Acta Physica Polonica B **56**, 8-A2 (2025)

THREE-BAND EXTENSION FOR THE GLASHOW–WEINBERG–SALAM MODEL

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Received 6 March 2025, accepted 17 June 2025, published online 1 August 2025

By analogy with the Ginzburg–Landau theory of multi-band superconductors with inner (interband) Josephson couplings, we formulate the three-band Glashow–Weinberg–Salam model with weak Josephson couplings between strongly asymmetrical condensates of scalar (Higgs) fields. Unlike the usual single-band model, we found three Higgs bosons corresponding to the three generations of particles. Moreover, the heaviest of these bosons corresponds to the already discovered H-boson and decays into fermions of only the third generation through the Yukawa interaction. The other two decay into fermions of the first and second generations, but they are difficult to observe due to very poor production conditions. We found two sterile ultra-light Leggett bosons, the Bose condensates of which form the dark halos of galaxies and their clusters (*i.e.* so-called Dark Matter). The masses of the Leggett bosons are determined by the coefficient of the interband coupling and can be arbitrarily small ($\sim 10^{-20}$ eV) due to non-perturbativeness of the interband coupling. Since propagation of the Leggett bosons is not accompanied by a current, these bosons are not absorbed by gauge fields, unlike the common-mode Goldstone bosons. Three coupled condensates of the scalar fields are related to the existence of three generations of leptons, where each generation interacts with the corresponding condensate getting mass. The interflavor mixing between the generations of active neutrinos and sterile right-handed neutrinos in the three-band system causes the existence of mass states of neutrino without interaction with the Higgs condensates.

DOI:10.5506/APhysPolB.56.8-A2

1. Introduction

The Standard Model (SM) is an $\mathrm{SU}(3)_c \otimes \mathrm{SU}(2)_I \otimes \mathrm{U}(1)_Y$ gauge theory. Here, $\mathrm{SU}(3)_c$ is the symmetry of the strong color interaction of quarks and gluons. The group of the weak isospin I and the weak hypercharge Y, $\mathrm{SU}(2)_I \otimes \mathrm{U}(1)_Y$, describes the electroweak interaction of quarks and leptons mediated by the corresponding gauge bosons $\vec{A}_\mu, B_\mu$. Due to the coefficient $a<0$ in the potential for the scalar field $a\varphi^+\varphi+\frac{b}{2}(\varphi^+\varphi)^2$, the complex scalar field $\varphi=|\varphi|\mathrm{e}^{i\theta}$ acquires a nonzero vacuum expectation value, which can be supposed as $|\langle 0|\varphi|0\rangle|=\sqrt{\frac{|a|}{b}}\equiv\varphi_0$, and the $\mathrm{SU}(2)_I\otimes\mathrm{U}(1)_Y$ electroweak symmetry is spontaneously broken down to the $\mathrm{U}(1)_Q$ gauge symmetry of electromagnetism with the electrical charge $Q=I_z+\frac{Y}{2}$. Here, the Higgs mechanism takes place: the phase θ is absorbed by the gauge fields, and while three linear combinations of the gauge fields interact with the condensate φ_0 and become massive (*i.e.* W^+, W^-, Z bosons), the photon $\gamma_\mu=A_{z\mu}\sin\alpha+B_\mu\cos\alpha$ remains massless $\frac{g^2}{4}\varphi_0^2(A_{x\mu}A_x^\mu+A_{y\mu}A_y^\mu)+\frac{1}{4}\varphi_0^2\left(g^2A_{z\mu}A_z^\mu+f^2B_\mu B^\mu\right)=\frac{g^2}{2}\varphi_0^2W_\mu W^{*\mu}+\frac{\widetilde{g}^2}{4}\varphi_0^2Z_\mu Z^\mu$ (here, $\sin\alpha=0.47$ is the Weinberg angle, $e=1/\sqrt{137}$ is an elementary charge in the Gaussian system of units, $g=\frac{e}{\sin\alpha}$, $f=\frac{e}{\cos\alpha}$, $\widetilde{g}^2=g^2+f^2$). In addition, the Dirac fields ψ (spinor) interact with the condensate by the Yukawa interaction $\chi(\bar{\psi}_\mathrm{L}\varphi\psi_\mathrm{R}+\bar{\psi}_\mathrm{R}\varphi^+\psi_\mathrm{L})$, and, as the result, leptons obtain masses $m_{\mathrm{D}i}=\chi_i\varphi_0$ (where $i=e,\mu,\tau$ — electron, muon, tauon); it is analogously for quarks, however neutrino remains strictly massless, and it is supposed that the right-handed neutrino ν_R and the left-handed antineutrino $\nu_\mathrm{L}^\mathrm{C}$ are absent [1, 2], but in various extensions of SM, the existence of additional neutrinos with different parameters is allowed, for example, the neutrino minimal Standard Model (νMSM) supposes the existence of three sterile right-handed neutrinos ν_R [3]. It should be noted that the lepton mixing and the quark mixing occur in such a way that some elements of the mixing matrices, *i.e.* the PMNS matrix for neutrino mixing and the CKM matrix for quark mixing, are complex (presence of phase multipliers $\mathrm{e}^{\pm i\delta_\mathrm{CP}}$), which results in the CP violation [4–9].

SM with its minimal Higgs structure successfully describes the nature of fundamental particles. Especially, the Glashow–Weinberg–Salam (GWS) model of the electroweak interaction provides an extremely successful description of the observed electroweak phenomena. However, SM in its present form is unable to describe a number of extremely important phenomena. In the present work, we would like to discuss some of them.

1.1. Dark Matter

At present time, it is well known that the total mass–energy of the observable universe consists of 5% ordinary matter (baryonic, leptonic, photonic), 26% Dark Matter (DM), and 69% in the form of energy known as the dark energy [10]. Thus, DM constitutes 81% of the total mass. Thus, the total mass of the Milky Way taking into account DM is estimated as $M\sim 0.8\ldots 1.2\times 10^{12}M_\odot$ and the radius of the DM halo is estimated as

$r_0 \sim 120$ kpc [11]. On the contrary, the mass of baryonic matter in the Milky Way is estimated as $M_B \sim 5 \ldots 7 \times 10^{10} M_\odot$, and radius of the disk is estimated as $r_B \sim 25$kpc. Thus, DM constitutes 94% of the total mass of the Milky Way and the region occupied by the relatively dense baryonic matter is a very small region in a central part of the DM halo. Thus, the Milky Way (in the same way as other galaxies and galaxy clusters) is immersed in an almost homogeneous cloud of DM as illustrated in Fig. 1. Moreover, density perturbations in the baryon–electron–photon plasma before recombination do not grow due to high light pressure. Instead, the perturbations produce sound waves that propagate in the plasma. Since DM particles do not interact with photons, there is nothing to prevent them from forming self-gravitating clusters. After recombination, baryons fall into potential wells formed by DM. Galaxies form in those regions where DM originally formed self-gravitating clusters [12]. Thus, without DM, no structures would have been formed, no galaxies, no life.

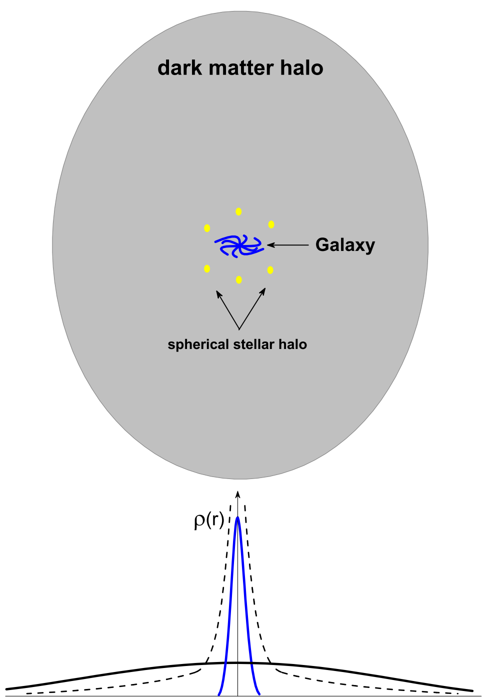

Fig. 1. Top figure: the region of the DM halo compared with the size of the galaxy and its stellar halo. Lower figure: corresponding profiles $\rho(r)$ of DM density (dark line) and baryonic matter density (blue line). The dashed line is the result of numerical simulations of the distribution of DM density, where we can see a singularity — "cusp".

Thus, we have a situation, when SM does not describe 81% of matter in the universe. Attempts have been made repeatedly to expand the SM so that it would include particles of DM. Since such particles do not manifest themselves in any way except through gravity (do not absorb, radiate or scatter electromagnetic waves and do not cause any significant nuclear reactions), they must be almost sterile: they do not interact with photons and do not participate in strong interactions, only the weak interactions are allowed. Therefore, they have been proposed as candidates for DM particles, for example, sterile (right-chiral) neutrinos [13–16] with a mass of ~ 1 keV, neutralinos (as WIMP) with a mass of $> 10^2$ GeV [17, 18], axions with a mass of $\sim 10^{-2}$ eV [17, 19], light scalaron of $f(R)$ gravity with a mass of $\sim 10^{-3}$ eV [20], and many others [17, 21]. At present, no DM candidate particles have been detected.

In order to form potential wells, the DM particles must be nonrelativistic, because relativistic particles travel through gravitational wells instead of being trapped there. On the other hand, according to numerical simulations, a DM halo should tend to produce densities in galactic centers as $\rho \sim r^{\alpha}$ with $\alpha \approx -1$: the so-called cusp in density profile [22–25]. At the same time, the observed distributions of the DM halo is almost flat in the centre of a DM cloud $\rho \sim r^0$. For example, distributions of mass in a DM halo profile and in ordinary baryonic matter are schematic shown in Fig. 1. The cuspy halo problem is proposed to be solve by heating the DM gas in the central region as, for example, is proposed in [26]. Another solution to this problem is, instead of proposing a complicated mechanism for heating the DM gas, to assume a property of the DM particles, which makes impossible formation of a cusp. If DM is composed of some kind of ultra-light bosons ($10^{-24} \lesssim m \lesssim 1$ eV), then such a Bose gas can form a Bose–Einstein condensate [25, 27–29]. The latest state of development of this hypothesis is presented in the review [30]. Due to the uncertainty principle, the central cusp is washed out to the flat profile, moreover, the formation of small structures (galaxy satellites) is suppressed, many of which are predicted by the cold DM theory. Such a model has different names in the literature, such as Fuzzy Dark Matter (FDM), ultra-light DM, BEC Dark Matter, wave DM, scalar field DM, and others. Estimation of the ultra-light boson masses lies within a wide range — from $\sim 10^{-24}$ eV, which was obtained by comparing the de-Broglie wave length of DM to the typical size of the DM halo in galaxies (~ 100 kpc) [27]. If we suppose that the DM halo has some structure: a core of size ~ 1 kpc from BEC and a Bose gas behaving as the cold DM, then a mass of $\sim 10^{-22}$ eV [28–32] is assumed. At the same time, observations of stellar kinematics in dwarf galaxies give a mass of $\sim 10^{-22} \ldots 10^{-20}$ eV [33–35]. Obviously, in these models, the ultra-light bosons are assumed to be noninteracting or to interact very weakly with each other. If we suppose a strong interaction between bosons, then they can form a superfluid Bose liquid (as HeII). In this case, the mass of the boson can be ~ 1 eV [30]. However, obviously, in such a model, in addition to unknown particles, there is also an interaction of unknown nature. Thus, we can see that this hypothesis about FDM adequately describes the dark halo, despite some backlash in boson masses. However, the nature of the ultra-light weakly interacting (or even sterile) bosons remains unknown: these bosons do not fit into the framework of SM.

1.2. Neutrino masses

Observation of the neutrino oscillations in vacuum means the presence of mass of neutrinos [36–41], but only the differences in the squares of the masses can be measured: $|\Delta m_{23}^2| \equiv |m_3^2 - m_2^2| \approx 2.51 \times 10^{-3}$ eV^2, $|\Delta m_{12}^2| \approx 7.41 \times 10^{-5}$ eV^2 [42, 43], and the upper limits of the masses $\sqrt{m_{\nu e}^2}, \sqrt{m_{\nu\mu}^2}, \sqrt{m_{\nu\tau}^2}$ can be determined experimentally from the β-decay of tritium, pion decay, τ-decays into multi-pion final states, respectively [44, 45]. Cosmological data (anisotropy of cosmic microwave background radiation, formation of structures, *etc.*) impose restrictions on masses: $\sum_\nu m_\nu < 0.19$ eV [46], $\sum_\nu m_\nu < 0.28$ eV [47]. We can formally write the Dirac mass term (Yukawa interaction) for both the charged lepton and the neutrino

$$\begin{aligned} \mathcal{U}_{\mathrm{D}} &= \chi_l \left(\bar{\psi}_{\mathrm{L}} \Psi l_{\mathrm{R}} + l_{\mathrm{R}} \Psi^+ \psi_{\mathrm{L}}\right) + \chi_\nu \left(\bar{\psi}_{\mathrm{L}} \widetilde{\Psi} \nu_{\mathrm{R}} + \bar{\nu}_{\mathrm{R}} \widetilde{\Psi}^+ \psi_{\mathrm{L}}\right) \\ &= m_{\mathrm{D}l} \left(\bar{l}_{\mathrm{L}} l_{\mathrm{R}} + \bar{l}_{\mathrm{R}} l_{\mathrm{L}}\right) + m_{\mathrm{D}\nu} \left(\bar{\nu}_{\mathrm{L}} \nu_{\mathrm{R}} + \bar{\nu}_{\mathrm{R}} \nu_{\mathrm{L}}\right), \end{aligned} \tag{1}$$

where the isospinor $\psi_{\mathrm{L}} = \begin{pmatrix} \nu_{\mathrm{L}} \\ l_{\mathrm{L}} \end{pmatrix}$ is a left-handed dublet, l_{R} are ν_{R} right-handed singlets (here, ν_{L} is a spinor of the active neutrino, ν_{R} is a spinor of the hypothetical sterile neutrino, $l_{\mathrm{L,R}}$ are spinors of charged lepton), $\Psi = \begin{pmatrix} 0 \\ \varphi \end{pmatrix}$, $\widetilde{\Psi} = i\tau_y \Psi = \begin{pmatrix} \varphi \\ 0 \end{pmatrix}$ are isospinors, where φ is scalar field with condensate $\langle\varphi\rangle = \varphi_0 \neq 0$, τ_y is a Pauli matrix. $m_{\mathrm{D}l} = \chi_l \varphi_0$ and $m_{\mathrm{D}\nu} = \chi_\nu \varphi_0$ are Dirac masses of the charged lepton and the neutrino, respectively. However, the problem is the unnatural difference in the Yukawa constants

$$\chi_\nu \sim 10^{-11} \ll \chi_l \sim 10^{-6}, \tag{2}$$

unlike, for example, top and bottom rows of quarks, where their masses are not very different.

