

REPRESENTATION STRUCTURE OF THE $SL(2, \mathbb{C})$ ACTING IN THE HILBERT SPACE OF THE QUANTUM COULOMB FIELD

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We give a complete description of the representation of $SL(2, \mathbb{C})$ acting in the Hilbert space of the quantum Coulomb field and a constructive consistency proof of the axioms of the quantum theory of the Coulomb field.

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1. Introduction

In paper [1], we have announced the following:

Theorem. *Let $U|_{\mathcal{H}_n}$ be the restriction of the unitary representation U of $SL(2, \mathbb{C})$ in the Hilbert space of the quantum phase field S to the invariant eigenspace $\mathcal{H}_n$ of the total charge operator Q corresponding to the eigenvalue ne for some integer n . Then, for all n such that*

$$|n| > \sqrt{\frac{\pi}{\epsilon^2}},$$

the representations $U|_{\mathcal{H}_n}$ are unitarily equivalent

$$U|_{\mathcal{H}_n} \cong_U U|_{\mathcal{H}_{n'}}, \quad \text{whenever} \quad |n| > \sqrt{\frac{\pi}{\epsilon^2}}, \quad |n'| > \sqrt{\frac{\pi}{\epsilon^2}}.$$

If the two integers n, n' have different absolute values $|n| \neq |n'|$ and

$$|n| < \sqrt{\frac{\pi}{\epsilon^2}}, \quad |n'| < \sqrt{\frac{\pi}{\epsilon^2}},$$

then the representations $U|_{\mathcal{H}_n}$ and $U|_{\mathcal{H}_{n'}}$ are inequivalent. Each representation $U|_{\mathcal{H}_n}$ contains a unique discrete supplementary component if and only if

$$|n| < \sqrt{\frac{\pi}{\epsilon^2}},$$

and the supplementary components contained in $U|_{\mathcal{H}_n}$ with different values of $|n|$ fulfilling the last inequality are inequivalent.

We have based the proof on the false lemma asserting that the representation $U|_{\mathcal{H}_n}$ of $G = \text{SL}(2, \mathbb{C})$ acting in the eigenspace $\mathcal{H}_n$ corresponding to the eigenvalue $n\epsilon$ of the total charge Q has the tensor product structure, $U|_{\mathcal{H}_n} = U|_{|u\rangle} \otimes U|_{\mathcal{H}_0}$, where $U|_{|u\rangle}$ is the cyclic subrepresentation with the cyclic spherically symmetric vector $|u\rangle = e^{-inS(u)}|0\rangle$. However, this (false) lemma was inferred from calculations containing a computational error, as was recognized only after publication and announced in [2]. In this paper, we give a proof of the above theorem going along a different line. The general idea of the proof consists in decomposition of the cyclic representations with cyclic vectors

$$x = c_{\alpha_1}^+ \dots c_{\alpha_q}^+ e^{-inS(u)}|0\rangle \in \mathcal{H}_n, \quad q = 1, 2, \dots \quad (1)$$

and then by recovering inclusion relations between the cyclic representations, where $c_{\alpha_i}^+$ are the creation operators of the transversal infrared photons, and $S(u)$ is the phase operator of [3], all computed in the reference frame with the time-like unit vector u . For the construction of decompositions of the cyclic representations, we generalize the method of [4]. Contrary to the cyclic state $|u\rangle$ of the cyclic representation investigated in [4], which is spherically symmetric, our cyclic states are not spherically symmetric. Therefore, the reproducing kernel Hilbert spaces, corresponding to our cyclic representations must necessarily be constructed on the whole group $\text{SL}(2, \mathbb{C})$, and not just on the Lobachevsky space $\text{SL}(2, \mathbb{C})/\text{SU}(2, \mathbb{C})$. Except for this difference, our method of constructing decomposition of the cyclic representation is the same as that of [4]. The lack of tensor product structure is compensated for by the analyticity of the Fourier transform of the kernels, induced by the cyclic representations with the cyclic vectors (1). The basic properties of decomposition may be read off from the orthogonality relations of the matrix elements of the unitary irreducible representations and properties of the poles of the Fourier transforms of the positive definite functions associated with the cyclic representations. The orthogonality and analyticity are also used to investigate the mutual inclusion relations of the cyclic representations. We obtain results going further than is asserted by the above theorem. In particular, we obtain a complete description of the representation U , as well as the action of the operators $c_{\alpha_i}^+$, $S(u)$ on the subspaces invariant for U .

For example, let (l_0, l_1) be the unitary irreducible representations of [5], then we show that

$$U|_{\mathcal{H}_n} = \bigoplus_{l_0 \in \mathbb{Z}} \int_0^{+\infty} (l_0, i\rho) \nu(\rho, z) d\rho \bigoplus \nu(z) (l_0 = 0, 1 - z), \quad z = \frac{n^2 \epsilon^2}{\pi}, \quad (2)$$

where for each $z > 0$, the weight $\nu(\rho, z)$ is almost everywhere > 0 and with a positive weight $\nu(z)$ of the supplementary component $(l_0, l_1) = (0, 1 - z)$, nonzero if and only if $0 < z < 1$. From (2) the theorem follows. Decomposition (2) can be also used to solve the problems concerning U and raised in [2].

In [6], we have given a proof that the invariant kernel

$$\langle u|v \rangle = \langle 0|e^{inS(u)}e^{-inS(v)}|0 \rangle$$

on the Lobachevsky space is positive definite, independently of the theory of the quantum Coulomb field [3]. In paper [6], we have written one sentence (repeated twice) that the proof presented there “gives us a proof of (relative) consistency of the theory [3]”. This sentence was based on the above-mentioned false lemma. Therefore, in Subsection 2.1, we are giving a constructive consistency proof of the axioms, finishing a proof initiated in the works [3, 4, 7, 8], by adding one last step: we compute explicitly the vector-valued function B on G , which respect, together with the matrix-valued function A (both introduced in [7]), the requirements formulated in [4, 7, 8]. Another, more general, consistency proof was given in [2], including a proof of existence, indecomposability and positivity of a one-parameter family of states of the C^* -algebra $\mathcal{A}$ of [2], with unitarily implementable Lorentz automorphisms, among them, the state giving the theory [3], which is the subject of our paper.

Section 2 has preparatory character, and gives all ingredients, which are then used in the proof. In Subsection 2.1, we define A, B . In Subsection 2.2, we are giving explicit formulas for A, B . In Subsection 2.3, we give explicit formulas for the matrix elements of all unitary irreducible representations (l_0, l_1) of G , and point out the properties of these matrices, which we then use in the proof. In Subsection 2.4, we give general formulas for the Fourier transforms of the positive definite functions canonically associated with the cyclic representations with cyclic vectors (1), and construct their analytic continuations. We give there decomposition of these cyclic representations. In Section 3, we give the method for investigation of the inclusion relation of the cyclic subspaces. This method is then applied in Sections 4 and 5, in which a complete description of $U|_{\mathcal{H}_n}$ is given.

Summing up, in this paper, we present a constructive consistency proof of the axioms of [3], then give a proof of (2), and the theorem as a corollary of (2).

Contribution of the second-named author is the derivation of the series expansion (74) of the Fourier transforms, valid for the whole class of the Fourier transforms of the positive definite functions on $\mathrm{SL}(2, \mathbb{C})$, which are defined by the cyclic representations with cyclic vectors (1).

2. Cyclic subrepresentations

In the sequel $\mathcal{H}$ be the Hilbert space of the operators $S, Q, c_\alpha, c_\alpha^+$, $\alpha = (l, m)$, $l = 1, 2, \dots$, $-l \leq m \leq l$, introduced in [3]. Let for $n \in \mathbb{Z}$, $n \neq 0$, $\mathcal{H}_n$ be the eigenspace of the total charge Q corresponding to the eigenvalue $n\mathfrak{e}$. We will use the notation of [4, 7–9]. Let $G = \mathrm{SL}(2, \mathbb{C})$ and let U be the unitary representation of G acting in $\mathcal{H}$. We consider cyclic subrepresentation of U , with the invariant subspace $\mathcal{H}_x \subset \mathcal{H}_n$ generated by a single cyclic $x \in \mathcal{H}_n$, i.e., with $\mathcal{H}_x$ spanned by Lorentz transforms $|g\rangle = U(g)x \in \mathcal{H}_n$, $g \in G$. Below, we consider special x of the form (1) or $x = e^{-inS(u)}|0\rangle = |u\rangle$, with $\alpha_i = (l_i, m_i)$, $i, l = 1, 2, \dots$, $-l_i \leq m_i \leq l_i$. With this convention, x will sometimes be denoted by $x = U(e)x = |e\rangle$. In what follows e stands for the basis of natural logarithms or the unit element in G , and $\mathfrak{e}^2$ in $z = n^2 \mathfrak{e}^2 / \pi$ stands for the square of the elementary charge, the experimental value of which is approximately equal to $1/137$ in units in which $\hbar = c = 1$.

To compute the decomposition of the cyclic subrepresentation with cyclic vector (1), we consider the corresponding invariant subspace $\mathcal{H}_x$ as the reproducing kernel Hilbert space with the corresponding left-invariant kernel

$$\langle g|h \rangle = \langle U(g)x | U(h)x \rangle$$

on G . Next, we use the Fourier transform (Gelfand–Neumark–Plancherel formula on G , [10]) and compute the Fourier transform of the corresponding left-invariant kernel.

Let $U^{(l_0, l_1)}$ be the irreducible unitary representation (l_0, l_1) of G [5, 10], with l_0 any half-integer or integer, and $l_1 = i\rho$, $\rho \in \mathbb{R}$ (principal series) or $l_0 = 0, l_1 \in (-1, 1)$ (supplementary series). The representations (l_0, l_1) , $(-l_0, -l_1)$ are not only equivalent, but the matrices of these representations, found in [5, 10], are identical. Recall that the representation of the subgroup $\mathrm{SU}(2, \mathbb{C}) \subset G$, defined by restriction of $U^{(l_0, l_1)}$ to $\mathrm{SU}(2, \mathbb{C})$, is equal to the direct sum of the standard irreducible representations with the weight $l = |l_0|, |l_0| + 1, \dots$, each entering with multiplicity one, and with $|l_0|$ being the lowest weight. We have the following range of the matrix component indices $\alpha = (l, m)$: $l \in \{|l_0|, |l_0| + 1, \dots\}$, $-l \leq m \leq l$ of the representations (l_0, l_1) . Thus, the only irreducible unitary representations containing

nonzero $SU(2, \mathbb{C})$ -invariant state are those with $l_0 = 0$, and are called spherical. Representations of the principal and supplementary series exhaust all equivalence classes of unitary irreducible representations of G [10]. Representations $(l_0, l_1 = i\rho)$, $\rho \geq 0$, exhaust all equivalence classes of the principal series. Equivalently, we will also use $(l_0, l_1 = i\rho)$, with $l_0 > 0$, $\rho \in \mathbb{R}$ or with $l_0 = 0$, $\rho \geq 0$, as the representants of the principal series. We found the following coordinate system $(\theta_1, \varphi_1, \vartheta_1, \vartheta, \varphi, \lambda)$ on G useful. Let $g \in G$. Let $g_{03}(\lambda) \in G$ be the (matrix representing) hyperbolic rotation along the 03 plane with the hyperbolic angle λ . Let $g_{ik}(\theta)$, for $i, k \in \{1, 2, 3\}$, be the (matrix representing) spatial rotation along the ik plane with the rotation angle θ . Then each $g \in G$ can be uniquely decomposed as follows:

$$\begin{aligned} g &= g_{12}(\theta_1)g_{13}(\varphi_1)g_{12}(\vartheta_1)g_{12}(-\vartheta)g_{13}(-\varphi)g_{03}(\lambda)g_{13}(\varphi)g_{12}(\vartheta) \\ &= a_1(\theta_1, \varphi_1, \vartheta_1)^* a_2(\vartheta, \varphi)^* g_{03}(\lambda) a_2(\vartheta, \varphi) \end{aligned} \quad (3)$$

with the invariant measure on G

$$\begin{aligned} dg &= \frac{1}{8\pi^2} \sin \varphi_1 d\theta_1 d\varphi_1 d\vartheta_1 \pi^2 \sinh^2 \lambda \sin \varphi d\lambda d\vartheta d\varphi \\ &= da_1 \pi^2 \sinh^2 \lambda \sin \varphi d\lambda d\vartheta d\varphi \end{aligned} \quad (4)$$

normalized as in [10], where da_1 is the invariant measure on $SU(2, \mathbb{C})$ normalized to unity. Here,

$$0 \leq \varphi_1, \varphi \leq \pi, \quad 0 \leq \vartheta_1, \vartheta \leq 2\pi, \quad 0 \leq \lambda < \infty.$$

2.1. Definition of A and B . Consistency

Before we continue, let us introduce the basic quantities A, B used in the proof and give a constructive consistency proof of the axioms of [3]. Consistency, for each $\epsilon^2 \geq 0$, has in principle been almost completely shown already in works [3, 7, 8]. We finish the proof initiated there. As remarked in [3], in a concrete Lorentz frame with the time-like unit vector $u = (1, 0, 0, 0)$, the operators $S_0 = S(u), Q, c_\alpha, c_\alpha^+$ can be constructed on the Hilbert space tensor product $\mathcal{H} = L^2(\mathbb{S}^1) \otimes \Gamma(H_{1,0})$ of the Hilbert space $L^2(\mathbb{S}^1, d\phi)$ of square summable functions on the unit circle with the standard invariant measure $d\phi$, with the bosonic Fock space $\Gamma(H_{1,0})$ over the single-particle Hilbert space $H_{1,0}$ of infrared electric-type transversal states, putting, respectively, $S_0 = S'_0 \otimes \mathbf{1}, Q = Q' \otimes \mathbf{1}, c_\alpha = \mathbf{1} \otimes c'_\alpha, c_\alpha^+ = \mathbf{1} \otimes c'^+_\alpha$. Here, $H_{1,0}$ is the Hilbert space of the unitary irreducible representation $(l_0, l_1) = (1, 0)$ of $SL(2, \mathbb{C})$. Q' is the self-adjoint extension of the operator $i\epsilon \frac{d}{d\phi}$ on $L^2(\mathbb{S}^1)$. S'_0 is the operator of multiplication by the (periodic) angle ϕ acting on $L^2(\mathbb{S}^1)$. c'_α, c'^+_α are the standard annihilation-creation operators in the Fock space $\Gamma(H_{1,0})$,

of the single-particle states ξ_α , $\alpha = (l, m)$, which are the normalized eigenstates of the operators $\Delta = M_{23}^2 + M_{31}^2 + M_{12}^2$, M_{12} : $\Delta \xi_{l,m} = l(l+1) \xi_{l,m}$, $M_{12} \xi_{l,m} = m \xi_{l,m}$, $l \in \{l_0 = 1, l_0 + 1 = 2, \dots, -l \leq m \leq l\}$. $|0\rangle = v_0 \otimes \frac{1}{\sqrt{2\pi}} 1_{\mathbb{S}^1}$, where v_0 is the vacuum in $\Gamma(H_{1,0})$ and $\frac{1}{\sqrt{2\pi}} 1_{\mathbb{S}^1} \in L^2(\mathbb{S}^1)$ is the normalized constant function equal to $\frac{1}{\sqrt{2\pi}}$ for each angle in $\mathbb{S}^1$. For a rigorous construction of the Fock space with the creation–annihilation operators, compare *e.g.* [11]. In the construction of [11], we are using the single-particle Gelfand triple $E \subset H_{1,0} \subset E^*$, determined by the standard operator Δ on $H_{1,0}$, and its Fock lifting $(E) \subset \Gamma(H_{1,0}) \subset (E)^*$, determined by the standard operator $\Gamma(\Delta)$ on $\Gamma(H_{1,0})$. For details, we refer to [11]. For the dense nuclear domain $(\mathcal{E})$ of the operators Q, c_α, c_α^+ , we are using the (unique) tensor product of the Hida test space $(E) \subset \Gamma(H_{1,0})$, with the dense nuclear subspace $\mathcal{C}^\infty(\mathbb{S}^1) = \bigcap_{n \in \mathbb{N}} \text{Dom } Q^n \subset L^2(\mathbb{S}^1)$. $S(u)$ being bounded has the whole Hilbert space $\mathcal{H}$ as its domain. By construction, $|0\rangle \in (\mathcal{E})$. The operators $e^{-iS(u)}, Q, c_\alpha, c_\alpha^+$ transform continuously $(\mathcal{E})$ into itself with respect to the nuclear countably normed topology, making $(\mathcal{E})$ a standard countably Hilbert nuclear space [11]. Thus, the existence of the operators $S_0 = S(u), Q, c_\alpha, c_\alpha^+$ fulfilling the commutation rules, with a cyclic vacuum in their Hilbert space, is clear. Let us mention that irreducibility of the algebra generated by $S_0 = S(u), Q, c_\alpha, c_\alpha^+$ is an almost immediate corollary of the irreducibility theorem, *i.e.* Corollary 4.7 of [12], and the construction of $S_0 = S(u), Q, c_\alpha, c_\alpha^+$ given here.

As shown in [7], any representation U of $\text{SL}(2, \mathbb{C})$, giving transformation rule $S(u)' = US(u)U^{-1}, Q' = UQU^{-1} = Q, \dots, c_\alpha'^+ = Uc_\alpha U^{-1}$, of the operators $S(u), Q, c_\alpha, c_\alpha^+$, preserving the commutation rules

$$[Q, S(u)] = i\epsilon, \quad [Q, c_\alpha] = [S(u), c_\alpha] = 0, \quad [c_\alpha, c_\beta^+] = 4\pi\epsilon^2 \delta_{\alpha\beta}, \quad (5)$$

$$c_\alpha|0\rangle = \langle 0|c_\alpha^+ = Q|0\rangle = \langle 0|Q = 0 \quad (6)$$

of the theory [3], necessary has the general form

$$c'_\alpha = \sum_\beta c_\beta A_{\beta\alpha} + B_\alpha Q, \quad S(u)' = S(u) - \frac{1}{4\pi i\epsilon} \sum_{\alpha, \beta} [c_\alpha A_{\alpha\beta} \overline{B_\beta} - c_\alpha^+ \overline{A_{\alpha\beta}} B_\beta], \quad (7)$$

where A is a unitary matrix-valued function, and B vector-valued function on G preserving the conditions

$$(I) \quad A(gh) = A(g)A(h),$$

$$(II) \quad B(gh) = B(g)A(h) + B(h), \quad g, h \in G,$$

$$(III) \quad B(g) = 0, \quad g \in \text{SU}(2, \mathbb{C}) \subset G,$$

with the products understood as the ordinary matrix products of a matrix A with a vector B . Conditions (I)–(II) immediately follow from the assumed representation (or homomorphism) property of U , for (III), compare [8]. (Here, we are using the convention with right multiplication by A in (II), in order to keep the ordinary representation property of A . In the convention used in [7], A is a representation of the group opposite to G .) The second formula in (7) holds for $S(u)' = U(g)S(u)U(g)^{-1}$ with $g \in G$ only if $g \notin \text{SU}(2, \mathbb{C})$. Next, we observe that preservation of the commutation rules by U implies preservation of the orthogonality of the complete system of vectors (1) and of their norms, and thus implies unitarity of U . It was shown in [8] that $A = U^{(l_0=1, l_1=0)}$ is the matrix of the irreducible unitary representation $(l_0 = 1, l_1 = 0)$ of [5]. It remains to determine B . Consistency will be proved if we construct explicitly B on G , which together with A preserves conditions (I)–(III). The equation

$$\frac{d}{d\lambda} B_{l,m}(\lambda = 0) = \epsilon \sqrt{\frac{8}{3}} \delta_{l1} \delta_{m0} \quad (8)$$

for $B(\lambda) = B(g_{03}(\lambda))$ was found in [8], with the initial conditions

$$B_{l,m}(\lambda = 0) = 0, \quad l = 1, 2, \dots, \quad -l \leq m \leq l, \quad (9)$$

and the square

$$\|B(g)\|^2 = \sum_{\alpha} |B(g)_{\alpha}|^2 = 8\epsilon^2(\lambda \coth \lambda - 1) \quad (10)$$

of the norm of B , computed in [3], for hyperbolic rotation g with hyperbolic angle λ . To continue our proof, and for the further part of the paper, we need the formulas ($l, l' \in \{l_0 = 1, l_0 + 1, \dots\}$, $-l \leq m \leq l$, $-l' \leq m' \leq l'$)

$$\begin{aligned} A_{l_m l' m'}(g_{03}(\lambda)) &= A_{l_m l' m'}(\lambda) = U_{l_m l' m'}^{(l_0=1, l_1=0)}(g_{03}(\lambda)) \\ &= \delta_{m, m'} \sqrt{\frac{l(l+1)(2l+1)(l-m)!(2l'+1)(l'-m')!}{l'(l'+1)2(l+m)!2(l'+m')!}} \int_{-1}^1 P_{l,m}(y) P_{l',m'}\left(\frac{\tanh(\lambda)+y}{1+\tanh(\lambda)y}\right) dy \end{aligned}$$

and

$$U_{l_m l' m'}^{(l_0, l_1)}(a) = \delta_{l l'} T_{m m'}^l(a) \quad \text{for } a \in \text{SU}(2, \mathbb{C}). \quad (12)$$

Here,

$$\begin{aligned} T_{m m'}^l(a) &= (-1)^{2l-m-m'} \sqrt{\frac{(l-m)!(l+m)!}{(l-m')!(l+m')!}} \sum_{\alpha=\max\{0, -m-m'\}}^{\min\{l-m, l-m'\}} \\ &\quad \times \left[\binom{l-m'}{\alpha} \binom{l+m'}{l-m-\alpha} (a_{11})^{\alpha} (a_{12})^{l-m-\alpha} (a_{21})^{l-m'-\alpha} (a_{22})^{m+m'+\alpha} \right], \end{aligned}$$

for each fixed l , are the standard matrices of irreducible representations of weight l of $SU(2, \mathbb{C})$, [5, 10]. Formula (11) uses the realization $\alpha \mapsto f(\Lambda(\alpha)^{-1}p)$ of the representation $(l_0 = 1, l_1 = 0)$ in the space of classes modulo constant, homogeneous of degree zero functions f , on the cone $p \cdot p = 0, p_0 > 0$, thus living effectively on the unit $\mathbb{S}^2$ sphere in the cone, with the standard homomorphism $\alpha \mapsto \Lambda(\alpha)$ into the Lorentz group acting in $\mathbb{R}^4$, with the orthonormal basis given by the representants $\frac{1}{\sqrt{l(l+1)}}Y_{lm}$, $l = 1, 2, \dots, -l \leq m \leq l$, and with the Lorentz invariant inner product

$$(f, g) = - \int_{\mathbb{S}^2} \bar{f} \Delta_{\mathbb{S}^2} g \, d\mu_{\mathbb{S}^2}$$

with an ordinary Laplace operator and $SU(2, \mathbb{C})$ -invariant measure on $\mathbb{S}^2$. A coincides with the matrix of $(l_0 = 1, l_1 = 0)$, which can be checked by comparing with the matrix of $(l_0 = 1, l_1 = 0)$, given in [5, 10]. In the sequel, we are using the orthogonality properties of the matrix elements $T_{m' m}^l$ regarded as functions on $SU(2, \mathbb{C})$ (Peter–Weyl theorem). Integration of (II) along the one-parameter subgroup $g_{03}(\lambda)$ of hyperbolic rotations, using (8) and the initial condition (9) gives B along this subgroup. *I.e.*, writing $B(g_{03}(\lambda)) = B(\lambda)$

$$B_{l,m}(\lambda' + \lambda) = \sum_{l', m'} B_{l', m'}(\lambda') A_{l', m' \quad l, m}(\lambda) + B_{l, m}(\lambda).$$

Differentiating $\partial_{\lambda'}$ at $\lambda' = 0$ and using (8) we obtain

$$\frac{dB_{l,m}(\lambda)}{d\lambda} = i\epsilon \sqrt{\frac{8}{3}} A_{1,0 \quad l, m}(\lambda),$$

which, together with the initial conditions (9), determines $B(\lambda)$ uniquely. Similarly, using (III), we have the trivial zero for B along any one-parameter subgroup of unitary elements of G . Using decomposition (3), we immediately see that conditions (I)–(III) imply the value of B at a general element (3) to be equal to

$$B_{l,m}(g) = B_{l,0}(\lambda) A_{l,0 \quad l, m} \left(g_{13}(\varphi) g_{12}(\vartheta) \right) = (B(\lambda) A(a_2))_{lm}, \quad (13)$$

$$\begin{aligned} B_{l,m}(g^{-1}) &= (-1)^l B_{l,0}(\lambda) A_{l,0 \quad l, m} (g_{13}(\varphi) g_{12}(\vartheta) g_{12}(-\vartheta_1) g_{13}(-\varphi_1) g_{12}(-\theta_1)) \\ &= (B(-\lambda) A(a_2 a_1^*))_{lm}, \end{aligned} \quad (14)$$

$$B_{l,m}(g_{03}(\lambda)) = B_{l,m}(\lambda) = (-1)^l B_{l,m}(-\lambda) = i\epsilon \sqrt{\frac{8}{3}} \int_0^\lambda A_{1,0 \quad l, m}(\lambda') d\lambda'. \quad (15)$$

Note that $B_{l,m}(\lambda) = 0$ for $m \neq 0$, by the property

$$A_{l',0 \ l,m}(\lambda) = \delta_{m0} A_{l',0 \ l,m}(\lambda)$$

of the matrix $A(\lambda)$. Here, $P_{l,m}$ are the associated Legendre “polynomials”

$$P_{l,m}(y) = (-1)^m 2^l (1 - y^2)^{m/2} \sum_{k=m}^l \frac{k!}{(k-m)!} \binom{l}{k} \binom{l+k-1}{l} y^{k-m}$$

with the generalized binomial symbol

$$\binom{w}{k} = \frac{w(w-1)(w-2)\dots(w-k+1)}{k(k-1)\dots 1}, \quad w \in \mathbb{R}, k \in \mathbb{N}.$$

To finish the proof, we have to show that A, B given by formulas (11), (13), (15) respect conditions (I)–(III). (I) and (III) are trivially fulfilled. In order to show (II), we note the parity property $A_{l,0l',0}(-\lambda) = (-1)^{l+l'} A_{l,0l',0}(\lambda)$ of the matrix $A(\lambda)$ for the hyperbolic rotations parallel to the 03 hyperplane. This parity property can be easily seen by the exponentiation of the generator M_{03} of the representation ($l_0 = 1, l_1 = 0$) given explicitly in [5, 10], and which immediately gives the series expansion at $\lambda = 0$

$$A_{l',0 \ l,0}(\lambda) = a_{l',l,0} \lambda^{|l-l'|} + a_{l',l,2} \lambda^{|l-l'|+2} + a_{l',l,4} \lambda^{|l-l'|+4} + \dots, \quad (16)$$

$$a_{l',l,0} = \frac{(-1)^{l-l'}}{(l-l')!} \prod_{j=0}^{l-l'-1} \left[(l-j) \sqrt{\frac{(l-j)^2-1}{4(l-j)^2-1}} \right], \quad l' < l, \quad (17)$$

$$a_{l,l',r} = (-1)^{l+l'} a_{l,l',r}.$$

The explicit value of the coefficients (17), will be used only in the further part of the paper. Using the parity property of $A(\lambda)$ and representation property of A , we can easily see that for any $g \in G$ of the form (3) and for $B(g)$ given by (13) and (15),

$$B(g)A(g^{-1}) = -B(-\lambda)A(a_2 a_1^*) = -B(g^{-1}). \quad (18)$$

From (18), it follows that condition (II) is fulfilled with h of the form $g^{-1}k$, for any $g, k \in G$, and for $B(g)$ defined by (13), (15), because the right-hand side of (II) for $h = g^{-1}k$ is equal to

$$B(g)A(g^{-1}k) + B(g^{-1})A(k) + B(k) = B(k)$$

by (18), and thus equal to the left-hand side of (II). Putting $k = gh$, we get (II) for all $g, h \in G$, which proves the consistency of the theory. For another consistency proof, compare [2, 13].

