OPTIMAL INVESTMENT FOR THE DEFINED-CONTRIBUTION PENSION BASED ON NON-EXTENSIVE STATISTICAL MECHANICS AND EXPONENTIAL UTILITY FUNCTION

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Under the framework of non-extensive statistical mechanics, the optimal fund allocation of pension is studied by maximizing the expected utility of terminal wealth. In order to accurately approximate the actual financial market, the non-extensive statistical theory is employed to model prices of the risky asset, which can describe the high peak and fat tail characteristics of returns. Based on the criterion of maximizing the expected utility of terminal wealth, the Hamilton–Jacob–Bellman equation is established under the condition of the exponential utility function. Furthermore, using the duality theory, an analytical solution for the optimal investment strategy is obtained. Finally, the influence of the main model parameters on the optimal investment strategy is analyzed through numerical methods.

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1. Introduction

With the improvement of healthcare and the acceleration of aging population, the pressure on pension payments is increasing. How to optimize the allocation of pension funds in the financial market to maintain and increase their value is a very urgent and important issue, because it is related to the retirement security of pension holders and the harmony and stability of society.

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At present, there are two main types of pension plans: one is a defined-benefit (DB) pension, the other is a defined-contribution (DC) pension. A defined-benefit pension plan assumes that the benefits of pension holders are predetermined after retirement. However, the contribution rate can be changed, and the risk generated during the investment process is borne by the pension plan manager. In a defined-contribution pension, the contribution rate is predetermined, and the investment risk is borne by pension plan holders. Defined-contribution pension plans are more suitable for the pension management mechanism than defined-benefit pension plans and many countries have adopted them. Therefore, the optimal allocation of the defined-contribution pension has become a new research hotspot in the financial field. The optimal investment problem of the defined-contribution pension in a multi-period and discrete-time framework was first studied by Vigna and Haberman [1]. Considering personal investment risk and annualized fund risk, the optimal investment strategy for the defined-contribution pension was obtained by using a dynamic programming method. Boulier *et al.* [2] studied the optimal allocation of the defined-contribution pension in a continuous-time framework, based on a constant relative risk aversion (CRRA) utility function and constant interest rate. Devolder *et al.* [3] aimed at maximizing the utility of terminal wealth. Under the constant absolute risk aversion (CARA) and constant relative risk aversion (CRRA) utility functions, the optimal investment strategy of the defined-contribution pension in two stages before and after retirement was obtained. Gerrard *et al.* [4] studied the investment of the defined-contribution pension in a complete market, and analyzed the impact of interest rates on the optimal investment strategy. Cairns *et al.* [5] added random wages to Gerrard's research and obtained the analytical solution of the optimal strategy under the logarithmic utility function. Vigna [6], He and Liang [7], and Menoncin and Vigna [8] derived the optimal investment strategy for the defined-contribution pension under the mean-variance criterion. The above research results provide an important theoretical and practical support for the improvement of the pension system.

A large number of empirical results show that due to the complexity of asset-price fluctuations, using the classical Brownian motion for modeling the stock-price variation cannot closely match the actual financial market. The fact that the stock price follows the Brownian motion means that its return is a normal distribution, but the actual return has the characteristics of a high peak and fat tail. This phenomenon is common in the financial markets of many countries [9–12]. To make pricing and investment more accurate, several price models different from the Brownian motion have been developed. For example, Gu *et al.* [13] and Dufera [14] employed the fractional Brownian motion to model the stock-price variation, which can characterize the phenomenon of self-similarity and long-term correlation in stock prices. Carr *et al.* [15] and Kim *et al.* [16] used Lévy processes, which can describe the peak and thick tail characteristics of stock prices, while Duffie *et al.* [17] and Wang *et al.* [18] employed jump-diffusion processes.

Recently, many scholars have used statistical physics to study financial problems, resulting in a new discipline called Econophysics [19]. Especially in 1988, Tsallis [20] proposed non-extensive statistical mechanics, which is a generalization of Boltzmann–Gibbs statistics. It has been widely applied in the field of finance, thus promoting the development of Econophysics. For example, using the non-extensive statistical approach, Tsallis *et al.* [21], Rak *et al.* [22], Senapati and Karameshu [23], and Duarte Queirós [24] studied the variation of stock prices. Drożdż *et al.* [25] studied the fluctuation of foreign exchange rates. Borland [26], Nayak *et al.* [27], and William *et al.* [28] explored the pricing problem of options. In addition, Trindade *et al.* [29] considered the optimal investment in the stock market.

In this study, to get close to the actual financial market, the non-extensive statistical mechanics is employed to describe the fluctuation of risky asset prices. Then we use the expected utility maximization method to study the investment problem of the defined-contribution pension. Furthermore, under the condition of the exponential utility function, we obtain an analytical solution for the optimal investment strategy of defined-contribution pension using the dynamic programming method and duality theory. Finally, the effects of main parameters on investment strategies are illustrated by numerical methods.

The study is organized as follows. In Section 2, the non-extensive statistical mechanics is employed to model the risky asset price. The price model can describe the high peak and fat tail characteristics of returns. In Section 3, we derive the wealth equations of the defined-contribution pension in two different periods before and after retirement. In Section 4, an optimal investment model for the defined-contribution pension is established under the criterion of maximizing expected utility. In Section 5, the analytical solution of the optimal investment strategy for the defined-contribution pension is obtained using dynamic programming, Legendre transformation, and duality theory. In Section 6, the summary of this study is provided.