In SM, the right-handed neutrinos ν_{R} are absent, hence $m_{\mathrm{D}\nu} = 0$. There are several opportunities for the extension of SM, where the small neutrino mass appears. For example, following review in [38], it should be noted that in the Gelmini–Roncadelli model, where an extension of the model with the single-scalar field to the scalar doublet has been proposed, the additional vacuum condensate φ_1 appears, so that $\varphi_1 \ll \varphi_0 \sim 250$ GeV. In this model, the neutrino interacts only with the last $\chi\bar{\nu}_{\mathrm{L}}^{\mathrm{C}}\varphi_1\nu_{\mathrm{L}}$ (where "C" is a charge conjugation), so the neutrino mass can be much smaller than the electron mass. Another example is the well-known "see-saw" mechanism, in which there are two scales of mass $m_{\mathrm{D}} \ll m_{\mathrm{M}} \sim 10^{14}$ GeV, so that $m_\nu \sim \frac{m_{\mathrm{D}}^2}{m_{\mathrm{M}}} \ll m_{\mathrm{D}}$ as a result of diagonalization of the mass matrix. However, these models assume that the neutrino is a Majorana neutrino which results in the neutrinoless double β-decay, but this has not been yet observed. Thus, origin of the neutrino mass remains unknown.

1.3. The absence of experimentally detected decays of the Higgs boson into fermions of the second and first generations

There are many types of H-boson decay channels [48–50]. Due to the Yukawa coupling, the H-boson can decay into quark–antiquark pairs (all quarks except t-quarks, because $m_t > m_H$) and into lepton–antilepton pairs as illustrated in Fig. 2. According to SM, the H-boson should decay as follows: $H \to b\bar{b}$ with a probability of 57.5%, $H \to \tau\bar{\tau}$ with a probability of 6.30%, $H \to c\bar{c}$ with a probability of 2.90%, and $H \to \mu\bar{\mu}$ with a probability of $\lesssim 0.022\%$ [50]. At the same time, there has been no quite reliable experimental evidence found in direct searches by the ATLAS and CMS collaborations [51, 52] for an H-boson decaying into a charm quark–antiquark pair, a strange quark–antiquark, an electron–positron pair, and a muon–antimuon pair. This fact is usually associated with the small Yukawa constant for the first and second generations of fermions. However, the decay rate into a pair of c-quarks is not much smaller than the decay rate into a pair of τ-leptons (the decay probabilities are 2.9% and 6.4%, respectively). On the other hand, such very rare decays as two-photon decay $H \to \gamma\gamma$ with a probability of $\approx 0.2\%$ have been detected. Thus, in our opinion, the absence of experimentally detected decays of the H-boson into fermions of the second and first generations can point to New Physics, in the sense that several Higgs fields can exist, so that the mass of each generation is caused by the corresponding Higgs field.

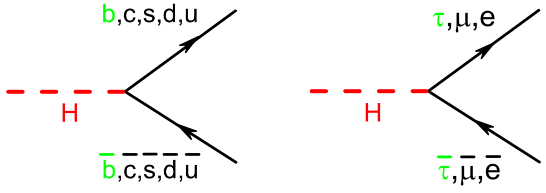

Fig. 2. Theoretical decays of Higgs boson into quark–antiquark pairs and lepton–antilepton pairs due to the Yukawa coupling. The green font denotes experimentally observed decays with a significance grater than 5σ.

It should be noted that the CMS Collaboration reported on the $H \to \mu\bar{\mu}$ decay with a significance of 3σ [53]. At the same time, the ATLAS Collaboration reported on $H \to \gamma\mu\bar{\mu}, \gamma e\bar{e}$ decays, which occur through many intermediate channels due to various interactions (via virtual photons, Z-, W-bosons, quarks) with a significance of 3.2σ [54]. Thus, $H \to \mu\bar{\mu}$ and $H \to \gamma\mu\bar{\mu}$ decays still need to be securely separated. Unlike $H \to \gamma\gamma$, it is difficult to discover the signal of $H \to c\bar{c}$, because the background from QCD is several orders of magnitude larger than the signal. Thus, LHC is not well suited to these problems, but multi-TeV lepton–antilepton colliders would be more suitable.

1.4. Why are three generations of fermions needed, the problem of the hierarchy of their masses and lepton oscillations

As is well known, all fundamental fermions are divided into three generations, that is, three sets of particles with identical interactions but with very different masses (except neutrinos): the first — u, d-quarks, e, ν_e-leptons (electron and electron neutrino), the second — c, s-quarks, μ, ν_μ-leptons (muon and muon neutrino), the third — t, b-quarks, τ, ν_τ-leptons (tauon and tau neutrino). However, the first generation is sufficient for the substance and it is unclear why the other two are needed. Thus, in Ref. [55], the model with two heavy right-handed neutrinos is proposed in order to provide a generation of baryon asymmetry in the early universe and one sterile right-handed neutrino which makes up DM. However, this model requires the "see-saw" mechanism. The origin of the mass hierarchy is unknown at this time. Indeed, for instance, the electron ($m_e = 0.511$ MeV), the muon ($m_e = 105.7$ MeV), and the tauon ($m_e = 1777$ MeV) carry identical gauge quantum numbers, but their masses differ by orders of magnitude (this means that their Yukawa constants $\chi_e, \chi_\mu, \chi_\tau$ differ by orders of magnitude, because $m_{\mathrm{D}i} = \chi_i \varphi_0$). As stated in the review [56], an explanation of the hierarchy requires extra spatial dimensions. Moreover, the neutrino oscillations take place with large mixing angles ($\sim \pi/4$), however for charged leptons (electron–muon–tauon), the mixing is absent. It should be noted that purely quantum-mechanical reasons for the absence of oscillations of charged leptons associated with the processes of their detection were expressed [57, 58]. When the production of more than one type of mass-eigen-state charged leptons is kinematically allowed, the charged lepton states are either produced as incoherent mixtures of e, μ, and τ, or they lose their coherence over microscopic distances due to the large difference in the masses of the basis states $m(\tau) - m(e) \gg m(\nu_\tau) - m(\nu_e)$, except at extremely high energies, not accessible to present experiments. However, this does not exclude others fundamental reasons for the absence of the mixing of charged leptons.

To solve these and other problems of SM, a two-Higgs-doublet model (2HDM) as a simple extension of SM is used [59–63]. This model supposes a two-doublet scalar potential

$$
\begin{aligned}
V_{\rm 2HDM} \; = \; & m_{11}^2 \Psi_1^+ \Psi_1 + m_{22}^2 \Psi_2^+ \Psi_2 - m_{12}^2 \left(\Psi_1^+ \Psi_2 + \Psi_2^+ \Psi_1\right) \\
& + \frac{1}{2}\lambda_1 \left(\Psi_1^+ \Psi_1\right)^2 + \frac{1}{2}\lambda_2 \left(\Psi_2^+ \Psi_2\right)^2 + \lambda_3 \left(\Psi_1^+ \Psi_1\right)\left(\Psi_2^+ \Psi_2\right) \\
& + \lambda_4 \left(\Psi_1^+ \Psi_2\right)\left(\Psi_2^+ \Psi_1\right) + \frac{1}{2}\lambda_5 \left(\left(\Psi_1^+ \Psi_2\right)^2 + \left(\Psi_2^+ \Psi_1\right)^2\right) \\
& + \lambda_6 \left(\Psi_1^+ \Psi_1\right)\left(\Psi_1^+ \Psi_2 + \Psi_2^+ \Psi_1\right) + \lambda_7 \left(\Psi_2^+ \Psi_2\right)\left(\Psi_1^+ \Psi_2 + \Psi_2^+ \Psi_1\right) \, . \qquad (3)
\end{aligned}
$$

Here, we restrict to the CP-conserving models in which all λ_i and m_{ij}^2 are real, at least one of $m_{ii}^2 < 0$ and $\lambda_{1,2} > 0$. For illustration and simplicity, an exact Z_2 discrete symmetry can be imposed, *i.e.* $\Psi_1 \to -\Psi_1, \Psi_2 \to \Psi_2$. Then $m_{12} = 0, \lambda_{6,7} = 0$. The fields $\Psi_{1,2}$ are SU(2) isospinors

$$
\begin{aligned}
\Psi_{1,2} \; &= \; \begin{pmatrix} \phi_{1,2}^+ \\ (v_{1,2} + \rho_{1,2} + i\eta_{1,2})/\sqrt{2} \end{pmatrix} , \\
\Psi_{1,2}^+ \; &= \; \begin{pmatrix} \phi_{1,2}, & (v_{1,2} + \rho_{1,2} - i\eta_{1,2})/\sqrt{2} \end{pmatrix} , \qquad (4)
\end{aligned}
$$

where scalar vacuum condensates $v_{1,2}$ are such that $\sqrt{v_1^2 + v_2^2} = 246$ GeV, $\langle \rho_{1,2} \rangle = \langle \eta_{1,2} \rangle = \langle \phi_{1,2} \rangle = 0$. There are 8 fields

$$
\begin{pmatrix} H \\ h \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} \; \text{— two neutral scalars (neutral Higgs bosons)} \, , \qquad (5)
$$

$$
\begin{pmatrix} G^0 \\ A \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \; \text{— two neutral pseudoscalars} \, , \qquad (6)
$$

$$
\begin{pmatrix} G^\pm \\ H^\pm \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \phi_1^\pm \\ \phi_2^\pm \end{pmatrix} \; \text{— two charged scalars} \, , \qquad (7)
$$

where G^0 and $G^\pm$ are Goldstone bosons which are absorbed as longitudinal components of the $W^\pm, Z$, $\tan\beta \equiv \frac{v_2}{v_1}$, α is an angle.

Masses of fermions (quarks and leptons) are the result of Yukawa interaction: coupling of left-handed Dirac fields $q_{\rm L} \equiv \begin{pmatrix} u_{\rm L} \\ d_{\rm L} \end{pmatrix}$, $l_{\rm L} \equiv \begin{pmatrix} \nu_{e\rm L} \\ e_{\rm L} \end{pmatrix}$ with right-handed Dirac fields $u_{\rm R}, d_{\rm R}, e_{\rm R}$ via isospinor fields $\Psi = \Psi_1, \Psi_2$

$$
\begin{aligned}
U_{\rm D} \; = \; & \sqrt{2}\chi_u \left(\bar{q}_{\rm L} \widetilde{\Psi} u_{\rm R} + \bar{u}_{\rm R} \widetilde{\Psi}^+ q_{\rm L}\right) + \sqrt{2}\chi_d \left(\bar{q}_{\rm L} \Psi d_{\rm R} + \bar{d}_{\rm R} \Psi^+ q_{\rm L}\right) \\
& + \sqrt{2}\chi_e \left(\bar{l}_{\rm L} \Psi e_{\rm R} + \bar{e}_{\rm R} \Psi^+ l_{\rm L}\right) \, , \qquad (8)
\end{aligned}
$$

where $\bar{\psi} = \psi^{+}\gamma_0$ is Dirac conjugated spinor, $\widetilde{\Psi} = i\tau_y\Psi$; χ_u, χ_d, χ_e are Yukawa constants for u-quark, d-quark, and electron, respectively. The neutrino ν_e remains massless. Since there are two fields Ψ_1, Ψ_2, four options of interaction with fermions (u, d quarks and electron e) are possible, which is illustrated in Table 1. It is analogously for the second c, s, μ, ν_μ and third t, b, τ, ν_τ generations.

Table 1. The four independent types of the Yukawa interaction for 2HDM scalar doublets.

	u	d	e
Type I	Ψ_1	Ψ_1	Ψ_1
Type II	Ψ_1	Ψ_2	Ψ_2
Lepton-specific	Ψ_1	Ψ_1	Ψ_2
Flipped	Ψ_1	Ψ_2	Ψ_1

Let $m_{11}^2 < 0, m_{22}^2 > 0$, then we should choose the vacuum as $v = v_1 = 246$ GeV, $v_2 = 0$ [60, 65], and the expressions for the boson masses take the simple form [60]

$$m_h^2 = \lambda_1 v^2 = -2m_{11} = (126\ \text{GeV})^2\,, \qquad m_H^2 = m_A^2 + \lambda_5 v^2\,,$$
$$m_{H^\pm}^2 = m_{22} + \frac{\lambda_3}{2}v^2\,, \qquad m_A^2 = m_{H^\pm}^2 + \frac{\lambda_4 - \lambda_5}{2}v^2\,, \quad (9)$$

where the h-boson is associated with the observed Higgs boson. Due to the exact Z_2 symmetry, the lightest neutral component H or A is stable and may be considered as a DM candidate. If taking H as DM, it requires $\lambda_5 < 0, \lambda_4 - |\lambda_5| < 0$. If taking A as DM, it requires $\lambda_5 > 0, \lambda_4 - \lambda_5 < 0$. However, the model requires [60]

$$m_A + m_H \ >\ m_Z\,, \qquad 2m_{H^\pm} > m_Z\,, \qquad m_A + m_{H^\pm} > m_W\,,$$
$$m_H + m_{H^\pm} \ >\ m_W \Longrightarrow m_A\,, \qquad m_H \sim 10\ldots 100\ \text{GeV}\,. \quad (10)$$

As we can see, particles that are candidates for the ultra-light DM should have a mass of $m_{\text{DM}} \sim 10^{-24}\ldots 1$ eV. Obviously, that H-, A-bosons are not suitable for this role.

We can go another way. In Ref. [64], a model containing two scalar doublets, Ψ_1 and Ψ_2, and a real scalar singlet Ψ_{S} with a specific discrete symmetry $\Psi_1 \to \Psi_1, \Psi_2 \to -\Psi_2, \Psi_{\text{S}} \to -\Psi_{\text{S}}$ has been constructed

$$
\begin{aligned}
V_{\rm 2HDM+S} \;=\;& m_{11}^2\Psi_1^+\Psi_1 + m_{22}^2\Psi_2^+\Psi_2 + \frac{1}{2}m_{\rm S}^2\Psi_{\rm S}^2 + \Psi_{\rm S}\left(A\Psi_1^+\Psi_2 + A^*\Psi_2^+\Psi_1\right) \\
&+\frac{1}{2}\lambda_1\left(\Psi_1^+\Psi_1\right)^2 + \frac{1}{2}\lambda_2\left(\Psi_2^+\Psi_2\right)^2 + \lambda_3\left(\Psi_1^+\Psi_1\right)\left(\Psi_2^+\Psi_2\right) \\
&+\lambda_4\left(\Psi_1^+\Psi_2\right)\left(\Psi_2^+\Psi_1\right) + \frac{1}{2}\lambda_5\left(\left(\Psi_1^+\Psi_2\right)^2 + \left(\Psi_2^+\Psi_1\right)^2\right) \\
&+\frac{1}{4}\lambda_6\Psi_{\rm S}^4 + \frac{1}{2}\lambda_7\Psi_1^+\Psi_1\Psi_{\rm S}^2 + \frac{1}{2}\lambda_8\Psi_2^+\Psi_2\Psi_{\rm S}^2\,. \qquad (11)
\end{aligned}
$$

All fermion fields are considered to be neutral under this symmetry. As such, only the doublet Ψ_1 couples to fermions. Thus, DM can be attached to 2HDM Lagrangian as excitations of the neutral $\Psi_{\rm S}$ field (which does not interact with either fermions or gauge bosons). Thus, we can obtain the desired mass of DM by choosing the appropriate values for the coefficients $m_{\rm S}^2, \lambda_6, \lambda_7, \lambda_8$.