From the general formulas (11), (15), it follows formula (10) and

$$\langle u|u' \rangle = \langle u|gu \rangle = \langle 0|e^{inS(u)}e^{-inS(gu)}|0 \rangle = e^{-z(\lambda \coth \lambda - 1)}, \quad (19)$$

for general g of the form (3). Of course, the consistency of the theory, implying unitarity of U , implies also positive definiteness of the kernel $\langle g|h \rangle$ associated to the cyclic representation associated with any cyclic vector, in particular with the cyclic vector of the form (1).

2.2. Explicit formulas for A, B

For further purposes, we give explicit forms of $B_{l,0}(\lambda)$ and $A_{l,m,l',m}(\lambda)$. In what follows, we are using the convention that the sum $\Sigma \dots$ is zero whenever the upper summation limit is less than the lower summation limit, and analogously for the product $\Pi(\dots)$, which by definition is equal to 1, whenever the upper product limit is less than the lower product limit. The integral (11) representing $A_{l,0,l,0}(\lambda)$ can be explicitly computed in terms of the functions $Q_l(-1/t)$, $t = \tanh \lambda$, closely related to the Legendre functions of the second kind. Namely, we use the identity

$$\int_{-1}^1 \frac{1}{1+tx} P_l(x) dx = -\frac{2}{t} Q_l\left(-\frac{1}{t}\right), \quad -1 < t < 1, \quad (20)$$

where

$$Q_l(x) = P_l(x) \frac{1}{2} \log \frac{x+1}{x-1} - \sum_{k=1}^l \frac{1}{k} P_{k-1}(x) P_{l-k}(x) = P_l(x) \frac{1}{2} \log \frac{x+1}{x-1} - W_{l-1}(x),$$

which can be proved using Bonnet's recursion formula for the Legendre polynomials $P_l = P_{l,0}$. Q_l are closely related to the Legendre functions of the second kind¹ which can be obtained from Q_l by replacing the argument of $\log$ with $(x+1)/(1-x)$ in Q_l [14], 8.831. From Bonnet's recursion formula

$$(2l+1)xP_l(x) - (l+1)P_{l+1}(x) - lP_{l-1}(x) = 0$$

for P_l , it follows that Q_l and W_{l-1} respect the same Bonnet's recurrence [15]

¹ We should note that there are two versions of the Legendre functions of the second kind which are in use. In fact, our Q_l constitute one of these versions and are called *Legendre functions of degree l of the second kind*, e.g., in [15], p. 316, and are also named Legendre functions of the second kind in [16], cited below by us, and denoted there by Q_l . Unfortunately, these versions are sometimes mixed, e.g. the formula 8.825 of [14] is in agreement with our Q_l , and with that used in [15, 16], but the formula 8.827 of [14] gives the other version of the Legendre functions of the second kind, with the other form of the argument in $\log$.

$$\begin{aligned}(2l+1)xQ_l(x) - (l+1)Q_{l+1}(x) - lQ_{l-1}(x) &= 0, \\ (2l+1)xW_{l-1}(x) - (l+1)W_l(x) - lW_{l-2}(x) &= 0.\end{aligned}$$

Let us compute $B_{l,0}(\lambda)$ explicitly. Applying (20) and Bonnet's recursion for P_l to (11), and then Bonnet's recursion for Q_l , we get

$$\begin{aligned}A_{l,0 \ 1,0}(\lambda) &= c[l]2 \left[-Q_l \left(-\frac{1}{t}\right) + \frac{(l+1)}{2l+1} \left(-\frac{1}{t}\right) Q_{l+1} \left(-\frac{1}{t}\right) + \frac{l}{2l+1} \left(-\frac{1}{t}\right) Q_{l-1} \left(-\frac{1}{t}\right) \right] \\ &= -2c[l] \left(1 - \frac{1}{t^2}\right) Q_l \left(-\frac{1}{t}\right) = -(-1)^{l+1} 2c[l] \left(1 - \frac{1}{t^2}\right) Q_l \left(\frac{1}{t}\right),\end{aligned}\quad (21)$$

where $c[l] = \sqrt{\frac{3}{8}l(l+1)(2l+1)}$. Performing the integration (15) with $A_{1,0l,0}(\lambda) = (-1)^{l+1}A_{l,0 \ 1,0}(\lambda)$, we obtain

$$\begin{aligned}B_{l,0}(\lambda) &= i\epsilon\sqrt{\frac{8}{3}}(-1)^{l+1} \int_0^\lambda A_{l,0 \ 1,0}(\lambda') d\lambda' = I(\lambda) + b[l], \\ I(\lambda) &= i\epsilon\sqrt{\frac{8}{3}}(-1)^{l+1} \int A_{l,0 \ 1,0}(\lambda) d\lambda.\end{aligned}\quad (22)$$

Comparing (21) and (22), we can see that the computation of the indefinite integral $I(\lambda)$ is reduced to the computation of the indefinite integrals $I[k, \lambda] = \int t^{-k} d\lambda$ and $J[k, \lambda] = \int t^{-k} \lambda d\lambda$. Then we compute the limit of B at infinity, which determines the integration constant $b[l]$. We use the variable $u = e^{-2\lambda}$ in all the indefinite integrals. The last two indefinite integrals can be rewritten as $I[k, \lambda] = -\frac{1}{2}\mathbb{F}_k(u)$, $J[k, \lambda] = \frac{1}{4}\mathbb{G}_k(u)$, where $u = e^{-2\lambda}$ and

$$\mathbb{F}_k(u) = \int \frac{(1+u)^k}{(1-u)^{k_u}} du, \quad \mathbb{G}_k(u) = \int \frac{(1+u)^k}{(1-u)^{k_u}} \log u du.$$

Using the following identities

$$\frac{u^j}{(1-u)^k} = \sum_{s=0}^j \frac{\binom{j}{s}(-1)^s}{(1-u)^{k-s}}, \quad j < k, \quad \frac{1}{(1-u)^{k_u}} = \sum_{s=1}^k \frac{1}{(1-u)^s} + \frac{1}{u},$$

and integration by parts in case $\mathbb{G}_k(u)$, we compute the indefinite integrals

$$\begin{aligned}
 \mathbb{F}_k(u) &= \int \frac{(1+u)^k}{(1-u)^k u} du = \sum_{s=2}^k \frac{1}{(s-1)} \frac{1}{(1-u)^{s-1}} \\
 &+ \sum_{j=1}^{k-1} \sum_{s=0}^{j-1} \binom{k}{j} \binom{j-1}{s} \frac{(-1)^s}{(k-s-1)} \frac{1}{(1-u)^{k-s-1}} \\
 &+ \sum_{s=0}^{k-2} \binom{k-1}{s} \frac{(-1)^s}{(k-s-1)} \frac{1}{(1-u)^{k-s-1}} + ((-1) - (-1)^{k-1}) \log(1-u) + \log u \\
 &= F[k, u] + ((-1) - (-1)^{k-1}) \log(1-u) + \log u,
 \end{aligned}$$

$$\begin{aligned}
 \mathbb{G}_k(u) &= \int \frac{(1+u)^k}{(1-u)^k u} \log u du = F[k, u] \log u \\
 &- \sum_{s=2}^k \sum_{r=2}^{s-1} \frac{1}{(s-1)(r-1)} \frac{1}{(1-u)^{r-1}} - \sum_{j=1}^{k-1} \sum_{s=0}^{j-1} \binom{k}{j} \binom{j-1}{s} \frac{(-1)^s}{(k-s-1)} \sum_{r=2}^{k-s-1} \frac{1}{(r-1)} \frac{1}{(1-u)^{r-1}} \\
 &- \sum_{s=0}^{k-2} \binom{k-1}{s} \frac{(-1)^s}{(k-s-1)} \sum_{r=2}^{k-s-1} \frac{1}{(r-1)} \frac{1}{(1-u)^{r-1}} \\
 &+ ((-1) - (-1)^{k-1}) \log(1-u) \log u + \frac{1}{2} \log^2 u \\
 &- F[k] (\log(1-u) - \log u) + ((-1) - (-1)^{k-1}) \text{Li}_2 u \\
 &= G[k, u] + ((-1) - (-1)^{k-1}) \log(1-u) \log u + \frac{1}{2} \log^2 u \\
 &- F[k] (\log(1-u) - \log u) + ((-1) - (-1)^{k-1}) \text{Li}_2 u,
 \end{aligned}$$

where $F[k] = -F[k, u=0]$.

Let us introduce the coefficients $p_{l,k}$ and $w_{l-1,k}$ of the polynomials

$$P_l(x) = P_{l,0}(x) = \sum_{k=0}^l p_{l,k} x^k, \quad W_{l-1}(x) = \sum_{k=0}^{l-1} w_{l-1,k} x^k.$$

We have the following identity:

$$\sum_{k=0}^l p_{l,k} (F[k] - F[k+2]) = 0. \quad (23)$$

To prove (23), we notice that $F[k]$, defined as above, can be simplified to the following form:

$$F[k] = -2 \left(1 + (-1)^k \right) \sum_{j=1}^{\lfloor k/2 \rfloor} \frac{1}{2j-1}. \quad (24)$$

To prove (24) we apply to $F[k]$ the following standard identities. First, the standard identity

$$\sum_{s=1}^{k-1} \binom{k-1}{s} \frac{(-1)^s}{s} = - \sum_{s=1}^{k-1} \frac{1}{s}$$

coming from the Stern series for the digamma function and the relation of digamma to the harmonic numbers, we apply to the last sum in $F[k]$, converting it into the following form:

$$\sum_{s=0}^{k-2} \binom{k-1}{s} \frac{(-1)^s}{(k-s-1)} = (-1)^k \sum_{s=1}^{k-1} \frac{1}{s}.$$

Next, the standard identity

$$\sum_{s=0}^{j-1} \binom{j-1}{s} \frac{(-1)^s}{k-s-1} = \frac{(-1)^{j+1}}{j \binom{k-1}{j}},$$

coming from the fact that the l.h.s. is equal to value $-B(j, -k+1)$ of the Euler beta function B , we apply to the first summation of the double summation contribution to $F[k]$, converting it into the following form:

$$\sum_{j=1}^{k-1} \sum_{s=0}^{j-1} \binom{k}{j} \binom{j-1}{s} \frac{(-1)^s}{k-s-1} = \sum_{j=1}^{k-1} \frac{k(-1)^{j+1}}{(k-j)j} = - \left(1 + (-1)^k\right) \sum_{j=1}^{k-1} \frac{(-1)^j}{j}.$$

The identity (23) and the fact that the polynomials $(1-x^2)P_l(x)$, $(1-x^2)W_{l-1}(x)$ are zero at $x = \pm 1$ imply that the total contribution to the linear combination $I(\lambda)$ of the indefinite integrals, $I[k, \lambda] = \mathbb{F}_k(u)$ and $J[k, \lambda] = \mathbb{G}_k(u)$ comes only from the parts $F[k, u]$ and $G[k, u]$ of the indefinite integrals $\mathbb{F}_k(u)$ and $\mathbb{G}_k(u)$, and the remaining terms of the indefinite integrals $\mathbb{F}_k(u)$, $\mathbb{G}_k(u)$, not included in $F[k, u]$ or, respectively, in $G[k, u]$, drop out. Therefore, in our computation we may replace the indefinite integrals $\mathbb{F}_k(u)$, $\mathbb{G}_k(u)$, with $F[k, u]$, $G[k, u]$. We thus obtain $I(\lambda)$ in terms of $F[k, u]$, $G[k, u]$ and the coefficients of the polynomials P_l , W_{l-1} entering Q_l in (21)

$$\begin{aligned} B_{l,0}(\lambda) = & -\frac{ie\sqrt{l(l+1)(2l+1)}}{2} \left[\sum_{k=0}^l p_{l,k} (G[k, u] - G[k+2, u]) \right. \\ & \left. + 2 \sum_{k=0}^{l-1} w_{l-1,k} (F[k, u] - F[k+2, u]) + b'[l] \right], \quad u = e^{-2\lambda}. \end{aligned} \quad (25)$$

To fix $b'[l]$, we compute the limit

$$\lim_{\lambda \rightarrow +\infty} B_{l,0}(\lambda) = i\epsilon \sqrt{\frac{8}{3}} (-1)^{l+1} \int_0^{\infty} A_{l,0-1,0}(\lambda) d\lambda.$$

Inserting the last form of the integrand (21) and using the new variable x

$$\lambda = -\frac{1}{2} \log \frac{x+1}{x-1},$$

we get

$$\lim_{\lambda \rightarrow +\infty} B_{l,0}(\lambda) = i\epsilon 2c[l] \sqrt{\frac{8}{3}} (-1)^{l+1} \int_{-\infty}^{-1} Q_l(x) dx = i\epsilon 2 \sqrt{\frac{2l+1}{l(l+1)}},$$

where we have used the identity

$$\int_{-\infty}^{-1} Q_l(x) dx = \frac{(-1)^{l+1}}{l(l+1)},$$

immediately following from formula (18), p. 324, of [16], and the parity property $Q_l(-x) = (-1)^{l+1} Q_l(x)$.

In our proof, we use the fact that the limit of $B(\lambda)$ at $\lambda = +\infty$ is nonzero, but we do not need the explicit form of the constant $b[l]$ or $b'[l]$. But, concerning $b'[l]$, let us note the following. We have the following identities:

$$\sum_{k=0}^{l-1} w_{l,k} (F[k] - F[k+2]) = \frac{-4((-1)^l - 1)}{l(l+1)} \quad (26)$$

and

$$\sum_{k=0}^l p_{l,k} (\mathcal{G}[k] - \mathcal{G}[k+2]) = \frac{-4((-1)^l - 1)}{l(l+1)}, \quad (27)$$

with $\mathcal{G}[k] = \mathcal{G}[k, u = 0]$, where $\mathcal{G}[k, u] = G[k, u] - F[k, u] \log u$ is that part of $G[k, u]$ which does not contain the $\log u$ -factor. From (23), it follows at once that the function

$$\sum_{k=0}^l p_{l,k} (F[k, u] - F[k+2, u])$$

of u is of the order greater than zero at $u = 0$ in the variable u . From this and by the L'Hôpital rule, the limit $u \rightarrow 0$ of

$$\sum_{k=0}^l p_{l,k} (F[k, u] - F[k+2, u]) \log u$$

is zero and, thus, the limit $u \rightarrow 0$ of the first sum in (25) is equal to the l.h.s. of (27). It is immediately seen that the limit $u \rightarrow 0$ of the second sum in (25) is equal to the l.h.s. of (26). Comparing with the limit of $B(\lambda)$ at $\lambda = +\infty$, obtained above, we finally get

$$B_{l,0}(\lambda) = -\frac{i\epsilon\sqrt{l(l+1)(2l+1)}}{2} \left[\sum_{k=0}^l p_{l,k} (G[k, u] - G[k+2, u]) + 2 \sum_{k=0}^{l-1} w_{l-1,k} (F[k, u] - F[k+2, u]) - \frac{4(-1)^l}{l(l+1)} \right], \quad u = e^{-2\lambda}. \quad (28)$$

Formula (28) can be rewritten in the following form

$$B_{2k+1,0}(\lambda) = \frac{1}{\sinh^{2k+2}\lambda} \left[\lambda \sum_{j=0}^k \mathfrak{b}_j \cosh(2j\lambda) + \sum_{j=1}^{k+1} \mathfrak{b}_{k+j} \sinh(2j\lambda) \right], \quad (29)$$

$$B_{2k,0}(\lambda) = \frac{1}{\sinh^{2k+1}\lambda} \left[\lambda \sum_{j=0}^{k-1} \mathfrak{b}'_j \cosh^{2j+1}\lambda + \sum_{j=0}^k \mathfrak{b}'_{k+j} \sinh^{2j+1}\lambda \right] \quad (30)$$

with nonzero constants $(\mathfrak{b}_0, \dots, \mathfrak{b}_{2k+1})$ and $(\mathfrak{b}'_0, \dots, \mathfrak{b}'_{2k})$. We have the expansion

$$\frac{n}{4\pi\epsilon} B_{l,0}(-\lambda) = b_{l,0} \lambda^l + b_{l,2} \lambda^{l+2} + b_{l,4} \lambda^{l+4} + \dots, \quad (31)$$

$$b_{l,0} = i \frac{n}{4\pi} \sqrt{\frac{8}{3}} \frac{(-1)^l}{l} a_{1, l, 0}, \quad (32)$$

following from (15), (16), and (17). Thus, $B_{l,0}(\lambda)$ is a function of the order l in λ at $\lambda = 0$ and tends to a nonzero constant $2^l \mathfrak{b}_l = 2^{2k+1} \mathfrak{b}_{2k+1}$ or, respectively, $\mathfrak{b}'_l = \mathfrak{b}'_{2k}$, at infinity

$$\lim_{\lambda \rightarrow +\infty} B_{l,0}(\lambda) = i\epsilon 2 \sqrt{\frac{2l+1}{l(l+1)}} = \begin{cases} 2^l \mathfrak{b}_l, & \text{if } l \text{ is odd} \\ \mathfrak{b}'_l, & \text{if } l \text{ is even} \end{cases}. \quad (33)$$

Application of the identities

$$P_{l,m}(y) = (-1)^{m/2} (1-y^2)^{\frac{d^m P_l(y)}{dy^m}}, \quad y^n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(2n-4k+1)n!}{2^k k! (2n-2k+1)!!} P_{n-2k}(y) \quad (34)$$

to the integrand of (11), and formula (20), gives

$$A_{l,m,l',m}(\lambda) = \frac{1}{\sinh^{l+l'+1}\lambda} \left[\sum_{\text{odd } j=0}^{l+l'-||m|-1|} \alpha_j \sinh^j \lambda \cosh^{l+l'-||m|-1|-j} \lambda \right. \\ \left. + \lambda \sum_{\text{even } j=0}^{l+l'-||m|-1} \alpha_j \sinh^j \lambda \cosh^{l+l'-||m|-1|-j} \lambda \right], \quad (35)$$

if $l + l' - ||m| - 1|$ is odd, and the upper limits in the sums are exchanged with each other, if $l + l' - ||m| - 1|$ is even. Thus, we have the asymptotic $A_{l,m,l',m}(\lambda) \sim e^{-(||m|-1|+1)\lambda}$ at infinity. The constants $\alpha_0, \dots, \alpha_{l+l'-||m|-1|}$ are nonzero. From (11), it follows that the matrix elements of $A(\lambda)$ are real, and $B(\lambda)$ pure imaginary.

In what follows, regarding the constants α_j , we use only the fact that the constant α_0 is nonzero. To see it, let $m \geq 0$ in (11). Note that by performing the integration (11), we get $(1 - t^2)^{m/2} [W(1/t)\lambda + R(1/t)]$, $t = \tanh \lambda$, with some polynomials W, R . The constant α_0 is the highest-degree coefficient of the polynomial W , which is of degree $l + l' + 1$. To show $\alpha_0 \neq 0$, it is sufficient to compute only the contribution proportional to $(1 - t^2)^{m/2} \lambda / t^{l+l'+1}$, and show that it is nonzero. Denoting the constant in front of the integral (11) by $c[l, l', m]$, and using the first formula of (34), we see that (11) can be rewritten as

$$A_{l,m,l',m}(\lambda) = (1 - t^2)^{m/2} c[l, l', m] \int_{-1}^1 \left(\frac{1-y^2}{1+ty} \right)^m P_{l'}^{(m)} \left(\frac{y+t}{1+ty} \right) P_l^{(m)}(y) dy \\ = (1 - t^2)^{m/2} c[l, l', m] \int_{-1}^1 \frac{P(y)}{(1+ty)^{l'}} dy, \quad (36)$$

where $P(y) = p_{l+l'} y^{l+l'} + p_{l+l'-1} y^{l+l'-1} + \dots$ is a polynomial of degree $l + l'$, with the coefficients p_k being polynomials in t , and with the highest degree coefficient equal

$$p_{l+l'} = (-1)^m p_{l,l-m}^{(m)} \sum_{j=0}^{l'-m} p_{l',j}^{(m)} t^{l'-m-j},$$

where $p_{l',j}^{(m)}$ are the coefficients of $P_{l'}^{(m)}$

$$P_{l'}^{(m)}(t) = \sum_{j=0}^{l'-m} p_{l',j}^{(m)} t^j.$$

P can be represented as a linear combination of the Legendre polynomials,

$$P(y) = \frac{p_{l+l'}}{p_{l+l', l+l'}} P_{l+l'}(y) + \dots, \quad (37)$$

where dots denote the linear combination of the Legendre polynomials of degree $< l + l'$. Repeated differentiation of (20) with respect to t gives

$$\int_{-1}^1 \frac{1}{(1+tx)^k} P_l(x) dx = \frac{1}{(k-1)!} \frac{1}{t^k} \left[t^2 \frac{d}{dt} \right]^{k-1} \left(-2Q_l \left(-\frac{1}{t} \right) \right). \quad (38)$$

The operator $t^2 \frac{d}{dt}$, acting on a polynomial function of the variable $-1/t$ of degree d , gives again a polynomial of $-1/t$ of degree $d - 1$. Thus, applying (38) to (36), we see that in order to obtain the contribution proportional to $(1 - t^2)^{m/2} \lambda / t^{l+l'+1}$ with the highest possible degree $(1/t)^{l+l'+1}$, we can discard not only the contributions to P denoted by dots in (37), but also all contributions to the polynomial $p_{l+l'}$ of t which are of the positive order in t , and compute only

$$A_{l, m \ l', m}(\lambda) = (1 - t^2)^{m/2} c[l, l', m] \int_{-1}^1 \frac{(-1)^m p_{l, l-m}^{(m)} p_{l', l'-m}^{(m)} P_{l+l'}(y) + \dots}{(1+ty)^{l'} p_{l+l', l+l'}} dy,$$

discarding remaining terms, denoted by dots. Applying (38) we obtain

$$\begin{aligned} A_{l, m \ l', m}(\lambda) &= \frac{2c[l, l', m] (-1)^{l+m+1} p_{l, l-m}^{(m)} p_{l', l'-m}^{(m)} p_{l+l', l+l}^{(l'-1)}}{(l'-1)! p_{l+l', l+l'}} \frac{(1-t^2)^{m/2} \lambda}{t^{l+l'+1}} \\ &\quad + (1 - t^2)^{m/2} K(1/t) \lambda + (1 - t^2)^{m/2} R(1/t), \end{aligned}$$

where K is a polynomial of degree $< l + l' + 1$, and R is a polynomial. Therefore,

$$\mathfrak{a}_0 = \frac{2c[l, l', m] (-1)^{l+m+1} p_{l, l-m}^{(m)} p_{l', l'-m}^{(m)} p_{l+l', l+l}^{(l'-1)}}{(l'-1)! p_{l+l', l+l'}} \neq 0.$$

We obtain the same $\mathfrak{a}_0$ for $A_{l, -m \ l', -m}(\lambda) = A_{l, m \ l', m}(\lambda)$.

