $t = -8$, $t = 0$, and $t = 8$, respectively. The parameters are the same as those in Fig. 2.

Funded by the National Natural Science Foundation of China (grant No. 12271490).

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