As a further generalization, the three-Higgs-doublet model (3HDM) can be formulated [65, 66]. The maximal symmetry for such a model is $\mathrm{U}(1)\otimes\mathrm{U}(1)$. The corresponding potential V_0 is invariant under any phase rotation

$$
\begin{aligned}
V_0 \;=\;& \sum_{i=1}^{3}\left[m_{ii}^2\Psi_i^+\Psi_i + \frac{1}{2}\lambda_{ii}\left(\Psi_i^+\Psi_i\right)^2\right] \\
&+\sum_{i=1,i\neq j}^{3}\left[\lambda_{ij}\left(\Psi_i^+\Psi_i\right)\left(\Psi_j^+\Psi_j\right) + \lambda'_{ij}\left(\Psi_i^+\Psi_j\right)\left(\Psi_j^+\Psi_i\right)\right]. \qquad (12)
\end{aligned}
$$

This potential gives three massive neutral scalars $H_{1,2,3}$, two massive charged scalars $H^{\pm}_{1,2}$, and one massless charged scalar $H^{\pm}_3$, two massive neutral pseudo-scalars $A_{1,2}$ and one massless neutral pseudo-scalar A_3. In the general case, the 3HDM potential symmetric under a group G can be written as

$$
V_0 = V_0 + V_G\,, \qquad (13)
$$

where V_G is a collection of extra terms ensuring the symmetry group G, which can be both continuous and discrete symmetries, both Abelian and non-Abelian symmetries. The classification of symmetric 3HDM potentials and the corresponding Higgs and Goldstone particles is presented in [65]. For clarity, in Appendix A, we present some invariant potentials under the simplest transformations. Finally, an n-Higgs-doublet model (nHDM, $n>3$) can be formulated [67], where the number of scalar, pseudoscalar, and charged bosons will be even larger. The lightest of the neutral massive bosons (H-, A- or S-types) can be a candidate for the role of DM.

Despite the fact that in the nHDM or nHDM+S ($n \geq 2$) models and in many others, the particle candidates for the role of DM appear (the lightest of massive neutral H-, A- or S-bosons), these models generate a lot of other particles (which can be numbered in tens in multiplets). These particles can be both electrically neutral and charged, both massless and massive, and have not yet been detected in collider experiments or in cosmic rays. In addition, even the proposed DM particles are weakly interacting (as WIMPs), that is, they are not completely sterile, hence could have been detected as well. In the future, with more in-depth research, the discovery of these particles, of course, cannot be ruled out.

Historically, the GWS theory arose as a field-theoretic, dynamic, relativistic, group (from the U(1) symmetry to the SU(2) ⊗ U(1) symmetry) generalization of the Ginzburg–Landau (GL) theory for superconductors. Attractive forces act between electrons with opposite spins due to the exchange of phonons, overpowering Coulomb repulsion. As a result, electrons bind into effective pairs (so-called Cooper pairs), which at low temperatures condense into the same quantum state (similar to a Bose–Einstein condensate). The resulting coherent state of a collective of Cooper pairs can be described with the many-particles wave function

$$\varphi(\boldsymbol{r}) = |\varphi(\boldsymbol{r})| \mathrm{e}^{i\theta(\boldsymbol{r})}, \tag{14}$$

where both the module $|\varphi|$ and the phase θ are functions of spatial coordinates $\boldsymbol{r}$, moreover, the module determines the density of superconducting electrons $n_{\mathrm{s}} = 2|\varphi|^2$, and the gradient of the phase determines the current $\boldsymbol{J} = \frac{e\hbar}{m}|\varphi|^2 \nabla\theta$. The density of free energy is

$$\mathcal{F} = \frac{\hbar^2}{4m}\left(\nabla - \frac{i2e}{\hbar c}\boldsymbol{A}\right)\varphi\left(\nabla + \frac{i2e}{\hbar c}\boldsymbol{A}\right)\varphi^{+} + a\,|\varphi|^2 + \frac{b}{2}\,|\varphi|^4 + \frac{(\nabla \times \boldsymbol{A})^2}{8\pi}, \tag{15}$$

where $a < 0$, $b > 0$, $\boldsymbol{A}$ is a vector potential of magnetic field, $2m$ and $2e$ are the mass and charge of a Cooper pair, respectively. Then the current is

$$\boldsymbol{J} = \frac{e\hbar}{m}\varphi_0^2\left(\nabla\theta - \frac{2e}{\hbar c}\boldsymbol{A}\right), \tag{16}$$

where

$$\varphi_0 = \sqrt{\frac{-a}{b}} \tag{17}$$

is an equilibrium magnitude of the module of the field φ. Free energy (15) and current (16) are invariants under the U(1) gauge transformation, *i.e.* when the phase is rotated by a certain angle $\delta\theta$: $\theta \to \theta + \delta\theta$, which is a function of a point $\delta\theta(\boldsymbol{r})$ in the general case

$$\mathcal{F}\left(\varphi\to\varphi\,\mathrm{e}^{i\delta\theta},\varphi^{+}\to\varphi^{+}\,\mathrm{e}^{-i\delta\theta},\boldsymbol{A}\to\boldsymbol{A}+\frac{\hbar c}{2e}\nabla\delta\theta\right) = \mathcal{F}\left(\varphi,\varphi^{+},\boldsymbol{A}\right), \tag{18}$$

$$\boldsymbol{J}\left(\theta\to\theta+\delta\theta,\boldsymbol{A}\to\boldsymbol{A}+\frac{\hbar c}{2e}\nabla\delta\theta\right) = \boldsymbol{J}\left(\theta,\boldsymbol{A}\right). \tag{19}$$

This means that any phase rotations do not change either the energy of the system or the current flowing through the superconductor. This symmetry is illustrated schematically in Fig. 3 (a). Moreover, the equation for the magnetic field has the form (in the gauge $\nabla\cdot\boldsymbol{A}=0$)

$$\nabla\times\frac{\partial\mathcal{F}}{\partial(\nabla\times\boldsymbol{A})}-\frac{\partial\mathcal{F}}{\partial\boldsymbol{A}}=0\Rightarrow\Delta\boldsymbol{A}=\frac{8\pi e^{2}\varphi_{0}^{2}}{mc^{2}}\boldsymbol{A}\equiv\frac{1}{\lambda^{2}}\boldsymbol{A}\propto m_{A}^{2}\boldsymbol{A}\,, \tag{20}$$

where the value reciprocal of the magnetic penetration depth λ plays the role of a photon mass m_A. The dynamics generalization of the GL theory has been done in Ref. [68], where it has been demonstrated that the Higgs mass in such a system is

$$m_H=\sqrt{2}\kappa m_A\propto\frac{1}{\xi}\,, \tag{21}$$

where $\kappa\equiv\lambda/\xi$ is a GL parameter, ξ is a coherence length. Then for type-I superconductors, $m_H<m_A$, and for type-II superconductors, $m_H>m_A$.

Now, let us cut our superconductor into two parts and place them far apart. We obtain two independent condensates

$$\varphi_1=|\varphi_1|\mathrm{e}^{i\theta_1}\,,\qquad \varphi_2=|\varphi_2|\mathrm{e}^{i\theta_2}\,. \tag{22}$$

Then, let us bring them closer to a distance of the order of the coherence length $\xi\sim 1/m_H$. The remaining slit can be filled, for example, with an insulator as demonstrated in Fig. 3 (b) and (c). A Copper pair from bank 1 with condensate φ_1 can tunnel to bank 2 with condensate φ_2, which is described by nondiagonal matrix elements [69]

$$H_{12}=\int\varphi_1^{+}\widehat{H}\varphi_2\mathrm{d}V\,,\qquad H_{21}=\int\varphi_2^{+}\widehat{H}\varphi_1\mathrm{d}V\,,\qquad K\equiv|H_{12}|=|H_{21}|\,. \tag{23}$$

The value K is determined by the properties of the junction. Such a device is called the Josephson junction and matrix elements (23) are the Josephson coupling. Then the current through the junction is

$$J=\frac{4K\varphi_0^2}{\hbar}\sin(\theta_2-\theta_1)\,. \tag{24}$$

It is not difficult to see that the Josephson coupling breaks the U(1) gauge invariance, because the current (24) depends on the phase differences $\theta_2-\theta_1$.

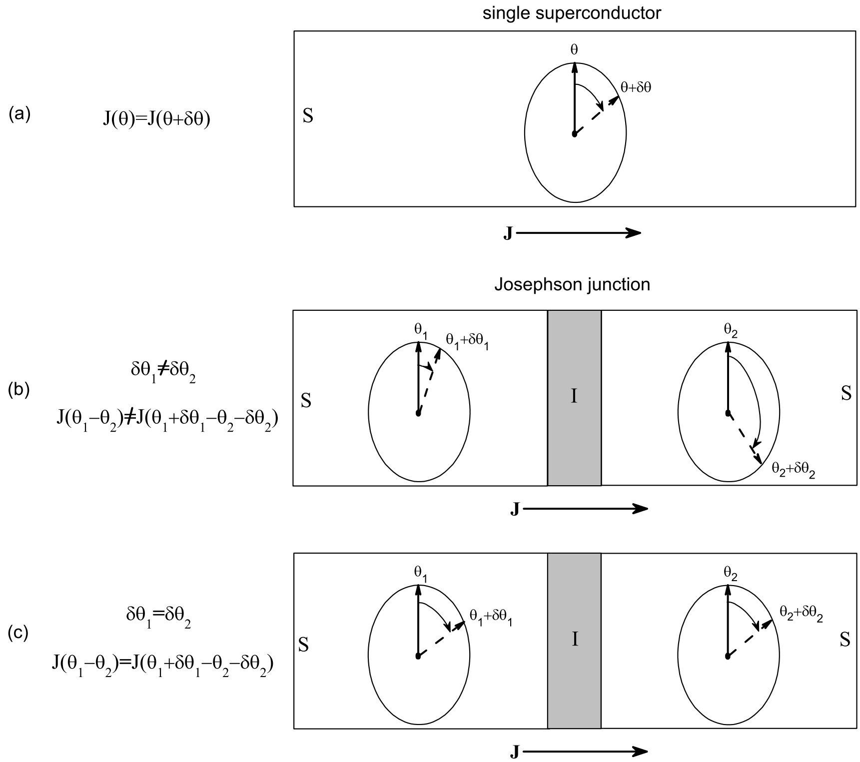

Fig. 3. (a) U(1) symmetry of a one-piece superconducting sample: any phase rotations do not change the current J. (b) Independent phase rotations in superconductors separated by a thin insulator with a thickness of the order of the coherence length (S–I–S Josephson junction) change the current through the junction. (c) Synchronous phase rotations (so that $\theta_2 - \theta_1 = \text{const.}$) do not change the current.

Thus, if we rotate phases θ_1 and θ_2 in each bank independently, then the current J changes. In order to keep the current constant, we must rotate the phases synchronously, *i.e.* so that $\theta_2 - \theta_1 = \text{const}$.

The Josephson junction can also be realized in the momentum space: if in some material two conduction bands take place (for example, in magnesium diboride MgB_2, nonmagnetic borocarbides $LuNi_2B_2C$, YNi_2B_2C, and some oxypnictide compounds), then in each band the condensate of Cooper pairs φ_1 and φ_2 can exist. In a bulk isotropic s-wave superconductor, the GL free energy functional can be written as [70–75]

$$F = \int \mathrm{d}^3r \Bigg[\frac{\hbar^2}{4m_1} |\nabla\varphi_1|^2 + \frac{\hbar^2}{4m_2} |\nabla\varphi_2|^2 + a_1 |\varphi_1|^2 + a_2 |\varphi_2|^2 + \frac{b_1}{2} |\varphi_1|^4 + \frac{b_2}{2} |\varphi_2|^4 + \epsilon \left(\varphi_1^+\varphi_2 + \varphi_1\varphi_2^+\right) \Bigg], \quad (25)$$

where $m_{1,2}$ denotes the effective mass of carriers in the corresponding band, the coefficients $a_{1,2}$ are given as $a_i = \gamma_i(T - T_{ci})$, where γ_i are some constants, the coefficients $b_{1,2}$ are independent of temperature, the quantity ϵ describes the interband mixing of the two condensates: the proximity effect or the internal Josephson effect. If we switch off the interband interaction $\epsilon = 0$, then we will have two independent superconductors with different critical temperatures T_{c1} and T_{c2}, because the intraband interactions can be different. Thus, a two-band superconductor is understood as two single-band superconductors with the corresponding condensates of Cooper pairs φ_1 and φ_2 (so that densities of superconducting electrons are $n_{s1} = 2|\varphi_1|^2$ and $n_{s2} = 2|\varphi_2|^2$, respectively), but these two condensates are coupled by the internal proximity effect $\epsilon\left(\varphi_1^+\varphi_2 + \varphi_1\varphi_2^+\right)$.

Minimization of the free energy functional with respect to the amplitudes of condensates, if $\nabla\varphi_{1,2} = 0$, gives

$$\left\{ \begin{array}{c} a_1\varphi_1 + \epsilon\varphi_2 + b_1\varphi_1^3 = 0 \\ a_2\varphi_2 + \epsilon\varphi_1 + b_2\varphi_2^3 = 0 \end{array} \right\}, \quad (26)$$

where the equilibrium values $\varphi_{1,2}$ are assumed to be real (*i.e.* the phases $\theta_{1,2}$ are 0 or π) in the absence of a current and magnetic field. Near the critical temperature T_c, we have $\varphi_{1,2}^3 \to 0$, hence we can find the critical temperature by equating to zero the determinant of the linearized system (26)

$$a_1a_2 - \epsilon^2 = \gamma_1\gamma_2(T_c - T_{c1})(T_c - T_{c2}) - \epsilon^2 = 0\,. \quad (27)$$

By solving this equation, we find $T_c > T_{c1}, T_{c2}$, moreover, the solution does not depend on the sign of ϵ. The sign determines the equilibrium phase difference of the condensates $|\varphi_1|\mathrm{e}^{i\theta_1}$ and $|\varphi_2|\mathrm{e}^{i\theta_2}$

$$\begin{aligned} \cos(\theta_1 - \theta_2) &= 1 \quad \text{if} \quad \epsilon < 0\,, \\ \cos(\theta_1 - \theta_2) &= -1 \quad \text{if} \quad \epsilon > 0\,, \end{aligned} \quad (28)$$

that follows from Eq. (26). The $\epsilon < 0$ case corresponds to an attractive interband interaction (for example, in MgB_2, where s^{++} wave symmetry occurs), the $\epsilon > 0$ case corresponds to a repulsive interband interaction (for example, in iron-based superconductors, where s^{+-} wave symmetry occurs) [71]. The solutions of Eq. (26) $\varphi_{01}, \varphi_{02}$ are illustrated in Fig. 4 for the case of strongly asymmetrical bands $T_{c1} \ll T_{c2}$. We can see that the effect of interband coupling $\epsilon \neq 0$, even if the coupling is weak $|\epsilon| \ll |a_1(0)|$, is *nonperturbative*: applying the interband coupling drags the smaller parameter φ_{01} up to the new critical temperature $T_c \gg T_{c1}$. At the same time, the effect on the larger parameter φ_2 is less significant — applying the interband coupling only slightly increases the critical temperature compared with T_{c2}: $T_c \gtrsim T_{c2}$.