Taking into account $\mathfrak{a}_0 \neq 0$, we observe the following. Because the components of $A(\lambda), B(\lambda)$ are analytic also at $\lambda = 0$ and we have denominators that have zero of positive order at $\lambda = 0$ in (35), (29), and (30), then it is easily seen that the sums, say $F_0(\lambda)$, in the numerators in (35), (29), (30), which are not multiplied by λ , must be equal to zero at $\lambda = 0$, and the sums, say $F_1(\lambda)$, in the numerators, which are multiplied by λ , must be nonzero

at $\lambda = 0$, because the lower order terms in these numerators must drop out, including the first-order term. Indeed, if we rewrite (28) (resp. (29), (30)) or (35) in the form

$$2^q \frac{f_0(\lambda) + \lambda f_1(\lambda)}{(1 - e^{-2\lambda})^q} = 2^q \frac{\sum_{j_0} a_{0,j_0} e^{-j_0\lambda} + \lambda \sum_{j_1} a_{1,j_1} e^{-j_1\lambda}}{(1 - e^{-2\lambda})^q}, \quad (39)$$

with $f_i(\lambda) = e^{-q\lambda} F_i(\lambda)$, $i = 0, 1$, denoting the corresponding sums, and with a finite linear combination of exponents $e^{j_p\lambda}$, $j_p \in \mathbb{N}$, in the numerator, then

$$f_0(0) = \sum_{j_0} a_{0,j_0} = 0, \quad (40)$$

and

$$\begin{aligned} f_1(0) &= \sum_{j=0}^k \mathfrak{b}_j = \sum_{j_1} a_{1,j_1} \neq 0, \quad \text{or} \quad f_1(0) = \sum_{j=0}^{k-1} \mathfrak{b}'_j = \sum_{j_0} a_{1,j_1} \neq 0, \\ f_1(0) &= \mathfrak{a}_0 = \sum_{j_1} a_{0,j_1} \neq 0, \end{aligned} \quad (41)$$

respectively, for B and A . $q = l + 1, l + l' + 1$, respectively, for B, A . (40) is obvious. (41) is obvious for A , because $\mathfrak{a}_0 \neq 0$. For B , the statement (41) can be seen by using (41) for $A_{l,0 \ 1,0}(\lambda)$, and the fact that $A_{l,0 \ 1,0}(\lambda)$ is, up to a nonzero factor, equal to the derivative of $B_{l,0}(\lambda)$ with respect to λ . Moreover, using this relation between A and B , we can easily see that

$$\begin{aligned} \sum_{j=0}^{(l-1)/2} \mathfrak{b}_j \quad \text{or, respectively,} \quad \sum_{j=0}^{l/2-1} \mathfrak{b}'_j &= (-1)^l i \epsilon \sqrt{\frac{8}{3}} \frac{1}{l+1} \mathfrak{a}_0 \\ &= -i \epsilon 2^{l+2} \binom{l+1/2}{l+1} \frac{l(l+1)}{2l+1}, \end{aligned} \quad (42)$$

where the coefficient $\mathfrak{a}_0$ refers to the matrix element $A_{l,0 \ 1,0}(\lambda)$. Because the first-order terms drop out in the numerator of (39), then from (40) and (41) we get

$$f_0^{(1)}(0) = - \sum_{j_0} a_{0,j_0} j_0 = -f_1(0), \quad (43)$$

so that f_1 is a zero-order function and f_0 is a first-order function at $\lambda = 0$, with $f_1(0) \neq 0, f_0^{(1)}(0) \neq 0$.

It is easily seen that the natural numbers j_0, j_1 in formula (39) representing $B_{l,0}(\lambda)$ are always even, independently of the parity of l , and have the parity the same as m , in formula (39) representing $A_{l,m \ l',m}(\lambda)$. We will use these simple facts in Subsection 4.1.

In order to prove our theorem, in addition to the above-mentioned properties of A, B , we need to know explicitly the coefficient

$$\mathfrak{a}_0 = (-1)^l l! l' 2^{l+2l'} \frac{\sqrt{(2l+1)(2l'+1)}}{(l+1)!} \binom{l+l'-1/2}{l+l'} \binom{l'-1/2}{l'} \prod_{j=0}^{l'-1} \frac{l+j+1}{2(l+j)+1} \prod_{r=0}^{l'-2} (l+l'-r) \quad (44)$$

for the matrix element $A_{l, -1 \ l', -1}(\lambda)$.

2.3. Explicit formulas for $U_{l, m \ l', m'}^{(l_0, l_1)}(g_{03}(\lambda))$

In what follows, we will use the formulas (let us recall that $l, l' \in \{l_0, l_0 + 1, \dots\}$, $-l \leq m \leq l$, $-l' \leq m' \leq l'$)

$$\begin{aligned} U_{l, m \ l', m'}^{(l_0, l_1)}(g_{03}(\lambda)) &= U_{l, m \ l', m'}^{(l_0, l_1=i\rho)}(\lambda) = \delta_{m m'} \frac{1}{2} \sqrt{(2l'+1)(2l+1)} (\cosh \lambda)^{i\rho-1} \\ &\times \int_{-1}^1 (1-ty)^{i\rho-1} \overline{P_{l_0 m'}^{l'}(y)} P_{l_0 m}^l \left(\frac{y-t}{1-ty} \right) dy. \end{aligned} \quad (45)$$

Here, $t = \tanh \lambda$ and $P_{m n}^l(\cos \varphi) = e^{im\phi} T_{m n}^l(a) e^{in\vartheta}$ i.e., $P_{m n}^l(\cos \varphi)$ is given by the above formula for $T_{m n}^l(a)$ in which we substitute $\cos(\varphi/2)$ for a_{11} and a_{22} and $i \sin(\varphi/2)$ for a_{12} and a_{21} . $P_{m n}^l$ are closely related to the Jacobi polynomials. These formulas are obtained by changing the coordinate system on G in the corresponding formulas for $U^{(l_0, l_1)}$ given in [10]. We need an explicit form of the matrix elements (45) with the integer l_0 . Performing integration (45) by parts and using recurrence rules for the Jacobi polynomials, we get by induction

$$\begin{aligned} &U_{l, m \ l', m}^{(l_0, l_1=i\rho)}(\lambda) \\ &= \frac{1}{\rho Q(\rho) \sinh^{l+l'+1} \lambda} \left[e^{-i\rho\lambda} \sum_{j=0}^{l+l'-||m|-|l_0||} p_j^- \sinh^j \lambda \cosh^{l+l'-||m|-|l_0||-j} \lambda \right. \\ &\quad \left. + e^{i\rho\lambda} \sum_{j=0}^{l+l'-||m|-|l_0||} p_j^+ \sinh^j \lambda \cosh^{l+l'-||m|-|l_0||-j} \lambda \right], \end{aligned} \quad (46)$$

for non-negative m . In the formula for non-positive m , the upper summation ranges are exchanged with each other. The matrix elements (46) with $l = l'$ and opposite m are mutually complex conjugated. Here, p_j^-, p_j^+ are polynomials in ρ of degree j and parity coinciding with the parity of j , depending

on l_0, l, l', m . Thus, $U_{l,m,l',m}^{(l_0,l_1=i\rho)}(\lambda) \sim e^{-(||m|-|l_0||+1)\lambda}$ at infinity. Q is a polynomial depending only on l, l' . We use Q so normalized that the highest-degree coefficient in Q is to equal one, *i.e.*, our Q in (46) is monic

$$Q(\rho) = (-i + \rho)(i + \rho)(-2i + \rho)(2i + \rho)(-3i + \rho) \dots$$

with the number of factors equal to $l + l'$, and with the numbers $-i, i, -2i, 2i, \dots$ in them increasing in absolute value but alternating in sign, except for the last $|l' - l|$ factors, in which these numbers continue to increase in absolute value, but with the constant sign $+$, when $l > l'$, or with the constant sign $-$, when $l < l'$. For example for $l = 2, l' = 5$ and any $-2 \leq m \leq 2$, and any integer l_0

$$Q(\rho) = (-i + \rho)(i + \rho)(-2i + \rho)(2i + \rho)(-3i + \rho)(-4i + \rho)(-5i + \rho) \dots$$

With this normalization of Q , we can fix the coefficients of the polynomials p_j^-, p_j^+ . We need the highest-degree coefficient $p_{j,j}^\pm$ of the polynomials

$$p_j^\pm(\rho) = p_{j,0}^\pm + p_{j,1}^\pm \rho + \dots p_{j,j}^\pm \rho^j$$

with maximal

$$j = j_{\max} = l + l' - ||m| - |l_0||$$

for the matrix elements $U_{l,m=\pm l_0,l',m=\pm l_0}^{(l_0,i\rho)}(\lambda)$ and $U_{l,m=0,l',m=0}^{(l_0=1,i\rho)}(\lambda)$. The computation of the general formula for the term of the highest order in ρ can be simplified. Indeed, from formula (46) it follows that this term determines the leading contribution

$$\frac{p_{j_{\max},j_{\max}}^\mp e^{\mp i\rho\lambda}}{\sinh \lambda} \frac{1}{\rho}, \quad \text{for } U_{l,\pm l_0,l',\pm l_0}^{(l_0,i\rho)}(\lambda), \quad (47)$$

$$\left(\frac{p_{j_{\max},j_{\max}}^+ e^{i\rho\lambda}}{\sinh^{l_0+1} \lambda} + \frac{p_{j_{\max},j_{\max}}^- e^{-i\rho\lambda}}{\sinh^{l_0+1} \lambda} \right) \frac{1}{\rho^{l_0+1}}, \quad \text{for } U_{l,0,l',0}^{(l_0,i\rho)}(\lambda), \quad (48)$$

to the asymptotics $\rho \rightarrow +\infty$ of the matrix (46). It follows from (46) that $p_{j_{\max},j_{\max}}^+ = 0$, if $|m| - |l_0| \neq 0$, $m > 0$, and $p_{j_{\max},j_{\max}}^- = 0$, if $|m| - |l_0| \neq 0$, $m < 0$. Using the integral (45), we can compute the asymptotic $\rho \rightarrow +\infty$ expansion by application of the saddle point method. In fact, for $l_0 = 0$, we can use an integral involving only the associated Legendre polynomials

$$\begin{aligned} & U_{l,m,l',m}^{(0,i\rho)}(\lambda) \\ &= (-1)^{l+l'} c_{l,l',m} \cosh^{i\rho-1} \lambda \int_{-1}^1 P_{l',m}(y) (1 + y \tanh \lambda)^{i\rho-1} P_{l,m} \left(\frac{y + \tanh \lambda}{1 + y \tanh \lambda} \right) dy \end{aligned} \quad (49)$$

with

$$c_{l,l',m} = \sqrt{\frac{(2l+1)(2l'+1)(l-m)!(l'-m)!}{2(l+m)!2(l'+m)!}}.$$

Formula (49) follows from the relation between the Jacobi and the associated Legendre polynomials. Independently, (49) also follows from the realization $\alpha \mapsto f(\Lambda(\alpha)^{-1}p)$ of the representation $(l_0 = 0, l_1 = i\rho)$ in the space of homogeneous of degree $i\rho - 1$ functions f on the cone, thus living effectively on the unit $\mathbb{S}^2$ sphere in the cone, with the inner product

$$(f, g) = \int_{\mathbb{S}^2} \bar{f} g \, d\mu_{\mathbb{S}^2}$$

invariant under the action naturally induced by the Lorentz transforms in the ambient Minkowski space, and with ordinary invariant measure $\mu_{\mathbb{S}^2}$ on $\mathbb{S}^2$, and the orthonormal basis Y_{lm} , $l = 0, 1, \dots$, $-l \leq m \leq l$, [17]. The matrix (49) coincides with (45) for $l_0 = 0$. The integrals (45) and (49) can be rewritten as

$$U_{l,m \, l',m}^{(l_0, i\rho)}(\lambda) = \frac{c e^{i\rho \ln(\cosh \lambda)}}{\cosh \lambda} \int_{-1}^1 f(y) e^{i\rho S(y)} \, dy$$

with the phase, respectively, equal to $S(y) = \ln(1 \pm y \tanh \lambda)$, with “+” sign for (45) and “−” sign for (49), and with

$$f(y) = \overline{P_{l_0,m}^{l'}}(y) P_{l_0,m}^l \left(\frac{y - \tanh \lambda}{1 - y \tanh \lambda} \right) \frac{1}{1 - y \tanh \lambda}$$

for (45), and

$$f(y) = P_{l',m}(y) P_{l,m} \left(\frac{y + \tanh \lambda}{1 + y \tanh \lambda} \right) \frac{1}{1 + y \tanh \lambda}$$

for (49). $S'(y) \neq 0$ for $y \in [-1, 1]$, and the function f is analytic in an open set containing the closed interval $[-1, 1]$. This is evident if m, l_0 are of the same parity in (45), or even m in (49), because in this case, each factor P in f separately, is analytic. But it can be seen that this is generally the case, by noticing that $P_{l,m}$ and, respectively, $P_{l_0,m}^{l'}$, can be written as the product of a polynomial and, eventually, the square root factor $(1 - y^2)^{1/2}$. Then, it is easily seen that the square roots in the product f drop out. The polynomial which arises after extracting the (eventual) square root $(1 - y^2)^{1/2}$ in $P_{l_0,m}^{l'}$ is a Jacobi polynomial multiplied by a natural power of $(1 - x)$ and a natural

power of $(1+x)$. Thus, theorem 1, §44.1, p. 410 of [18] can be applied, and gives the following asymptotic expansion:

$$U_{l,m,l',m}^{(l_0,i\rho)}(\lambda) \stackrel{\rho \rightarrow +\infty}{\sim} \frac{c}{\cosh \lambda} \sum_{n=0}^{+\infty} \left(e^{-i\rho\lambda} b_n - e^{i\rho\lambda} a_n \right) (i\rho)^{-n-1},$$

with

$$c = \frac{1}{2} \sqrt{(2l'+1)(2l+1)} = c_{l,l',0}$$

for (45) or, respectively, $c = (-1)^{l+l'} c_{l,l',m}$ for (49), and with

$$a_n = (-1)^n M^n \left(\frac{f(y)}{S'(y)} \right) \Big|_{y=-1}, \quad b_n = (-1)^n M^n \left(\frac{f(y)}{S'(y)} \right) \Big|_{y=1}, \quad M = \frac{1}{S'(y)} \frac{d}{dy},$$

respectively, in each case. In particular, for $l_0 = m = 0$, we get the leading order term

$$U_{l,0,l',0}^{(0,i\rho)}(\lambda) \stackrel{\rho \rightarrow +\infty}{\sim} \frac{(-1)^{l+l'} c_{l,l',0}}{\sinh \lambda} \left((-1)^{l+l'} e^{i\rho} - e^{-i\rho} \right) \frac{1}{i\rho}.$$

For $l_0 > 0$ and $m = \pm l_0$, respectively, we get

$$\begin{aligned} U_{l,l_0,l',l_0}^{(l_0,i\rho)}(\lambda) &\stackrel{\rho \rightarrow +\infty}{\sim} -\frac{c_{l,l',0}}{\sinh \lambda} e^{-i\rho} \frac{1}{i\rho}, \\ U_{l,-l_0,l',-l_0}^{(l_0,i\rho)}(\lambda) &\stackrel{\rho \rightarrow +\infty}{\sim} (-1)^{l+l'} \frac{c_{l,l',0}}{\sinh \lambda} e^{i\rho} \frac{1}{i\rho}. \end{aligned}$$

For $l_0 = 1$ and $m = 0$, we get

$$U_{l,0,l',0}^{(1,i\rho)}(\lambda) \stackrel{\rho \rightarrow +\infty}{\sim} -\frac{\sqrt{l(l+1)l'(l'+1)}}{4} \frac{c_{l,l',0}}{\sinh^2 \lambda} \left((-1)^{l+l'} e^{i\rho} - e^{-i\rho} \right) \frac{1}{\rho^2}.$$

Comparing with (47) or, respectively, with (48), we see that

$$p_{j_{\max},j_{\max}}^+ = p_{l+l',l+l'}^+ = \frac{(-1)^{l+l'} c_{l,l',0}}{i} \quad \text{in } U_{l,-l_0,l',-l_0}^{(l_0,i\rho)}(\lambda), \quad (50)$$

$$p_{j_{\max},j_{\max}}^- = p_{l+l',l+l'}^- = -\frac{c_{l,l',0}}{i} \quad \text{in } U_{l,l_0,l',l_0}^{(l_0,i\rho)}(\lambda), \quad (51)$$

$$p_{j_{\max},j_{\max}}^+ = p_{l+l',l+l'-1}^+ = -\frac{(-1)^{l+l'} c_{l,l',0} \sqrt{l(l+1)l'(l'+1)}}{4} \quad \text{in } U_{l,0,l',0}^{(1,i\rho)}(\lambda), \quad (52)$$

$$p_{j_{\max},j_{\max}}^- = p_{l+l',l+l'-1}^- = -\frac{c_{l,l',0} \sqrt{l(l+1)l'(l'+1)}}{4} \quad \text{in } U_{l,0,l',0}^{(1,i\rho)}(\lambda) \quad (53)$$

for the maximal degree coefficients of the maximal degree polynomials $p_j^\pm$, and

$$\begin{aligned} p_{j_{\max},j_{\max}}^+ &= p_{l+l',l+l'}^+ = \frac{(-1) c_{l,l',0}}{i}, \\ p_{j_{\max},j_{\max}}^- &= p_{l+l',l+l'}^- = \frac{(-1)^{l'+l+1} c_{l,l',0}}{i} \end{aligned} \quad (54)$$

for the maximal degree coefficients of the maximal degree polynomials $p_j^\pm$ in $U_{l,0 \ l',0}^{(0,i\rho)}(\lambda)$. Note that the coefficients (50)–(53) are always nonzero, with (50) and (51) purely imaginary and with (52) and (53) real. Using the saddle point method, as above, one can easily compute the coefficients (48) for $l_0 > 1$, and show that they are all nonzero, but in this paper, we do not need the explicit values of these coefficients for $l_0 > 1$. We note here that for fixed l, l' , all polynomials $p_j^\pm$ in the matrix elements $U_{l,m \ l',m}^{(0,i\rho)}(\lambda)$ with $m \neq 0$, have maximal degree $l + l' - |m|$, which is less than the maximal degree $j_{\max} = l + l'$ of the polynomials $p_{j_{\max}}^\pm$ in $U_{l,0 \ l',0}^{(0,i\rho)}(\lambda)$. More generally, for fixed l, l' , the matrix elements $U_{l,\pm l_0 \ l',\pm l_0}^{(l_0,i\rho)}(\lambda)$ are precisely these that contain the polynomial $p_j^\pm$ of the highest degree $l + l'$ among all matrix elements $U_{l,m \ l',m}^{(l_0,i\rho)}(\lambda)$, $-\min\{l, l'\} \leq m \leq \min\{l, l'\}$. All other matrix elements $U_{l,m \ l',m}^{(l_0,i\rho)}(\lambda)$ contain polynomials in ρ of degree strictly smaller. We will use these facts in Subsection 4.1.

Let us remark, that the higher-order terms in the asymptotic $\rho \rightarrow +\infty$ expansion of the matrix elements (46) are determined by the lower-order coefficients of the polynomials $p_j^\pm$ of lower order, and can also be determined by the saddle point method shown above.

The general formula for (46) with $l_0 = m = l = 0$ and for any l'

$$\begin{aligned}
 U_{0,0 \ 0,0}^{(0,i\rho)}(\lambda) &= \frac{\sqrt{2l+1}(-1)^{l+1}}{2 \prod_{s=0}^l (-is+\rho) \sinh^{l+1} \lambda} \sum_{k=0}^l p_{l,k} \sum_{j=1}^{k+1} \frac{i^j k!}{(k-j+1)!} \\
 &\times \left[\sinh^{l+1-j} \lambda (\cosh \lambda + \sinh \lambda)^{j-1} \prod_{s=j}^l (-is + \rho) e^{i\rho \lambda} \right. \\
 &\left. - \sinh^{l+1-j} \lambda (\cosh \lambda - \sinh \lambda)^{j-1} (-1)^{k-j+1} \prod_{s=j}^l (-is + \rho) e^{-i\rho \lambda} \right] \quad (55)
 \end{aligned}$$

can easily be computed on repeated application of integration by parts in (49). Here $p_{l,k}$ are the coefficients of the Legendre polynomial P_l (Subsections 2.1 and 2.2).

If we write $U_{l,m \ l',m}^{(l_0,i\rho)}(\lambda)$ in the form

$$\frac{1}{\rho Q(\rho)} \frac{\sum_{j \geq -1} \left[a_j^+ e^{-j\lambda} e^{i\rho \lambda} + a_j^- e^{-j\lambda} e^{-i\rho \lambda} \right]}{(1 - e^{-2\lambda})^q}, \quad (56)$$

with polynomials $a_j^\pm$ of ρ , then the natural numbers j have the parity the same as the number $-|m| - |l_0| - 1$.

In the sequel, we use the fact that the general matrix elements (46) are quantities of the order of $|l' - l|$ at $\lambda = 0$, with the following expansion at $\lambda = 0$

$$U_{l',m,l,m}^{(l_0,l_1=i\rho)}(\lambda) = u_0 \lambda^{|l-l'|} + u_1 \lambda^{|l-l'|+1} + \dots, \quad (57)$$

$$u_0 = \frac{(-1)^{l-l'}}{(l-l')!} \prod_{j=0}^{l-l'-1} C_{l-j,m}, \quad (58)$$

$$u_0 = \frac{1}{(l-l')!} \prod_{j=0}^{l-l'-1} [C_{l+j+1,m}], \quad l' > l, \quad (59)$$

$$C_{l,m} = \frac{\sqrt{l^2-m^2}}{l} \sqrt{\frac{(l^2-l_0^2)(l^2-l_1^2)}{4l^2-1}}, \quad (60)$$

in which $u_0 \neq 0$ in the leading order terms. One can easily convince himself of the validity of (57)–(60) by the exponentiation of the generator M_{03} of the representation (l_0, l_1) given in [5, 10].

2.4. Decomposition of a cyclic representation. Example with $x = c_{1,0}^+ |u\rangle$

Now we pass to the decomposition of the cyclic representation with the cyclic vector (1), and thus to the Fourier analysis of the associated invariant kernel $\langle g|h\rangle$, and thus, Fourier transform of the associated positive definite function $\varphi(h) = \langle e|h\rangle$ on G .

The effectiveness of the Fourier analysis comes from the fact that the matrix elements $U_{l,m,l',m'}^{(l_0,l_1=i\rho)}$ of the principal series, regarded as functions on G , compose a complete system (in the generalized sense) of generalized functions on G . In general, they are not square-summable on G . But by using them, we can construct square summable functions on G with the required support of their Fourier transform in the space of unitary irreducible equivalence classes, represented by (l_0, l_1) . In particular, integrating any matrix function $U_{lm,l'm'}^{(l_0,l_1=i\rho)}(g)$ with respect to ρ over any interval $I \subset \mathbb{R}_+$, we get square summable function

$$U_{lm,l'm'}^{(l_0,iI)}(g) = \int_I U_{lm,l'm'}^{(l_0,l_1=i\rho)}(g) d\rho$$

on G (recall oscillatory character $\cos(\rho\lambda), \sin(\rho\lambda)$ of matrix elements), with the Fourier transform supported at (l_0, l_1) with $l_1 \in iI$. Any two such square

summable packets are orthogonal in $L^2(G)$ whenever the corresponding intervals I have an empty intersection, or whenever the corresponding l_0 are different. Moreover, even if the corresponding intervals I and l_0 coincide, the packets remain orthogonal if any of their corresponding matrix coefficients do not coincide

$$\int_G \overline{U_{lm' l'm'}^{(l_0, iI)}(g)} U_{l''m'' l'''m'''}^{(l_0, iI')}(g) dg = \begin{cases} c(l_0, I, I') \delta_{l_0 l'_0} \delta_{l'' l'_0} \delta_{m m''} \delta_{l' l'''} \delta_{m' m'''} \cdot \\ = 0, \text{ if } I \cap I' = \emptyset. \end{cases} \quad (61)$$

Orthogonality of the matrix elements corresponding to non-equivalent representations comes from the fact that they compose generalized eigenstates of the two Casimir operators acting in $L^2(G)$, associated with different generalized eigenvalues. Orthogonality relations of the matrix functions corresponding to the same representation classes cannot be justified in this manner. But in this case, we use the coordinate system on G determined by (3), representation property of $U_{lm' l'm'}^{(l_0, l_1=i\rho)}$, and perform the integration iteratively, in which the integration with respect to ρ must necessarily be performed first. Finally, using the orthogonality relations for the matrix elements $T_{m m'}^l(a)$, $a \in SU(2, \mathbb{C})$, we obtain the orthogonality relations (61), compare the analogue orthogonality for the $SL(2, \mathbb{R})$ in [19]. The supplementary series does not enter the Plancherel formula for the group G , *i.e.* it is absent in the decomposition of $L^2(G)$. Matrix elements $U_{lm' l'm'}^{(l_0=0, l_1=1-s)}(g)$, $0 < s < 1$, of the supplementary series decrease slower at infinity, *e.g.* $U_{2,0 \ 2,0}^{(0, l_1=1-s)}(\lambda) \sim e^{-s\lambda}$ at infinity, and are still not square integrable even after integration over an interval in the real parameter s . But $G = SL(2, \mathbb{C})$ is complex analytic, and the matrix elements of the representations (l_0, l_1) are analytic, both as functions of $g \in G$ and as functions of the continuous parameter l_1 of both series, principal and supplementary, with the matrix elements of the supplementary series $(l_0 = 0, s)$, $-1 < s < 1$, being the analytic continuations of the corresponding matrix elements of the principal series $(l_0 = 0, l_1 = i\rho)$. We will use these facts in what follows.

For $x = |e\rangle = c_\alpha^+ e^{-inS(u)} |0\rangle = c_\alpha^+ |u\rangle$, we have

$$\langle g|h\rangle = 4\pi\epsilon^2 \left[\overline{A_{\alpha\alpha}(g^{-1}h)} + \frac{z}{4}\overline{B_{\alpha}(g^{-1}h)}\frac{1}{\epsilon}B_{\alpha}(h^{-1}g) \right] e^{-z(\lambda\coth\lambda-1)}, \quad (62)$$

for $g^{-1}h$ with decomposition (3), in which λ can be interpreted as the hyperbolic angle between u and $g^{-1}hu$.

Each of the invariant kernels $\langle g|h\rangle$, corresponding to cyclic vectors (1), can be written in the form of a function depending only on g, h, z (up to a constant factor depending on ϵ). Each $\langle g|h\rangle$ depends on n only through $z = n^2\epsilon^2/\pi$, where $n\epsilon$ is the eigenvalue of the total charge Q . We have already seen the simplest examples (19) and (62) for $x = |u\rangle, c_\alpha^+ |u\rangle$.

The cases with cyclic $x = |u\rangle, c_{1,0}^+|u\rangle$ are exceptional, so we will continuously be giving explicit formulas for the concrete example of representation with cyclic vector $x = c_{1,0}^+|u\rangle$. The exceptional case $x = |u\rangle$ was worked out in [4].