2. Asset price model

In our model, as in the literature [4–6], the pension fund is invested in two assets, a risk-free asset and a risky asset. Let the price $B(t)$ of the risk-free asset satisfy the following differential equation:

$$\begin{cases} \mathrm{d}B(t) = rB(t)\mathrm{d}t\,, \\ B(0) = B_0\,, \end{cases} \tag{1}$$

where r is a positive constant called the risk-free return rate. Let the price $S(t)$ of the risky asset satisfy the following differential equation:

$$\begin{cases} \mathrm{d}S(t) = S(t)(\mu\mathrm{d}t + \sigma\mathrm{d}\Omega(t))\,, \\ S(0) = S_0\,, \end{cases} \tag{2}$$

where $\mu > r$ is an expected return rate, and σ is a volatility of the risky asset. Moreover, let the random variable $\Omega(t)$ satisfy

$$\mathrm{d}\Omega(t) = P(\Omega,t)^{\frac{1-q}{2}}\mathrm{d}W(t)\,, \tag{3}$$

where $\{W(t)\}_{t\geq 0}$ is the standard Brownian motion defined on the probability space $(\mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$. Moreover, $P(\Omega,t)$ is a Tsallis distribution with the index q satisfying the following equation:

$$P(\Omega,t) = \frac{1}{z(t)}\left[1-\beta(t)(1-q)\Omega^2\right]^{\frac{1}{1-q}}\,, \tag{4}$$

with

$$z(t) = [(2-q)(3-q)kt]^{\frac{1}{3-q}}\,, \tag{5}$$

$$\beta(t) = k^{\frac{1-q}{3-q}}[(2-q)(3-q)t]^{\frac{2}{q-3}}\,, \tag{6}$$

$$k = \frac{\pi}{q-1}\frac{\Gamma^2\left(\frac{1}{q-1}-\frac{1}{2}\right)}{\Gamma^2\left(\frac{1}{q-1}\right)}\,. \tag{7}$$

In the $q \to 1$ limit (see [26]), equation (4) recovers a Gaussian distribution. When $q > 1$, the $P_q(\Omega)$ distribution exhibits the characteristics of high peak and fat tail, which can better fit the actual market than the Gaussian distribution. That comes from the fact that, for the correlated random variables of price returns, the q-Gaussian distributions are stable, and their tail behavior can resemble power laws. In fact, there has been some evidence that the q-Gaussian distributions can approximate the empirical return distributions (see [22, 30–35]).

3. Wealth model

In this study, we divide the defined-contribution pension plan into two stages: before and after retirement. Then, the wealth value process of pension fund investors can be correspondingly divided into two periods. We reasonably assume that the payment method after retirement is an annuity, and the amount is predetermined by the pension fund manager. In addition, the death of the policyholder is not considered. Let T be the time point of retirement. Let N be the payment cycle of the annuity after retirement. Suppose the total amount of the pension fund is used to invest in the risky asset and risk-free asset. Let $\pi(t)$ and $1-\pi(t)$ be the proportion of the wealth invested in the risky asset and the risk-free asset at time t. Let $V(T)$ be the wealth value of the pension fund investors.

3.1. Wealth model before retirement

Let c be the pension contribution rate which is a positive constant. Let the salary be taken as 1. Then the wealth value process $V(t)$ before retirement can be written as the following stochastic differential equation:

$$\begin{cases} \mathrm{d}V(t) = [1-\pi(t)]V(t)\frac{\mathrm{d}B(t)}{B(t)} + \pi(t)V(t)\frac{\mathrm{d}S(t)}{S(t)} + c\,\mathrm{d}t\,, \\ V(0) = V_0\,, \end{cases} \tag{8}$$

where V_0 is the initial wealth. Substituting (1) and (2) into equation (8), we can obtain

$$\begin{cases} \mathrm{d}V(t) = [\pi(t)V(t)\mu + (1-\pi(t))V(t)r + c]\,\mathrm{d}t \\ \qquad\qquad +\pi(t)V(t)\sigma P(\Omega,t)^{\frac{1-q}{2}}\mathrm{d}W(t)\,, \\ V(0) = V_0\,. \end{cases} \tag{9}$$

3.2. Wealth model after retirement

Suppose the pension accumulated from $t=0$ to $t=T$ is all used to purchase annuities. Let D be the pension fund paid when purchasing the N-term annuity, which obviously satisfies $D \le V(T)$. Let $\bar{D} = D/\bar{a}_{\bar{N}|}$ be the payment amount at time t after retirement, where $\bar{a}_{\bar{N}|} = (1-\mathrm{e}^{-\delta N}/\delta)$ and δ is a continuous technical rate. Then the wealth value process $V(t)$ after retirement can be written as the following stochastic differential equation:

$$\begin{cases} \mathrm{d}V(t) = \left[\pi(t)V(t)\mu + (1-\pi(t))V(t)r - \bar{D}\right]\mathrm{d}t \\ \qquad\qquad +\pi(t)V(t)\sigma P(\Omega,t)^{\frac{1-q}{2}}\mathrm{d}W(t)\,, \\ V(0) = V_0\,. \end{cases} \tag{10}$$

4. Optimal investment problem

In the field of economics, expected utility is commonly used to describe investors' sense of gain. Moreover, pension fund investors are generally conservative and belong to risk-averse investors. Therefore, we choose the exponential function as the expected utility function for investors.

Definition 1. Suppose the expected utility function of pension fund investors is an exponential function as follows:

$$U(x) = -\frac{1}{\beta}\,\mathrm{e}^{-\beta x}\,, \qquad \beta > 0\,. \tag{11}$$

Then, they are called exponential utility function investors.

Definition 2. If the investment strategy $\pi(t)$ is the solution of the stochastic differential equation (9), then the investment strategy $\pi(t)$ is feasible. Note that the set of all feasible solutions is $L^2_{\mathrm{F}}(0,T;R)$, then $\pi(t) \in L^2_{\mathrm{F}}(0,T;R)$.