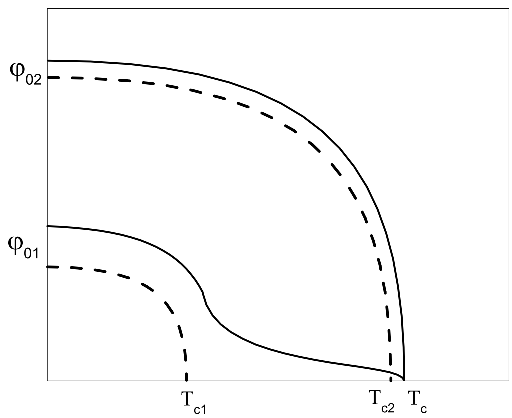

Fig. 4. The amplitudes of the condensates $\varphi_{01}(T)$ and $\varphi_{02}(T)$ as solutions of Eq. (26), if the interband coupling is absent, *i.e.* $\epsilon = 0$ (dashed lines), and if the interband coupling is weak, *i.e.* $\epsilon \neq 0$, $|\epsilon| \ll |a_1(0)|$ (solid lines). Applying the weak interband coupling drags the smaller parameter φ_{01} up to a new critical temperature $T_c \gg T_{c1}$. The effect on the larger parameter φ_{02} is less significant.

In the module-phase representation (22), the interband mixing takes the form

$$\epsilon \left(\varphi_1^+ \varphi_2 + \varphi_1 \varphi_2^+\right) = 2\epsilon |\varphi_1||\varphi_2| \cos(\theta_1 - \theta_2)\,. \tag{29}$$

Thus, the Josephson term describes interference between Cooper pairs condensates φ_1 and φ_2. As in the Josephson junction, the Josephson term breaks the U(1) gauge invariance, because this term depends on the phase differences $\theta_1 - \theta_2$. In Ref. [74], the normal oscillations of the internal degrees of freedom (the Higgs and Goldstone modes) of two-band superconductors using the dynamical generalization of GL theory have been investigated, which was formulated in Ref. [68]. It is demonstrated that, due to the internal proximity effect, the Goldstone modes from each band transform to normal oscillations for all bands: common mode oscillations with an acoustic spectrum, which are absorbed by the gauge field because propagation of these collective excitations is accompanied by a current; and anti-phase oscillations with an energy gap in the spectrum (mass) determined by the interband coupling $m_L \sim \sqrt{|\epsilon|}$, which can be associated with the Leggett mode. Propagation of the Leggett mode is not accompanied by the current, hence this mode "survives". Analogously, for three-band superconductors [75], it has been demonstrated that the Goldstone modes from each band transform to normal oscillations for all bands: common mode oscillations with an acoustic spectrum, which are absorbed by the gauge field, and two massive modes for anti-phase oscillations which are analogous to the Leggett mode and are determined by the coefficients of the interband coupling $\epsilon_{12}, \epsilon_{13}, \epsilon_{23}$.

The free energy functional $F = \int d^3 r \mathcal{F}$ can be written in a general n-band system, where the potential has the form

$$V = V_0 + \sum_{i<k}^{n} \epsilon_{ik} \left(\varphi_i^+ \varphi_k + \varphi_i \varphi_k^+ \right) , \tag{30}$$

and the potential

$$V_0 = \sum_{i=1}^{n} a_i \left|\varphi_i\right|^2 + \frac{b_i}{2} \left|\varphi_i\right|^4 \tag{31}$$

is a sum of independent potentials of each condensate. The potential V_0 is invariant under any phase rotation. Since the condensates in a three-band system are coupled by the Josephson terms $\epsilon_{ik} \left(\varphi_i^+ \varphi_k + \varphi_i \varphi_k^+ \right) = \epsilon_{ik} |\varphi_i||\varphi_k| \cos(\theta_i - \theta_k)$, the spontaneously broken U(1) symmetry of the ground state is shared throughout the system: the presence of the condensate $\langle \varphi_i \rangle \neq 0$ in a band induces the condensation in other bands $\langle \varphi_k \rangle \neq 0$, that is the internal proximity effect takes place. At the same time, the global gauge symmetry $\mathrm{U}(1)^{n-1}$ of the potential V_0 with $n > 1$ is broken down by the Josephson terms [65], because these terms depend on the phase differences $\theta_i - \theta_k$. In the n-band case, we have $n - 1$ phase-difference (Leggett) modes. These modes acquire masses because the phase differences are fixed near the minimums of the potential V. In Ref. [76], the total rule has been formulated: in the n-band system, the global symmetry $\mathrm{U}(1)^{n-1}$ is broken down by the Josephson terms to the $\mathrm{U}(1)^{n-3}$ symmetry. Thus, in $n > 3$-band system, $n - 3$ massless Leggett modes must be present. Ultimately, the system with potential (30) is invariant under the synchronic U(1) gauge transformation, *i.e.* when each scalar field is turned by the same phase θ: $\varphi_k \to \varphi_k e^{i\theta}$. Hence, as demonstrated for two- and three-band superconductors in Refs. [74, 75], the common mode phase oscillations are absorbed by the gauge field, however, oscillations of the phase differences $\theta_i - \theta_k$ occur.

Proceeding from aforesaid, we can use the analogy with multi-band superconductors to formulate the appropriate extension of SM, formalizing the superconducting order parameter φ as a scalar field. Such a model allows us to obtain particle candidates for the role of DM — an analog of Leggett modes, because *(i)* masses of these bosons can be arbitrarily small due to the nonperturbativness of interband coupling $m_{\mathrm{L}} \sim \sqrt{|\epsilon|}$; *(ii)* since propagation of the Leggett mode is not accompanied by a current, then they can be "sterile" in the field theory. However, the symmetry of the GL free energy is $\mathrm{U}(1)_Q$, but the symmetry of GWS Lagrangian is $\mathrm{SU}(2)_I \otimes \mathrm{U}(1)_Y$. Accordingly, instead of the scalar field φ, we have the isospinor Ψ similar to Eq. (4). Hence, we must try to represent the interband coupling $\epsilon\left(\Psi_1^+\Psi_2 + \Psi_1\Psi_2^+\right)$ in the form of interference between the fields Ψ_1 and Ψ_2, similar to Eq. (29). Then, we can assume that the coefficients $\lambda_{n>2} = 0$ in Lagrangians (3), (11) or that the coefficients $\lambda_{i\neq j} = 0$ in Lagrangian (12). This approach relieves us of a large number of other particles (for example, charged Higgs bosons $H^{\pm}$) which could be easily detected experimentally. However, the purpose of formulation of the model that differs from SM is not so much in solving the DM problem, but in solving a whole complex of problems. Thus, except for the DM problem, we propose the nature of oscillations and masses of neutrinos, leaving them as Dirac fermions. At the same time, we demonstrate why oscillations of charged leptons (electron–muon–tauon) are absent, why masses of such leptons differ by orders, and why three generations of fermion are needed. The model proposes three neutral H-bosons that explain the absence of experimentally detected decays of the already discovered H-boson into fermions of the second and first generations, but these two additional H-bosons interact very weekly with gauge and Dirac fields which makes their detection difficult, but still possible. This could be an experimental test.

Our paper is organized in the following way. In Section 2, we formulate a model with three scalar fields (bands) with spontaneous breaking of the U(1) gauge symmetry in each field and with the Josephson couplings between them. In such a system, we obtain both the Higgs and Goldstone modes, and introduce the concept of band states and flavor states of the scalar fields. In Section 3, the Higgs effect on the Abelian (electromagnetic) field in the three-band system is considered. In Section 4, we connect the three-bandness with three generations of fermions, and we consider the band states and flavor states of the Dirac fields. In Section 5 and Section 6, we consider the three-band system with spontaneous breaking of the $\mathrm{SU}(2)_I$ and $\mathrm{SU}(2)_I \otimes \mathrm{U}(1)_Y$ gauge symmetries, respectively, and with the Josephson couplings between the bands. The Higgs effect on both the Abelian and Yang–Mills gauge fields is considered. In Section 7, the lepton mixing is described and the mechanism of origin of neutrino "masses" is proposed. In Section 8, we summarize the results of the three-band GWS model as the system of elementary particles, where the particles that make up DM are present. Moreover, we propose two additional neutral H-bosons, estimate their masses, and analyze their production and decays. The mechanism of the fermions mass hierarchy is proposed. In Section 9, we estimate the masses of L-bosons as DM particles and demonstrate that such ultra-light bosons solve the central cusp problem. In Section 10, we consider the masses of H-bosons at critical temperature.

2. Spontaneous breaking of the U(1) gauge symmetry in the three-band system with the Josephson couplings

2.1. The three-band Lagrangian with the Josephson terms

Let us have three complex scalar fields, which are equivalent to two real scalar fields each: the modulus $|\varphi(x)|$ and the phase $\theta(x)$ (the modulus–phase representation)

$$\varphi_1(x)=|\varphi_1(x)|\,\mathrm{e}^{i\theta_1(x)}\,,\quad \varphi_2(x)=|\varphi_2(x)|\,\mathrm{e}^{i\theta_2(x)}\,,\quad \varphi_3(x)=|\varphi_2(x)|\,\mathrm{e}^{i\theta_3(x)}. \tag{32}$$

Here, $x\equiv(t,\boldsymbol{r})$, and we will use the system of units, where $c=\hbar=1$. These fields should minimize some action S in the Minkowski space

$$S=\int\mathcal{L}\left(\varphi_1,\varphi_2,\varphi_3,\varphi_1^+,\varphi_2^+,\varphi_3^+\right)\mathrm{d}^4x\,, \tag{33}$$

where the Lagrangian $\mathcal{L}$ is a sum of three gauge-invariant Lagrangians (ordinary single-band Lagrangians) and Josephson terms (the interband two-by-two coupling of the scalar fields $\varphi_i\varphi_j^+ + \varphi_i^+\varphi_j$)

$$\begin{aligned}\mathcal{L}=&\,\partial_\mu\varphi_1\partial^\mu\varphi_1^+ + \partial_\mu\varphi_2\partial^\mu\varphi_2^+ + \partial_\mu\varphi_3\partial^\mu\varphi_3^+\\ &-a_1|\varphi_1|^2-a_2|\varphi_2|^2-a_3|\varphi_3|^2-\frac{b_1}{2}|\varphi_1|^4-\frac{b_2}{2}|\varphi_2|^4-\frac{b_3}{2}|\varphi_3|^4\\ &-\epsilon\left(\varphi_1^+\varphi_2+\varphi_1\varphi_2^+\right)-\epsilon\left(\varphi_1^+\varphi_3+\varphi_1\varphi_3^+\right)-\epsilon\left(\varphi_2^+\varphi_3+\varphi_2\varphi_3^+\right)\,,\end{aligned} \tag{34}$$

where $\partial_\mu\equiv\frac{\partial}{\partial x^\mu}\equiv\left(\frac{\partial}{\partial t},\nabla\right)$, $\partial^\mu\equiv\frac{\partial}{\partial x_\mu}\equiv\left(\frac{\partial}{\partial t},-\nabla\right)$ are covariant and contravariant differential operators, respectively. The $a_{1,2,3}<0$ and $b_{1,2,3}>0$ coefficients belong to the corresponding band. The $\epsilon<0$ case corresponds to the attractive interband interaction, the $\epsilon>0$ case corresponds to the repulsive interband interaction. If we switch off the interband interaction $\epsilon=0$, then we will have three independent scalar fields φ_i. It should be noted that the considered model is similar to 3HDM [65], but without any specific symmetry in the sense of Appendix A, except for symmetry under the synchronic U(1)-transformation

$$\mathcal{L}\left(\varphi_1\to\varphi_1\mathrm{e}^{i\delta\theta},\varphi_2\to\varphi_2\,\mathrm{e}^{i\delta\theta},\varphi_3\to\varphi_3\,\mathrm{e}^{i\delta\theta}\right)=\mathcal{L}\left(\varphi_1,\varphi_2,\varphi_3\right)\,, \tag{35}$$

i.e. all phases $\theta_1,\theta_2,\theta_3$ must be rotated equally, so that $\theta_2-\theta_1=\text{const.}$, $\theta_3-\theta_1=\text{const.}$, $\theta_3-\theta_1=\text{const.}$

The Lagrange equations for functional (33) are

$$\begin{aligned}&\partial^\mu\partial_\mu\varphi_1+a_1\varphi_1+\epsilon\varphi_2+\epsilon\varphi_3+b_1|\varphi_1|^2\varphi_1=0\,,\\ &\partial^\mu\partial_\mu\varphi_2+a_2\varphi_2+\epsilon\varphi_1+\epsilon\varphi_3+b_2|\varphi_2|^2\varphi_2=0\,,\\ &\partial^\mu\partial_\mu\varphi_3+a_3\varphi_3+\epsilon\varphi_1+\epsilon\varphi_2+b_3|\varphi_3|^2\varphi_3=0\,,\end{aligned} \tag{36}$$

where $\partial^\mu\partial_\mu = \partial_\mu\partial^\mu = \frac{\partial^2}{\partial t^2} - \Delta$. The current for such a Lagrangian is

$$\begin{aligned} J^\mu &= \sum_{j=1}^{3} \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \varphi_j\right)}(-i\varphi_j) + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \varphi_j^+\right)}\left(i\varphi_j^+\right) = i\sum_{j=1}^{3}\left(\varphi_j^+\partial^\mu\varphi_j - \varphi_j\partial^\mu\varphi_j^+\right) \\ &= -2\sum_{j=1}^{3}|\varphi_j|^2\partial^\mu\theta_j\,, \end{aligned} \tag{37}$$

where we have used the modulus-phase representation (32). Using equations of motion (36), it can be shown that $\partial_\mu J^\mu = 0$.

Let us consider stationary and spatially homogeneous case, *i.e.* $\partial_t\varphi = 0$, $\nabla\varphi = 0$. Then from Eqs. (36), we obtain

$$\left\{ \begin{array}{c} a_1\varphi_1 + \epsilon\varphi_2 + \epsilon\varphi_3 + b_1|\varphi_1|^2\varphi_1 = 0 \\ a_2\varphi_2 + \epsilon\varphi_1 + \epsilon\varphi_3 + b_2|\varphi_2|^2\varphi_2 = 0 \\ a_3\varphi_3 + \epsilon\varphi_1 + \epsilon\varphi_2 + b_3|\varphi_3|^2\varphi_3 = 0 \end{array} \right\}, \tag{38}$$

which can be rewritten in the form

$$\left\{ \begin{array}{c} a_1|\varphi_1| + \epsilon|\varphi_2|\mathrm{e}^{i(\theta_2-\theta_1)} + \epsilon|\varphi_3|\mathrm{e}^{i(\theta_3-\theta_1)} + b_1|\varphi_1|^3 = 0 \\ a_2|\varphi_2| + \epsilon|\varphi_1|\mathrm{e}^{i(\theta_1-\theta_2)} + \epsilon|\varphi_3|\mathrm{e}^{i(\theta_3-\theta_2)} + b_2|\varphi_2|^3 = 0 \\ a_3|\varphi_3| + \epsilon|\varphi_1|\mathrm{e}^{i(\theta_1-\theta_3)} + \epsilon|\varphi_2|\mathrm{e}^{i(\theta_2-\theta_3)} + b_3|\varphi_3|^3 = 0 \end{array} \right\}, \tag{39}$$

or in an expanded form

$$\left\{ \begin{array}{c} a_1|\varphi_1| + \epsilon|\varphi_2|\cos(\theta_2-\theta_1) + \epsilon|\varphi_3|\cos(\theta_3-\theta_1) + b_1|\varphi_1|^3 = 0 \\ a_2|\varphi_2| + \epsilon|\varphi_1|\cos(\theta_1-\theta_2) + \epsilon|\varphi_3|\cos(\theta_3-\theta_2) + b_2|\varphi_2|^3 = 0 \\ a_3|\varphi_3| + \epsilon|\varphi_1|\cos(\theta_1-\theta_3) + \epsilon|\varphi_2|\cos(\theta_2-\theta_3) + b_3|\varphi_3|^3 = 0 \\ |\varphi_2|\sin(\theta_2-\theta_1) + |\varphi_3|\sin(\theta_3-\theta_1) = 0 \\ |\varphi_1|\sin(\theta_1-\theta_2) + |\varphi_3|\sin(\theta_3-\theta_2) = 0 \\ |\varphi_1|\sin(\theta_1-\theta_3) + |\varphi_2|\sin(\theta_2-\theta_3) = 0 \end{array} \right\}. \tag{40}$$

In the $\epsilon > 0$ case for absolutely symmetrical bands $a_1 = a_2 = a_3$, $b_1 = b_2 = b_3$, we obtain $\cos(\theta_j - \theta_k) = -\frac{1}{2}$. In the $\epsilon < 0$ case, we obtain $\cos(\theta_j - \theta_k) = 0$ for any bands. Possible configurations corresponding to some limit cases are illustrated in Fig. 5. As an approximation in the case of weak coupling $\epsilon \ll |a_1|, |a_2|, |a_3|$, we can assume $|\varphi_i| = \sqrt{\frac{|a_i|}{b_i}}$ and then substitute them into Eq. (40) to find the angles $\theta_i - \theta_k$.