The positive definite function $k(h) = \langle e|h\rangle$, corresponding to the left invariant kernel $\langle g|h\rangle$ on G , is in $L^2(G)$ only if $z > 1$, and only in this case, we can use the Plancherel formula and the inverse Fourier transform formula for G to decompose $\langle g|h\rangle$. In the case of $0 < z < 1$, the decomposition will additionally involve the supplementary component which cannot be inferred immediately from the Plancherel formula. We use the method [4] of analytic continuation to obtain the formula valid in the domain $0 < z < 1$, which is based on the fact that $\langle g|h\rangle$ is analytic in z . In the case of $z > 1$, from Lebesgue's dominated convergence principle, it easily follows that the integrals

$$\left[K(l_0, l_1 = i\rho; z) \right]_{\beta \gamma} = \int_G \langle e|h\rangle U_{\beta \gamma}^{(l_0, l_1 = i\rho)}(h) dh, \quad z > 1 \quad (63)$$

are convergent and represent analytic functions of z, ρ . Here, in principle, $\beta, \gamma \in \{(l_0, m_0), (l_1, m_1) = (l_0 + 1, m_1), \dots\}$, $-l_i \leq m_i \leq l_i$ and all representations of the principal series are considered in this integral. But for the cyclic (1), the integral (63) is nonzero only for $l_0 \leq l(\alpha_1) + \dots l(\alpha_q)$ by the product formula and orthogonality relations of the matrix elements of the standard unitary representations of $SU(2, \mathbb{C})$ (Peter–Weyl theorem) and the structure (12) of representations (l_0, l_1) . For example, from formula (19), it follows that $\langle e|h\rangle = \langle u|hu\rangle$ does not depend on the angle coordinates $(\theta_1, \varphi_1, \vartheta_1, \vartheta, \varphi)$, thus, for the kernel $\langle g|h\rangle = \langle gu|hu\rangle$ corresponding to the cyclic vector $|u\rangle$ and $z > 1$, the Fourier transform (63) is nonzero only for $U_{0,0 \ 0,0}^{(l_0=0, l_1=i\rho)}$, independent of the angle coordinates, by the structure (12) of the restriction of the representations (l_0, l_1) of G to $SU(2, \mathbb{C})$ and orthogonality relations (Peter–Weyl theorem for $SU(2, \mathbb{C})$). Since for general $g \in G$ of the form (3)

$$U_{0,0 \ 0,0}^{(l_0=0, l_1=i\rho)}(g) = \frac{\sin(\rho\lambda)}{\rho \sinh(\lambda)},$$

then for the kernel $\langle g|h\rangle = \langle gu|hu\rangle$ corresponding to the cyclic vector $|u\rangle$ and $z > 1$, the only nonzero matrix element of $K(l_0 = 0, l_1 = i\rho; z)$ is the diagonal element

$$\left[K(l_0 = 0, l_1 = i\rho; z) \right]_{0,0 \ 0,0} = \frac{4\pi^3 e^z}{\rho} \int_0^{+\infty} \sinh(\lambda) \sin(\rho\lambda) e^{-z\lambda \coth \lambda} d\lambda, \quad (64)$$

and $K(l_0, l_1 = i\rho; z) = 0$ for $l_0 \neq 0$, which, up to the constant factor π^2 , agrees with the weight K of the cyclic representation with the cyclic vector $x = |u\rangle$, found in [4]. (The various constant factors come from the free choice in the normalization of the invariant measure on G and of the induced invariant measure on the Lobachevsky space $G/SU(2, \mathbb{C})$, irrelevant in our analysis.) The only nonzero element of the matrix K for the kernel (62) is the diagonal element

$$\begin{aligned} \left[K(l_0, i\rho; z) \right]_{\alpha \alpha} &= \\ & \left(\frac{4\pi\epsilon}{2l+1} \right)^2 e^z \int_0^{+\infty} \left[\sum_{n=-l}^l \overline{A_{l,n \ l,n}(\lambda)} U_{l,n \ l,n}^{(l_0, i\rho)}(\lambda) + \frac{z}{4} (-1)^l \left| \frac{1}{\epsilon} B_{l,0}(\lambda) \right|^2 U_{l,0 \ l,0}^{(l_0, i\rho)}(\lambda) \right] \\ & \times e^{-z\lambda \coth \lambda} \pi^2 \sinh^2 \lambda \, d\lambda, \end{aligned} \quad (65)$$

where $\alpha = (l, m)$ is the same which is present in $\langle g|h \rangle$ in (62) and in $x = |e\rangle$ defining $\langle g|h \rangle$, and is independent of the azimuthal number m . To see it, let $\beta = (L, M)$ and $\gamma = (L_1, M_1)$, with $L \leq L_1$ in (63). We decompose the general argument $h \in G$ of the integrand in (63) in the form $h = u_1 u_2^* g_{03}(\lambda) u_2$, according to (3), where $u_1 = g_{12}(\theta_1) g_{13}(\varphi_1) g_{12}(\vartheta_1) \in SU(2, \mathbb{C})$, $u_2 = g_{13}(\varphi) g_{12}(\vartheta) \in SU(2, \mathbb{C})$. We use the homomorphism property of the matrices of the representations (l_0, l_1) , the property (12) of their restriction to the subgroup $SU(2, \mathbb{C})$, and formulas (13), (14), to compute each of the terms of $\langle e, h \rangle$, given by (62), with $g = e$ in (62) and $U_{L,M \ L_1,M_1}^{(l_0, l_1=i\rho)}(h)$. Inserting the integrand computed in this way and the invariant measure (4), we get, by performing the integration du_1 over $SU(2, C)$ first, and using the $SU(2, C)$ -invariance of du_1

$$\begin{aligned} K_{L,M \ L_1,M_1} &= 4\pi\epsilon^2 \sum_{\substack{-l \leq m' \leq l \\ -L \leq M' \leq L}} \int_{SU(2, \mathbb{C})} \overline{T_{m \ m'}^l(u_1)} T_{M \ M'}^L(u_1) \, du_1 \\ & \times \int_{\mathbb{S}^2} \overline{T_{m' \ m}^l(g_{13}(\varphi) g_{12}(\vartheta))} T_{M' \ M_1}^{L_1}(g_{13}(\varphi) g_{12}(\vartheta)) \sin \varphi \, d\varphi \, d\vartheta \\ & \times \int_0^{+\infty} \overline{A_{l,m' \ l,m'}(\lambda)} U_{L,M' \ L_1,M'}^{(l_0, i\rho)}(\lambda) e^{-z(\lambda \coth \lambda - 1)} \pi^2 \sinh^2 \lambda \, d\lambda \\ & + 4\pi\epsilon^2 (-1)^l \frac{z}{4} \sum_{-L \leq M' \leq L} \int_{SU(2, \mathbb{C})} \overline{T_{m \ 0}^l(u_1)} T_{M \ M'}^L(u_1) \, du_1 \end{aligned}$$

$$\begin{aligned}
& \times \int_{\mathbb{S}^2} \overline{T_{0\ m}^l(g_{13}(\varphi)g_{12}(\vartheta))} T_{M'\ M_1}^{L_1}(g_{13}(\varphi)g_{12}(\vartheta)) \sin \varphi d\varphi d\vartheta \\
& \times \int_0^{+\infty} \left| \frac{1}{\epsilon} B_{l,0}(\lambda) \right|^2 U_{L,M'\ L_1,M'}^{(l_0,i\rho)}(\lambda) e^{-z(\lambda \coth \lambda - 1)} \pi^2 \sinh^2 \lambda d\lambda.
\end{aligned}$$

Now we use the first one of the following orthogonality relations:

$$\begin{aligned}
& \int_{\mathrm{SU}(2,\mathbb{C})} \overline{T_{m_1\ n_1}^{l_1}(a)} T_{m_2\ n_2}^{l_2}(a) da = \frac{1}{2l_1+1} \delta_{l_1\ l_2} \delta_{m_1\ m_2} \delta_{n_1\ n_2}, \\
& \int_{\mathrm{SU}(2,\mathbb{C})/\mathbb{S}^1} \overline{T_{m_1\ n_1}^{l_1}(g_{13}(\varphi)g_{12}(\vartheta))} T_{m_2\ n_2}^{l_2}(g_{13}(\varphi)g_{12}(\vartheta)) \sin \varphi d\varphi d\vartheta \\
& = \frac{4\pi}{2l_1+1} I_{m_1 m_2}^{l_1 l_2} \delta_{n_1 n_2}, \quad I_{mm}^{l_1 l_2} = \delta^{l_1 l_2}, \tag{66}
\end{aligned}$$

where

$$I_{m_1 m_2}^{l_1 l_2} = \frac{2l_1+1}{2} \int_0^\pi \overline{P_{m_1\ n_1}^{l_1}(\cos \varphi)} P_{m_2\ n_1}^{l_2}(\cos \varphi) \sin \varphi d\varphi.$$

We obtain

$$\begin{aligned}
K_{L,M\ L_1,M_1} &= \delta_{l\ L} \delta_{M\ m} 4\pi \epsilon^2 \sum_{-l \leq m' \leq l} \frac{1}{2l+1} \\
& \times \int_{\mathbb{S}^2} \overline{T_{m'\ m}^l(g_{13}(\varphi)g_{12}(\vartheta))} T_{m'\ M_1}^{L_1}(g_{13}(\varphi)g_{12}(\vartheta)) \sin \varphi d\varphi d\vartheta \\
& \times \int_0^{+\infty} \overline{A_{l,m'\ l,m'}(\lambda)} U_{L,m'\ L_1,m'}^{(l_0,i\rho)}(\lambda) e^{-z(\lambda \coth \lambda - 1)} \pi^2 \sinh^2 \lambda d\lambda \\
& + \delta_{l\ L} \delta_{M\ m} 4\pi \epsilon^2 (-1)^l \frac{z}{4} \frac{1}{2l+1} \\
& \times \int_{\mathbb{S}^2} \overline{T_{0\ m}^l(g_{13}(\varphi)g_{12}(\vartheta))} T_{0\ M_1}^{L_1}(g_{13}(\varphi)g_{12}(\vartheta)) \sin \varphi d\varphi d\vartheta \\
& \times \int_0^{+\infty} \left| \frac{1}{\epsilon} B_{l,0}(\lambda) \right|^2 U_{l,0\ L_1,0}^{(l_0,i\rho)}(\lambda) e^{-z(\lambda \coth \lambda - 1)} \pi^2 \sinh^2 \lambda d\lambda.
\end{aligned}$$

Using the second orthogonality relation (66) with $m_1 = m_2$, we obtain

$$\begin{aligned} K_{L, M \quad L_1, M_1} &= \left(\frac{4\pi\epsilon}{2l+1} \right)^2 e^z \delta_{l \quad L} \delta_{M \quad m} \delta_{L_1 \quad l} \delta_{M_1 \quad m} \\ &\times \left[\sum_{-l \leq m' \leq l} \int_0^{+\infty} \overline{A_{l, m' \quad l, m'}(\lambda)} U_{l, m \quad l, m}^{(l_0, i\rho)}(\lambda) e^{-z\lambda \coth \lambda} \pi^2 \sinh^2 \lambda \, d\lambda \right. \\ &\left. + (-1)^{l \frac{z}{4}} \int_0^{+\infty} \left| \frac{1}{\epsilon} B_{l, 0}(\lambda) \right|^2 U_{l, 0 \quad l, 0}^{(l_0, i\rho)}(\lambda) e^{-z\lambda \coth \lambda} \pi^2 \sinh^2 \lambda \, d\lambda \right], \end{aligned}$$

i.e., (65) as the only nonzero coefficient of the matrix K . Moreover, from the homomorphism property of $U^{(l_0, l_1=i\rho)}$ and orthogonality properties of $T_{m \quad m'}^l$, it follows that (65) is nonzero for integer l_0 only, and (65) is zero for $|l_0| > l(\alpha)$, where for $\alpha = (l, m)$, $l(\alpha) = l$. In particular, for α with $l(\alpha) = 1$, the only possible representations $(l_0, i\rho)$ for which (65) is nonzero, are the representations $(l_0 = 0, l_1 = i\rho)$ and $(l_0 = \pm 1, l_1 = i\rho) = (1, \pm i\rho)$, $\rho > 0$, of the principal series.

The integral (65) is convergent for $z > 0$ if $|l_0| > 0$. It follows from the fact that $B_{l, 0}(\lambda)$ tends to a nonzero constant for λ going to infinity, compare (33), and by the asymptotic behavior $U_{l, 0 \quad l', 0}^{(l_0, l_1=i\rho)} \sim e^{-(|l_0|+1)\lambda}$ at infinity. This is generally true for the integral (63).

As we have already mentioned, K is nothing else but the Hermitian conjugation of the Fourier transform of the complex conjugation $\bar{k}$ of the positive definite function $k(h) = \langle e|h \rangle$ corresponding to the positive definite kernel $\langle g|h \rangle$ on G . From the invariance it easily follows

$$\int_G \langle g|h \rangle U^{(l_0, l_1)}(h) \, dh = U^{(l_0, l_1)}(g) \int_G \langle e|h \rangle U^{(l_0, l_1)}(h) \, dh, \quad g \in G.$$

Let $f \in L^2(G)$. We have the following Fourier transform and its inverse on G [10]:

$$\begin{aligned} \mathcal{F}f(l_0, l_1 = i\rho) &= \int_G f(g) U^{(l_0, l_1=i\rho)*}(g) \, dg, \\ f(g) &= \sum_{l_0=-\infty}^{+\infty} \int_0^{+\infty} \text{Tr} \left[\mathcal{F}f(l_0, l_1 = i\rho) U^{(l_0, l_1=i\rho)}(g) \right] \frac{l_0^2 + \rho^2}{2\pi^4} \, d\rho \end{aligned}$$

$$\begin{aligned}
&= \int_0^{+\infty} \text{Tr} \left[\mathcal{F}f(0, l_1 = i\rho) U^{(l_0, l_1 = i\rho)}(g) \right] \frac{\rho^2}{2\pi^4} d\rho \\
&\quad + \sum_{l_0 > 0} \int_{-\infty}^{+\infty} \text{Tr} \left[\mathcal{F}f(l_0, l_1 = i\rho) U^{(l_0, l_1 = i\rho)}(g) \right] \frac{l_0^2 + \rho^2}{2\pi^4} d\rho,
\end{aligned}$$

where only the representations of the principal series are present here (recall $U^{(l_0, l_1)} = U^{(-l_0, -l_1)}$). The star denotes the Hermitian conjugation. Due to the last three formulas and because the integral (63) is convergent ($z > 1$), changing the order of integration is legitimate in deriving the following formula obtained by inserting the inverse Fourier formula:

$$\begin{aligned}
\langle f | f' \rangle &= \int_{G \times G} dg dh \langle g | h \rangle f(h) \overline{f'(g)} \\
&= \sum_{l_0 = -l_{\max}}^{l_{\max}} \int_0^{+\infty} \frac{l_0^2 + \rho^2}{2\pi^4} d\rho \text{Tr} \\
&\quad \times [\mathcal{F}f(l_0, l_1 = i\rho) K(l_0, l_1 = i\rho; z) \mathcal{F}f'(l_0, l_1 = i\rho)^*] . \quad (67)
\end{aligned}$$

As we will show, only the contribution $l_0 = 0$ is nonzero for the kernel (62) with $l(\alpha) = 1$. Formula (67), valid if $z > 1$, gives the decomposition of the cyclic representation with the cyclic (1) into the direct integral of the principal series representations (l_0, l_1) , each entering with multiplicity equal to the rank of $K(l_0, l_1 = i\rho; z)$, which in this case is equal to 1. This is the case for cyclic (1) with $\mathfrak{q} = 1$ and, generally for $\mathfrak{q} > 1$, it follows from the relations between the cyclic spaces proved in Section 5. The multiplicity of (l_0, l_1) , equal to the rank of $K(l_0, l_1 = i\rho; z)$, is equal to 1, because the corresponding representation is cyclic. Positivity of the operator matrix $K(l_0, l_1 = i\rho; z)$ (for almost all ρ) follows from the positivity of the kernel $\langle g | h \rangle$.

In passing to the domain $0 < z < 1$, we first observe that also in this domain, formula (67) remains meaningful, except for the term with $l_0 = 0$, as the integral (63) is convergent for $|l_0| > 0$. The integral (63) with $l_0 = 0$ becomes divergent for $0 < z < 1$, and so the term with $l_0 = 0$ in (67) becomes divergent for $0 < z < 1$. Using the symmetry $U^{(l_0=0, l_1)} = U^{(l_0=0, -l_1)}$, the contribution with $l_0 = 0$ in (67) can be written as

$$\begin{aligned}
& \frac{1}{2} \int_{-\infty}^{+\infty} d\rho \frac{\rho^2}{2\pi^4} \text{Tr} \left[\mathcal{F}f(0, i\rho) K(0, i\rho; z) \mathcal{F}f'(0, i\rho)^* \right] \\
&= \int_{G \times G} dg dh \langle g|h \rangle f(h) \overline{f'(g)} \frac{1}{2} \int_{-\infty}^{+\infty} d\rho \frac{\rho^2}{2\pi^4} \text{Tr} \left[U^{(0, i\rho)*}(h) K(0, i\rho; z) U^{(0, i\rho)}(g) \right],
\end{aligned}$$

where the last integral $\frac{1}{2} \int d\rho \dots = \int d\rho F(\rho, z)$ is treated (compare [4]) as a contour integral in the complex ρ -plane. In passing to the domain $0 < z < 1$ we are using the analytic continuation in z of the $l_0 = 0$ contribution. Its correct construction is summarized in the following prescription [4]: we replace the contour $d\rho$ integral with the $d\rho$ -integral in which the integrand $F(\rho, z)$ is replaced by its analytic continuation in the variables ρ, z (*i.e.* we replace $K(0, l_1 = i\rho; z)$ with its analytic continuation), then add to it the sum of all residues of the analytic continuation of $F(\rho, z)$ multiplied by $2\pi i$, which crosses the contour going “top down” when z is passing from the domain $z > 1$ to the domain $0 < z < 1$, and finally subtract the sum of all residues of the analytic continuation of $F(\rho, z)$ multiplied by $2\pi i$, which crosses the contour going “down top” when z is passing from the domain $z > 1$ to the domain $0 < z < 1$. For the general $\alpha = (l, m_l)$, $-l \leq m_l \leq l$, $l \geq 1$, in (62), the only residues in the complex ρ -plane of $F(\rho, z)$ crossing the contour are located at $\pm i(1 - z)$. At the same time, the only poles in the complex ρ -plane of the analytic continuation of $K(0, l_1 = i\rho; z)$ which crosses the contour, when z changes the domains $z > 1$ and $0 < z < 1$, are the simple poles, *i.e.* the residues at $\pm i(1 - z)$. This is generally the case for the kernel $\langle g|h \rangle$ corresponding to the general cyclic vector (1), and follows from the fact that the functions $U_{l', 0, l, 0}^{l_0, i\rho}$ behave at infinity as $\sim e^{-(|l_0|+1)\lambda}$ in the coordinates given by decomposition (3), so that the integrands of the particular contributions to the matrix element of $K(l_0, l_1 = i\rho; z)$, in the $d\lambda$ -integral, behave as $\sim e^{-s\lambda}$ at infinity, with $s = 1 - z + |l_0|, 1 - z + |l_0| + 1, 1 - z + |l_0| + 2, \dots$, for the essential contribution which does not contain the A -factors, and $s = z, z+1, z+2, \dots$, for the part of contributions containing A -factors, compare our discussion below. Taking these circumstances into account, the analytic continuation of the $l_0 = 0$ contribution in (67) has the form

$$\begin{aligned}
& \int_0^{+\infty} d\rho \frac{\rho^2}{2\pi^4} \text{Tr} \left[\mathcal{F}f(l_0 = 0, l_1 = i\rho) K(0, i\rho; z) \mathcal{F}f'(l_0 = 0, l_1 = i\rho)^* \right] \\
& + \text{Tr} \left[\mathcal{F}f(l_0 = 0, l_1 = 1 - z) \kappa(z) \mathcal{F}f'(l_0 = 0, l_1 = 1 - z)^* \right], \quad (68)
\end{aligned}$$

where the second term coming from the residues is present only if $0 < z < 1$,

and where $K(0, i\rho; z)$ is the analytic continuation of $K(l_0 = 0, i\rho; z)$ given by (63) or, in more explicit form, by (65), *etc.* κ is a finite dimensional positive definite matrix $\kappa_{\beta\gamma}$ which, in the case of the kernel (62), has only one nonzero matrix element $\kappa_{\alpha\alpha}$, with α being the same as the α in the definition of the cyclic vector $x = c_{\alpha}^+ e^{-inS(u)}|0\rangle$ and in the invariant kernel (62). In general, we have (68) and

$$\begin{aligned} \kappa(z) = & \frac{1}{2} 2\pi i \operatorname{res}_{\rho=-i(1-z)} \left[\frac{\rho^2}{2\pi^4} K(0, i\rho; z) \right] \\ & - \frac{1}{2} 2\pi i \operatorname{res}_{\rho=i(1-z)} \left[\frac{\rho^2}{2\pi^4} K(0, i\rho; z) \right]. \end{aligned} \quad (69)$$

In (68) and (69), we have used the fact that the only poles of $K(0, l_1 = i\rho; z)$ crossing the contour are the simple poles at $\rho = \mp i(1 - z)$.

Concerning analytic continuation, we perhaps should make the following remark: Suppose we have an analytic function f of complex z in a domain $D \subset \mathbb{C}$. Let Ω be an open, simply connected, subset of $\mathbb{C}$ with compact closure $\overline{\Omega}$ and with a regular boundary $\Gamma = \partial\Omega$. Let moreover Γ be an oriented cycle, inducing orientation on Ω through the Cauchy integral formula, so that $\Omega \cup \Gamma$ compose an oriented chain. Let W be an analytic function of complex $\rho, z \in D$, except for poles, which has no poles in the ρ -variable which cross the closed contour Γ when z varies within a subdomain $D_1 \subset D$. Suppose that for $z \in D_1$, f has a contour integral form

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} W(\rho, z) d\rho = \sum_i \operatorname{ind}_{\Gamma} \rho_i \operatorname{res}_{\rho_i} W_z, \quad W_z(\rho) = W(\rho, z) \quad (70)$$

equal, by the Cauchy theorem, to the sum of all residues in Ω , counted with the indices induced by the orientation of Γ (with residues, which should be understood as residues of the one-form $W_z d\rho$ rather than of the function W_z , in order for this construction to make sense on any Riemann surface). It is obvious that the same integral is in general not analytic in the whole D , and cannot represent f in the whole domain D , as for z going outside D_1 , some residues of the one-form $W_z d\rho$, will move outside Γ , and others will move inside Γ , causing the jump precisely equal to the difference between the sum of the residues going inside and the sum of residues, which goes outside. By definition, ρ lies inside or outside Γ , iff $\rho \in \Omega$ or $\rho \notin \overline{\Omega}$, respectively. In order to restore the analytic continuation of $f(z)$ for z outside D_1 , we need to add to the integral (70) all residues that come out, and subtract all that come in, of course, all weighted with one and the same index fixed by the chosen $\Omega \cup \Gamma$ -orientation. In this way, we recover the initial sum (70) of residues, which represented the integral (70) for z inside D_1 , but now with the argument z placed outside D_1 . That this sum represents analytic

continuation of f immediately follows from the fact that the residua are, by assumption, analytic functions of z . These considerations are valid for any Riemann surface, not just for $\mathbb{C}$. Passing to our case, our contour can be considered as a closed contour on the Riemann sphere $\mathbb{C} \cup \infty$. Next, we use the fact that only a finite number of poles cross our contour when z goes outside the domain $D_1 : \operatorname{Re} z > 1, -\epsilon < \operatorname{Im} z < \epsilon$. We separate our analytic function $F : \rho, z \mapsto F(\rho, z)$ into two analytic parts: $F = W_1 + W$, with W containing a finite number of poles, including all poles crossing the contour, and W_1 , subsuming the rest of poles. W can be chosen in the form of a finite sum of rational functions of the variables ρ, z (the reader will find specific formulas below). Finally, we observe that the integral

$$\int_{-\infty}^{+\infty} W_1(\rho, z) d\rho$$

with the poles of W_1 separated from our contour, represents an analytic function of z in the full domain $\operatorname{Re} z > 0, -\epsilon < \operatorname{Im} z < \epsilon$. To the part

$$\int_{-\infty}^{+\infty} W(\rho, z) d\rho,$$

we apply the same reasoning as to (70), because W , being a rational function, can be lifted to a meromorphic function on the Riemann sphere. As for the rule of signs, they are determined by the orientation of our contour and by the Cauchy theorem. In turn, the orientation of our contour follows from the construction of the integral $\int d\rho F(\rho, z)$ through a one-dimensional version of Stokes' theorem, in which $d\rho$ becomes the one-form, determined by the coordinate (or variable) ρ , on the one-dimensional cycle (after adjusting ∞) with the orientation of the cycle directed from $-\infty$ to $+\infty$. This fixes the signs in the invariant form independent of the choice of the open set Ω , which, when regarded as a subset of the Riemann sphere $\mathbb{C} \cup \infty$, can be chosen as the upper open hemisphere (with $\operatorname{Im} \rho > 0$) or as the lower open hemisphere (with $\operatorname{Im} \rho < 0$). For the upper hemisphere Ω , the residues in Ω get the index $+1$, and thus all residues have to be counted with the weight $+1$ in the corresponding sum (70). Therefore, the residues coming out of Ω ("top down"), have to be added. If we choose the lower hemisphere for Ω , the residues in Ω will get the index -1 , and thus all residues in the corresponding sum (70) have to be counted with the weight -1 . Thus, the residues coming in Ω ("top down") have to be added, as before. We see that changing the hemisphere does not change the rule of signs given above.

We base our analysis of the cyclic representations on the general properties of the matrix elements of the kernel K of decomposition (67) and its analytic continuation to the domain $0 < z < 1$ for the kernel $\langle g|h \rangle$ associated to the cyclic vector of the general form (1). These properties are deeply related to their analyticity. Before we formulate them, let us give explicit formulas for the particular kernel (62) with $\alpha = (1, 0)$ in (62). We compute explicitly (65) for $l_0 = 0$, $\alpha = (1, 0)$, and $z > 1$, and then compute the analytic continuation of such $K(l_0 = 0, l_1; z)$. We show that $K(l_0, l_1; z) = 0$, for $|l_0| > 0$, $z > 0$, $\alpha = (1, 0)$ in (62). As we have already mentioned, the case of $|l_0| > 1$, $z > 0$ follows from the orthogonality properties of the matrix elements of the irreducible representations of the $SU(2, \mathbb{C})$ group (Peter–Weyl theorem). It remains to show that in the case of $\alpha = (1, 0)$, $K(l_0, l_1; z) = 0$ also for $|l_0| = 1$, $z > 0$.