4.1. Optimal investment model before retirement

Based on maximizing the expected utility criterion, we can write the optimal investment problem for defined-contribution pension before retirement as

$$\begin{cases} \max\limits_{\pi(t)} & E[U(v)]\,, \\ \text{s.t.} & \pi(t) \in L^2_{\mathrm{F}}(0,T;R)\,. \end{cases} \tag{12}$$

4.2. Optimal investment model after retirement

Similarly, we can easily provide the mathematical expression of the optimal investment problem for the defined-contribution pension after retirement under the maximizing expected utility criterion as

$$\begin{cases} \max\limits_{\pi(t)} & E[U(v)]\,, \\ \text{s.t.} & \pi(t) \in L^2_{\mathrm{F}}(T,T+N;R)\,. \end{cases} \tag{13}$$

5. Model solution

To obtain the optimal investment strategy, we will apply a dynamic programming principle to transform the stochastic differential equation of the optimization problem into the corresponding Hamilton–Jacobi–Bellman equation. Then, the non-linear quadratic partial differential equation is derived. Furthermore, using the Legendre transformation and duality theory, we change the non-linear quadratic partial differential equation into a linear quadratic partial differential equation. Finally, the analytical solution is obtained by solving the linear partial differential equation.

5.1. Solving before retirement

Theorem 1. *When the expected utility of investors is the exponential function (11), the optimal investment strategy for the defined-contribution pension before retirement is*

$$\pi_t^* = \frac{\mu - r}{\beta v \sigma^2 P^{(1-q)}} \, \mathrm{e}^{-r(T-t)} \,. \tag{14}$$

Proof. According to the dynamic programming principle, we define the value function of the optimization problem before retirement (12) as

$$H(t,s,v) = \sup_{\pi(t)} E\{U(V(T))|S(t)=s, V(t)=v\}\,, \qquad 0<t<T\,, \tag{15}$$

where $H(T,s,v) = U(v)$. Then the corresponding Hamilton–Jacobi–Bellman equation can be written as

$$\begin{aligned} &H_t + \mu s H_s + (rv+c)H_v + \frac{1}{2}\sigma^2 P^{(1-q)} s^2 H_{ss} \\ &+ \max_{\pi(t)} \left[\pi_t(\mu - r) v H_v + \pi_t \sigma^2 P^{(1-q)} s v H_{sv} + \frac{1}{2}\pi_t^2 \sigma^2 P^{(1-q)} v^2 H_{vv}\right] = 0\,, \end{aligned} \tag{16}$$

where $H_t, H_s, H_v, H_{sv}, H_{ss}$, and H_{vv} are the first-order and second-order partial derivatives of time t, asset price s, and pension wealth v. Solving the partial derivative of the above equation with respect to the investment strategy π_t and making it equal to zero, we get

$$\pi_t^* = -\frac{(\mu - r)H_v + \sigma^2 P^{(1-q)} s H_{sv}}{\sigma^2 P^{(1-q)} v H_{vv}} \,. \tag{17}$$

Substituting (17) into (16), we obtain

$$\begin{aligned} &H_t + \mu s H_s + (rv+c)H_v + \frac{1}{2}\sigma^2 P^{(1-q)} s^2 H_{ss} \\ &- \frac{\left((\mu - r)H_v + \sigma^2 P^{(1-q)} s H_{sv}\right)^2}{2\sigma^2 P^{(1-q)} H_{vv}} = 0\,. \end{aligned} \tag{18}$$

$H(t,s,v)$ can be obtained by solving equation (18). Then by substituting $H(t,s,v)$ into equation (17), the optimal solution of the investment strategy can be derived. Based on the analysis above, we will first solve equation (18).

Definition 3. Suppose $f : R^n \to R$ is a convex function, and $z > 0$. Then the Legendre transformation is defined as

$$L(z) = \max_x \left(f(x) - zx\right). \tag{19}$$

$L(z)$ is called the Legendre dual function of $f(x)$ (see [36]). According to Definition 3, applying the Legendre transformation to the value function $H(t, s, v)$, we can obtain

$$\widetilde{H}(t, s, z) = \sup_{v>0}\{H(t, s, v) - zv | v > 0\}, \qquad z > 0, \qquad 0 < t < T, \quad (20)$$

where z and v are dual variables. The value of v where this optimum is attained is denoted by $g(t, s, z)$, so that (see [36])

$$g(t, s, z) = \inf_{v>0}\left\{v | H(t, s, v) \geq zv + \widetilde{H}(t, s, z)\right\}, \qquad 0 < t < T. \quad (21)$$

The function $\widetilde{H}(t, s, z)$ is related to $g(t, s, z)$ by

$$g(t, s, z) = -\widetilde{H}(t, s, z). \quad (22)$$

Both $g(t, s, z)$ and $\widetilde{H}(t, s, z)$ are dual functions of $H(t, s, v)$. Moreover, there is a correlation as follows:

$$\widetilde{H}(t, s, z) = H(t, s, g) - zg, \qquad g(t, s, z) = v, \qquad H_v = z. \quad (23)$$

Let $\widetilde{U}(z) = \sup_{x>0}\{U(v) - zv | v > 0\}$ and $G(z) = \sup_{v>0}\{v | U(v) \geq zv + \widetilde{U}(z)\}$. Then there is $G(z) = (U')^{-1}(z)$. At the terminal time T, since $H(T, s, v) = U(v)$, we can obtain $g(T, s, z) = \inf_{v>0}\{v | U(v) \geq zv + \widetilde{H}(T, s, z)\}$, $\widetilde{H}(T, s, z) = \sup_{v>0}\{U(v) - zv\}$, and $g(T, s, z) = (U')^{-1}(z)$. Solving the derivatives of variables t, s, and z for equation (23), the derivative relation between the value function H and its dual function $\widetilde{H}$ is obtained as follows:

$$\begin{aligned} H_v &= z, & H_t &= \widetilde{H}_t, & H_s &= \widetilde{H}_s, \\ H_{ss} &= \widetilde{H}_{ss} - \frac{\widetilde{H}_{sz}^2}{\widetilde{H}_{zz}}, & H_{sv} &= -\frac{\widetilde{H}_{sz}}{\widetilde{H}_{zz}}, & H_{vv} &= -\frac{1}{\widetilde{H}_{zz}}. \end{aligned} \quad (24)$$