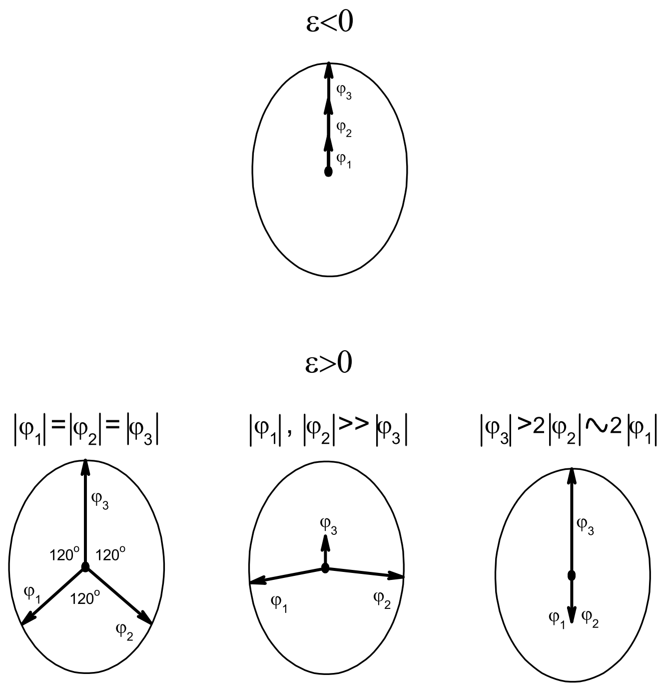

Fig. 5. The possible configurations of the mutual arrangement of the scalar fields $\varphi_1, \varphi_2, \varphi_3$ corresponding to some limit cases as solutions of Eq. (40).

Substituting representation (32) into Lagrangian (34), we obtain

$$\begin{aligned}\mathcal{L} \;=\; & \partial_\mu|\varphi_1|\partial^\mu|\varphi_1| + \partial_\mu|\varphi_2|\partial^\mu|\varphi_2| + \partial_\mu|\varphi_3|\partial^\mu|\varphi_3| \\ & +|\varphi_1|^2\partial_\mu\theta_1\partial^\mu\theta_1 + |\varphi_2|^2\partial_\mu\theta_2\partial^\mu\theta_2 + |\varphi_3|^2\partial_\mu\theta_3\partial^\mu\theta_3 \\ & -a_1\,|\varphi_1|^2 - \frac{b_1}{2}\,|\varphi_1|^4 - a_2\,|\varphi_2|^2 - \frac{b_2}{2}\,|\varphi_2|^4 - a_3\,|\varphi_3|^2 - \frac{b_3}{2}\,|\varphi_3|^4 \\ & -2\epsilon|\varphi_1||\varphi_2|\cos\theta_{12} - 2\epsilon|\varphi_1||\varphi_3|\cos\theta_{13} - 2\epsilon|\varphi_2||\varphi_3|\cos\theta_{23}\,. \end{aligned} \tag{41}$$

Let us consider small variations of the modules from their equilibrium values: $|\varphi_{1,2,3}| = \varphi_{01,02,03} + \phi_{1,2,3}$, where $|\phi_{1,2,3}| \ll \varphi_{01,02,03}$. Then, $|\varphi|^2 \approx \varphi_0^2 + 2\varphi_0\phi+\phi^2$, $|\varphi|^4 \approx \varphi_0^4+4\varphi_0^3\phi+6\varphi_0^2\phi^2$, $|\varphi_1||\varphi_2| \approx \varphi_{01}\varphi_{02}+\varphi_{01}\phi_2+\varphi_{02}\phi_1+\phi_1\phi_2$, and Lagrangian (41) takes the form

$$
\begin{aligned}
\mathcal{L} &= \partial_\mu|\phi_1|\partial^\mu|\phi_1| + \partial_\mu|\phi_2|\partial^\mu|\phi_2| + \partial_\mu|\phi_3|\partial^\mu|\phi_3| + \varphi_{01}^2\partial_\mu\theta_1\partial^\mu\theta_1 \\
&+\varphi_{02}^2\partial_\mu\theta_2\partial^\mu\theta_2 + \varphi_{03}^2\partial_\mu\theta_3\partial^\mu\theta_3 - \phi_1^2\left(a_1+3b_1\varphi_{01}^2\right) - \phi_2^2\left(a_2+3b_2\varphi_{02}^2\right) \\
&-\phi_3^2\left(a_3+3b_3\varphi_{03}^2\right) - 2\epsilon\phi_1\phi_2\cos\theta_{12} - 2\epsilon\phi_1\phi_3\cos\theta_{13} - 2\epsilon\phi_2\phi_3\cos\theta_{23} \\
&-2\phi_1\left(\epsilon\varphi_{02}\cos\theta_{12} + \epsilon\varphi_{03}\cos\theta_{13} + a_1\varphi_{01} + b_1\varphi_{01}^3\right) \\
&-2\phi_2\left(\epsilon\varphi_{01}\cos\theta_{12} + \epsilon\varphi_{03}\cos\theta_{23} + a_2\varphi_{02} + b_2\varphi_{02}^3\right) \\
&-2\phi_3\left(\epsilon\varphi_{01}\cos\theta_{13} + \epsilon\varphi_{02}\cos\theta_{23} + a_3\varphi_{03} + b_3\varphi_{03}^3\right) \\
&-2\epsilon\varphi_{01}\varphi_{02}\cos\theta_{12} - 2\epsilon\varphi_{01}\varphi_{03}\cos\theta_{13} - 2\epsilon\varphi_{02}\varphi_{03}\cos\theta_{23} \\
&-a_1\varphi_{01}^2 - \frac{b_1}{2}\varphi_{01}^4 - a_2\varphi_{02}^2 - \frac{b_2}{2}\varphi_{02}^4 - a_3\varphi_{03}^2 - \frac{b_3}{2}\varphi_{03}^4 \,. \qquad (42)
\end{aligned}
$$

We can consider small variations of the phase differences from their equilibrium values: $\cos\theta_{ik} = \cos(\theta_{ik} - \theta_{ik}^0 + \theta_{ik}^0) = \cos(\theta_{ik} - \theta_{ik}^0)\cos\theta_{ik}^0 - \sin(\theta_{ik} - \theta_{ik}^0)\sin\theta_{ik}^0 \approx \left(1 - \frac{(\theta_{ik}-\theta_{ik}^0)^2}{2}\right)\cos\theta_{ik}^0 - (\theta_{ik} - \theta_{ik}^0)\sin\theta_{ik}^0$. Then the potential energy in Lagrangian (42) takes the form

$$
\begin{aligned}
\mathcal{U} &\approx \mathcal{U}_\phi + \mathcal{U}_\theta + \mathcal{U}_{\phi\theta} + a_1\varphi_{01}^2 + \frac{b_1}{2}\varphi_{01}^4 + a_2\varphi_{02}^2 + \frac{b_2}{2}\varphi_{02}^4 + a_3\varphi_{03}^2 + \frac{b_3}{2}\varphi_{03}^4 \\
&\quad +2\epsilon\cos\theta_{12}^0\varphi_{01}\varphi_{02} + 2\epsilon\cos\theta_{13}^0\varphi_{01}\varphi_{03} + 2\epsilon\cos\theta_{23}^0\varphi_{02}\varphi_{03}\,, \qquad (43)
\end{aligned}
$$

where the last nine terms determine global potential (as the "Mexican hat"), $\mathcal{U}_\phi$ determines a potential for the module excitations $\phi_{1,2,3}$

$$
\begin{aligned}
\mathcal{U}_\phi &= \phi_1^2\left(a_1+3b_1\varphi_{01}^2\right) + \phi_2^2\left(a_2+3b_2\varphi_{02}^2\right) + \phi_3^2\left(a_2+3b_3\varphi_{03}^2\right) \\
&+\phi_1\phi_2 2\epsilon\cos\theta_{12}^0 + \phi_1\phi_3 2\epsilon\cos\theta_{13}^0 + \phi_2\phi_3 2\epsilon\cos\theta_{23}^0 \\
&+2\phi_1\left(\epsilon\cos\theta_{12}^0\varphi_{02} + \epsilon\cos\theta_{13}^0\varphi_{03} + a_1\varphi_{01} + b_1\varphi_{01}^3\right) \\
&+2\phi_2\left(\epsilon\cos\theta_{12}^0\varphi_{01} + \epsilon\cos\theta_{23}^0\varphi_{03} + a_2\varphi_{02} + b_2\varphi_{02}^3\right) \\
&+2\phi_3\left(\epsilon\cos\theta_{13}^0\varphi_{01} + \epsilon\cos\theta_{23}^0\varphi_{02} + a_3\varphi_{03} + b_3\varphi_{03}^3\right)\,. \qquad (44)
\end{aligned}
$$

The terms at $\phi_{1,2,3}$ have to be zero, then

$$
\left\{
\begin{array}{l}
\epsilon\cos\theta_{12}^0\varphi_{02} + \epsilon\cos\theta_{13}^0\varphi_{03} + a_1\varphi_{01} + b_1\varphi_{01}^3 = 0 \\
\epsilon\cos\theta_{12}^0\varphi_{01} + \epsilon\cos\theta_{23}^0\varphi_{03} + a_2\varphi_{02} + b_2\varphi_{02}^3 = 0 \\
\epsilon\cos\theta_{13}^0\varphi_{01} + \epsilon\cos\theta_{23}^0\varphi_{02} + a_3\varphi_{03} + b_3\varphi_{03}^3 = 0
\end{array}
\right\} \qquad (45)
$$

corresponds to the first three equations in Eq. (40). $\mathcal{U}_\theta$ determines a potential for the phase excitations $\theta_{1,2,3}$

$$
\begin{aligned}
\mathcal{U}_\theta &= -2\epsilon\varphi_{01}\varphi_{02}\frac{\left(\theta_{12}-\theta_{12}^0\right)^2}{2}\cos\theta_{12}^0 - 2\epsilon\varphi_{01}\varphi_{03}\frac{\left(\theta_{13}-\theta_{13}^0\right)^2}{2}\cos\theta_{13}^0 \\
&-2\epsilon\varphi_{02}\varphi_{03}\frac{\left(\theta_{23}-\theta_{23}^0\right)^2}{2}\cos\theta_{23}^0 - 2\epsilon\varphi_{01}\varphi_{02}\left(\theta_{12}-\theta_{12}^0\right)\sin\theta_{12}^0 \\
&-2\epsilon\varphi_{01}\varphi_{03}\left(\theta_{13}-\theta_{13}^0\right)\sin\theta_{13}^0 - 2\epsilon\varphi_{02}\varphi_{03}\left(\theta_{23}-\theta_{23}^0\right)\sin\theta_{23}^0 . \qquad (46)
\end{aligned}
$$

In order for the linear terms $(\theta_{ij}-\theta_{ij}^0)$ do not affect the equations of motion, the following condition must be satisfied:

$$
\left\{
\begin{array}{c}
\varphi_{02}\sin\theta_{12}^0+\varphi_{03}\sin\theta_{13}^0=0 \\
\varphi_{01}\sin\theta_{12}^0+\varphi_{03}\sin\theta_{32}^0=0 \\
\varphi_{01}\sin\theta_{13}^0+\varphi_{02}\sin\theta_{23}^0=0
\end{array}
\right\} \qquad (47)
$$

that corresponds to the second three equations in Eq. (40). $\mathcal{U}_{\phi\theta}$ determines interaction between the module excitations and the phase excitations

$$
\begin{aligned}
\mathcal{U}_{\phi\theta} &= -\phi_1\phi_2\epsilon\left(\left(\theta_{12}-\theta_{12}^0\right)^2\cos\theta_{12}^0+2\left(\theta_{12}-\theta_{12}^0\right)\sin\theta_{12}^0\right) \\
&-\phi_1\phi_3\epsilon\left(\left(\theta_{13}-\theta_{13}^0\right)^2\cos\theta_{13}^0+2\left(\theta_{13}-\theta_{13}^0\right)\sin\theta_{13}^0\right) \\
&-\phi_2\phi_3\epsilon\left(\left(\theta_{23}-\theta_{23}^0\right)^2\cos\theta_{23}^0+2\left(\theta_{23}-\theta_{23}^0\right)\sin\theta_{23}^0\right) \\
&-\phi_1\epsilon\left(\left(\theta_{12}-\theta_{12}^0\right)^2\cos\theta_{12}^0\varphi_{02}+\left(\theta_{13}-\theta_{13}^0\right)^2\cos\theta_{13}^0\varphi_{03}\right) \\
&-\phi_2\epsilon\left(\left(\theta_{12}-\theta_{12}^0\right)^2\cos\theta_{12}^0\varphi_{01}+\left(\theta_{23}-\theta_{23}^0\right)^2\cos\theta_{23}^0\varphi_{03}\right) \\
&-\phi_3\epsilon\left(\left(\theta_{13}-\theta_{13}^0\right)^2\cos\theta_{13}^0\varphi_{01}+\left(\theta_{23}-\theta_{23}^0\right)^2\cos\theta_{23}^0\varphi_{02}\right) \\
&-2\phi_1\epsilon\left(\left(\theta_{12}-\theta_{12}^0\right)\sin\theta_{12}^0\varphi_{02}+\left(\theta_{13}-\theta_{13}^0\right)\sin\theta_{13}^0\varphi_{03}\right) \\
&-2\phi_2\epsilon\left(\left(\theta_{12}-\theta_{12}^0\right)\sin\theta_{12}^0\varphi_{01}+\left(\theta_{23}-\theta_{23}^0\right)\sin\theta_{23}^0\varphi_{03}\right) \\
&-2\phi_3\epsilon\left(\left(\theta_{13}-\theta_{13}^0\right)\sin\theta_{13}^0\varphi_{01}+\left(\theta_{23}-\theta_{23}^0\right)\sin\theta_{23}^0\varphi_{02}\right) . \qquad (48)
\end{aligned}
$$

We can see that the first six terms are of the third $\phi_i\phi_k(\theta_{ik}-\theta_{ik}^0)$, $\phi_i(\theta_{ik}-\theta_{ik}^0)^2$ and the forth $\phi_i\phi_k(\theta_{ik}-\theta_{ik}^0)^2$ order, hence they can be neglected. At the same time, the last three terms are of the second order $\phi_i(\theta_{ik}-\theta_{ik}^0)$. In the $\epsilon<0$ case, we have all $\theta_{ik}^0=0$, that is $\sin\theta_{ik}^0=0$, hence the oscillations of modules and phases are not hybridized. Additionally, if $\theta_{ik}-\theta_{ik}^0=0$, that takes place for the common mode oscillations (the Goldstone mode with an acoustic spectrum), therefore in this case, the hybridization is also absent. Thus, the Leggett and Higgs modes are hybridized only in the $\epsilon>0$ case, that is the phase-amplitude modes can take place. However, as it will be demonstrated in Section 4, only the $\epsilon<0$ case has a physical sense, hence, we will consider the normal oscillations without the phase-amplitude hybridization further.