We compute the analytic continuation first. Up to the overall factor $\mathfrak{e}^2$ in front of (62) and (65) we have

$$[K(0, i\rho, z)]_{\alpha \alpha} = \frac{8\pi^4 e^z}{\rho^2 + 1} f_1(\rho, z) + \frac{16\pi^4 e^z}{\rho^2 + 1} f_2(\rho, z) + \frac{2\pi^4 z e^z}{\rho^2 + 1} f_3(\rho, z) \quad (71)$$

for each α with $l(\alpha) = 1$, where

$$\begin{aligned} f_1(\rho, z) &= \int_0^{+\infty} \left(\frac{\sinh(2\lambda) - 2\lambda}{\sinh \lambda} \right) \left(\frac{\cosh \lambda \sin(\rho \lambda)}{\rho \sinh \lambda} - \cos(\rho \lambda) \right) \frac{e^{-z\lambda \coth \lambda}}{\sinh^2 \lambda} d\lambda, \\ f_2(\rho, z) &= \int_0^{+\infty} \frac{[(\rho^2 + 1) \sinh \lambda \sin(\rho \lambda) + 2\rho \cosh \lambda \cos(\rho \lambda) - 2 \cosh \lambda \coth \lambda \sin(\rho \lambda)] (\lambda \coth \lambda - 1) e^{-z\lambda \coth \lambda}}{\rho \sinh^2 \lambda} d\lambda, \\ f_3(\rho, z) &= \int_0^{+\infty} \frac{[(\rho^2 + 1) \sinh \lambda \sin(\rho \lambda) + 2\rho \cosh \lambda \cos(\rho \lambda) - 2 \cosh \lambda \coth \lambda \sin(\rho \lambda)] (\sinh(2\lambda) - 2\lambda)^2 e^{-z\lambda \coth \lambda}}{\rho \sinh^4 \lambda} d\lambda. \end{aligned}$$

We construct the analytic continuation in two steps. In the first step, we construct a series of simple fractions which pointwisely (in the variables ρ, z , within the domain $|\operatorname{Im} \rho| < \epsilon$, $z > 1 + \epsilon$, for any positive ϵ) converges to K . In the second step, we prove the absolute and almost uniform convergence of this series for $(\rho, z) \in \mathbb{C} \times \mathbb{C}$, which proves that indeed it represents the required analytic continuation of K , regarded as an analytic function of the two complex variables ρ, z . We do it separately for each integral f_1, f_2, f_3 using a method which can be applied to more general K associated with cyclic (1). In order to illustrate the general method used in the first step,

let us consider the simple integral

$$f_0(z) = \int_0^{+\infty} e^{-z\lambda \coth \lambda} d\lambda.$$

Then we use the new variable $t = e^{-z\lambda}$, which converts the integral into the following

$$\frac{1}{z} \int_0^1 \exp \left[\frac{2t^{2/z}}{1-t^{2/z}} \ln t \right] dt,$$

in which the function of t in the argument of $\exp$ is bounded over the closed integration interval $[0, 1]$. Because the exponential series is uniformly convergent over any bounded domain, and the integration domain is bounded, then insertion of the exponential series and changing the order of summation and integration operations is allowed

$$f_0(z) = \frac{1}{z} \sum_{n=0}^{+\infty} \frac{1}{n!} \int_0^1 \left[\frac{2t^{2/z}}{1-t^{2/z}} \ln t \right]^n dt.$$

In the last integrals we use the new variable $s = -\ln t$, which, eventually, after simple rescaling of the variable, converts these integrals into

$$\int_0^1 \left[\frac{2t^{2/z}}{1-t^{2/z}} \ln t \right]^n dt = (-2)^n z^{n+1} \int_0^{+\infty} s^n e^{-(2n+z)s} \frac{1}{(1-e^{-2s})^n} ds.$$

In this integral, we use the convergence

$$\frac{1}{(1-e^{-2s})^n} = \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} e^{-2ks}$$

for all $s > 0$. Finally, the Lebesgue-dominated convergence principle is applicable to the last integral with the above series representation of the last factor in it, which again allows for changing the integration and summation over k . Therefore, using the identity

$$n! = \int_0^{+\infty} s^n e^{-s} ds,$$

we finally get

$$f_0(z) = \frac{1}{z} + \sum_{n=1}^{+\infty} \sum_{k=0}^{+\infty} \binom{k+n-1}{n-1} \frac{(-2z)^n}{(2k+2n+z)^{n+1}} = \frac{1}{z} - 2z \sum_{m=1}^{+\infty} \frac{(2m-z)^{m-1}}{(2m+z)^{m+1}},$$

where in this case, the absolute and almost uniform convergence of the double series is easily seen, and which allows reorganized summation (first with respect to all pairs (n, k) for which $n + k = m$ and then with respect to m) giving the last equality. Exactly the same changing of variables, with the same argument, allowing for changing the order of integration over a compact set and summation for uniformly convergent series, and dominated convergence principle, is applicable to our general integrals (including f_k , $k = 0, 1, 2, 3$)

$$f(\rho, z) = \int_0^{+\infty} g(\rho, z, \lambda) e^{-z\lambda \coth \lambda} d\lambda, \quad (72)$$

representing matrix elements of K , and thus, $\text{Tr } K$, associated with general cyclic (1) in which the function g can be represented as

$$g(\rho, z, \lambda) = \frac{\sum_{p \geq 0, j_p \geq -1} \left[a_{p, j_p}^+ \lambda^p e^{-j_p \lambda} e^{i \rho \lambda} + a_{p, j_p}^- \lambda^p e^{-j_p \lambda} e^{-i \rho \lambda} \right]}{(1 - e^{-2\lambda})^{q+1}}, \quad (73)$$

with integer $q \geq -1$ and finite sum over integer $p \geq 0$ and $j_p \geq -1$, and with $a_{p, j_p}^\pm$ being polynomial functions of ρ, z . The coefficients $a_{p, j_p}^\pm$ and powers of the exponent $e^{-\lambda}$ are such that g is analytic in ρ, λ , which, in some nonempty domain $|\text{Im } \rho| \leq \epsilon$, $\text{Re } z > 1 + \epsilon$ of the convergence of the integral, it has no singular points and the asymptotic $\sim e^{s\lambda}$, $s \leq 1 + \epsilon$ in λ at infinity, so that the zero of the order of $q + 1$ in λ at $\lambda = 0$ in the denominator cancels out with the zero of order $\geq q + 1$ of the numerator, and the above construction of the series expansion for (72) can be applied. Indeed, (29), (30) and the matrix elements (35), (46) all have the form (73) with $a_{p, j_p}^\pm$ being polynomial functions of ρ, z , divided by the common polynomial $\rho Q(\rho)$ in ρ , coming from the matrix elements (46). Their products, containing at most one factor (46), again have the same form (73), with $a_{p, j_p}^\pm$ being polynomial functions of ρ, z , divided by the polynomial $\rho Q(\rho)$, coming from the matrix elements (46). But any matrix element of K corresponding to a cyclic (1) is given by the integral (72) with g equal to a finite sum of products of the functions (29), (30), (35), (46), and $\sinh^2 \lambda$, with the factor of the type (46) entering each summand exactly once. Thus, g in (72) representing the matrix element of K has the form (73), divided by the polynomial $\rho Q(\rho)$.

For (73), the integral (72) has the following series representation:

$$f(\rho, z) = \sum_{n=0}^{+\infty} \sum_{k=0}^{+\infty} b_{n,k}(\rho, z) = \sum_{n=0}^{+\infty} \sum_{k=0}^{+\infty} (-2z)^n \binom{k+n+q}{n+q} \\ \times \sum_{p \geq 0, j_p \geq -1} \left[\frac{a_{p,j_p}^+ (n+1) \dots (n+p)}{(2k+2n+j_p+z-i\rho)^{n+p+1}} + \frac{a_{p,j_p}^- (n+1) \dots (n+p)}{(2k+2n+j_p+z+i\rho)^{n+p+1}} \right], \quad (74)$$

where the product $(n+1) \dots (n+p)$ is by definition equal to 1 for $p = 0$.

Returning to (62) with $l(\alpha) = 1$, we observe that the integrals f_1, f_2 are convergent for all $z > 0$, so they will have no contribution to the residue crossing the contour.

Lemma 1. *Each of the double series (74), representing matrix elements of K , is absolutely and uniformly convergent on any closed domain in $\mathbb{C} \times \mathbb{C}$ of the complex variables ρ, z in which $|j_p + z \mp i\rho| \leq R$, for any finite $R > 0$, except eventually the finite number of terms of the series, which have a pole in this domain or at its boundary. In particular, the series (74) representing K are analytic functions of ρ, z , having poles of finite order, without essential singularities.*

Proof. We ignore in the proof the common factor $\frac{1}{\rho Q(\rho)}$, coming from $U^{l_0, i\rho}$, in $b_{n,k}(\rho, z)$, which is irrelevant for the convergence claimed in the lemma. General term $b_{n,k}$ of the series is of the form

$$(-2z)^n \binom{k+n+q}{n+q} [\dots], \quad (75)$$

where dots denote the corresponding finite sum of simple fractions with denominators $(2k+2n+j_p+z \mp i\rho)^{n+p}$, $p = 1, 2, \dots, p_{\max}$. Let us replace the expression $2k+2n$ in each denominator with a variable x . Then, after reducing the sum in $[\dots]$ to a common denominator D , $b_{n,k}$ can be written as

$$(-2z)^n \binom{k+n+q}{n+q} \left[\frac{N(x=2k+2n)}{D(x=2k+2n)} \right]$$

with some polynomials N and D . If

$$\text{degree}[D(x)] - \text{degree}[x^{n+q}N(x)] \geq 2, \quad (76)$$

with x^{n+q} in the second term coming from the estimation

$$\binom{n}{k}^k < \binom{n}{k} < \left(\frac{ne}{k}\right)^k, \quad (77)$$

then the remainder, counting all terms of the series $|b_{n,k}|$ with $n+k \geq R$, is easily seen to be bounded by a remainder of a convergent series with the

bound independent of the particular choice of ρ, z in the assumed domain. The degree condition (76) can be checked “by hand on paper” only for the simplest double series, *e.g.* the first three simplest subseries in f_3 . It would be difficult to check it “by hand” even for some of the subseries in f_3 , and for the general term of the series (74). But we can use the observation: the condition (76) follows from the pointwise convergence of the series in the domain $|\operatorname{Im} \rho| < \epsilon$, $z > 1$, where the integral, represented by the series, is convergent. In this domain of the variables ρ, z , $|b_{n,k}(\rho, z)|$ goes to zero faster than $\frac{1}{n^2} \frac{1}{(n+k+q)^3}$. Indeed, let $\mathfrak{l}(\rho, z, \lambda)$ be the numerator of g in the integral representation (72) of the series (74). Then $b_{n,k}(\rho, z)$ is given by the Laplace integral

$$b_{n,k}(\rho, z) = \frac{(-2z)^n}{n!} \binom{k+n+q}{n+q} \int_0^{+\infty} \mathfrak{l}(\rho, z, \lambda) \lambda^n e^{-(z+2n+2k)\lambda} d\lambda,$$

in which $\mathfrak{l}$ has zero of the order of $\geq q+3$ at $\lambda = 0$, and, thus, $\mathfrak{l} \cdot \lambda^n$ has zero of the order of $\geq n+q+3$. Therefore, we have the asymptotic ([18], Theorem 1, p. 398)

$$b_{n,k}(\rho, z) \stackrel{n, k \rightarrow +\infty}{\sim} \frac{1}{(z+2n+2k)^{n+q+3}} \frac{(-2z)^n}{n!} \binom{k+n+q}{n+q},$$

if ρ, z are in the domain of pointwise convergence of the integral (72). Using this asymptotic and the estimation (77), it is easily seen that $|b_{n,k}(\rho, z)|$ goes to zero faster than $\frac{1}{n^2} \frac{1}{(n+k+q)^3}$, if ρ, z are in the domain of pointwise convergence of the integral (72). Using this, we proceed as follows. Let L_d be the coefficient of degree d of the polynomial $L(x) = x^{n+q}N(x)$, with $d \geq \deg[D(x)] - 1$. We observe that if $L_d \neq 0$, the components $b_{n,k}$ would be going to zero slower than $1/k^3$ for each fixed n —contradiction. Therefore, $L_d = 0$ for all $z > 1$, in the domain of convergence of the integral. L_d has the form of a polynomial in ρ, z . For each fixed ρ , L_d is a polynomial in z , with coefficients polynomially depending on ρ . A polynomial can have a continuum many zeros only if it is identically zero. Thus, for each $|\operatorname{Im} \rho| < \epsilon$, all coefficients of the polynomial L_d in z , depending on ρ , are zero. Therefore, $L_d(\rho, z) = 0$ identically for all $\rho, z \in \mathbb{C}$. $\square$

We have finished the construction of the analytic continuation of $K(0, i\rho; z)$ applicable generally to all $K(l_0, i\rho; z)$ corresponding to cyclic (1).

Now, we show that, in the case of $l(\alpha) = 1$, $K(l_0, l_1; z) = 0$ also for $|l_0| = 1$, and for all $z > 0$. Up to the overall factor ϵ^2 in front of (62) and (65), we have

$$[K(1, i\rho, z)]_{\alpha \alpha} = \int_0^{+\infty} g(\rho, z; \lambda) d\lambda, \quad l(\alpha) = 1$$

convergent for all $z > 0$, with the integrand

$$g(\rho, z; \lambda) = \frac{2\pi^4 e^z}{\rho(\rho^2+1)} \left[\frac{(\sinh(2\lambda)-2\lambda)(-2\sin(\rho\lambda)+2\rho^2\sinh^2\lambda\sin(\rho\lambda)+\rho\sinh(2\lambda)\cos(\rho\lambda))}{\sinh^2\lambda} \right. \\ \left. + \left(8(\lambda\coth\lambda - 1) - z \left(\frac{\sinh(2\lambda)-2\lambda}{\sinh\lambda} \right)^2 \right) (\coth\lambda\sin(\rho\lambda) - \rho\cos(\rho\lambda)) \right] \frac{e^{-z\lambda\coth\lambda}}{\sinh^2\lambda}$$

which has the antiderivative of the form

$$G(\rho, z; \lambda) = \frac{8\pi^4 e^z}{\rho(\rho^2+1)} e^{-z\lambda\coth\lambda} \left[\sin(\rho\lambda) + \frac{\rho\lambda\cos(\rho\lambda)+\sin(\rho\lambda)}{\sinh^2\lambda} \right. \\ \left. - \coth\lambda \left(\rho\cos(\rho\lambda) + \frac{\lambda\sin(\rho\lambda)}{\sinh^2\lambda} \right) \right] + C,$$

with C being arbitrary constant. Since

$$\lim_{\lambda \rightarrow 0} G(\rho, z; \lambda) = \lim_{\lambda \rightarrow +\infty} G(\rho, z; \lambda) = C,$$

then

$$[K(1, i\rho, z)]_{\alpha \alpha} = \int_0^{+\infty} g(\rho, z; \lambda) d\lambda = 0,$$

for all $z > 0$, $\rho \in \mathbb{R}$ and $l(\alpha) = 1$. Since this holds both, for non-negative and negative ρ , and because $U^{(1, i\rho)} = U^{(-1, -i\rho)}$, we have, therefore, shown that the Fourier transform $K(l_0, i\rho, z)$, i.e. (65), of the kernel (62) with $l(\alpha) = 1$, is equal to zero for all $z > 0$, $\rho \geq 0$, $|l_0| > 0$. Now, we are ready to formulate

Lemma 2. *Let $\langle g|h \rangle$ be the kernel (62) with $l(\alpha) = 1$ and f, f' smooth functions of compact support on G . We have the following decomposition valid for all $z > 0$:*

$$\begin{aligned} \langle f|f' \rangle &= \int_{G \times G} dg dh \langle g|h \rangle f(h) \overline{f'(g)} \\ &= \int_0^{+\infty} \frac{d\rho \epsilon^2 \rho^2}{2\pi^4} \text{Tr} [\mathcal{F}f(0, i\rho) K(0, i\rho; z) \mathcal{F}f'(0, i\rho)^*] \\ &\quad + \frac{2\epsilon^2 \pi e^z (1-z)(3-z)}{2-z} \text{Tr} [\mathcal{F}f(0, 1-z) \kappa \mathcal{F}f'(0, 1-z)^*], \quad (78) \end{aligned}$$

where the last term is present only if $0 < z < 1$, and where the only nonzero component of the matrix κ is the component $\kappa_{\alpha\alpha} = 1$, and similarly the only nonzero component of the positive matrix $K(0, i\rho; z)$ is the component (71) with f_1, f_2, f_3 given by the above sums of absolutely and almost uniformly convergent, in $\rho, z \in \mathbb{C}$, series.

We have also proved a lemma more general than Lemma 2, valid for the representations with cyclic vectors (1), but in general, decomposition will also contain non-spherical representations:

Lemma 3. *Let $\langle g|h \rangle$ be the invariant kernel of the representation with cyclic vector (1) and $q = 1$, $\alpha_1 = \alpha$. Then,*

$$\begin{aligned} \langle f|f' \rangle &= \int_{G \times G} dg dh \langle g|h \rangle f(h) \overline{f'(g)} \\ &= \sum_{l_0=-l(\alpha)}^{l(\alpha)} \int_0^{+\infty} d\rho \frac{l_0^2 + \rho^2}{2\pi^4} \text{Tr} [\mathcal{F}f(l_0, i\rho) K(l_0, i\rho; z) \mathcal{F}f'(l_0, i\rho)^*] \\ &\quad + \text{Tr} [\mathcal{F}f(0, 1-z) \kappa(z) \mathcal{F}f'(0, 1-z)^*] , \end{aligned} \quad (79)$$

with the sum over integer l_0 . The last term is present only if $0 < z < 1$. $K(l_0, l_1 = i\rho; z), \kappa(z)$ are finite dimensional positive definite matrices with the only nonzero coefficient $K_{\alpha\alpha}, \kappa_{\alpha\alpha}$, analytic in ρ, z . The nonzero matrix element $[K(l_0, l_1 = i\rho; z)]_{\alpha\alpha}$ is given by the absolutely and almost uniformly convergent, in $\rho, z \in \mathbb{C}$, series of the general form (74), divided by $\rho Q(\rho)$, where $Q(\rho)$ is the polynomial in ρ , coming from the denominator in formula (46) with $l = l(\beta), l' = l(\gamma)$. $\kappa(z)$ is given by the residue formula (69).

3. Relation between the cyclic representations.

Example of cyclic $x = |u\rangle$ and $x = c_{1,0}^+ |u\rangle$

From Lemma 2 it follows that the representation with the cyclic vector $x = |e\rangle = c_{\alpha}^+ e^{-inS(u)} |0\rangle$, and with $\alpha = (1, 0)$ decomposes into direct integral of the spherical representations ($l_0 = 0, l_1 = i\rho$), $\rho > 0$, of the principal series if $z > 1$, and, in addition, it has the supplementary component ($l_0 = 0, l_1 = 1 - z$) entering as a discrete direct summand, if $0 < z < 1$. The absence of non-spherical representations in Lemma 2 independently follows from

Lemma 4.

$$c_{1,0}^+ |u\rangle \in \mathcal{H}_{|u\rangle} .$$

Proof. Let $U(g_{03}(\lambda)) = U(\lambda)$. From the Baker–Campbell–Hausdorff formula, commutation rules (5), (6), and the second transformation rule in (7), we get

$$\begin{aligned} U(\lambda)|u\rangle &= U(\lambda)e^{-inS(u)}|0\rangle \\ &= e^{-\frac{n^2}{8\pi}\|B(\lambda)\|^2} e^{-inS(u)} \exp \left[\frac{+n}{4\pi\epsilon} \sum_l B_{l,0}(-\lambda) c_{l,0}^+ \right] |0\rangle. \end{aligned} \quad (80)$$

Differentiating (80) at $\lambda = 0$, we obtain a nonzero vector proportional to $c_{1,0}^+|u\rangle \in \mathcal{H}_{|u\rangle}$, due to the expansion (31). $\square$

Lemma 4 can still be strengthened using orthogonality relations of the matrix elements of the representations (l_0, l_1) .

Lemma 5. *The representations with cyclic $x = |u\rangle$ and $x = c_{1,0}^+|u\rangle$ coincide.*

Proof. We “smear out” the Lorentz transformed cyclic state $|g\rangle = U(g)c_{1,0}^+|u\rangle$ with a function f , Fourier transform $\mathcal{F}f$ of which has prescribed support. In fact, in the case of $z > 1$, we can use generalized state f with a single-point support $\{(l_0, l_1)\}$. Namely, we first consider

$$|f\rangle = \int f(g)|g\rangle dg, \quad f(g) = \int_{I_\epsilon} U_{0,0 \ 1,0}^{(l_0=0, l_1=i\rho)}(g) d\rho, \quad I_\epsilon = [\rho - \epsilon/2, \rho + \epsilon/2].$$

This state, decomposed accordingly to the last lemma, has nonzero decomposition components only at $(l_0, l_1) = (0, i\rho)$, $\rho \in I_\epsilon$. Next, we compute

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \langle u|f\rangle = \frac{4\pi^{7/2}\sqrt{z}e^z}{3} \int_0^{+\infty} \overline{B_{1,0}(\lambda)} U_{0,0 \ 1,0}^{(l_0=0, i\rho)}(\lambda) e^{-z\lambda \coth \lambda} \sinh^2 \lambda d\lambda, \quad (81)$$

obtaining analytic function of ρ, z of the same kind as the matrix elements of K , if we ignore the, irrelevant here, factor $4\pi^{7/2}\sqrt{z}e^z/5$ in front of the integral (81). We have used $z > 1$, allowing for exchanging the ϵ -limit with integration (dominated convergence principle), commutation rules (5), (6), transformation rule (7) and orthogonality (66). (86) can be represented by almost uniformly and absolutely convergent double series of finite sums of simple fractions of the general form (74), divided by the polynomial $\rho Q(\rho)$ in ρ , and which can explicitly be computed with the general method given above. This function has the property that it is not identically equal to zero as a function of ρ for each $z \neq 0$, in particular for each $z > 0$, understood as the analytic continuation for $0 < z < 1$. This can immediately be seen

from the value of the residues of (86) at $\rho = \pm i(z - 1)$, which, in turn, can easily be read off from the general series (74) representing (86), with the irrelevant factor in front of the integral in (86) ignored. Being analytic, it is, for each $z > 0$, nonzero for almost all ρ . Therefore, the decomposition component of the projection of the state $|u\rangle$ on the domain $\mathcal{H}_{c_{1,0}^+|u}$ of the cyclic representation with cyclic vector $c_{1,0}^+|u\rangle$ is nonzero for $(l_0 = 0, l_1 = i\rho)$ with almost all ρ . It means that the subspace spanned by the spherical representations in $\mathcal{H}_{c_{1,0}^+|u}$ is contained in $\mathcal{H}_{|u\rangle}$, if $z > 1$. In the more general case of $z > 0$, it means that the subspace of $\mathcal{H}_{c_{1,0}^+|u}$ spanned by the spherical representations, which is orthogonal to the subspace spanned by the supplementary series, is contained in the subspace of $\mathcal{H}_{|u\rangle}$, orthogonal to the subspace spanned by the supplementary series. Analogously, we “smear out” the state $|g\rangle = U(g)|u\rangle$, and consider the “smeared” state

$$|f\rangle = \int f(g)|g\rangle dg, \quad f(g) = \int_{I_\epsilon} U_{1,0}^{(l_0=0, l_1=i\rho)}(g) d\rho, \quad I_\epsilon = [\rho - \epsilon/2, \rho + \epsilon/2]$$

with the corresponding projection function

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \langle u | c_{1,0} | f \rangle = -\frac{4\pi^{7/2}\sqrt{ze^z}}{3} \int_0^{+\infty} B_{1,0}(\lambda) U_{1,0}^{(l_0=0, i\rho)}(\lambda) e^{-z\lambda \coth \lambda} \sinh^2 \lambda d\lambda, \quad (82)$$

and show that the decomposition components of the projection of the state $c_{1,0}^+|u\rangle$ on the domain $\mathcal{H}_{|u\rangle}$ of the cyclic representation with the cyclic vector $|u\rangle$ are nonzero for almost all ρ in the decomposition found in [4] (with the parameter ρ denoted by ν in [4]). It means that the subspace of $\mathcal{H}_{|u\rangle}$ spanned by the spherical representations, which is orthogonal to the subspace spanned by the supplementary series, is contained in the subspace of $\mathcal{H}_{c_{1,0}^+|u}$, orthogonal to the subspace spanned by the supplementary series. Therefore, the spherical parts of the representations with cyclic vectors $c_{1,0}^+|u\rangle$ and $|u\rangle$, orthogonal to the subspaces spanned by the supplementary series, do actually coincide for all $z > 0$. Since the non-spherical components of these cyclic representations are zero, these representations do actually coincide if $z > 1$. From $\mathcal{H}_{c_{1,0}^+|u} \subset \mathcal{H}_{|u\rangle}$ and Lemma 2, it follows a still stronger statement, that they coincide for all $z > 0$. $\square$

This is the method that we apply to the analysis of cyclic representations with cyclic vectors (1) and for their comparison.

Let us remark that, in principle, we can use the analytic continuations of the projection functions (81) and (82) to compare also the supplementary components, without adhering to the expansions (80) and (31). To this end, we continue analytically (81) and (82) to the value $\rho = -i(z-1)$ or $\rho = i(z-1)$, with $0 < z < 1$. The subtle point is that these values of ρ are, in general, among the places where the projection functions (81) and (82) (and their analogues with $c_{1,0}$ replaced with $c_{i,0}$) have residues. This reflects the fact that the matrix elements of the supplementary series are not square summable on G , even after integration over the continuous parameter of the series. Equivalently, the “smeared out” state, $|f\rangle$, as it stands in (81) and (82), is not normalizable for the values of ρ lying in neighborhoods of the points $\pm i(z-1)$, with $0 < z < 1$, understood as the analytic continuation of the above-given formula for $|f\rangle$. This shortcoming can be, as can easily be seen, repaired by multiplying the state $|f\rangle$, or the “smearing” function f , by $(\rho + i(z-1))$ or, respectively, by $(\rho - i(z-1))$, which removes the corresponding singularity and gives a normalizable state $|(\rho \pm i(z-1))f\rangle$ in a vicinity of the corresponding $\rho = \mp i(z-1)$. In other words,

Remark 1. *For the cyclic representations with cyclic vectors, respectively, $x = |u\rangle$ and $x = c_{i,0}^+|u\rangle$, the mutual projections of the supplementary components are nonzero, and thus these components coincide, iff the residue at $\rho = -i(z-1)$ or at $\rho = i(z-1)$, with $0 < z < 1$, of the analytic continuation of the projection functions, (86) and (87), is nonzero.*

In the next section we show that these residues are all nonzero.