Substituting (24) into (18), we have

$$\begin{aligned} &\widetilde{H}_t + \mu s \widetilde{H}_s + rzv + \frac{1}{2}\sigma^2 P^{(1-q)} s^2 \widetilde{H}_{ss} + \frac{(\mu - r)^2}{2\sigma^2 P^{(1-q)}} z^2 \widetilde{H}_{zz} \\ &-(\mu - r) sz \widetilde{H}_{sz} = 0. \end{aligned} \quad (25)$$

Solving the derivative of variable z for (25) and considering $v = g = -\widetilde{H}_z$, the partial differential equation of g is obtained

$$\begin{aligned} &g_t + rsg_s - rg - c + \frac{1}{2}\sigma^2 P^{(1-q)} s^2 g_{ss} + \left(\frac{(\mu - r)^2}{\sigma^2 P^{(1-q)}} - r\right) z g_z \\ &+ \frac{(\mu - r)^2}{2\sigma^2 P^{(1-q)}} z^2 g_{zz} - (\mu - r) sz g_{sz} = 0. \end{aligned} \quad (26)$$

Thus, the non-linear quadratic partial differential equation (18) is transformed into a linear partial differential equation (26). According to equation (17), the optimal investment strategy represented by the dual function g is

$$\begin{aligned}
\pi_t^* &= -\frac{(\mu - r)H_v + \sigma^2 P^{(1-q)} s H_{sv}}{\sigma^2 P^{(1-q)} v H_{vv}} \\
&= -\frac{(\mu - r)z - \sigma^2 P^{(1-q)} s \left(\widetilde{H}_{sz} / \widetilde{H}_{zz} \right)}{\sigma^2 P^{(1-q)} v \widetilde{H}_{zz}^{-1}} \\
&= -\frac{(\mu - r)z\widetilde{H}_{zz} - \sigma^2 P^{(1-q)} s \widetilde{H}_{sz}}{\sigma^2 P^{(1-q)} v} \\
&= \frac{-(\mu - r)z g_z + \sigma^2 P^{(1-q)} s g_s}{\sigma^2 P^{(1-q)} g} \,.
\end{aligned} \tag{27}$$

Therefore, we only need to solve the linear partial differential equation (26) of g, and then substitute its solution into equation (27) to obtain the optimal investment strategy for pension. Considering the exponential utility function (11) and $g(T, s, z) = (U')^{-1}(z)$, the boundary condition for the terminal time T is obtained

$$g(T, s, z) = -\frac{1}{\beta} \ln z \,. \tag{28}$$

Suppose the form of the solution to the partial differential equation (26) is

$$g(t, s, z) = -\frac{1}{\beta}[m(t)(\ln z + n(t, s))] + h(t) \,. \tag{29}$$

Moreover, the boundary conditions are $m(T) = 1$, $h(T) = 0$, and $n(T, s) = 0$. By solving the derivatives of equation (29), we can obtain

$$\begin{aligned}
g_s &= -\frac{1}{\beta} m(t) n_s \,, \\
g_t &= -\frac{1}{\beta} \left[m'(t)(\ln z + n(t, s)) + m(t) n_t \right] + h'(t) \,, \\
g_z &= -\frac{1}{\beta z} m(t) \,, \\
g_{zz} &= \frac{1}{\beta z^2} m(t) \,, \\
g_{ss} &= -\frac{1}{\beta} m(t) n_{ss} \,, \\
g_{sz} &= 0 \,.
\end{aligned}$$

Substituting them into equation (26), we have

$$\begin{aligned}&\big[m'(t) - rm(t)\big]\ln z + \big[c + rh(t) - h'(t)\big]\,\beta \\ &+ \left[n_t + rsn_s + \frac{1}{2}\sigma^2 P^{(1-q)} s^2 n_{ss} + \frac{(\mu - r)^2}{2\sigma^2 P^{(1-q)}} - rn + \frac{m'(t)}{m(t)} n - r\right] m(t) = 0\,.\end{aligned} \tag{30}$$

Then, we split equation (30) into the following three equations:

$$m'(t) - rm(t) \;=\; 0\,, \tag{31}$$

$$c + rh(t) - h'(t) \;=\; 0\,, \tag{32}$$

$$n_t + rsn_s + \frac{1}{2}\sigma^2 P^{(1-q)} s^2 n_{ss} + \frac{(\mu - r)^2}{2\sigma^2 p^{(1-q)}} - rn + \frac{m'(t)}{m(t)} n - r \;=\; 0\,. \tag{33}$$

Using the boundary condition $m(T) = 1$ and $h(T) = 0$, we can obtain

$$m(t) \;=\; \mathrm{e}^{-r(T-t)}\,, \tag{34}$$

$$h(t) \;=\; -c\bar{h}_{\overline{T-t}|}\,, \tag{35}$$

where $h(t) = -c\bar{h}_{\overline{T-t}|}$ is the continuous annuity during the $T - t$ period. Now, we only need to solve equation (33). Let $x = \ln s$ and $n(t, s) = u(t, x)$. Then, we can get $u(T, x) = 0$, $n_t = u_t$, $g_s = \frac{1}{s} u_x$, and $g_{ss} = \frac{1}{s^2}(u_{xx} - u_x)$. Substituting them into equation (33), we have

$$u_t + rs\frac{1}{s}u_x + \frac{1}{2}\sigma^2 P^{(1-q)} s^2 \frac{1}{s^2}(u_{xx} - u_x) + \frac{(\mu - r)^2}{2\sigma^2 p^{(1-q)}} - r = 0\,. \tag{36}$$

Combining the terms of u_x and u_{xx} in equation (36), we obtain

$$u_t + \left(r - \frac{1}{2}\sigma^2 P^{(1-q)}\right) u_x + \frac{1}{2}\sigma^2 P^{(1-q)} u_{xx} + \frac{(\mu - r)^2}{2\sigma^2 p^{(1-q)}} - r = 0\,. \tag{37}$$