2.2. Goldstone modes

Let us consider the movement of the phases $\theta_{1,2,3}$. The corresponding Lagrange equations for Lagrangian (42) are

$$\begin{aligned}
&\varphi_{01}^2\partial_\mu\partial^\mu\theta_1-\varphi_{01}\varphi_{02}\epsilon\sin(\theta_1-\theta_2)-\varphi_{01}\varphi_{03}\epsilon\sin(\theta_1-\theta_3)=0\,,\\
&\varphi_{02}^2\partial_\mu\partial^\mu\theta_2+\varphi_{01}\varphi_{02}\epsilon\sin(\theta_1-\theta_2)-\varphi_{02}\varphi_{03}\epsilon\sin(\theta_2-\theta_3)=0\,,\\
&\varphi_{03}^2\partial_\mu\partial^\mu\theta_3+\varphi_{01}\varphi_{03}\epsilon\sin(\theta_1-\theta_2)+\varphi_{02}\varphi_{03}\epsilon\sin(\theta_1-\theta_3)=0\,.
\end{aligned}\tag{49}$$

The phases can be written in the form of harmonic oscillations

$$\begin{aligned}
\theta_1&=\theta_1^0+A\,\mathrm{e}^{i(\boldsymbol{qr}-\omega t)}\equiv\theta_1^0+A\,\mathrm{e}^{-iq_\mu x^\mu}\,,\\
\theta_2&=\theta_2^0+B\,\mathrm{e}^{i(\boldsymbol{qr}-\omega t)}\equiv\theta_2^0+B\,\mathrm{e}^{-iq_\mu x^\mu}\,,\\
\theta_3&=\theta_3^0+C\,\mathrm{e}^{i(\boldsymbol{qr}-\omega t)}\equiv\theta_3^0+C\,\mathrm{e}^{-iq_\mu x^\mu}\,,
\end{aligned}\tag{50}$$

where $q_\mu=(\omega,-\boldsymbol{q})$, $x^\mu=(t,\boldsymbol{r})$, $\theta_{1,2,3}^0$ are equilibrium phases. Equation (49) can be linearized assuming $\cos\theta_{ik}\approx\cos\theta_{ik}^0$, $\sin\theta_{ik}=\sin(\theta_{ik}-\theta_{ik}^0+\theta_{ik}^0)\approx(\theta_{ik}-\theta_{ik}^0)\cos\theta_{ik}^0+\sin\theta_{ik}^0$, and using Eq. (47)

$$\begin{aligned}
&\varphi_{01}^2\partial_\mu\partial^\mu\theta_1-\varphi_{01}\varphi_{02}\epsilon\cos\theta_{12}^0\left(\theta_{12}-\theta_{12}^0\right)-\varphi_{01}\varphi_{03}\epsilon\cos\theta_{13}^0\left(\theta_{13}-\theta_{13}^0\right)=0\,,\\
&\varphi_{02}^2\partial_\mu\partial^\mu\theta_2+\varphi_{01}\varphi_{02}\epsilon\cos\theta_{12}^0\left(\theta_{12}-\theta_{12}^0\right)-\varphi_{02}\varphi_{03}\epsilon\cos\theta_{23}^0\left(\theta_{23}-\theta_{23}^0\right)=0\,,\\
&\varphi_{03}^2\partial_\mu\partial^\mu\theta_3+\varphi_{01}\varphi_{03}\epsilon\cos\theta_{13}^0\left(\theta_{13}-\theta_{13}^0\right)+\varphi_{02}\varphi_{03}\epsilon\cos\theta_{23}^0\left(\theta_{23}-\theta_{23}^0\right)=0\,.
\end{aligned}\tag{51}$$

Substituting Eq. (50) into Eq. (51), we obtain equations for the amplitudes A,B,C

$$\begin{aligned}
&A\left(-\frac{\varphi_{02}}{\varphi_{01}}\epsilon\cos\theta_{12}^0-\frac{\varphi_{03}}{\varphi_{01}}\epsilon\cos\theta_{13}^0-q_\mu q^\mu\right)+B\frac{\varphi_{02}}{\varphi_{01}}\epsilon\cos\theta_{12}^0+C\frac{\varphi_{03}}{\varphi_{01}}\epsilon\cos\theta_{13}^0=0\,,\\
&A\frac{\varphi_{01}}{\varphi_{02}}\epsilon\cos\theta_{12}^0+B\left(-\frac{\varphi_{01}}{\varphi_{02}}\epsilon\cos\theta_{12}^0-\frac{\varphi_{03}}{\varphi_{02}}\epsilon\cos\theta_{23}^0-q_\mu q^\mu\right)+C\frac{\varphi_{03}}{\varphi_{02}}\epsilon\cos\theta_{23}^0=0\,,\\
&A\frac{\varphi_{01}}{\varphi_{03}}\epsilon\cos\theta_{13}^0+B\frac{\varphi_{02}}{\varphi_{03}}\epsilon\cos\theta_{23}^0+C\left(-\frac{\varphi_{01}}{\varphi_{03}}\epsilon\cos\theta_{13}^0-\frac{\varphi_{02}}{\varphi_{03}}\epsilon\cos\theta_{23}^0-q_\mu q^\mu\right)=0\,.
\end{aligned}\tag{52}$$

Setting the determinant of the system (52) equal to zero, we find a dispersion equation

$$(q_\mu q^\mu)^3+(q_\mu q^\mu)^2\,b+(q_\mu q^\mu)\,c=0\,,\tag{53}$$

where

$$
\begin{aligned}
b &= \epsilon\Bigg[\left(\frac{\varphi_{01}}{\varphi_{03}}\cos\theta_{13}+\frac{\varphi_{02}}{\varphi_{03}}\cos\theta_{23}\right)+\left(\frac{\varphi_{01}}{\varphi_{02}}\cos\theta_{12}+\frac{\varphi_{03}}{\varphi_{02}}\cos\theta_{23}\right)\\
&\quad+\left(\frac{\varphi_{02}}{\varphi_{01}}\cos\theta_{12}+\frac{\varphi_{03}}{\varphi_{01}}\cos\theta_{13}\right)\Bigg],\\
c &= \epsilon^2\Bigg[\left(\frac{\varphi_{01}}{\varphi_{02}}\cos\theta_{12}+\frac{\varphi_{03}}{\varphi_{02}}\cos\theta_{23}\right)\left(\frac{\varphi_{01}}{\varphi_{03}}\cos\theta_{13}+\frac{\varphi_{02}}{\varphi_{03}}\cos\theta_{23}\right)\\
&\quad+\left(\frac{\varphi_{02}}{\varphi_{01}}\cos\theta_{12}+\frac{\varphi_{03}}{\varphi_{01}}\cos\theta_{13}\right)\left(\frac{\varphi_{01}}{\varphi_{03}}\cos\theta_{13}+\frac{\varphi_{02}}{\varphi_{03}}\cos\theta_{23}\right)\\
&\quad+\left(\frac{\varphi_{02}}{\varphi_{01}}\cos\theta_{12}+\frac{\varphi_{03}}{\varphi_{01}}\cos\theta_{13}\right)\left(\frac{\varphi_{01}}{\varphi_{02}}\cos\theta_{12}+\frac{\varphi_{03}}{\varphi_{02}}\cos\theta_{23}\right)\Bigg]. \qquad (54)
\end{aligned}
$$

From Eq. (53) we can see that one of dispersion relations is

$$q_\mu q^\mu = 0 \Rightarrow \omega^2 = q^2\,, \qquad (55)$$

wherein $A = B = C$, thus this mode is common mode oscillations as the Goldstone mode in the single-band GWS model. There are other oscillation modes with such spectra that

$$m_{\mathrm{L}1}^2 = q_\mu q^\mu = \frac{1}{2}\left(-b-\sqrt{b^2-4c}\right)\,, \qquad (56)$$

$$m_{\mathrm{L}2}^2 = q_\mu q^\mu = \frac{1}{2}\left(-b+\sqrt{b^2-4c}\right)\,, \qquad (57)$$

i.e. two massive modes, wherein

$$A\varphi_{01}^2 + B\varphi_{02}^2 + C\varphi_{03}^2 = 0\,. \qquad (58)$$

These modes are analogous to the Leggett modes in multi-band superconductors [68, 74, 75]. It should be noted that if we assume that $\epsilon = 0$, then $b = c = 0$ and the dispersion equation will be $(q_\mu q^\mu)^3 = 0$, that is we obtain independent common mode oscillations in each band. From Eqs. (54), (56), and (57), we can see that the squared masses of the L-bosons are proportional to the interband coupling $m_{\mathrm{L}1,2}^2 \sim |\epsilon|$.

For example, let us consider a symmetrical three-band system, *i.e.* $\varphi_{01} = \varphi_{02} = \varphi_{03}$. Then masses of both L-bosons are equal ($b^2 = 4c$)

$$
\begin{aligned}
m_{\mathrm{L}1} = m_{\mathrm{L}2} &= \sqrt{\frac{3}{2}\epsilon}\,, \quad \text{when} \quad \epsilon > 0 \Rightarrow \cos\theta_{12} = \cos\theta_{13} = \cos\theta_{23} = -\frac{1}{2}\,,\\
m_{\mathrm{L}1} = m_{\mathrm{L}2} &= \sqrt{3|\epsilon|}\,, \quad \text{when} \quad \epsilon < 0 \Rightarrow \cos\theta_{12} = \cos\theta_{13} = \cos\theta_{23} = 1\,.
\end{aligned} \qquad (59)
$$

rys.6 Amplitudes of the modes (56), (57) relate as $A = -C$, $B = 0$, and $A = C$, $B = -(A + C)$, respectively. These three Goldstone modes (the acoustic mode (55) and the Leggett modes (56), (57)) are shown in Fig. 6. If we have the case of strongly asymmetrical bands $\varphi_{01} \ll \varphi_{02} \ll \varphi_{03}$, then the masses of L-bosons are

$$m_{\mathrm{L}1}^2 \sim \min\left\{-\frac{\varphi_{03}}{\varphi_{01}}\epsilon\cos\theta_{13}, -\frac{\varphi_{03}}{\varphi_{02}}\epsilon\cos\theta_{23}, -\frac{\varphi_{02}}{\varphi_{01}}\epsilon\cos\theta_{12}\right\},$$
$$m_{\mathrm{L}2}^2 \sim \max\left\{-\frac{\varphi_{03}}{\varphi_{01}}\epsilon\cos\theta_{13}, -\frac{\varphi_{03}}{\varphi_{02}}\epsilon\cos\theta_{23}, -\frac{\varphi_{02}}{\varphi_{01}}\epsilon\cos\theta_{12}\right\}, \quad (60)$$

where we suppose that all $-\epsilon\cos\theta_{ij} > 0$.

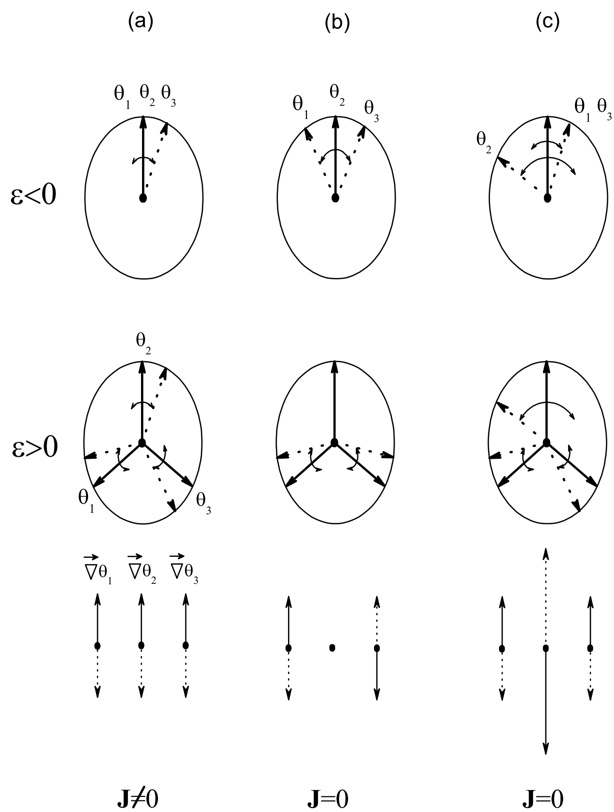

Fig. 6. Normal oscillations of the phases $\theta_1, \theta_2, \theta_3$ in the symmetrical three-band system $\varphi_{01} = \varphi_{02} = \varphi_{03}$, with the attractive interband interactions $\epsilon < 0$ and the repulsive interband interactions $\epsilon > 0$. (a) Common phase oscillations with the acoustic spectrum (55) accompanied by the nonzero current $\boldsymbol{J} = \varphi_{01}^2\nabla\theta_1 + \varphi_{02}^2\nabla\theta_2 + \varphi_{03}^2\nabla\theta_3 \neq 0$. (b), (c) Anti-phase oscillations with the massive spectrum (56) and (57), not accompanied by the current, *i.e.* $\boldsymbol{J} = 0$.

The phase oscillations (50) are accompanied by the current (37)

$$J^\mu = 2iq^\mu \mathrm{e}^{-iq_\mu x^\mu}\left(A\varphi_{01}^2 + B\varphi_{02}^2 + C\varphi_{03}^2\right) \Rightarrow \left[\begin{array}{ll} 2iAq^\mu \mathrm{e}^{-iq_\mu x^\mu}\left(\varphi_{01}^2 + \varphi_{02}^2 + \varphi_{03}^2\right) & \text{for the acoustic mode} \\ 0 & \text{for the Leggett modes} \end{array}\right], \quad (61)$$

where we have used Eq. (58). Thus, due to the internal proximity effect, the Goldstone modes from each band transform to common mode oscillations, where $\nabla\theta_1 = \nabla\theta_2 = \nabla\theta_3$, with the acoustic spectrum, see Eq. (55), and the oscillations of the relative phases $\theta_i - \theta_j$ between condensates with the energy gap in spectrum determined by the interband coupling ϵ, see Eqs. (56), (57), and (59), which can be identified as the Leggett mode by analogy with multi-band superconductors. Propagation of the acoustic Goldstone mode is accompanied by the current $J^\mu \neq 0$, propagation of the Leggett modes (the massive Goldstone modes) is not accompanied by the current $J^\mu = 0$. If we turn off the interband coupling $\epsilon = 0$, then we will have an ordinary Goldstone mode with an acoustic spectrum for each band. Transformation of Goldstone modes from each band into one common mode for all bands and two Leggett modes takes place even at the infinitely small coefficient ϵ: $|\epsilon| \ll |a_{1,2,3}(0)|$. Thus, the effect of interband coupling is nonperturbative.