Instead of calculating these residues, we can still use yet another method. Namely, in [9], a spherically symmetric normalized state

$$|u, \infty\rangle = \lim_{\lambda \rightarrow +\infty} c_\lambda \int_{\text{SU}(2, \mathbb{C})} da_1 U(a_1) U(\lambda) |u\rangle$$

was constructed, well-defined for real z if and only if $0 < z < 1$, and lying in the supplementary component of $\mathcal{H}_{|u\rangle}$, with the normalization factor c_λ equal to

$$c_\lambda = \frac{\sqrt{2} \sinh \lambda}{\left[\int_0^{2\lambda} dx \sinh x e^{-z(x \coth x - 1)} \right]^{1/2}}.$$

Instead of calculating the residues of Remark 1, we can calculate projections of the Lorentz transforms $U(\sigma) c_{i,0}^+ |u\rangle \in \mathcal{H}_{c_{i,0}^+ |u\rangle}$ of the cyclic vectors $c_{i,0}^+ |u\rangle$ on the normalized bound state $|u, \infty\rangle$. Here, $U(\sigma) = U(g_{03}(\sigma))$.

Lemma 6.

$$\langle u, \infty | U(\sigma) c_{i,0}^+ |u\rangle, \quad l = 1, 2, \dots, \quad 0 < z < 1$$

is nonzero for almost all hyperbolic angles σ .

Proof. It is not difficult to show that the one-parameter group $\sigma \mapsto U(\sigma)$ is strongly continuous. Because by construction, $|0\rangle, e^{iS(u)}|0\rangle$ belong to the domains of polynomials in $c_{l',0}^+$, $l' = 1, 2, \dots$, then also $c_{l,0}^+|u\rangle$ belongs to the domain of M_{03}^n , $n \in \mathbb{N}$, compare [8, 11], and beginning of Subsection 2.1. Using (5), (6), (7) and the exponentiation $U(\sigma)$ of $i\sigma M_{03}$, we show that $h_{\infty,l}(\sigma) = \langle u, \infty | U(\sigma) c_{l,0}^+ | u \rangle$ is an analytic function of σ , given by a power series around $\sigma = 0$ with the convergence radius $R > 0$. Let $|u_l\rangle = c_{l,0}^+|u\rangle$. Using (5), (6), and (7), we obtain analytic function

$$h_{l,l}(\sigma) = \langle u_l | U(\sigma) | u_l \rangle = \pi z e^{-z(\sigma \coth \sigma - 1)} \left[4\pi + (-1)^{l+1} B_{l,0}(\sigma)^2 \right], \quad (83)$$

represented by a power series around $\sigma = 0$, with the convergence radius equal to π . By theorem 13.35 of [20], $d^n h_{l,l} / d\sigma^n(0) = i^n \langle u_l | M_{03}^n | u_l \rangle$. Because the power series $h_{l,l}$ has the convergence radius π , we see that $\limsup_{n \rightarrow +\infty} (c'_n / n!)^{1/n} = 1/\pi$, on using the Cauchy–Hadamard formula for the convergence radius of a power series around $\sigma = 0$, where $c'_n = |\langle u_l | M_{03}^n | u_l \rangle|$. By the Cauchy–Schwartz inequality

$$c_n := |\langle u, \infty | M_{03}^n | u_l \rangle| \leq \|u, \infty\| \cdot |\langle u_l | M_{03}^{2n} | u_l \rangle|^{1/2} = |\langle u_l | M_{03}^{2n} | u_l \rangle|^{1/2} = \sqrt{c'_{2n}}.$$

From this, and by Stirling's approximation formula,

$$\begin{aligned} \limsup_{n \rightarrow +\infty} (c_n / n!)^{1/n} &\leq \limsup_{n \rightarrow +\infty} \left(\sqrt{c'_{2n}} / n! \right)^{1/n} \\ &= 2 \limsup_{n \rightarrow +\infty} (c'_{2n} / (2n)!)^{1/(2n)} = 2/\pi. \end{aligned}$$

Thus, by the Cauchy–Hadamard formula, the convergence radius R of the power series around $\sigma = 0$ for $h_{\infty,l}$ is greater than $\pi/2$.

To show our lemma, it is sufficient to determine the lowest-order terms in the expansion of $h_{\infty,l} = \langle u, \infty | U(\sigma) c_{l,0}^+ | u \rangle$ with respect to σ at $\sigma = 0$, and show that they are nonzero. In fact, it is sufficient to determine the lowest-order term and the behavior of $h_{\infty,l}$ at $\sigma = \infty$. Using the expansions (16), (31), and (80), the first transformation rule in (7), (66), and (33), and the product formula

$$T_{m_1 \ n_1}^{l_1}(a) T_{m_2 \ n_2}^{l_2}(a) = \sum_{l=|l_1-l_2|}^{l_1+l_2} \sum_{m,n=-l}^l C_{l_1, m_1 \ l_2, m_2}^{l, m} T_{m \ n}^l(a) C_{l_1, n_1 \ l_2, n_2}^{l, n},$$

with the Clebsch–Gordan coefficients $C_{l_1, n_1 \ l_2, n_2}^{l, n}$, and (16), (31), it is not difficult to see that the contribution of the lowest l -order term is equal to

$$\langle u, \infty | U(\sigma) c_{l,0}^+ | u \rangle = \text{sgn}(n) \sqrt{\pi} \sqrt{z} (1-z) b_{l,0} \left[1 + r_1(l)z + \dots + r_l(l)z^l \right] \sigma^l + \dots, \quad (84)$$

where the dots denote terms of the order of $> l$ in σ , and where

$$\begin{aligned} r_1(l) &= \sum_{j=1}^l \binom{l}{j} \frac{(-1)^j}{j(j+1)} = 1 - \gamma - \psi(l+2), \\ r_2(l) &= \frac{(-1)^l}{2!3!4^2} \sum_{j=0}^{l-2} \sum_{l_2=1}^{l-j-1} \frac{(-1)^j \frac{1}{i\epsilon} b_{l_2} \frac{1}{i\epsilon} b_{l-j-l_2} a_{l-j,l,0}}{\frac{1}{i\epsilon} b_l} \\ &\quad \times 2 \sqrt{\frac{2(l-j)+1}{(l-j)(l-j+1)}} 2 \sqrt{\frac{2l_2+1}{l_2(l_2+1)}} 2 \sqrt{\frac{2(l-j-l_2)+1}{(l-j-l_2)(l-j-l_2+1)}} \\ &\quad \times \frac{\langle u | c_{l-j,0} c_{l_2,0} c_{l-j-l_2,0} c_{l-j,0}^+ c_{l_2,0}^+ c_{l-j-l_2,0}^+ | u \rangle}{(4\pi\epsilon^2)^3} \int_{S^2} \frac{T_{0\ 0}^{l-j}(a) T_{0\ 0}^{l_2}(a) T_{0\ 0}^{l-j-l_2}(a)}{4\pi} da, \\ &\quad \vdots \\ r_{l-1}(l) &= \frac{(-1)^l}{(l-2)!2^l 4^{l-1}} \frac{\frac{1}{i\epsilon} b_2 \left(\frac{1}{i\epsilon} b_1 \right)^{l-2}}{\frac{1}{i\epsilon} b_l} 2 \sqrt{\frac{2l+1}{l(l+1)}} 2 \sqrt{\frac{5}{6}} \left(2 \sqrt{\frac{3}{2}} \right)^{l-2} \\ &\quad \times \frac{\langle u | c_{l,0} c_{2,0} (c_{1,0})^{l-2} c_{l,0}^+ c_{2,0}^+ (c_{1,0}^+)^{l-2} | u \rangle}{(4\pi\epsilon^2)^l} \int_{S^2} \frac{T_{0\ 0}^2(a) (T_{0\ 0}^1(a))^{l-2} T_{0\ 0}^l(a)}{4\pi} da \\ &\quad + \frac{(-1)^{l-1}}{(l-1)!2^l 4^{l-1}} \frac{\left(\frac{1}{i\epsilon} b_1 \right)^{l-1} a_{l-1,l,0}}{\frac{1}{i\epsilon} b_l} 2 \sqrt{\frac{2(l-1)+1}{(l-1)l}} \left(2 \sqrt{\frac{3}{2}} \right)^{l-1} \\ &\quad \times \frac{\langle u | c_{l-1,0} (c_{1,0})^{l-1} c_{l-1,0}^+ (c_{1,0}^+)^{l-1} | u \rangle}{(4\pi\epsilon^2)^l} \int_{S^2} \frac{(T_{0\ 0}^1(a))^{l-1} T_{0\ 0}^{l-1}(a)}{4\pi} da, \\ r_l(l) &= \frac{(-1)^l}{l!2^l 4^l} \frac{\left(\frac{1}{i\epsilon} b_1 \right)^l}{\frac{1}{i\epsilon} b_l} 2 \sqrt{\frac{2l+1}{l(l+1)}} \left(2 \sqrt{\frac{3}{2}} \right)^l \\ &\quad \times \frac{\langle u | c_{l,0} (c_{1,0})^l c_{l,0}^+ (c_{1,0}^+)^l | u \rangle}{(4\pi\epsilon^2)^{l+1}} \int_{S^2} \frac{(T_{0\ 0}^1(a))^l T_{0\ 0}^l(a)}{4\pi} da, \end{aligned}$$

with γ — the Euler gamma constant, ψ — digamma function. Here, $b_{l,0}$ is the coefficient of the lowest order in the expansion of $B_{l,0}(\lambda)$ at zero, and thus is equal to (32) multiplied by $\frac{4\pi\epsilon(-1)^l}{n}$, $a = g_{12}(\vartheta)g_{13}(\varphi)$, $da = \sin\varphi d\varphi d\vartheta$. A closer look at $r_k(l)$ shows that $r_k(l)$ are all negative and $r_1(l) \rightarrow -\infty$ if $l \rightarrow \infty$. For $l = 1, \dots, 4$, we have

l	$R_l(z) = 1 + r_1(l)z \dots + r_l(l)z^l$	roots $\in (0, 1)$
1	$1 - z/2$	no roots
2	$1 - 5z/6 - 3z^2/8 - z^3/6$	no roots
3	$1 - 13z/12 - 3z^2/8 - z^3/24$	≈ 0.726
4	$1 - 77z/60 - 71z^2/120 - 7z^3/60 - z^4/120$	≈ 0.596

Starting with $l = 3$, R_l has exactly one root in this interval, approximately equal to $-1/r_1(l)$ for large l , and converging to zero if $l \rightarrow \infty$.

To prove our lemma, it is sufficient to show, moreover, that

$$\begin{aligned}
& \lim_{\sigma \rightarrow \infty} c_\sigma \langle u, \infty | U(\sigma) c_{l,0}^+ | u \rangle = \text{sgn}(n) \sqrt{\pi} \sqrt{z} (1-z)^2 2 \sqrt{\frac{2l+1}{l(l+1)}} \\
& \times \left[1 + \sum_{k=2}^{\infty} \frac{(-z)^k}{k!^2} \sum_{l_1, \dots, l_k=1}^{\infty} \prod_{j=1}^k \left[\frac{2l_j+1}{l_j(l_j+1)} \right] \frac{\langle u | \prod_{j=1}^k [c_{l_j,0}] \prod_{s=1}^k [c_{l_s,0}^+] | u \rangle}{(4\pi \epsilon^2)^k} \int_{S^2} \prod_{s=1}^k \frac{T_{0\ 0}^{l_s}(a)}{4\pi} da \right] \\
& = \sqrt{\pi} \sqrt{z} (1-z)^2 2 \sqrt{\frac{2l+1}{l(l+1)}} \left[1 + \frac{z^2}{2} + \sum_{k=3}^{\infty} (-1)^k w_k z^k \right] \quad (85)
\end{aligned}$$

is nonzero in the interval $z \in (0, 0.73)$. Using the asymptotic properties of the Clebsch–Gordan coefficients, it can be shown that $0 < w_k < 1$ in the series (85), so that (85) is absolutely convergent for $0 < z < 1$, which legitimates exchanging the summation and transition to the limit, in obtaining the formulas (84) and (85) (averaging over $\text{SU}(2, \mathbb{C})$ intervenes substantially). The coefficient $w_2 = 1/2$ can be easily exactly computed, and we have estimated numerically more accurately the coefficients $w_3 \approx 32/375$, $w_4 \approx 259/4000$, so that

$$\begin{aligned}
& 1 + \sum_{k=2}^{\infty} (-1)^k w_k z^k > 1 + \frac{z^2}{2} - w_3 z^3 + w_4 z^4 - \frac{z^5}{1-z} \\
& > 1 + \frac{z^2}{2} - \frac{32z^3}{375} + \frac{259z^4}{4000} - \frac{z^5}{1-z} > 0, \quad \text{for } 0 < z < 0.77.
\end{aligned}$$

□

From Lemma 6 and Lemma 3, it follows that the supplementary component of the representation with cyclic vector $c_{l,0}^+ | u \rangle$ is nonzero for each $l = 1, 2, \dots$, and each of these supplementary components coincides with the supplementary component of the representation with the cyclic vector $| u \rangle$.

To implement generally the method used in the proof of Lemma 5, we prove, in Section 4, that the functions (65), and the analogues of the projection functions (81) and (82), joining the cyclic representations with cyclic

vectors $c_\alpha^+|u\rangle, c_{\alpha'}^+|u\rangle$, are nonzero functions of ρ for each positive z , if

$$\min(l(\alpha), l(\alpha')) > 1.$$

The case $l(\alpha) = 1, l_0 = 1$, in (65) is exceptional. The cyclic representation with the cyclic vector $c_\alpha^+|u\rangle$ and with $l(\alpha) = 1$ coincides with the representation with the cyclic vector $|u\rangle$. This makes it exceptional among the cyclic representations with cyclic vectors $c_\alpha^+|u\rangle$. As we have seen, (65) is an identically zero function for $l(\alpha) = 1$ and $l_0 = 1$ in (65), and the representation with the cyclic vector $c_\alpha^+|u\rangle$ and $l(\alpha) = 1$ decomposes into purely spherical representations, which is an exceptional property coming from the theory [3]. This is why we have treated the case with $l(\alpha) = 1$ in (65) explicitly. To investigate the relation between the representation spaces of representations with cyclic (1) in which $\mathfrak{q} > 1$, to the representations with cyclic (1) in which $\mathfrak{q} = 1$, we use the expansions (80) and (31), as in the proof that $c_{1,0}^+|u\rangle \in \mathcal{H}_{|u\rangle}$, given above, which allows us to reduce the problem to the mutual relations of the representations with cyclic (1) in which $\mathfrak{q} = 1$, *i.e.* with a cyclic vector of the form $c_\alpha^+|u\rangle$.

4. Poles and asymptotic of $\text{Tr } K$ and projection functions

For the proof of our theorem, we need the projection functions of the cyclic vectors — the analogues of (86) in which we replace the cyclic vectors $|u\rangle$ and $c_{1,0}^+|u\rangle$ with the more general cyclic vectors of the form $c_\alpha^+|u\rangle$ and the components $0, 0 \times 0, 1$ in the “smearing” with arbitrary components.

Namely, the first class of projective functions we need, connect the cyclic representations with cyclic vectors, respectively, equal to $x = |u\rangle$ and $x = c_{1,0}^+|u\rangle$, in the sense explained in the proof of Lemma 5. Thus, they arise by replacing the state $|g\rangle = U(g)c_{1,0}^+|u\rangle$ with $|g\rangle = U(g)c_{l,0}^+|u\rangle$ in (81) and with the “smeared” state

$$|f\rangle = \int f(g)|g\rangle dg, \quad f(g) = \int_{I_\epsilon} U_{0,0}^{(l_0=0, l_1=i\rho)}(g) d\rho, \quad I_\epsilon = [\rho - \frac{\epsilon}{2}, \rho + \frac{\epsilon}{2}]$$

and we introduce the following projection function

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \langle u|f\rangle = \frac{4\pi^{7/2}\sqrt{z}e^z}{2l+1} \int_0^{+\infty} \overline{B_{l,0}(\lambda)} U_{0,0}^{(l_0=0, i\rho)}(\lambda) e^{-z\lambda \coth \lambda} \sinh^2 \lambda d\lambda. \quad (86)$$

For the comparison of representations, we also need the corresponding projection function with the cyclic vectors reversed. Namely with $|g\rangle = U(g)|u\rangle$,

and with the “smeared” state

$$|f\rangle = \int f(g)|g\rangle dg, \quad f(g) = \int_{I_\epsilon} U_{l,0 \quad 0,0}^{(l_0=0, l_1=i\rho)}(g) d\rho, \quad I_\epsilon = [\rho - \frac{\epsilon}{2}, \rho + \frac{\epsilon}{2}]$$

and the projection function

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \langle u | c_{l,0} | f \rangle = \frac{4\pi^{7/2} \sqrt{z} e^z}{2l+1} (-1)^l \int_0^{+\infty} B_{l,0}(\lambda) U_{l,0 \quad 0,0}^{(l_0=0, i\rho)}(\lambda) e^{-z\lambda \coth \lambda} \sinh^2 \lambda d\lambda. \quad (87)$$

But for each fixed z , (86) is a nonzero function of ρ if and only if (87) is a nonzero function of ρ , and we need to analyze only one of the two (86) and (87). The same is true of the residues of (86) and (87) at $\rho = -i(z-1)$, or at $\rho = i(z-1)$, with $0 < z < 1$: the residue is nonzero for (86) if and only if it is nonzero for (87). We need the second class of projection functions, which connect the cyclic representations with cyclic vectors, respectively, equal to $x = c_{\alpha'}^+ |u\rangle$ and $x = c_\alpha^+ |u\rangle$ defined in the analogous way. Namely, let $|g\rangle = U(g)c_{\alpha'}^+ |u\rangle$ and $z > 1$. Let us consider

$$|f\rangle = \int f(g)|g\rangle dg, \quad f(g) = \int_{I_\epsilon} U_{\beta \quad \beta'}^{(l_0, i\rho)}(g) d\rho, \quad I_\epsilon = [\rho - \frac{\epsilon}{2}, \rho + \frac{\epsilon}{2}]$$

and the following projection functions:

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \langle u | c_\alpha | f \rangle &= \frac{\delta_{\beta \quad \alpha} \delta_{\beta' \quad \alpha'} (4\pi\epsilon)^2 e^z}{(2l+1)(2l'+1)} \int_0^{+\infty} \left[\sum_{n=-l}^l \overline{A_{l,n \quad l',n}}(\lambda) U_{l,n \quad l',n}^{(l_0, i\rho)}(\lambda) \right. \\ &\quad \left. + \frac{z}{4} (-1)^l \overline{\frac{1}{\epsilon} B_{l',0}}(\lambda) \frac{1}{\epsilon} B_{l,0}(\lambda) U_{l,0 \quad l',0}^{(l_0, i\rho)}(\lambda) \right] e^{-z\lambda \coth \lambda} \pi^2 \sinh^2 \lambda d\lambda, \end{aligned} \quad (88)$$

where $\alpha = (l, m)$, $\alpha' = (l', m')$, $l \leq l'$. Again, we have used in (86), (87) and (88), the fact that the integration and passing to the $\epsilon \rightarrow 0$ limit operations can be exchanged if $z > 1$, by the dominated convergence principle, and where we have used (66), commutation rules (5), (6), and transformation rule (7). (86), (87) and (65), (88) are analytic functions of $(\rho, z) \in \mathbb{C}^2$ and (up to the factors in front of the integrals) can be represented by almost uniformly and absolutely convergent double series of finite sums of simple fractions of the general form (74).

We first note the following.

Lemma 7. *The residue at $\rho = -i(z-1)$ or at $\rho = i(z-1)$, with $0 < z < 1$, of the analytic continuation of the projection functions, (86), (87), is nonzero for each $l = 1, 2, \dots$*

Proof. We need to analyze only (86). Having given (55), (29), and (30), we represent the function g of the integrand $ge^{-z\lambda\coth\lambda}$ in (86) in the general form (73) with the analytic continuation of the integral in (86) given by the corresponding series (74). It is easily seen that the residue of this series at $\rho = -i(z-1)$ or, respectively, at $\rho = i(z-1)$ is equal to

$$ia_{0,-1}^+(\rho = -i(z-1)) \quad \text{or, respectively,} \quad ia_{0,-1}^-(\rho = i(z-1)),$$

where $a_{0,-1}^\pm(\rho)$ are the polynomials in (73) and in the series (74) equal to the analytic continuation of the integral in (86). In these formulas for the residues, we ignore the factor in front of the integral (86). We will bring it back at the very end of the calculation. Using formulas (33) and (55), we get the following values:

$$\begin{aligned} & \frac{\epsilon 32\pi^{7/2} \sqrt{ze^z} i^{l+1}}{\sqrt{l(l+1)} \prod_{s=0}^l (-i(s+z-1))} \prod_{j=2}^{l+1} (z-j) \quad \text{or, respectively,} \\ & - \frac{\epsilon 8\pi^{7/2} \sqrt{ze^z} i^l}{\sqrt{l(l+1)} \prod_{s=0}^l (-i(s-z+1))} \prod_{j=2}^{l+1} (z-j), \end{aligned}$$

for the residues. It is evident that they are nonzero for each $0 < z < 1$ and $l = 1, 2, \dots$ $\square$

Lemma 8. *(65) is identically zero function of (z, ρ) if $l = l_0 = 1$ in (65). (88) is identically zero function of (z, ρ) , if $l = l_0 = 1$ or $l' = l_0 = 1$ in (88).*

Proof. The first statement has been already proved in Subsection 2.4. Because (65) is identically zero function of (z, ρ) , if $l = l_0 = 1$ in (65), then by definition, the projection functions (88) must be identically zero functions of (z, ρ) , if $l = l_0 = 1$ or $l' = l_0 = 1$ in (88). (Compare the proof of Lemma 5.) $\square$

One can also convince himself of the validity of this lemma for particular values of l or l' with $l_0 = 1$ and $l = 1$ or $l' = 1$, by explicit calculation. In these cases, the integrals (65) and (88) are elementary and have the primitives which are in the same class (73), multiplied by $e^{-z\lambda\coth\lambda}$, as the integrands themselves in (65) and (88), with the primitives which have the

same limit value at zero and at infinity, thus giving identically zero values for (65) and (88).

We show that for all remaining cases, (65) and (88), with $\alpha = \beta$, $\alpha' = \beta'$ in (88), are, for each fixed $z > 0$, nonzero functions of ρ , *i.e.* in this section, we give a proof that (65), (88), and (86) are, for each positive z , nonzero functions of ρ , except for $l = l' = l_0 = 1$ in (65) and except for $l_0 = l = 1$ or $l_0 = l' = 1$ in (88). Since the common factor, equal to the inverse of the polynomial $\rho Q(\rho)$, and coming from the denominator in formula (46), is nonzero for all real ρ , we ignore this common factor $\frac{1}{\rho Q(\rho)}$ in the proof, and the factors in front of the integrals in (65), (88), and (86). We give here a short outline of the proof. First, we show the statement that, under these restrictions, and any finite positive z , (analytic continuation of) (65), (88), (86) will have nontrivial poles, *i.e.* with nonzero coefficients $L_{m,s}(z)$ in the Laurent series, for which the distance $m + z$ of the pole from the real axis in the complex ρ -plane is arbitrarily large, the order s of the pole is arbitrarily large, and the ratio m/s of the distance m to the order s is arbitrarily large. In fact, we put $m = s^2$ with the order s going to infinity. We prove it in Subsection 4.1. As we have ignored the factor $\frac{1}{\rho Q(\rho)}$, and the factors in front of the integrals, the Laurent coefficients $L_{m,s}(z)$ are polynomial functions of z . We compute the asymptotic form of the polynomials $L_{m,s}(z)$ equal to the Laurent coefficients for large $m, s, m/s$, and show that for each $z > 0$, they are nonzero for large $m, s, m/s$, if $\min(l, l') > 1$ or $l_0 \neq 1$. This proves that (65), (88) and (86), are nonzero functions for each $z > 0$, except for $l = l' = l_0 = 1$ in (65) and except for $l_0 = l = 1$ or $l_0 = l' = 1$ in (88). In these exceptional cases, (65) and (88) are identically zero functions of z, ρ .

4.1. Asymptotic of the Laurent coefficient $L_{m,s}(z)$

The integrals in (65), (86) or (88), regarded as functions of ρ, z , have the asymptotic expansions of the general form ([18], Corollary 1, p. 408)

$$\underset{z \rightarrow +\infty}{\sim} e^{-z} \sum_{n=n_0}^{+\infty} c_n z^{-(n+1)/2} \quad (89)$$

with c_n depending rationally on ρ , and with n_0 greater than or equal to the order of zero of g_0 at $\lambda = 0$, where $g(\rho, z, \lambda) = \sum_{i=0}^q z^i g_i(\rho, \lambda)$ in the integrand in the general form (72) of these integrals. More generally, the integrals (72), giving, up to the factor e^z , the matrix elements of the Fourier transform (63), have the asymptotic (89). Here, we ignore the factors, respectively,

$$\left(\frac{4\pi\epsilon}{2l+1} \right)^2 e^z, \quad \frac{4\pi^{7/2} \sqrt{z} e^z}{2l+1} \quad \text{or} \quad \frac{\delta_\beta \alpha \delta_{\beta'} \alpha' (4\pi\epsilon)^2 e^z}{(2l+1)(2l'+1)}, \quad (90)$$

standing before the integrals in (65), (86) or (88). Therefore, in order to obtain the asymptotic of the full expressions (65), (86) or (88), we need to multiply (89), respectively, by (90). Similarly, we need to multiply (89) by e^z in order to obtain asymptotics of the matrix elements of K .

Applying Corollary 1, pp. 407, 408 of [18] to each integrand $g_i(\rho, \lambda)$ separately, and using the expansions (57)–(60) and (31), it is easily seen that for appropriately large z , (65), (86), and (88) are nonzero functions of ρ , if $l > 1$ in (65) and $\min\{l, l'\} > 1$ in (88). Therefore, only in the mentioned exceptional cases, these functions can be identically zero.