Using the homogenization method, we first solve the following auxiliary equation

$$\begin{cases} \widetilde{u}_t + \left(r - \frac{1}{2}\sigma^2 P^{(1-q)}\right) \widetilde{u}_x + \frac{1}{2}\sigma^2 P^{(1-q)} \widetilde{u}_{xx} = 0\,, \\ \widetilde{u}(t, x; \xi) = \frac{(\mu - r)^2}{2\sigma^2 p^{(1-q)}} - r\,, \end{cases} \tag{38}$$

where $\widetilde{u} = \widetilde{u}(t, x; \xi)$, and the solution $\widetilde{u}(t, x; \xi)$ of the auxiliary equation (38) has the relation with the solution $u(t, x)$ of the original equation (37) as follows:

$$u(t, x) = \int_t^T \widetilde{u}(t, x; \xi)\,\mathrm{d}\xi\,. \tag{39}$$

Let

$$\widetilde{u}(t,x;\xi) = \left[\frac{(\mu-r)^2}{2\sigma^2 p^{(1-q)}} - r\right] \mathrm{e}^{k_1(t;\xi)x+k_2(t;\xi)}\,, \tag{40}$$

where $k_1(\xi;\xi)=0$, $k_2(\xi;\xi)=0$, $\widetilde{u}_t = [\frac{(\mu-r)^2}{2\sigma^2 p^{(1-q)}} - r]\mathrm{e}^{k_1(t;\xi)x+k_2(t;\xi)}(\frac{\mathrm{d}k_1}{\mathrm{d}t}x + \frac{\mathrm{d}k_2}{\mathrm{d}t})$, $\widetilde{u}_x = [\frac{(\mu-r)^2}{2\sigma^2 p^{(1-q)}} - r]k_1\mathrm{e}^{k_1(t;\xi)x+k_2(t;\xi)}$, and $\widetilde{u}_{xx} = [\frac{(\mu-r)^2}{2\sigma^2 p^{(1-q)}} - r]k_1^2\mathrm{e}^{k_1(t;\xi)x+k_2(t;\xi)}$. Substituting them into equation (38), we have

$$\frac{\mathrm{d}k_1}{\mathrm{d}t}x + \frac{\mathrm{d}k_2}{\mathrm{d}t} + \frac{1}{2}\sigma^2 P^{(1-q)}k_1^2 + \left(r - \frac{1}{2}\sigma^2 P^{(1-q)}\right)k_1 = 0\,. \tag{41}$$

Using $k_1(\xi;\xi)=0$ and $k_2(\xi;\xi)=0$, we get

$$\begin{cases} \frac{k_1}{\mathrm{d}t} = 0\,, \\ k_1(\xi;\xi) = 0\,, \end{cases} \tag{42}$$

$$\begin{cases} \frac{k_2}{\mathrm{d}t} + \frac{1}{2}\sigma^2 P^{(1-q)}k_1^2 + \left(r - \frac{1}{2}\sigma^2 P^{(1-q)}\right)k_1 = 0\,, \\ k_2(\xi;\xi) = 0\,. \end{cases} \tag{43}$$

By solving them, we can obtain

$$k_1(t,\xi) = 0\,, \qquad k_2(t,\xi) = 0\,. \tag{44}$$

Substituting equation (44) into equation (40), we have

$$\widetilde{u}(t,x;\xi) = \frac{(\mu-r)^2}{2\sigma^2 p^{(1-q)}} - r\,. \tag{45}$$

Substituting equation (45) into equation (39), we get

$$\begin{aligned} u(t,x) &= \int_t^T \left(\frac{(\mu-r)^2}{2\sigma^2 P^{(1-q)}} - r\right)\mathrm{d}\xi\,, \\ &= \left(\frac{(\mu-r)^2}{2\sigma^2 P^{(1-q)}} - r\right)(T-t)\,. \end{aligned} \tag{46}$$

Thus, we can obtain

$$n(t,s) = u(t,x) = \left(\frac{(\mu-r)^2}{2\sigma^2 P^{(1-q)}} - r\right)(T-t)\,. \tag{47}$$

Substituting equations (34), (35), and (47) into equation (29), we have

$$g(t,s,z) = -\frac{1}{\beta}\left[\mathrm{e}^{-r(T-t)}\left(\ln z + \left(\frac{(\mu-r)^2}{2\sigma^2 P^{(1-q)}} - r\right)(T-t)\right)\right] - c\bar{h}_{\overline{T-t|}}\,. \tag{48}$$

Calculating the partial derivatives of equation (48), we obtain

$$g_s = 0\,, \tag{49}$$

$$g_z = -\frac{1}{\beta z}\,\mathrm{e}^{-r(T-t)}\,. \tag{50}$$

Substituting equations (49), (50), and $v = g = -\widetilde{H}_z$ into equation (27), we have

$$\pi_t^* = \frac{\mu - r}{\beta v \sigma^2 P^{(1-q)}}\,\mathrm{e}^{-r(T-t)}\,. \tag{51}$$

□

5.2. Solving after retirement

Theorem 2. *When the expected utility of investors is the exponential function (11), the optimal investment strategy for the defined-contribution pension after retirement is*

$$\pi_t^* = \frac{\mu - r}{\beta v \sigma^2 P^{(1-q)}}\,\mathrm{e}^{-r(T+N-t)}\,. \tag{52}$$

Proof. Similar to the solving method before retirement, we first define the value function of the optimization problem (13) after retirement as

$$H(t,s,v) = \sup_{\pi(t)} E\{U(V(T+N))|S(t)=s, V(t)=v\}\,, \qquad T < t \le T+N\,. \tag{53}$$