2.3. Higgs modes

Let us consider movement of the modules $|\varphi_{1,2,3}(t,\boldsymbol{r})| \approx \varphi_{01,02,03} + \phi_{1,2,3}(t,\boldsymbol{r})$. The corresponding Lagrange equations for Lagrangian (42) with accounting Eq. (45) are

$$\begin{aligned} \partial_\mu\partial^\mu\phi_1 + \alpha_1\phi_1 + \epsilon\cos\theta_{12}\phi_2 + \epsilon\cos\theta_{13}\phi_3 = 0\,, \\ \partial_\mu\partial^\mu\phi_2 + \alpha_2\phi_2 + \epsilon\cos\theta_{12}\phi_1 + \epsilon\cos\theta_{23}\phi_3 = 0\,, \\ \partial_\mu\partial^\mu\phi_3 + \alpha_3\phi_3 + \epsilon\cos\theta_{13}\phi_1 + \epsilon\cos\theta_{23}\phi_2 = 0\,, \end{aligned} \quad (62)$$

where we have introduced the following notes:

$$\alpha_1 \equiv a_1 + 3b_1\varphi_{01}^2\,, \qquad \alpha_2 \equiv a_2 + 3b_2\varphi_{02}^2\,, \qquad \alpha_3 \equiv a_3 + 3b_3\varphi_{03}^2\,. \quad (63)$$

Then, in the case of weak coupling $|\epsilon| \ll |a_1|, |a_2|, |a_3|$, where corresponding amplitudes of the condensates can be assumed as $\varphi_{0i} = \sqrt{\frac{|a_i|}{b_i}}$, we have

$$\alpha_i = -2a_i = 2|a_i|\,. \quad (64)$$

The fields $\phi_{1,2,3}$ can be written in a form of harmonic oscillations: $\phi_1 = A\mathrm{e}^{-iq_\mu x^\mu}$, $\phi_2 = B\mathrm{e}^{-iq_\mu x^\mu}$, $\phi_3 = C\mathrm{e}^{-iq_\mu x^\mu}$, where $q_\mu x^\mu = \omega t - \boldsymbol{q}\boldsymbol{r}$. Substituting them into Eq. (62), we obtain equations for the amplitudes A, B, C

$$\begin{aligned}
&A\left(\alpha_1 - q_\mu q^\mu\right) + B\epsilon\cos\theta_{12} + C\epsilon\cos\theta_{13} = 0\,,\\
&A\epsilon\cos\theta_{12} + B\left(\alpha_2 - q_\mu q^\mu\right) + C\epsilon\cos\theta_{23} = 0\,,\\
&A\epsilon\cos\theta_{13} + B\epsilon\cos\theta_{23} + C\left(\alpha_3 - q_\mu q^\mu\right) = 0\,.
\end{aligned} \tag{65}$$

Setting the determinant of the system (65) equal to zero, we find the dispersion equation

$$\left(q_\mu q^\mu\right)^3 + \left(q_\mu q^\mu\right)^2 b + \left(q_\mu q^\mu\right) c + d = 0\,, \tag{66}$$

where

$$\begin{aligned}
b &= -\alpha_1 - \alpha_2 - \alpha_3\,,\\
c &= \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 - \epsilon^2\left(\cos^2\theta_{12} + \cos^2\theta_{13} + \cos^2\theta_{23}\right)\,,\\
d &= -\alpha_1\alpha_2\alpha_3 - 2\epsilon^3\cos\theta_{12}\cos\theta_{13}\cos\theta_{23}\\
&\quad +\epsilon^2\left(\alpha_1\cos^2\theta_{23} + \alpha_2\cos^2\theta_{13} + \alpha_3\cos^2\theta_{12}\right)\,.
\end{aligned} \tag{67}$$

In real physical cases $|\varepsilon| < |a_{1,2,3}|$, hence $b < 0, c > 0, d < 0$. This cubic equation has three real positive roots $q_\mu q^\mu = m_H^2$ (squared masses of H-bosons). In the symmetrical case $\alpha_1 = \alpha_2 = \alpha_3 \equiv \alpha$, $\cos\theta_{12} = \cos\theta_{13} = \cos\theta_{23} \equiv \cos\theta$, and we obtain

$$m_H^2 = \alpha + 2\epsilon\cos\theta\,, \qquad \alpha - \epsilon\cos\theta\,, \qquad \alpha - \epsilon\cos\theta\,. \tag{68}$$

It should be noted that these three frequencies are normal modes, but not the frequencies of oscillations of each band separately. The amplitudes of these modes relate as, for example, $A = B = C$; $A = C$, $B = -(A + C)$, and $A = -C$, $B = 0$, respectively. We can see that in the case of weak interband coupling $|\epsilon| \ll |a_{1,2,3}|$, the masses of H-bosons are almost equal $m_H \approx \alpha = \sqrt{2|a|}$. These three Higgs modes are shown in Fig. 7 (a).

Let us consider the case of weakly coupled $|\epsilon| \ll \alpha_{1,2,3}$ and strongly asymmetrical bands, because, as we will see below, exactly this case corresponds to the real physical situation. Let us suppose

$$\varphi_{01} \ll \varphi_{02} \ll \varphi_{03}\,, \qquad \alpha_1 < \alpha_2 < \alpha_3\,, \tag{69}$$

where we assume that small changes in the Higgs mass $m_H = \sqrt{\alpha}$ correspond to large changes in the amplitude of the condensate φ_0. Similar behavior takes place in superconductor: $|a| \propto \mathcal{N}$, where $\mathcal{N}$ is the density of electron states on the Fermi surface, then in the case of weak electron–phonon coupling, we have $\varphi_0 \sim \Omega\exp(-1/g\mathcal{N})$. In the asymmetrical case, we can

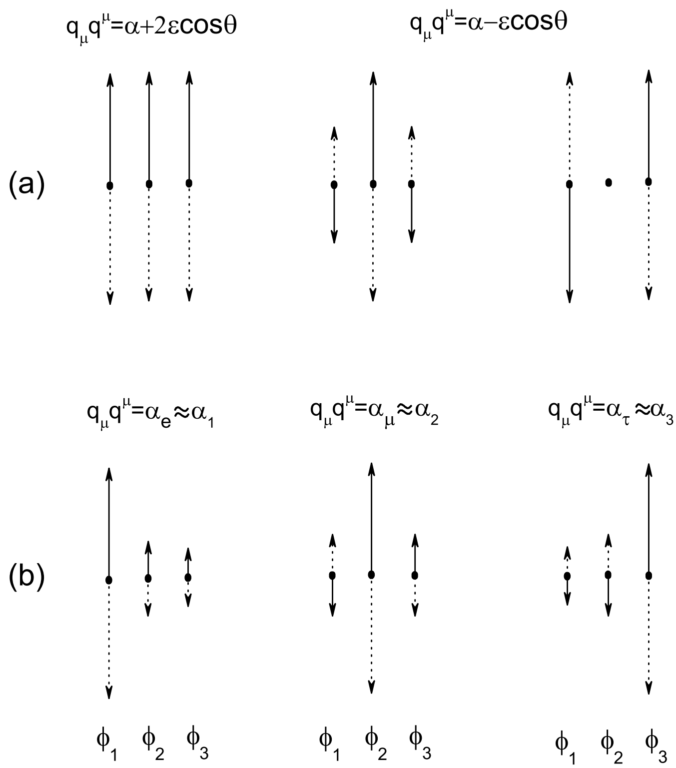

Fig. 7. Normal oscillations of the small variations of the modules of the scalar fields ϕ_1, ϕ_2, ϕ_3 in a symmetrical case $\alpha_1 = \alpha_2 = \alpha_3 \equiv \alpha$, $\cos\theta_{12} = \cos\theta_{13} = \cos\theta_{23} \equiv \cos\theta$ (a), and in the case of strongly asymmetrical bands $\alpha_1 < \alpha_2 < \alpha_3$ (b).

obtain masses of H-bosons $m_{He}, m_{H\mu}, m_{H\tau}$, *i.e.* frequencies of each normal mode

$$\begin{aligned} m_{He}^2 &\approx \alpha_1 - \frac{\epsilon^2\cos^2\theta_{12}}{\alpha_2-\alpha_1} - \frac{\epsilon^2\cos^2\theta_{13}}{\alpha_3-\alpha_1}\,, \\ m_{H\mu}^2 &\approx \alpha_2 - \frac{\epsilon^2\cos^2\theta_{12}}{\alpha_1-\alpha_2} - \frac{\epsilon^2\cos^2\theta_{23}}{\alpha_3-\alpha_2}\,, \\ m_{H\tau}^2 &\approx \alpha_3 - \frac{\epsilon^2\cos^2\theta_{13}}{\alpha_1-\alpha_3} - \frac{\epsilon^2\cos^2\theta_{23}}{\alpha_2-\alpha_3}\,, \end{aligned} \tag{70}$$

and the relations between the amplitudes of these modes

$$
\begin{aligned}
q_\mu q^\mu = \alpha_1 \;\Rightarrow\; B = -A\frac{\epsilon\cos\theta_{12}}{\alpha_2-\alpha_1}\,, \qquad C = -A\frac{\epsilon\cos\theta_{13}}{\alpha_3-\alpha_1}\,,\\
q_\mu q^\mu = \alpha_2 \;\Rightarrow\; A = -B\frac{\epsilon\cos\theta_{12}}{\alpha_1-\alpha_2}\,, \qquad C = -B\frac{\epsilon\cos\theta_{23}}{\alpha_3-\alpha_2}\,,\\
q_\mu q^\mu = \alpha_2 \;\Rightarrow\; A = -C\frac{\epsilon\cos\theta_{13}}{\alpha_1-\alpha_3}\,, \qquad B = -C\frac{\epsilon\cos\theta_{23}}{\alpha_2-\alpha_3}\,.
\end{aligned} \tag{71}
$$

We have written the index e for the lightest boson, the index τ for the heaviest boson, and the index μ for the boson of medium mass. These three Higgs modes are shown in Fig. 7 (b) for the case, where $\epsilon\cos\theta_{ij} < 0$ (as the rule).

Due to the weakness of interband coupling $|\epsilon| \ll \alpha_{1,2,3}$, we can write the following effective diagonalization of the potential energy in the sense that each *normal* mode $\phi_e, \phi_\mu, \phi_\tau$ is an oscillation of the corresponding *effective* band:

$$
|\varphi_e| \approx \varphi_{0e} + \phi_e(t,\boldsymbol{r})\,, \qquad |\varphi_\mu| \approx \varphi_{0\mu} + \phi_\mu(t,\boldsymbol{r})\,, \qquad |\varphi_\tau| \approx \varphi_{0\tau} + \phi_\tau(t,\boldsymbol{r}) \tag{72}
$$

so, that these effective bands are not coupled

$$
\begin{aligned}
\mathcal{U} \;&=\; a_1\,|\varphi_1|^2 + a_2\,|\varphi_2|^2 + a_3\,|\varphi_3|^2 + \frac{b_1}{2}\,|\varphi_1|^4 + \frac{b_2}{2}\,|\varphi_2|^4 + \frac{b_3}{2}\,|\varphi_3|^4\\
&\quad +2\epsilon\cos\theta^0_{12}|\varphi_1||\varphi_2| + 2\epsilon\cos\theta^0_{13}|\varphi_1||\varphi_3| + 2\epsilon\cos\theta^0_{23}|\varphi_2||\varphi_3|\\
&\approx\; a_e\,|\varphi_e|^2 + a_\mu\,|\varphi_\mu|^2 + a_\tau\,|\varphi_\tau|^2 + \frac{b_1}{2}\,|\varphi_e|^4 + \frac{b_2}{2}\,|\varphi_\mu|^4 + \frac{b_3}{2}\,|\varphi_\tau|^4\\
&\quad +\mathcal{O}\left(\frac{|\epsilon|}{m^2_{Hi}-m^2_{Hj}}\right)\,,
\end{aligned} \tag{73}
$$

where the strong band asymmetry (69) is assumed, and we have noted

$$
m_{He} = \sqrt{-2a_e}\,, \qquad m_{H\mu} = \sqrt{-2a_\mu}\,, \qquad m_{H\tau} = \sqrt{-2a_\tau}\,, \tag{74}
$$

that can be named as *the flavor masses* (*i.e.* eigen-frequencies — the masses of H-bosons), and

$$
m_{H1} = \sqrt{-2a_1}\,, \qquad m_{H2} = \sqrt{-2a_2}\,, \qquad m_{H2} = \sqrt{-2a_2}\,, \tag{75}
$$

that can be named as *the band masses* (*i.e.* frequencies if there was no interband coupling $\epsilon = 0$). Accordingly, the states $\left(\ \varphi_1,\quad \varphi_2,\quad \varphi_3\ \right)$ with equilibrium condensate amplitudes

$$
\varphi_{01} = \sqrt{\frac{|a_1|}{b_1}}\,, \qquad \varphi_{02} = \sqrt{\frac{|a_2|}{b_2}}\,, \qquad \varphi_{03} = \sqrt{\frac{|a_3|}{b_3}} \tag{76}
$$

can be named as *the band states*, and the states $\left(\begin{array}{ccc}\varphi_{e}, & \varphi_{\mu}, & \varphi_{\tau}\end{array}\right)$ with equalibrium condensate amplitudes

$$\varphi_{0e}\approx\sqrt{\frac{|a_e|}{b_1}}, \qquad \varphi_{0\mu}\approx\sqrt{\frac{|a_\mu|}{b_2}}, \qquad \varphi_{0\tau}\approx\sqrt{\frac{|a_\tau|}{b_3}} \tag{77}$$

can be named as *the flavor states*, *i.e.* they give normal oscillations of the multi-band system. For strongly asymmetrical bands with the weak interband coupling, the band masses and flavor masses are almost equal: $m_{He}\approx m_{H1}, m_{H\mu}\approx m_{H2}, m_{H\tau}\approx m_{H3}$. Moreover, the equilibrium amplitudes of the condensates of band states and flavor states are also almost equal: $\varphi_{0e}\approx\varphi_{01},\varphi_{0\mu}\approx\varphi_{02},\varphi_{0\tau}\approx\varphi_{03}$. Indeed, we could see that due to the strong band asymmetry (69) and the weak interband coupling $|\epsilon|\ll\alpha_{1,2,3}$, each collective mode in Eq. (70) is approximately an oscillation of a single band according to the following correspondence $\phi_e\approx\phi_1,\phi_\mu\approx\phi_2,\phi_\tau\approx\phi_3$, see Fig. 7 (b).