We represent (65) or (88) or, respectively, (86), in the form (72) with the integrand (73), with common denominator $(1 - e^{-2\lambda})^{q+1}$, with q equal $4l, 2(l + l')$, or $2l - 1$, respectively, for (65) and (88) or, respectively, (86). Using the parities of j_p, j in (39) and (56), we can easily see that the parities of the integer numbers $j_p \geq -1$ in formula (73) representing (65), (88) or (86), are the same as the parity of the number $l_0 - 1$, where l_0 is the number in $U^{(l_0, i\rho)}$ in (65), (88) or (86). In what follows, we ignore the common factor $\frac{1}{\rho Q(\rho)}$, coming from $U^{(l_0, i\rho)}$, and the factors (90) in formula (73) representing (65), (88) or (86).

Let us introduce the coefficients $a_{p, j_p, k}^{\pm}$ of the polynomials

$$a_{p, j_p}^{\pm}(\rho) = \sum_k a_{p, j_p, k}^{\pm} \rho^k$$

in (73). In the formula of the integrand (73) in (65) or in (88), we collect all terms $a_{p, j_p, k}^{\pm}$, containing linearly the parameter z , which represent the second summand of the integrand in (65) or (88), containing z as a factor, and denote them with the prime

$$a_{p, j_p, k}^{\pm} = z a'_{p, j_p, k}{}^{\pm}$$

and all remaining terms $a_{p, j_p, k}^{\pm}$ without z , and coming from the first summand of the integrand in (65) or (88), let us denote with the double prime

$$a_{p, j_p, k}^{\pm} = a''_{p, j_p, k}{}^{\pm}.$$

The coefficients will be denoted simply by $a_{p, j_p, k}^{\pm}$ without any primes, in the formula for the integrand (73) in the integral (72) in (86). Note that $p = 0, 1$ or 2 for the single primed coefficients, and $p = 0$ or 1 for the double primed in (73), representing (65) or (88), and $p = 0$ or 1 for $a_{p, j_p}^{\pm}$ in (73) representing (86).

Using the fact that f_0 is of the first-order, and f_1 is of zero order at $\lambda = 0$ in formula (39) for $B_{l,0}(\lambda)$ and $A_{l,m,l',m}(\lambda)$, and formulas (47), (48) for the coefficients of the maximal order in ρ in the numerator of (46), we easily see the following order behavior of the function in the numerator of (73). The sum of exponents of λ , regarded as a function $f(\lambda)$ of λ , multiplying the monomial $\lambda^0 \rho^{j_{\max}} e^{i\rho\lambda}$ of the maximal degree in ρ , which collects all single primed terms, is of $j_{\max} + 2 = l + l' - l_0 + 2$ -order at $\lambda = 0$ for (88), with $l = l'$ for (65). The function of λ multiplying the monomial $\lambda^1 \rho^{j_{\max}} e^{i\rho\lambda}$ of the maximal degree in ρ , which collects all single-primed terms, is of $j_{\max} + 1 = l + l' - l_0 + 1$ -order for (88), with $l = l'$ for (65). The function of λ multiplying the monomial $\lambda^2 \rho^{j_{\max}} e^{i\rho\lambda}$ of the maximal degree in ρ , which collects all single-primed terms, is of $j_{\max} = l + l' - l_0$ -order for (88), with $l = l'$ for (65). The function of λ multiplying the monomial $\lambda^0 \rho^{j_{\max}} e^{i\rho\lambda}$ of the maximal degree in ρ , which collects all double-primed terms, is of $j_{\max} + 2 = l + l' + 2$ -order for (88) with $l = l'$ for (65). The function of λ multiplying the monomial $\lambda^1 \rho^{j_{\max}} e^{i\rho\lambda}$ of the maximal degree in ρ , which collects all double-primed terms, is of $j_{\max} + 1 = l + l' + 1$ -order for (88), with $l = l'$ for (65). The function of λ multiplying the monomial $\lambda^0 \rho^{j_{\max}} e^{i\rho\lambda}$ of the maximal degree in ρ , which collects all unprimed terms, is of $j_{\max} + 1 = l + 1$ -order for (86). The function of λ multiplying the monomial $\lambda^1 \rho^{j_{\max}} e^{i\rho\lambda}$ of the maximal degree in ρ , which collects all unprimed terms, is of $j_{\max} = l$ -order for (86). The function(s) $f(\lambda)$ multiplying the monomial $\lambda^p \rho^{j_{\max}} e^{i\rho\lambda}$ has the k -order Taylor coefficient at $\lambda = 0$ equal to

$$\frac{1}{k!} f^{(k)}(0) = \frac{(-1)^k}{k!} \sum_{j_p} a'^+_{p,j_p,j_{\max}} j_p^k,$$

for the primed contribution, and analogously for the double primed contribution in the integrand (73), (88) or (65), and the unprimed $a'^+_{p,j_p,j_{\max}}$ in the integrand (73) of the integral (72) in (86). Therefore, this order behavior can be summarized by the following formulas

$$\begin{aligned} \sum_{j_0} a'^+_{0,j_0,l+l'-l_0} j_0^{l+l'-l_0+2} &\neq 0, \\ \sum_{j_0} a'^+_{0,j_0,l+l'-l_0} j_0^k &= 0, \quad k = 0, \dots, l + l' - l_0 + 1, \\ \sum_{j_1} a'^+_{1,j_1,l+l'-l_0} j_0^{l+l'-l_0+1} &\neq 0, \end{aligned}$$

$$\begin{aligned}
\sum_{j_0} a'_{1,j_1,l+l'-l_0}{}^+ j_1^k &= 0, \quad k = 0, \dots, l + l' - l_0, \\
\sum_{j_2} a'_{2,j_2,l+l'-l_0}{}^+ j_2^{l+l'-l_0} &\neq 0, \\
\sum_{j_0} a'_{2,j_2,l+l'-l_0}{}^+ j_2^k &= 0, \quad k = 0, \dots, l + l' - l_0 - 1,
\end{aligned} \tag{91}$$

for the single-primed coefficients,

$$\begin{aligned}
\sum_{j_0} a''_{0,j_0,l+l'}{}^+ j_0^{l+l'+2} &\neq 0, \\
\sum_{j_0} a''_{1,j_1,l+l'}{}^+ j_0^k &= 0, \quad k = 0, \dots, l + l' + 1, \\
\sum_{j_1} a''_{1,j_1,l+l'}{}^+ j_1^{l+l'+1} &\neq 0, \\
\sum_{j_1} a''_{1,j_1,l+l'}{}^+ j_1^k &= 0, \quad k = 0, \dots, l + l',
\end{aligned} \tag{92}$$

for the double-primed coefficients,

$$\begin{aligned}
\sum_{j_0} a_{0,j_0,l}{}^+ j_0^{l+1} &\neq 0, \\
\sum_{j_0} a_{0,j_0,l}{}^+ j_0^k &= 0, \quad k = 0, \dots, l + 1, \\
\sum_{j_1} a_{1,j_1,l}{}^+ j_1^l &\neq 0, \\
\sum_{j_1} a_{1,j_1,l}{}^+ j_1^k &= 0, \quad k = 0, \dots, l,
\end{aligned} \tag{93}$$

for the unprimed coefficients. In particular, using (44) and (50), we obtain

$$\begin{aligned}
&\frac{(-1)^{l+l'+1}}{(l+l'+1)!} \sum_{j_1} a''_{1,j_1,l+l'}{}^+ j_1^{l+l'+1} \\
&= \frac{1}{i} 2^{3l+4l'} (-1)^{2l+l'} l l' \frac{(2l+1)(2l'+1)}{(l'+1)!} \binom{l+l'-1/2}{l+l'} \binom{l'-1/2}{l'} \\
&\times \prod_{j=0}^{l'-1} \frac{l+j+1}{2(l+j)+1} \prod_{r=0}^{l'-2} (l + l' - r),
\end{aligned} \tag{94}$$

for $l_0 = 1$. Using (42) and (52), we get

$$\begin{aligned} & \frac{(-1)^{l+l'-l_0}}{(l+l'-l_0)!} \sum_{j_2} a_{2,j_2,l+l'-l_0}^{l'+} j_2^{l+l'-l_0} \\ &= (-1)^{2l+l'+1} 2^{4l+2l'+1} l(l+1)l'(l'+1) \binom{l'+1/2}{l'+1} \binom{l+1/2}{l+1} \end{aligned} \quad (95)$$

for $l_0 = 1$.

Next, we consider the Laurent coefficient $L_{m,s}(z)$ of the Laurent expansion of the series (74), representing (65), (88) or (86), at $\rho = -i(m+z)$, of the order of s , with the factor $\frac{1}{\rho Q(\rho)}$ and the factors (90) ignored. Therefore, in order to compute $L_{m,s}(z)$, we need to consider only the simple fractions in the series (74) in which $2k+2n+j_p = m$ is fixed and equal to m , with the parity of m the same as the parity of $l_0 - 1$. We compute $L_{m,s}(z)$ and its asymptotic for $s \rightarrow +\infty$, with $m = s^2$, for the general (74), with the restriction that the range of values of p is $0, 1, 2$, and with $a_{p,j_p}^{\pm}$ being polynomials in ρ , which is the case for (65), (88) or (86), because we have ignored the factor $\frac{1}{\rho Q(\rho)}$ and the factors (90). We compute separately the contributions to $L_{m,s}(z)$ coming from the simple fractions containing, respectively, the polynomials $a_{0,j_0}^{\pm}$, $a_{1,j_1}^{\pm}$, and separately $a_{2,j_2}^{\pm}$. Let $\mathfrak{d}_0, \mathfrak{d}_1, \mathfrak{d}_2$ be, respectively, the degrees of these polynomials.

Then the contribution to the Laurent coefficient $L_{m,s}(z)$, coming from the simple fractions containing the polynomials $a_{0,j_0}^{\pm}$, $a_{1,j_1}^{\pm}$ and, respectively, $a_{2,j_2}^{\pm}$ is equal, for large m, s , to

$$\sum_{\ell}^{\mathfrak{d}_0} i a_{0,j_0,\ell}^{+} (-2iz)^{s-1} \sum_{r=0}^{\ell} \binom{\ell}{r} (-i(m+z))^{\ell-r} (-2iz)^r \binom{(m-j_0)/2+q}{s+r+q-1}, \quad (96)$$

$$\sum_{\ell}^{\mathfrak{d}_1} i^2 a_{1,j_1,\ell}^{+} (-2iz)^{s-2} \sum_{r=0}^{\ell} \binom{\ell}{r} (-i(m+z))^{\ell-r} (-2iz)^r \binom{(m-j_1)/2+q}{s+r+q-2}, \quad (97)$$

and

$$\sum_{\ell}^{\mathfrak{d}_2} i^3 a_{2,j_2,\ell}^{+} (-2iz)^{s-3} \sum_{r=0}^{\ell} \binom{\ell}{r} (-i(m+z))^{\ell-r} (-2iz)^r \binom{(m-j_2)/2+q}{s+r+q-3}, \quad (98)$$

so that $L_{m,s}(z)$ is equal to the sum of (96), (97), and (98). The asymptotic expansion $\widetilde{s \rightarrow +\infty}$ of the contribution to the Laurent coefficient $L_{m=s^2,s}(z)$, coming from the simple fractions containing the polynomials $a_{0,j_0}^{\pm}$, $a_{1,j_1}^{\pm}$ and, respectively, $a_{2,j_2}^{\pm}$, is equal to

$$\begin{aligned}
& \widetilde{s \rightarrow +\infty} \frac{i(-2iz)^{s-1}}{e\sqrt{2\pi s}} e^{-\frac{2}{3s} - \frac{2}{3s^2} - \frac{4}{5s^3} - \frac{16}{15s^4} - \dots} \left(\frac{es}{2}\right)^s \left(\frac{s}{2}\right)^{q-1} \sum_{\ell}^{\mathfrak{d}_0} \sum_{r=0}^{\ell} \sum_{j_0}^{\ell} \\
& \times \left[\binom{\ell}{r} (-i(s^2 + z))^{\ell-r} \left(-\frac{2izs}{2}\right)^r a_{0,j_0,\ell}^+ \left[1 + \left(\frac{(-1)^1 j_0^1}{1!} + \dots\right) \frac{1}{s^1} \right. \right. \\
& \left. \left. + \left(\frac{(-1)^2 j_0^2}{2!} + \dots\right) \frac{1}{s^2} + \left(\frac{(-1)^3 j_0^3}{3!} + \dots\right) \frac{1}{s^3} + \left(\frac{(-1)^4 j_0^4}{4!} + \dots\right) \frac{1}{s^4} + \dots \right] \right], \\
& \tag{99}
\end{aligned}$$

$$\begin{aligned}
& \widetilde{s \rightarrow +\infty} \frac{i^2(-2iz)^{s-2}}{e\sqrt{2\pi s}} e^{-\frac{2}{3s} - \frac{2}{3s^2} - \frac{4}{5s^3} - \frac{16}{15s^4} - \dots} \left(\frac{es}{2}\right)^s \left(\frac{s}{2}\right)^{q-2} \sum_{\ell}^{\mathfrak{d}_1} \sum_{r=0}^{\ell} \sum_{j_1}^{\ell} \\
& \times \left[\binom{\ell}{r} (-i(s^2 + z))^{\ell-r} \left(-\frac{2izs}{2}\right)^r a_{1,j_1,\ell}^+ (r+s-1) \left[1 + \left(\frac{(-1)^1 j_1^1}{1!} + \dots\right) \frac{1}{s^1} \right. \right. \\
& \left. \left. + \left(\frac{(-1)^2 j_1^2}{2!} + \dots\right) \frac{1}{s^2} + \left(\frac{(-1)^3 j_1^3}{3!} + \dots\right) \frac{1}{s^3} + \left(\frac{(-1)^4 j_1^4}{4!} + \dots\right) \frac{1}{s^4} + \dots \right] \right], \\
& \tag{100}
\end{aligned}$$

and

$$\begin{aligned}
& \widetilde{s \rightarrow +\infty} \frac{i^3(-2iz)^{s-3}}{e\sqrt{2\pi s}} e^{-\frac{2}{3s} - \frac{2}{3s^2} - \frac{4}{5s^3} - \frac{16}{15s^4} - \dots} \left(\frac{es}{2}\right)^s \left(\frac{s}{2}\right)^{q-3} \sum_{\ell}^{\mathfrak{d}_2} \sum_{r=0}^{\ell} \sum_{j_2}^{\ell} \\
& \times \left[\binom{\ell}{r} (-i(s^2 + z))^{\ell-r} \left(-\frac{2izs}{2}\right)^r a_{2,j_2,\ell}^+ (r+s-1)(r+s-2) \right. \\
& \times \left[1 + \left(\frac{(-1)^1 j_2^1}{1!} + \dots\right) \frac{1}{s^1} + \left(\frac{(-1)^2 j_2^2}{2!} + \dots\right) \frac{1}{s^2} \right. \\
& \left. \left. + \left(\frac{(-1)^3 j_2^3}{3!} + \dots\right) \frac{1}{s^3} + \left(\frac{(-1)^4 j_2^4}{4!} + \dots\right) \frac{1}{s^4} + \dots \right] \right], \\
& \tag{101}
\end{aligned}$$

so that the total asymptotic expansion of $L_{m=s^2,s}(z)$ is equal to the sum of (99), (100), and (101). Dots in each term

$$\left(\frac{(-1)^k j_p^k}{k!} + \dots \right) \frac{1}{s^k}$$

denote polynomials in j_p, r, q , which in j_p are of degree stricly less than k . These polynomials are inessential for us due to the order identities (92)–(93).

In the derivation of the asymptotic expansion (99)–(101), we have used the asymptotic expansion $\xrightarrow{s \rightarrow +\infty}$ of the binomials

$$\binom{(s^2-j_0)/2+q}{s+r+q-1}, \quad \binom{(s^2-j_1)/2+q}{s+r+q-2}, \quad \binom{(s^2-j_2)/2+q}{s+r+q-3}$$

which are contained in the contributions to $L_{m=s^2,s}(z)$, coming from the simple fractions containing $a_{0,j_0}^\pm$, $a_{1,j_1}^\pm$ and, respectively, $a_{2,j_2}^\pm$.

Lemma 9. *Let $\alpha = \beta$ and $\alpha' = \beta'$ in (88). For each $z \neq 0$, analytic continuations of (65), (88), and (86) are nonzero functions of ρ , except for they are identically zero, which is the case if and only if $l = l_0 = 1$ in (65) and $l_0 = l = 1$ or $l_0 = l' = 1$ in (88).*

Proof. We apply the formulas (99)–(101) to our series (74), representing (65), (88) or (86), remembering that the single-primed coefficients $a'_{p,j_p}^\pm$, representing the contribution from the second integrand in (65) or (88), are in addition multiplied by z . We compute the leading order contribution to $L_{m=s^2,s}(z)$, with respect to s going to infinity. Taking into account the order identities (92)–(92), it is easily seen that for (74), representing (88), the leading order contributions to $L_{m=s^2,s}(z)$, in the asymptotic $\xrightarrow{s \rightarrow +\infty}$, are equal to

$$\begin{aligned} & \frac{i^2(-2iz)^{s-2}}{e\sqrt{2\pi s}} \left(\frac{es}{2}\right)^s \left(\frac{s}{2}\right)^{q-2} (-is^2)^{l+l'} s \sum_{j_1} a''^{+}_{1,j_1,l+l'} \frac{j_1^{l+l'+1} (-1)^{l+l'+1}}{(l+l'+1)! s^{l+l'+1}} \\ & + \frac{i^3(-2iz)^{s-3}z}{e\sqrt{2\pi s}} \left(\frac{es}{2}\right)^s \left(\frac{s}{2}\right)^{q-3} (-is^2)^{l+l'-1} s^2 \sum_{j_2} a'^{+}_{2,j_2,l+l'-l_0} \frac{j_2^{l+l'-l_0} (-1)^{l+l'-l_0}}{(l+l'+1)! s^{l+l'-l_0}} \end{aligned} \quad (102)$$

with $q = 2(l + l')$, and with $l = l'$ in this formula for the leading order contributions to $L_{m=s^2,s}(z)$ for (65). Similarly, using the identities (92), we see that for (86) the leading order contribution to $L_{m=s^2,s}(z)$, in the asymptotic $\xrightarrow{s \rightarrow +\infty}$, is equal to

$$\frac{i^2(-2iz)^{s-2}}{e\sqrt{2\pi s}} \left(\frac{es}{2}\right)^s \left(\frac{s}{2}\right)^{q-2} (-is^2)^l s \sum_{j_1} a_{1,j_1,l}^+ \frac{j_1^l (-1)^l}{l! s^l}, \quad q = 2l - 1. \quad (103)$$

From (92), it follows that for any fixed and nonzero z , (103) is nonzero for appropriately large s . Therefore, $L_{m=s^2,s}(z) \neq 0$ for all sufficiently large s . Thus (86) is a nonzero function of ρ for each nonzero value of z . In particular, for each $z > 0$, (86) is a nonzero function of ρ .

Let $z \neq 0$ in (102). From (92)–(92), it follows that the contributions in (102) can cancel out for all $s \rightarrow +\infty$, if and only if they are of the same order in s . This cancellation holds if and only if $l_0 = 1$ and

$$-i \frac{(-1)^{l+l'+1}}{(l+l'+1)!} \sum_{j_1} a''^+_{1,j_1,l+l'} j_1^{l+l'+1} = \frac{(-1)^{l+l'-1}}{(l+l'-1)!} \sum_{j_1} a'^+_{2,j_2,l+l'-1} j_2^{l+l'-1} \quad (104)$$

with $l = l'$ for (65). Therefore, if $l_0 \neq 1$, we are left with one of the two terms in (102), as the leading term. Therefore, if $l_0 \neq 1$, then for each nonzero z , in particular for each $z > 0$, (88) and (65) are nonzero functions of ρ . It remains to investigate the case $l_0 = 1$ for (88) and (65). Inserting (94) and (95) into (104), and using standard binomial identities, we easily see that (104) is equivalent to

$$\frac{2}{(l+l'+2)(l+l'+1)} \binom{l+l'+2}{l+1} = 1$$

which is equivalent to the assertion that $l = 1$ and l' is any natural ≥ 1 , or $l' = 1$ and l is any natural ≥ 1 . Therefore, if $l_0 = 1$, then the leading order term in the asymptotic $\widetilde{s \rightarrow +\infty}$ of $L_{m=s^2,s}(z)$ is zero for (88) or, respectively, for (65), if and only if l or $l' = 1$ in (88), respectively, in (65). Thus, $L_{m=s^2,s}(z) \neq 0$ for (88) and (65) if $l_0 = 1$ and $\min(l, l') > 1$, which shows that for each $z \neq 0$, (88) and (65) are nonzero functions of ρ , if $l_0 = 1$ and $\min(l, l') > 1$, accordingly with Lemma 8. $\square$

Note that Lemma 9 for (86) follows already from Lemma 7, so the essential part of Lemma 9 is the assertion concerning (65) and (88). In Lemmas 7–9, we have used the representations $U^{(l_0, i\rho)}$ with non-negative integer l_0 , but with any real ρ . With the convention $l_0 \in \mathbb{Z}$ and $\rho \geq 0$, numbering the equivalence classes of representations $(l_0, l_1 = i\rho)$ (used in Lemma 3), Lemma 9 says, that for each positive z , (65), (88), and (86) are nonzero functions of ρ except for $|l_0| = l = 1$ or $|l_0| = l' = 1$ in (88) and (65). With this convention Lemma 8 implies that (88) and (65) are identically zero functions of z, ρ , whenever $|l_0| = l = 1$ or $|l_0| = l' = 1$ in (88) and (65).

5. Relation between the subspaces with cyclic vectors

Lemma 10.

$$\mathcal{H}_{c_{\alpha'}^+|u\rangle} \subset \mathcal{H}_{c_{\alpha''}^+|u\rangle}, \quad \text{whenever } l(\alpha') \leq l(\alpha''),$$

and $\mathcal{H}_{|u\rangle} \subset \mathcal{H}_{c_{\alpha}^+|u\rangle}.$

Proof. This is a corollary of Lemmas 3, 7, 9. Let $l(\alpha') \leq l(\alpha'')$. We apply the same proof as that of Lemma 5 and Remark 1 (instead of Remark 1 and Lemma 7, one can use Lemma 6). From Lemmas 7, 9, it follows that the corresponding direct sum l_0 -components, for $-l(\alpha') \leq l_0 \leq l(\alpha')$, in the decomposition of Lemma 3 of the cyclic spaces $\mathcal{H}_{c_{\alpha'}^+|u}, \mathcal{H}_{c_{\alpha''}^+|u}$, coincide, including the supplementary component. By Lemma 3, decomposition of $\mathcal{H}_{c_{\alpha'}^+|u}$ does not contain direct sum components with $l_0 < -l(\alpha')$ and $l(\alpha') < l_0$, whence the inclusion. In particular, it follows from Lemma 7 that $\mathcal{H}_{|u} \subset \mathcal{H}_{c_{\alpha'}^+|u}$ for each $l(\alpha) = 1, 2, \dots$. In particular, the supplementary component of $\mathcal{H}_{c_{\alpha'}^+|u}$ is nonzero and is common for all $\mathcal{H}_{c_{\alpha'}^+|u}$ and $\mathcal{H}_{|u}$. $\square$

Let $U(g_{03}(\lambda)) = U(\lambda)$. Using (5), the first transformation rule in (7) and (80), we obtain the following generalization of the expansion: (80)

$$\begin{aligned} & U(\lambda) c_{\ell_1, m_1}^+ |u\rangle \\ &= \sum_{\ell'_1} \overline{A_{\ell'_1, m_1 \ell_1, m_1}(\lambda)} e^{-\frac{n^2}{8\pi} \|B(\lambda)\|^2} \exp \left[\frac{n}{4\pi\epsilon} \sum_l B_{l,0}(-\lambda) c_{l,0}^+ \right] c_{\ell'_1, m_1}^+ |u\rangle \\ &+ \delta_{m_1 0} n\epsilon \overline{B_{\ell_1,0}(\lambda)} e^{-\frac{n^2}{8\pi} \|B(\lambda)\|^2} \exp \left[\frac{n}{4\pi\epsilon} \sum_l B_{l,0}(-\lambda) c_{l,0}^+ \right] |u\rangle, \end{aligned} \quad (105)$$

and

$$\begin{aligned} & U(\lambda) c_{\ell_1, m_1}^+ \dots c_{\ell_q, m_q}^+ |u\rangle \\ &= \sum_{\ell'_1, \dots, \ell'_q} \prod_i^q \left[\overline{A_{\ell'_i, m_i \ell_i, m_i}(\lambda)} \right] e^{-\frac{n^2}{8\pi} \|B(\lambda)\|^2} \exp \left[\frac{n}{4\pi\epsilon} \sum_l B_{l,0}(-\lambda) c_{l,0}^+ \right] \prod_i^q c_{\ell'_i, m_i}^+ |u\rangle \\ &+ \dots \\ &+ \prod_i^q \left[\delta_{m_i 0} n\epsilon \overline{B_{\ell_i,0}(\lambda)} \right] e^{-\frac{n^2}{8\pi} \|B(\lambda)\|^2} \exp \left[\frac{n}{4\pi\epsilon} \sum_l B_{l,0}(-\lambda) c_{l,0}^+ \right] |u\rangle, \end{aligned} \quad (106)$$

where dots denote terms in which one of the factors $\overline{A_{\ell'_i, m_i \ell_i, m_i}(\lambda)} c_{\ell'_i, m_i}^+$ in the first sum is replaced with $\delta_{m_i 0} n\epsilon \overline{B_{\ell_i,0}(\lambda)}$, with the summation over the corresponding ℓ'_i withdrawn, and further terms, which arise from the first sum in which two

$$\overline{A_{\ell'_i, m_i \ell_i, m_i}(\lambda)} c_{\ell'_i, m_i}^+, \quad \overline{A_{\ell'_j, m_j \ell_j, m_j}(\lambda)} c_{\ell'_j, m_j}^+$$

of the factors in the first sum are replaced with

$$\left(\delta_{m_i 0} n\epsilon \overline{B_{\ell_i,0}(\lambda)} \right), \quad \left(n\epsilon \overline{\delta_{m_j 0} B_{\ell_j,0}(\lambda)} \right),$$

with the summation over the corresponding ℓ'_i, ℓ'_j , withdrawn, and so on, up to the last term written explicitly in (106). We use the expansions (105), (106) in the same way as the expansion (80) in Subsection 3, utilizing the fact that $B_{l,0}(\lambda)$ is of l -order at zero, and $A_{l,m}{}_{l',m}(\lambda) = U_{l,m}{}_{l',m}^{(1,0)}(\lambda)$ is of $|l - l'|$ -order at zero, compare (31) and (57), with $l_0 = 1, \rho = 0$, in (57).