Then, its corresponding Hamilton–Jacobi–Bellman equation is

$$H_t + \mu s H_s + \left(rv - \bar{D}\right) H_v + \frac{1}{2}\sigma^2 P^{(1-q)} s^2 H_{ss} + \max_{\pi(t)}\left[\pi_t(\mu-r)vH_v + \pi_t\sigma^2 P^{(1-q)} svH_{sv} + \frac{1}{2}\pi_t^2\sigma^2 P^{(1-q)} v^2 H_{vv}\right] = 0\,, \tag{54}$$

where $H(T,s,v) = U(v)$. $H_t, H_s, H_v, H_{sv}, H_{ss}$, and H_{vv} are the first-order and second-order partial derivative functions of time t, asset price s, and pension wealth v. Solving the partial derivative of the above equation with respect to the investment strategy π_t and making it equal to zero, we have

$$\pi_t^* = -\frac{(\mu-r)H_v + \sigma^2 P^{(1-q)} sH_{sv}}{\sigma^2 P^{(1-q)} vH_{vv}}\,. \tag{55}$$

Substituting (55) into (54), we obtain

$$
\begin{aligned}
&H_t + \mu s H_s + \left(rv - \bar{D}\right) H_v + \frac{1}{2}\sigma^2 P^{(1-q)} s^2 H_{ss} \\
&- \frac{\left((\mu - r)H_v + \sigma^2 P^{(1-q)} s H_{sv}\right)^2}{2\sigma^2 P^{(1-q)} H_{vv}} = 0 \,.
\end{aligned} \tag{56}
$$

Using Definition 2 for the value function $H(t, s, v)$, we get

$$
\widetilde{H}(t, s, z) = \sup_{v>0}\{H(t, s, v) - zv | v > 0\}\,, \qquad z > 0\,, \qquad T < t \le T + N\,, \tag{57}
$$

where z is the dual variable of v. Moreover, v satisfies the following equation:

$$
g(t, s, z) = \inf_{v>0} \left\{v | H(t, s, v) \ge zv + \widetilde{H}(t, s, z)\right\}\,, \qquad T < t \le T + N\,, \tag{58}
$$

where $\widetilde{H}(t, s, z)$ and $g(t, s, z)$ are both dual functions of $H(t, s, v)$. Moreover, they have the relation as follows:

$$
\widetilde{H}(t, s, z) = H(t, s, g) - zg\,, \qquad g(t, s, z) = v\,, \qquad H_v = z\,. \tag{59}
$$

Let $\widetilde{U}(z) = \sup_{x>0}\{U(x) - zv | v > 0\}$ and $G(z) = \sup_{v>0}\{v | U(v) \ge zv + \widetilde{U}(z)\}$. It is not difficult to obtain $G(z) = (U')^{-1}(z)$. At the terminal time $T + N$, using $H(T + N, s, v) = U(v)$, we have $g(T + N, s, z) = \inf_{v>0}\{v | U(v) \ge zv + \widetilde{H}(T + N, s, z)\}$, $\widetilde{H}(T + N, s, z) = \sup_{v>0}\{U(v) - zv\}$, and $g(T + N, s, z) = (U')^{-1}(z)$. Solving the derivatives of variables t, s, and z for (59), we have

$$
\begin{aligned}
H_v &= z\,, & H_t &= \widetilde{H}_t\,, & H_s &= \widetilde{H}_s\,, \\
H_{ss} &= \widetilde{H}_{ss} - \frac{\widetilde{H}_{sz}^2}{\widetilde{H}_{zz}}\,, & H_{sv} &= -\frac{\widetilde{H}_{sz}}{\widetilde{H}_{zz}}\,, & H_{vv} &= -\frac{1}{\widetilde{H}_{zz}}\,.
\end{aligned} \tag{60}
$$

Substituting (60) into (56) and using $v = g = -\widetilde{H}_z$, the partial differential equation of g is obtained

$$
\begin{aligned}
&g_t + rsg_s - rg + \bar{D} + \frac{1}{2}\sigma^2 P^{(1-q)} s^2 g_{ss} + \left(\frac{(\mu - r)^2}{\sigma^2 P^{(1-q)}} - r\right) z g_z \\
&+ \frac{(\mu - r)^2}{2\sigma^2 P^{(1-q)}} z^2 g_{zz} - (\mu - r) s z g_{sz} = 0\,.
\end{aligned} \tag{61}
$$

Meanwhile, substituting (60) into (55) and using $v = g = -\widetilde{H}_z$, we have

$$\begin{aligned}\pi_t^* &= -\frac{(\mu - r)H_v + \sigma^2 P^{(1-q)} s H_{sv}}{\sigma^2 P^{(1-q)} v H_{vv}} \\ &= -\frac{(\mu - r)z - \sigma^2 P^{(1-q)} s \left(\widetilde{H}_{sz}/\widetilde{H}_{zz}\right)}{\sigma^2 P^{(1-q)} v \left(1/\widetilde{H}_{vv}\right)} \\ &= \frac{-(\mu - r) z g_z + \sigma^2 P^{(1-q)} s g_s}{\sigma^2 P^{(1-q)} g} \,. \end{aligned} \tag{62}$$

Therefore, to obtain the optimal investment strategy of the defined-contribution pension, we only need to solve g through (61) and then substitute the solution into (62). Similar to the solution method before retirement, we suppose that the form of the solution of equation (61) is

$$g(t, s, z) = -\frac{1}{\beta}[m(t)(\ln z + n(t, s))] + h(t)\,. \tag{63}$$

Solving the derivatives of equation (63), we obtain

$$\begin{aligned} g_s &= -\frac{1}{\beta} m(t) n_s\,, \\ g_t &= -\frac{1}{\beta}\left[m'(t)(\ln z + n(t,s) + m(t) n_t\right] + h'(t)\,, \\ g_z &= -\frac{1}{\beta z} m(t)\,, \\ g_{zz} &= \frac{1}{\beta z^2} m(t)\,, \\ g_{ss} &= -\frac{1}{\beta} m(t) n_{ss}\,, \\ g_{sz} &= 0\,. \end{aligned}$$

Substituting them into equation (61), we have

$$\begin{aligned} &\left[m'(t) - r m(t)\right] \ln z + \left[r h(t) - h'(t) - \bar{D}\right] \beta + n_t + r s n_s \\ &+ \frac{1}{2}\sigma^2 P^{(1-q)} s^2 n_{ss} + \frac{(\mu - r)^2}{2\sigma^2 P^{(1-q)}} - r n + \frac{m'(t)}{m(t)} n - r m(t) = 0\,. \end{aligned} \tag{64}$$

We split equation (64) into the following three equations:

$$m'(t) - r m(t) = 0\,, \tag{65}$$

$$-\bar{D} + r h(t) - h'(t) = 0\,, \tag{66}$$

$$n_t + r s n_s + \frac{1}{2}\sigma^2 P^{(1-q)} s^2 n_{ss} + \frac{(\mu - r)^2}{2\sigma^2 p^{(1-q)}} - r n + \frac{m'(t)}{m(t)} n - r = 0\,. \tag{67}$$

Using the boundary conditions $m(T+N)=1$ and $h(T+N)=0$, we obtain

$$m(t) = \mathrm{e}^{-r(T+N-t)}, \tag{68}$$

$$h(t) = \bar{D}\,\bar{h}_{\overline{T+N-t}|}. \tag{69}$$

Then, we only need to solve equation (67). Let $x=\ln s$ and $n(t,s)=u(t,x)$. Using the boundary condition $n(T+N,s)=0$, we have $u(T+N,x)=0$, $n_t=u_t$, $g_s=\frac{1}{s}u_x$, and $g_{ss}=\frac{1}{s^2}(u_{xx}-u_x)$. Substituting them into equation (67), we get

$$u_t + rs\frac{1}{s}u_x + \frac{1}{2}\sigma^2 P^{(1-q)} s^2 \frac{1}{s^2}(u_{xx}-u_x) + \frac{(\mu-r)^2}{2\sigma^2 p^{(1-q)}} - r = 0. \tag{70}$$

Combining the terms of u_x and u_{xx} in equation (70), we obtain

$$u_t + \left(r - \frac{1}{2}\sigma^2 P^{(1-q)}\right)u_x + \frac{1}{2}\sigma^2 P^{(1-q)} u_{xx} + \frac{(\mu-r)^2}{2\sigma^2 p^{(1-q)}} - r = 0. \tag{71}$$

Using the homogenization method, we first solve the auxiliary equation (72) as follows:

$$\begin{cases} \widetilde{u}_t + \left(r - \frac{1}{2}\sigma^2 P^{(1-q)}\right)\widetilde{u}_x + \frac{1}{2}\sigma^2 P^{(1-q)}\widetilde{u}_{xx} = 0, \\ \widetilde{u}(t,x;\xi) = \frac{(\mu-r)^2}{2\sigma^2 p^{(1-q)}} - r, \end{cases} \tag{72}$$

where $\widetilde{u}=\widetilde{u}(t,x;\xi)$ and

$$u(t,x) = \int_t^{T+N} \widetilde{u}(t,x;\xi)\mathrm{d}\xi. \tag{73}$$

Let

$$\widetilde{u}(t,x;\xi) = \left[\frac{(\mu-r)^2}{2\sigma^2 p^{(1-q)}} - r\right]\mathrm{e}^{k_1(t;\xi)x + k_2(t;\xi)}. \tag{74}$$

Substituting equation (74) into equation (72), we have

$$\frac{\mathrm{d}k_1}{\mathrm{d}t}x + \frac{\mathrm{d}k_2}{\mathrm{d}t} + \frac{1}{2}\sigma^2 P^{(1-q)}k_1^2 + \left(r - \frac{1}{2}\sigma^2 P^{(1-q)}\right)k_1 = 0. \tag{75}$$

Using $k_1(\xi;\xi)=0$ and $k_2(\xi;\xi)=0$, we get

$$\begin{cases} \frac{k_1}{\mathrm{d}t} = 0, \\ k_1(\xi;\xi) = 0, \end{cases} \tag{76}$$

$$\begin{cases} \frac{k_2}{\mathrm{d}t} + \frac{1}{2}\sigma^2 P^{(1-q)}k_1^2 + \left(r - \frac{1}{2}\sigma^2 P^{(1-q)}\right)k_1 = 0, \\ k_2(\xi;\xi) = 0. \end{cases} \tag{77}$$

Furthermore, it can easily be obtained that

$$k_1(t,\xi) = 0, k_2(t,\xi) = 0\,. \tag{78}$$

Substituting equation (78) into equation (74), we have

$$\widetilde{u}(t,x;\xi) = \frac{(\mu - r)^2}{2\sigma^2 p^{(1-q)}} - r\,. \tag{79}$$

Substituting equation (79) into equation (73), we get

$$\begin{aligned} u(t,x) &= \int_t^{T+N} \left(\frac{(\mu - r)^2}{2\sigma^2 P^{(1-q)}} - r \right) \mathrm{d}\xi \\ &= \left(\frac{(\mu - r)^2}{2\sigma^2 P^{(1-q)}} - r \right) (T + N - t)\,. \end{aligned} \tag{80}$$

Furthermore, we have

$$n(t,s) = u(t,x) = \left(\frac{(\mu - r)^2}{2\sigma^2 P^{(1-q)}} - r \right) (T + N - t)\,. \tag{81}$$

Substituting (68), (69), and (81) into (63), the solution for $g(t,s,z)$ is obtained

$$\begin{aligned} &g(t,s,z) \\ &= -\frac{1}{\beta} \left[\mathrm{e}^{-r(T+N-t)} \left(\ln z + \left(\frac{(\mu - r)^2}{2\sigma^2 P^{(1-q)}} - r \right) (T + N - t) \right) \right] - c\bar{h}_{\overline{T+N-t}|}\,. \end{aligned} \tag{82}$$