Lemma 11.

$$c_{\ell_1, m_1}^+ c_{\ell_2, m_2}^+ \dots c_{\ell_q, m_q}^+ |u\rangle \in \mathcal{H}_{c_{\ell_0, 0}^+ |u\rangle}, \quad \text{whenever} \quad \sum_{j=1}^q \ell_j \leq \ell, \quad -\ell_i \leq m_i \leq \ell_i.$$

Proof. We proceed by induction with respect to q . Let $q = 1$. The cyclic subspace of the lemma is by definition invariant under U , and thus invariant under the generators M_{23}, M_{13} , and M_{31} of the action of the subgroup $SU(2, \mathbb{C}) \subset G$. The vector $|u\rangle$ is invariant under the action of this subgroup. From this and from the first transformation rule in (7), with $B(a) = 0$, $a \in SU(2, \mathbb{C})$, it follows that this subgroup acts on the vectors $\xi_{\ell_1, m_1} = c_{\ell_1, m_1}^+ |u\rangle$, in the cyclic subspace of the lemma with $q = 1$, by the standard representation, given by right multiplication by the matrix $\overline{A_{\ell'_1, m'_1}{}_{\ell_1, m_1}}(a) = \delta_{\ell_1 \ell'_1} \overline{T_{m'_1 m_1}^{\ell_1}}(a)$, $a \in SU(2, \mathbb{C})$, with

$$\begin{aligned} H_+ \xi_{\ell_1, m_1} &= \sqrt{(\ell_1 + m_1 + 1)(\ell_1 - m_1)} \xi_{\ell_1, m_1 + 1}, \\ H_- \xi_{\ell_1, m_1} &= \sqrt{(\ell_1 + m_1)(\ell_1 - m_1 + 1)} \xi_{\ell_1, m_1 - 1}, \end{aligned} \quad (107)$$

where $H_{\pm} = M_{23} \pm iM_{13}$. This representation reduces to the standard representation of weight ℓ_1 [5], on each invariant subspace, spanned by ξ_{ℓ_1, m_1} , $-\ell_1 \leq m_1 \leq \ell_1$. Lemma 11 for $q = 1$ follows from (107) and from Lemma 10.

Now we proceed to the proof of the inductive step: assuming the assertion of Lemma 11 for $q \leq q'$, we show the assertion of Lemma 11 for $q = q' + 1$.

We first prove the inductive step for $m_{q'+1} = 0$. We, in turn, divide the proof of this assertion into several cases: (1) $\ell_{q'+1} = 1$, (2) $\ell_{q'+1} = 2, \dots, \ell_{q'+1} \ell_{q'+1} = \sum \ell_i - \ell_{q'+1}$.

Let us consider now case (1). Let us denote by $|w\rangle^i$ the vector which arises from the vector

$$|w\rangle = c_{\ell_1, m_1}^+ c_{\ell_2, m_2}^+ \dots c_{\ell_{q'}, m_{q'}}^+ |u\rangle \in \mathcal{H}_{c_{\ell_0, 0}^+ |u\rangle} \subset \mathcal{H}_{c_{\ell+1, 0}^+ |u\rangle},$$

by removing in $|w\rangle$ the factor c_{ℓ_i, m_i}^+ . The inclusion follows from Lemma 10. By the inductive assumption,

$$|w\rangle^i \in \mathcal{H}_{c_{\ell-\ell_i, 0}^+ |u\rangle} \subset \mathcal{H}_{c_{\ell+1, 0}^+ |u\rangle}.$$

Let us denote by $|\ell_i^{-1}\rangle, |\ell_i^{+1}\rangle$, the vectors

$$\begin{aligned} |\ell_i^{-1}\rangle &= c_{\ell_1, m_1}^+ \dots c_{\ell_i-1, m_i}^+ \dots c_{\ell_{q'}, m_{q'}}^+ |u\rangle, \\ |\ell_i^{+1}\rangle &= c_{\ell_1, m_1}^+ \dots c_{\ell_i+1, m_i}^+ \dots c_{\ell_{q'}, m_{q'}}^+ |u\rangle. \end{aligned}$$

By the inductive assumption,

$$\begin{aligned} |\ell_i^{+1}\rangle &\in \mathcal{H}_{c_{\ell+1,0}^+ |u\rangle}, \\ |\ell_i^{-1}\rangle &\in \mathcal{H}_{c_{\ell-1,0}^+ |u\rangle} \subset \mathcal{H}_{c_{\ell+1,0}^+ |u\rangle}, \end{aligned}$$

with the last inclusion following from Lemma 10. To the vector $|w\rangle$ we apply $U(\lambda)$, using (106), and arriving with the expansion

$$\begin{aligned} &e^{\frac{n^2}{8\pi} \|B(\lambda)\|^2} U(\lambda) |w\rangle - |w\rangle \\ &= \frac{nB_{1,0}(-\lambda)}{4\pi\epsilon} c_{1,0}^+ |w\rangle + \frac{nB_{2,0}(-\lambda)}{4\pi\epsilon} c_{2,0}^+ |w\rangle + \dots \\ &+ \frac{1}{2!} \left[\left(\frac{nB_{1,0}(-\lambda)}{4\pi\epsilon} \right) \left(\frac{nB_{1,0}(-\lambda)}{4\pi\epsilon} \right) c_{1,0}^+ c_{1,0}^+ |w\rangle \right. \\ &+ \left(\frac{nB_{1,0}(-\lambda)}{4\pi\epsilon} \right) \left(\frac{nB_{2,0}(-\lambda)}{4\pi\epsilon} \right) c_{1,0}^+ c_{2,0}^+ |w\rangle \\ &+ \left. \left(\frac{nB_{2,0}(-\lambda)}{4\pi\epsilon} \right) \left(\frac{nB_{1,0}(-\lambda)}{4\pi\epsilon} \right) c_{2,0}^+ c_{1,0}^+ |w\rangle + \dots \right] \\ &+ \frac{1}{3!} \left[\left(\frac{nB_{1,0}(-\lambda)}{4\pi\epsilon} \right) \left(\frac{nB_{1,0}(-\lambda)}{4\pi\epsilon} \right) \left(\frac{nB_{1,0}(-\lambda)}{4\pi\epsilon} \right) c_{1,0}^+ c_{1,0}^+ c_{1,0}^+ |w\rangle + \dots \right] \\ &+ \sum_{i=1}^{q'} \overline{A_{\ell_i'+1, m_i} \ell_i, m_i}(\lambda) |\ell_i^{+1}\rangle + \sum_{i=1}^{q'} \overline{A_{\ell_i'-1, m_i} \ell_i, m_i}(\lambda) |\ell_i^{-1}\rangle \\ &+ \sum_{i=1}^{q'} \delta_{m_i 0} n\epsilon \overline{B_{\ell_i, 0}}(\lambda) |w^i\rangle + \dots, \end{aligned} \tag{108}$$

where dots denote higher-order terms in λ . We have used the fact that $B_{l,0}(-\lambda)$ is a quantity of the order of l , and $A_{l,m} \ell', m(\lambda)$ is of the order of

$|l - l'|$, at $\lambda = 0$. Thus, the vector

$$\begin{aligned} & \frac{1}{\frac{-n}{4\pi\epsilon} B_{1,0}(-\lambda)} \left(e^{\frac{n^2}{8\pi} \|B(\lambda)\|^2} U(\lambda) |w\rangle - |w\rangle \right. \\ & - \sum_{i=1}^{q'} \overline{A_{\ell'_i+1, m_i \ell_i, m_i}(\lambda)} \left| w^{\ell_i+1} \right\rangle - \sum_{i=1}^{q'} \overline{A_{\ell'_i-1, m_i \ell_i, m_i}(\lambda)} \left| w^{\ell_i-1} \right\rangle \\ & \left. - \sum_{i=1}^{q'} \delta_{m_i 0} n \epsilon \overline{B_{\ell_i, 0}(\lambda)} \left| w^i \right\rangle \right) \in \mathcal{H}_{c_{\ell+1, 0}^+ |u} \end{aligned}$$

has the expansion

$$\begin{aligned} & c_{1,0}^+ |w\rangle + \frac{B_{2,0}(-\lambda)}{B_{1,0}(-\lambda)} c_{2,0}^+ |w\rangle + \frac{B_{3,0}(-\lambda)}{B_{1,0}(-\lambda)} c_{3,0}^+ |w\rangle + \dots \\ & + \frac{1}{2!} \left[\left(\frac{n B_{1,0}(-\lambda)}{4\pi\epsilon} \right) c_{1,0}^+ c_{1,0}^+ |w\rangle \right. \\ & + \left(\frac{n B_{2,0}(-\lambda)}{4\pi\epsilon} \right) c_{1,0}^+ c_{2,0}^+ |w\rangle \\ & + \left. \left(\frac{n B_{2,0}(-\lambda)}{4\pi\epsilon} \right) c_{2,0}^+ c_{1,0}^+ |w\rangle + \dots \right] \\ & + \frac{1}{3!} \left[\left(\frac{n B_{1,0}(-\lambda)}{4\pi\epsilon} \right) \left(\frac{n B_{1,0}(-\lambda)}{4\pi\epsilon} \right) c_{1,0}^+ c_{1,0}^+ c_{1,0}^+ |w\rangle + \dots \right] \end{aligned}$$

converging to

$$c_{1,0}^+ |w\rangle \in \mathcal{H}_{c_{\ell+1, 0}^+ |u},$$

if $\lambda \rightarrow 0$, because $\mathcal{H}_{c_{\ell, 0}^+ |u}$ is closed and dots denote terms of the first order at least in λ . Thus, we can see that

$$c_{1,0}^+ |w\rangle = c_{\ell_1, m_1}^+ \dots c_{\ell_{q'}, m_{q'}}^+ c_{1,0}^+ |u\rangle \in \mathcal{H}_{c_{\ell+1, 0}^+ |u},$$

which proves the inductive step in case (1).

Now we proceed to case (2). We again use the inductive assumption and validity of the inductive step in case (1). Let $\ell_{q'+1} = 2$. In addition to $|w\rangle$, $|w^{\ell_i-1}\rangle$, $|w^{\ell_i+1}\rangle$ introduced in case (1), we need to introduce

$$\begin{aligned} \left| w^{\ell_i-2} \right\rangle &= c_{\ell_1, m_1}^+ \dots c_{\ell_i-2, m_i}^+ \dots c_{\ell_{q'}, m_{q'}}^+ |u\rangle, \\ \left| w^{\ell_i+2} \right\rangle &= c_{\ell_1, m_1}^+ \dots c_{\ell_i+2, m_i}^+ \dots c_{\ell_{q'}, m_{q'}}^+ |u\rangle, \end{aligned}$$

and $\left| \begin{smallmatrix} ij \\ w \end{smallmatrix} \right\rangle$, which arises from $|w\rangle$ by removing two operators: $c_{\ell_i, m_i}^+, c_{\ell_j, m_j}^+$. By the inductive assumption,

$$\begin{aligned} \left| \begin{smallmatrix} \ell_i+2 \\ w \end{smallmatrix} \right\rangle &\in \mathcal{H}_{c_{\ell+2,0}^+ |u\rangle}, \\ \left| \begin{smallmatrix} \ell_i-2 \\ w \end{smallmatrix} \right\rangle &\in \mathcal{H}_{c_{\ell-2,0}^+ |u\rangle} \subset \mathcal{H}_{c_{\ell+2,0}^+ |u\rangle}, \\ \left| \begin{smallmatrix} ij \\ w \end{smallmatrix} \right\rangle &\in \mathcal{H}_{c_{\ell-\ell_i-\ell_j,0}^+ |u\rangle} \subset \mathcal{H}_{c_{\ell+2,0}^+ |u\rangle}. \end{aligned}$$

Because

$$c_{1,0}^+ |w\rangle \in \mathcal{H}_{c_{\ell+1,0}^+ |u\rangle} \subset \mathcal{H}_{c_{\ell+2,0}^+ |u\rangle},$$

then,

$$\begin{aligned} &\frac{B_{1,0}(-\lambda)}{B_{2,0}(-\lambda)} \left(\frac{1}{\frac{-n}{4\pi\epsilon} B_{1,0}(-\lambda)} \left(e^{\frac{n^2}{8\pi} \|B(\lambda)\|^2} U(\lambda) |w\rangle - |w\rangle \right. \right. \\ &- \sum_{i=1}^{q'} \overline{A_{\ell'_i+1, m_i \ell_i, m_i}(\lambda)} \left| \begin{smallmatrix} \ell_i+1 \\ w \end{smallmatrix} \right\rangle - \sum_{i=1}^{q'} \overline{A_{\ell'_i-1, m_i \ell_i, m_i}(\lambda)} \left| \begin{smallmatrix} \ell_i-1 \\ w \end{smallmatrix} \right\rangle \\ &- \sum_{i=1}^{q'} \delta_{m_i 0} n\epsilon \overline{B_{\ell_i,0}(\lambda)} \left| \begin{smallmatrix} i \\ w \end{smallmatrix} \right\rangle \\ &- \sum_{i=1}^{q'} \overline{A_{\ell'_i+2, m_i \ell_i, m_i}(\lambda)} \left| \begin{smallmatrix} \ell_i+2 \\ w \end{smallmatrix} \right\rangle - \sum_{i=1}^{q'} \overline{A_{\ell'_i-2, m_i \ell_i, m_i}(\lambda)} \left| \begin{smallmatrix} \ell_i-2 \\ w \end{smallmatrix} \right\rangle \\ &\left. \left. - \sum_{i,j=1}^{q'} \delta_{m_i 0} n\epsilon \overline{B_{\ell_i,0}(\lambda)} \delta_{m_j 0} n\epsilon \overline{B_{\ell_j,0}(\lambda)} \left| \begin{smallmatrix} ij \\ w \end{smallmatrix} \right\rangle \right) - c_{1,0}^+ |w\rangle \right) \in \mathcal{H}_{c_{\ell+2,0}^+ |u\rangle}. \end{aligned}$$

From (108), it follows that this vector has the expansion

$$\begin{aligned} &c_{2,0}^+ |w\rangle + \frac{B_{3,0}(-\lambda)}{B_{2,0}(-\lambda)} c_{3,0}^+ |w\rangle + \frac{B_{4,0}(-\lambda)}{B_{2,0}(-\lambda)} c_{4,0}^+ |w\rangle + \dots \\ &+ \frac{1}{2!} \left[\frac{B_{1,0}(-\lambda)}{B_{2,0}(-\lambda)} \left(\frac{nB_{1,0}(-\lambda)}{4\pi\epsilon} \right) c_{1,0}^+ c_{1,0}^+ |w\rangle \right. \\ &+ \frac{B_{1,0}(-\lambda)}{B_{2,0}(-\lambda)} \left(\frac{nB_{2,0}(-\lambda)}{4\pi\epsilon} \right) c_{1,0}^+ c_{2,0}^+ |w\rangle \\ &+ \frac{B_{1,0}(-\lambda)}{B_{2,0}(-\lambda)} \left(\frac{nB_{2,0}(-\lambda)}{4\pi\epsilon} \right) c_{2,0}^+ c_{1,0}^+ |w\rangle + \dots \Big] \\ &+ \frac{1}{3!} \left[\frac{B_{1,0}(-\lambda)}{B_{2,0}(-\lambda)} \left(\frac{nB_{1,0}(-\lambda)}{4\pi\epsilon} \right) \left(\frac{nB_{1,0}(-\lambda)}{4\pi\epsilon} \right) c_{1,0}^+ c_{1,0}^+ c_{1,0}^+ |w\rangle + \dots \right] \\ &+ \dots \end{aligned}$$

which converges to

$$\begin{aligned} & c_{2,0}^+ |w\rangle + \frac{1}{2!} \lim_{\lambda \rightarrow 0} \frac{\frac{n}{4\pi\epsilon} B_{1,0}(-\lambda) \frac{n}{4\pi\epsilon} B_{1,0}(-\lambda)}{\frac{n}{4\pi\epsilon} B_{2,0}(-\lambda)} c_{1,0}^+ c_{1,0}^+ |w\rangle \\ & = c_{2,0}^+ |w\rangle + \frac{1}{2} \frac{b_{1,0} b_{1,0}}{b_{2,0}} c_{1,0}^+ c_{1,0}^+ |w\rangle \in \mathcal{H}_{c_{\ell+2,0}^+ |u\rangle}, \end{aligned}$$

if $\lambda \rightarrow 0$, because $\mathcal{H}_{c_{\ell,0}^+ |u\rangle}$ is closed. Here, $b_{l,0}$ are the coefficients of the lowest order in the expansion (31). By case (1), already proved,

$$c_{1,0}^+ c_{1,0}^+ |w\rangle \in \mathcal{H}_{c_{\ell+2,0}^+ |u\rangle},$$

therefore,

$$c_{2,0}^+ |w\rangle = c_{\ell_1, m_1}^+ \dots c_{\ell_{q'}, m_{q'}}^+ c_{2,0}^+ |u\rangle \in \mathcal{H}_{c_{\ell+2,0}^+ |u\rangle},$$

which proves validity of the inductive step in case (2).

It is not difficult to see, that we can continue in this way, proving the inductive step in $(j+1)$ -case, using the inductive assumption and all preceding cases, (1), (2), ..., (j), constructing in this way the corresponding vectors

$$\begin{aligned} & c_{j+1,0}^+ |w\rangle + \frac{1}{2!} \sum_{l_1+l_2=j+1} \frac{b_{l_1,0} b_{l_2,0}}{b_{j+1,0}} c_{l_1,0}^+ c_{l_2,0}^+ |w\rangle + \dots \\ & \dots + \frac{1}{(j+1)!} \sum_{l_1+\dots+l_{j+1}=j+1} \frac{b_{l_1,0} \dots b_{l_{j+1},0}}{b_{j+1,0}} c_{l_1,0}^+ \dots c_{l_{j+1},0}^+ |w\rangle \in \mathcal{H}_{c_{\ell+j+1,0}^+ |u\rangle}, \end{aligned}$$

where all terms, except the first, belong to $\mathcal{H}_{c_{\ell+j+1,0}^+ |u\rangle}$, by the preceding cases (1), ..., (j), and thus also with the first

$$c_{j+1,0}^+ |w\rangle = c_{\ell_1, m_1}^+ \dots c_{\ell_{q'}, m_{q'}}^+ c_{j+1,0}^+ |u\rangle \in \mathcal{H}_{c_{\ell+j+1,0}^+ |u\rangle},$$

for $\ell_{q'+1} = j+1$. Therefore, we have proved that if the assertion of Lemma 11 holds for $\mathbf{q} \leq \mathbf{q}'$, then it holds for $\mathbf{q}' + 1$, if $m_{\mathbf{q}'+1} = 0$, i.e. with

$$c_{\ell_1, m_1}^+ \dots c_{\ell_{q'}, m_{q'}}^+ c_{\ell_{q'+1}, 0}^+ |u\rangle \in \mathcal{H}_{c_{\Sigma \ell_i, 0}^+ |u\rangle}, \quad -\ell_i \leq m_i \leq \ell_i, \quad 1 \leq i \leq \mathbf{q}', \quad (109)$$

following from the assertion of Lemma 11 for $\mathbf{q} \leq \mathbf{q}'$. Here,

$$\Sigma \ell_i = \sum_{i=1}^{\mathbf{q}'+1} \ell_i.$$

Now, using (109) we show that also

$$c_{\ell_1, m_1}^+ \dots c_{\ell_{q'}, m_{q'}}^+ c_{\ell_{q'+1}, m_{q'+1}}^+ |u\rangle \in \mathcal{H}_{c_{\Sigma \ell_i, 0}^+ |u\rangle}, \quad -\ell_i \leq m_i \leq \ell_i, \quad 1 \leq i \leq q' + 1, \quad (110)$$

which will end the proof of the inductive step.

The subgroup $SU(2, \mathbb{C})$ acts on the states

$$c_{\ell_1, m_1}^+ \dots c_{\ell_q, m_q}^+ |u\rangle$$

through the symmetrized tensor product of the representation given by right multiplication by the matrix

$$\overline{A_{\ell_i, m_i} \ell'_i, m'_i}(a) = \delta_{\ell_i, \ell'_i} \overline{T_{m_i, m'_i}^{\ell_i}}(a),$$

with

$$H_{\pm} c_{\ell_1, m_1}^+ \dots c_{\ell_q, m_q}^+ |u\rangle = \sum_i \alpha_{\ell_i, m_i}^{\pm} c_{\ell_1, m_1}^+ \dots c_{\ell_i, m_i \pm 1}^+ \dots c_{\ell_q, m_q}^+ |u\rangle,$$

$$\alpha_{\ell_i, m_i}^+ = \sqrt{(\ell_i + m_i + 1)(\ell_i - m_i)}, \quad \alpha_{\ell_i, m_i}^- = \sqrt{(\ell_i + m_i)(\ell_i - m_i + 1)}.$$

It is not difficult to see that all vectors (110) can be expressed as linear combinations of vectors which arise by applications of powers of $H_{\pm}$ to the vectors (109). First, we show (110) for $m_{q'+1} = \pm 1$, and any $-\ell_1 \leq m_i \leq \ell_i$, $i \leq q'$ in (110). To this end, we observe that all vectors in

$$\begin{aligned} H_{\pm} c_{\ell_1, m_1}^+ \dots c_{\ell_{q'}, m_{q'}}^+ c_{\ell_{q'+1}, 0}^+ |u\rangle &= \alpha_{\ell_1, m_1}^{\pm} c_{\ell_1 \pm 1, m_1}^+ \dots c_{\ell_q, m_q}^+ c_{\ell_{q'+1}, 0}^+ |u\rangle \\ &+ \alpha_{\ell_2, m_2}^{\pm} c_{\ell_1, m_1}^+ c_{\ell_2 \pm 1, m_2}^+ \dots c_{\ell_{q'}, m_{q'}}^+ c_{\ell_{q'+1}, 0}^+ |u\rangle + \dots \\ &\dots + \alpha_{\ell_{q'+1}, 0}^{\pm} c_{\ell_1, m_1}^+ c_{\ell_2, m_2}^+ \dots c_{\ell_{q'}, m_{q'}}^+ c_{\ell_{q'+1} \pm 1, \pm 1}^+ |u\rangle, \end{aligned} \quad (111)$$

except for the last one, are of the form (109), having at least one $m_i = 0$, and by (109), they all belong to $\mathcal{H}_{c_{\Sigma \ell_i, 0}^+ |u\rangle}$. Since the vector (111) belongs to

$\mathcal{H}_{c_{\Sigma \ell_i, 0}^+ |u\rangle}$ then,

$$c_{\ell_1, m_1}^+ c_{\ell_2, m_2}^+ \dots c_{\ell_{q'}, m_{q'}}^+ c_{\ell_{q'+1}, \pm 1}^+ |u\rangle \in \mathcal{H}_{c_{\Sigma \ell_i, 0}^+ |u\rangle}, \quad (112)$$

thus showing that the vector (110) belongs to $\mathcal{H}_{c_{\Sigma \ell_i, 0}^+ |u\rangle}$ if any of m_i is equal to ± 1 . We continue similarly, applying $H_{\pm}$ to (112), and showing that the vector (110) belongs to $\mathcal{H}_{c_{\Sigma \ell_i, 0}^+ |u\rangle}$ if any of m_i is equal to ± 2 , and so on. $\square$

5.1. Relation of the space $\mathcal{H}_n$ with the cyclic subspaces $\mathcal{H}_x \subset \mathcal{H}_n$.
Proof of theorem

Consider the set of theoretical sums

$$\left(\mathcal{H}_{|u\rangle} = \mathcal{H}_{c_{1,0}^+|u\rangle} \right) \cup \left(\mathcal{H}_{c_{2,0}^+|u\rangle} \cup \mathcal{H}_{c_{1,0}^+ c_{1,0}^+|u\rangle} \right) \\ \cup \left(\mathcal{H}_{c_{3,0}^+|u\rangle} \cup \mathcal{H}_{c_{2,0}^+ c_{1,0}^+|u\rangle} \cup \mathcal{H}_{c_{1,0}^+ c_{1,0}^+ c_{1,0}^+|u\rangle} \right) \cup \dots$$

of the domains of the cyclic representations with the cyclic vectors $|u\rangle$, $c_{\alpha_i}^+|u\rangle, \dots$ of the form (1) with the azimuthal numbers $m_i = 0$ all equal to zero in $\alpha_i = (l_i, m_i)$. From Lemma 11, it follows that this sum contains the linear span of the orthogonal complete system (1). Therefore, the closure of this sum is equal to $\mathcal{H}_n$. From Lemma 11, it follows that the closure of

$$\mathcal{H}_{c_{1,0}^+|u\rangle} \cup \mathcal{H}_{c_{2,0}^+|u\rangle} \cup \mathcal{H}_{c_{3,0}^+|u\rangle} \cup \dots$$

is equal to $\mathcal{H}_n$ and

$$\mathcal{H}_{c_{l,0}^+|u\rangle} \subset \mathcal{H}_{c_{l+1,0}^+|u\rangle}$$

by Lemma 10.

Let (l_0, l_1) be the irreducible representations of [5], and let, for simplicity of notation, the direct sum/integral decomposition of Lemma 3 for $x = c_{l,0}^+|u\rangle$ be denoted by

$$U \Big|_{\mathcal{H}_{c_{l,0}^+|u\rangle}} = \bigoplus_{l_0=-l}^l \int_0^{+\infty} (l_0, i\rho) \nu(\rho, z) d\rho \bigoplus \nu(z) (l_0 = 0, 1 - z), \quad z = \frac{n^2 \epsilon^2}{\pi},$$

with the corresponding positive weights $\nu(\rho, z), \nu(z)$, with $\nu(\rho, z) > 0$ a.e. for each $z > 0$, and $\nu(z) > 0$ for $0 < z < 1$, and $\nu(z) = 0$ for $z \geq 1$, with the standard Lebesgue measure $d\rho$ on $\mathbb{R}$. Using again Lemma 10, we arrive with decomposition (2). Our theorem follows from (2).

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