Solving the derivatives of (82), we get

$$g_s = 0\,, \tag{83}$$

$$g_z = -\frac{1}{\beta z} \mathrm{e}^{-r(T+N-t)}\,. \tag{84}$$

Substituting (83) and (84) into (62) and using $v = g = -\widetilde{H}_z$, we have

$$\pi_t^* = \frac{\mu - r}{\beta v \sigma^2 P^{(1-q)}} \mathrm{e}^{-r(T+N-t)}\,. \tag{85}$$

□

According to Theorem 1 and Theorem 2, when $q \to 1$, we can easily derive the optimal investment strategy under the condition that the underlying asset obeys the Brownian motion as follows:

Before retirement:

$$\pi_t^* = \frac{\mu - r}{\beta v \sigma^2} \, \mathrm{e}^{-r(T-t)} \,, \tag{86}$$

after retirement:

$$\pi_t^* = \frac{\mu - r}{\beta v \sigma^2} \, \mathrm{e}^{-r(T+N-t)} \,. \tag{87}$$

6. Numerical results

This section provides some numerical calculations to illustrate the applicability of the model and the dynamic behavior of the optimal strategy. The daily closing data of the Shanghai Composite Index is selected as analyzed data sets. The time span of the data is from 01/04/2022 to 07/30/2024. In practice, we adopt the usual logarithmic return form. Denoting the closing index on the days as $x(t)$, the daily index return is defined by $R(t) = \ln[x(t)] - \ln[x(t-1)]$.

From Table 1, we find that the kurtosis value of the daily return distribution is 33.1689 which is much greater than that of the Gaussian distribution (the kurtosis value of the Gaussian distribution is 3). Moreover, the value of the J–B test is 24269 and the test probability is 0.0010 (the significance level is set as 0.05), which means the J–B test rejects the null hypothesis that the daily return distribution of the Shanghai Composite Index follows the Gaussian distribution.

Table 1. The statistical characteristics of daily returns of the Shanghai Composite Index.

Mean	Standard deviation	Kurtosis	J–B	P
−0.0000569	0.0108	33.1689	24269	0.0010

Figure 1 illustrates that both the histogram of daily returns and the Tsallis distribution have greater kurtosis than the Gaussian distribution. Furthermore, the Tsallis distribution with parameter $q = 1.45$ fits the empirical density distribution of the daily returns more accurately than the Gaussian distribution.

Figure 2 shows that the optimal investment strategy for risky assets is a decreasing function of wealth. However, at the same level of wealth, a greater parameter q corresponds to a greater optimal investment strategy, which means that investors need to take higher risks to obtain expected returns when the market fluctuates greatly.

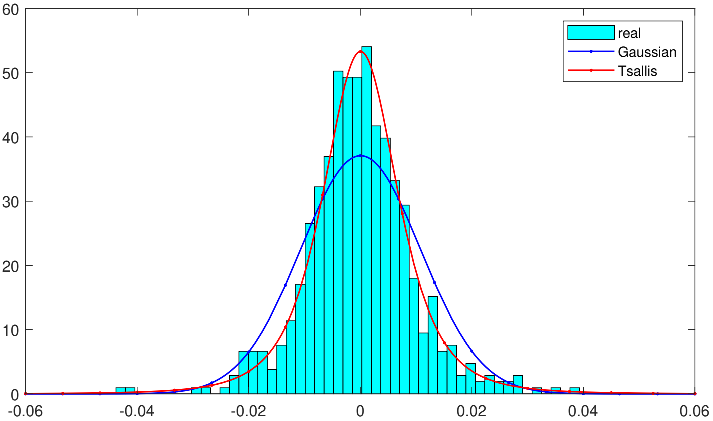

Fig. 1. Comparison of fitting of the daily return empirical distribution for the empirical distribution, Gaussian distribution, and Tsallis distribution ($q = 1.45$).

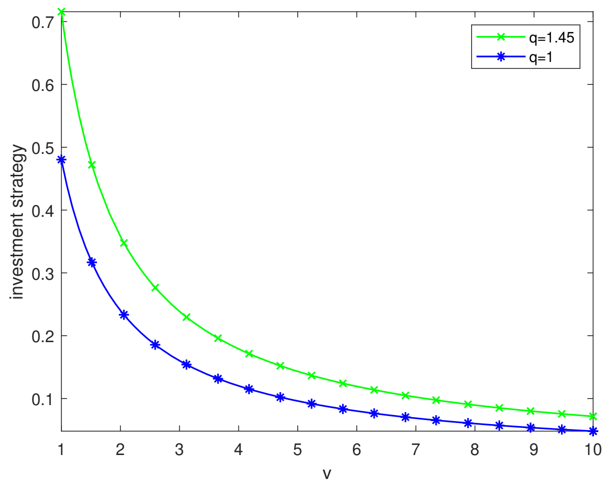

Fig. 2. Comparison of the optimal investment strategies for the Gaussian ($q = 1$) and Tsallis distribution. ($\beta = 2$, $\mu = 0.08$, $\sigma = 0.2$, $r = 0.04$, $v_0 = 1$, $c = 0.1$, $q = 1.45$, $\Delta t = 1$).

7. Summary

Accurately fitting the volatility of asset prices is the foundation for investors to make investment decisions. An increasing number of research results indicate that asset prices often exhibit long-range memory and fat tails, which are not suitable for modeling using the classical Brownian motion. In this paper, we use the non-extensive statistical theory to establish the asset-price model, which can more accurately fit the fluctuations of asset prices. Moreover, based on this, we study the optimal investment problem of pension and obtain an analytical solution for the optimal investment strategy. In future work, we can apply this optimal investment model to study the investment decision-making problems of other financial derivatives.

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