Thus, the above transition from the coupled scalar fields ϕ_1,ϕ_1,ϕ_1 to the normal oscillations $\phi_e,\phi_\mu,\phi_\tau$ with frequencies $m_{He},m_{H\mu},m_{H\tau}$ (the masses of H-bosons), see Eq. (70), can be considered as diagonalization of the "potential" energy (44)

$$\begin{aligned}
\mathcal{U}_\phi &= \phi_1^2\alpha_1+\phi_2^2\alpha_2+\phi_3^2\alpha_3+\phi_1\phi_2 2\epsilon\cos\theta_{12}^0+\phi_1\phi_3 2\epsilon\cos\theta_{13}^0+\phi_2\phi_3 2\epsilon\cos\theta_{23}^0\\
&=\alpha_e|\phi_e|^2+\alpha_\mu|\phi_\mu|^2+\alpha_\tau|\phi_\tau|^2\\
&=\left(\begin{array}{ccc}\phi_1, & \phi_2, & \phi_3\end{array}\right)\left(\begin{array}{ccc}\alpha_1 & \epsilon\cos\theta_{12}^0 & \epsilon\cos\theta_{13}^0\\ \epsilon\cos\theta_{12}^0 & \alpha_2 & \epsilon\cos\theta_{23}^0\\ \epsilon\cos\theta_{13}^0 & \epsilon\cos\theta_{23}^0 & \alpha_3\end{array}\right)\left(\begin{array}{c}\phi_1\\ \phi_2\\ \phi_3\end{array}\right)\\
&=\left(\begin{array}{ccc}\phi_e, & \phi_\mu, & \phi_\tau\end{array}\right)\left(\begin{array}{ccc}\alpha_e & 0 & 0\\ 0 & \alpha_\mu & 0\\ 0 & 0 & \alpha_\tau\end{array}\right)\left(\begin{array}{c}\phi_e\\ \phi_\mu\\ \phi_\tau\end{array}\right)\\
&\equiv\langle\phi_{123}|M_{123}|\phi_{123}\rangle=\langle\phi_{e\mu\tau}|M_{e\mu\tau}|\phi_{e\mu\tau}\rangle\,.
\end{aligned} \tag{78}$$

Obviously, $\alpha_e,\alpha_\mu,\alpha_\tau$ are eigen-values of the matrix M_{123}: $M_{e\mu\tau}=\mathrm{diag}(M_{123})$, in addition, the band H-bosons and the flavor H-bosons are connected by the unitary transformation: $|\varphi_{e\mu\tau}\rangle=U|\varphi_{123}\rangle$, $|\varphi_{123}\rangle=U^{\mathrm{T}}|\varphi_{e\mu\tau}\rangle$, where U is a unitary matrix $U^{-1}=U^{\mathrm{T}}$, which can be written via the mixing angles $\alpha_{12},\alpha_{13},\alpha_{23}$

$$U=\left(\begin{array}{ccc}1 & 0 & 0\\ 0 & c_{23} & s_{23}\\ 0 & -s_{23} & c_{23}\end{array}\right)\left(\begin{array}{ccc}c_{13} & 0 & s_{13}\\ 0 & 1 & 0\\ -s_{13} & 0 & c_{13}\end{array}\right)\left(\begin{array}{ccc}c_{12} & s_{12} & 0\\ -s_{12} & c_{12} & 0\\ 0 & 0 & 1\end{array}\right), \tag{79}$$

$$U^{\mathrm{T}}=\left(\begin{array}{ccc}c_{12} & -s_{12} & 0\\ s_{12} & c_{12} & 0\\ 0 & 0 & 1\end{array}\right)\left(\begin{array}{ccc}c_{13} & 0 & -s_{13}\\ 0 & 1 & 0\\ s_{13} & 0 & c_{13}\end{array}\right)\left(\begin{array}{ccc}1 & 0 & 0\\ 0 & c_{23} & -s_{23}\\ 0 & s_{23} & c_{23}\end{array}\right), \tag{80}$$

where $c_{ik} = \cos\alpha_{ik}$, $s_{ik} = \sin\alpha_{ik}$. Then, we obtain an equation for the mixing angles α_{ik}

$$M_{e\mu\tau} = UM_{123}U^{\mathrm{T}} \quad \text{or} \quad M_{123} = U^{\mathrm{T}}M_{e\mu\tau}U\,. \tag{81}$$

Assuming the independent mixing for each pair of bands $1 \leftrightarrow 2$, $1 \leftrightarrow 3$, $2 \leftrightarrow 3$, we obtain (for example, for $1 \leftrightarrow 2$, besides, $a_{1,2} < 0, |\epsilon| \ll |a_1| < |a_2|$)

$$\tan 2\alpha_{12} = \frac{2\epsilon\cos\theta^0_{12}}{\alpha_1 - \alpha_2}\,, \qquad \sin 2\alpha_{12} = \frac{2\epsilon\cos\theta^0_{12}}{\alpha_e - \alpha_\mu}\,,$$

$$\begin{array}{c} (\alpha_e - \alpha_\mu)^2 = (\alpha_1 - \alpha_2)^2 + 4\epsilon^2\cos^2\theta^0_{12} \\ \alpha_e + \alpha_\mu = \alpha_1 + \alpha_2 \end{array} \Rightarrow \alpha_e \approx \alpha_1 + \frac{\epsilon^2\cos^2\theta^0_{12}}{\alpha_1 - \alpha_2}\,,$$

$$\alpha_\mu \approx \alpha_2 - \frac{\epsilon^2\cos^2\theta^0_{12}}{\alpha_1 - \alpha_2}\,, \tag{82}$$

which is approximately consistent with Eq. (70). In the case of weak inter-band coupling $|\epsilon| \ll \alpha_1, \alpha_2, \alpha_3$ and asymmetrical bands $\alpha_1 < \alpha_2 < \alpha_3$, the mixing angles are very small $|\tan\alpha_{ik}| \ll 1$. This means that the flavor states almost coincide with the band states (as we can see in Fig. 7 (b)). Let us estimate the mixing angle α_{ik}. In Section 9, it will be demonstrated that $\epsilon \sim 10^{-40}$ eV^2. Since $\alpha_i = m^2_{Hi}$, then $\alpha_2 - \alpha_1 = m^2_{H2} - m^2_{H1} \sim m^2_H \sim 10^4$ GeV^2. Hence

$$\alpha_{ik} \sim 10^{-62}\,. \tag{83}$$

Thus, oscillations of H-bosons (unlike the neutrino oscillations) are negligible. On the contrary, in the symmetrical case (68), we have

$$\alpha_{ik} = \frac{\pi}{4}\,, \qquad a_{e,\mu,\tau} - a \sim \epsilon\,. \tag{84}$$

Thus, in the symmetrical case, each flavor state is the complete mixing of all band states (as we can see in Fig. 7 (a)).

3. The Higgs effect for the Abelian gauge field

Let us consider the interaction of the scalar fields $\varphi_{1,2,3}$, spontaneously breaking the gauge U(1) symmetry, with the gauge field A_μ in its simplest Abelian (Maxwell) form. The corresponding gauge-invariant Lagrangian has the form

$$\begin{aligned} \mathcal{L} \;=\; & (\partial_\mu + ieA_\mu)\varphi_1(\partial^\mu - ieA^\mu)\varphi_1^+ + (\partial_\mu + ieA_\mu)\varphi_2(\partial^\mu - ieA^\mu)\varphi_2^+ \\ & +(\partial_\mu + ieA_\mu)\varphi_3(\partial^\mu - ieA^\mu)\varphi_3^+ \\ & -a_1|\varphi_1|^2 - a_2|\varphi_2|^2 - a_3|\varphi_3|^2 - \frac{b_1}{2}|\varphi_1|^4 - \frac{b_2}{2}|\varphi_2|^4 - \frac{b_3}{2}|\varphi_3|^4 \\ & -\epsilon\left(\varphi_1^+\varphi_2 + \varphi_1\varphi_2^+\right) - \epsilon\left(\varphi_1^+\varphi_3 + \varphi_1\varphi_3^+\right) - \epsilon\left(\varphi_2^+\varphi_3 + \varphi_2\varphi_3^+\right) \\ & -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}\,, \end{aligned} \tag{85}$$

where $A_\mu = (\varphi, -\boldsymbol{A}), A^\mu = (\varphi, \boldsymbol{A})$ are covariant and contravariant potentials of the electro-magnetic field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Faraday tensor. The corresponding Lagrange equation

$$\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} - \frac{\partial \mathcal{L}}{\partial A_\mu} = 0 \tag{86}$$

and the Maxwell equation $\partial_\nu F^{\mu\nu} = -4\pi J^\mu$ give the current

$$J^\mu = -2e\left[|\varphi_1|^2(\partial^\mu\theta_1 + eA^\mu) + |\varphi_2|^2(\partial^\mu\theta_2 + eA^\mu) + |\varphi_3|^2(\partial^\mu\theta_3 + eA^\mu)\right]. \tag{87}$$

The potential can be transformed as

$$A'_\mu = A_\mu + \frac{1}{e}(\alpha\partial_\mu\theta_1 + \beta\partial_\mu\theta_2 + \gamma\partial_\mu\theta_3), \tag{88}$$

where

$$\begin{aligned}\alpha &= \frac{|\varphi_1|^2}{|\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2}, \qquad \beta = \frac{|\varphi_2|^2}{|\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2}, \\ \gamma &= \frac{|\varphi_3|^2}{|\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2},\end{aligned} \tag{89}$$

so that

$$\alpha + \beta + \gamma = 1, \qquad |\varphi_2|^2|\varphi_3|^2\alpha = |\varphi_1|^2|\varphi_3|^2\beta = |\varphi_1|^2|\varphi_2|^2\gamma. \tag{90}$$

Then Eq. (87) is reduced to the "London law"

$$J^\mu = -2e^2\left(|\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2\right)A^\mu \equiv -\frac{1}{4\pi\lambda^2}A^\mu, \tag{91}$$

where

$$\lambda = \frac{1}{\sqrt{8\pi e^2(|\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2)}} \tag{92}$$

is the "penetration depth" — the length of interaction mediated by the gauge bosons A_μ. Thus, screening of the electro-magnetic field by the scalar fields $\varphi_{1,2,3}$, spontaneously breaking the gauge U(1) symmetry, is analogous to the response of the single-band system, but with a contribution from each band $|\varphi_i|^2$. It should be noted that in Eqs. (89)–(92), the field modules $|\varphi_1|^2, |\varphi_2|^2, |\varphi_3|^2$ should be replaced with their equilibrium values $\varphi_{01}^2, \varphi_{02}^2, \varphi_{03}^2$, respectively.

The modulus-phase representations (32) can be considered as the local gauge U(1) transformations. Then, the covariant derivative is transformed by the follows:

$$(\partial_\mu + ieA_\mu)\varphi_j = \mathrm{e}^{i\theta_j}(\partial_\mu + i\partial_\mu\theta_j + ieA_\mu)|\varphi_j|. \tag{93}$$

Applying the transformation (88), we can transform Lagrangian (85) to the following form:

$$\begin{aligned}\mathcal{L}=&\,\partial_\mu|\varphi_1|\partial^\mu|\varphi_1|+\partial_\mu|\varphi_2|\partial^\mu|\varphi_2|+\partial_\mu|\varphi_3|\partial^\mu|\varphi_3|\\&+e^2\left(|\varphi_1|^2+|\varphi_2|^2+|\varphi_3|^2\right)A_\mu A^\mu\\&-2\epsilon|\varphi_1||\varphi_2|\cos(\theta_1-\theta_2)-2\epsilon|\varphi_1||\varphi_3|\cos(\theta_1-\theta_3)-2\epsilon|\varphi_2||\varphi_3|\cos(\theta_2-\theta_3)\\&+\left(|\varphi_1|^2\beta^2+|\varphi_2|^2\alpha^2\right)\partial_\mu(\theta_1-\theta_2)\partial^\mu(\theta_1-\theta_2)\\&+\left(|\varphi_1|^2\gamma^2+|\varphi_3|^2\alpha^2\right)\partial_\mu(\theta_1-\theta_3)\partial^\mu(\theta_1-\theta_3)\\&+\left(|\varphi_2|^2\gamma^2+|\varphi_3|^2\beta^2\right)\partial_\mu(\theta_2-\theta_3)\partial^\mu(\theta_2-\theta_3)\\&-|\varphi_1|^2 2\gamma\beta\partial_\mu(\theta_1-\theta_2)\partial^\mu(\theta_1-\theta_3)-|\varphi_2|^2 2\alpha\gamma\partial_\mu(\theta_1-\theta_2)\partial^\mu(\theta_2-\theta_3)\\&-|\varphi_3|^2 2\alpha\beta\partial_\mu(\theta_1-\theta_3)\partial^\mu(\theta_2-\theta_3)\\&-a_1|\varphi_1|^2-a_2|\varphi_2|^2-a_3|\varphi_3|^2-\frac{b_1}{2}|\varphi_1|^4-\frac{b_2}{2}|\varphi_2|^4-\frac{b_3}{2}|\varphi_3|^4\\&-\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}\,.\end{aligned}\tag{94}$$

We can see that the phases $\theta_1,\theta_2,\theta_3$ have been excluded from the Lagrangian individually leaving only their differences: $\theta_1-\theta_2,\theta_1-\theta_3,\theta_2-\theta_3$. Thus, the gauge field A_μ absorbs the Goldstone boson (*i.e.* the common mode oscillations, where $\nabla\theta_1=\nabla\theta_2=\nabla\theta_3$) with an acoustic spectrum (55). At the same time, the L-bosons (*i.e.* the oscillations of the phases differences $\theta_i-\theta_j$) with massive spectra (56), (57) "survive". This "survival" can be explained as follows. Each phase oscillation θ_i is absorbed by the gauge field, but such mutual oscillations of θ_i and θ_k exist that the gauge fields from each oscillation cancel each other out due to interference, so that the Leggett modes "survive". The phase differences are not normal coordinates, because, firstly, they are not independent: we can assume, for example, $\theta_2-\theta_3=\theta_1-\theta_3-(\theta_1-\theta_2)$; secondly, we can see from Eq. (94), that there are off-diagonal kinetic terms, as $\partial_\mu(\theta_1-\theta_2)\partial^\mu(\theta_1-\theta_3)$. Thus, diagonalizing Lagrangian (94) and noticing that $\theta_{23}=\theta_{13}-(\theta_{12})$, we can obtain the Leggett modes (56), (57) again.

Substituting the calibrated Lagrangian (94) into the Eq. (86), we obtain the equation for the field A_μ

$$\partial_\nu F^{\nu\mu}+\frac{1}{\lambda^2}A^\mu=0\,,\tag{95}$$

where

$$\frac{1}{\lambda^2}=8\pi e^2\left(\varphi_{01}^2+\varphi_{02}^2+\varphi_{03}^2\right)\equiv m_A^2\tag{96}$$

is the squared mass of the gauge boson A^μ, which is the squared reciprocal "penetration depth" (92) in the "London law" (91). The scalar field φ can be written in a dimensionless form: $\varphi=\varphi_0\widetilde{\varphi}$, where $\varphi_0=\sqrt{\frac{-a}{b}}$ is the equilibrium value. Then the Lagrangian takes the form

$$\mathcal{L} = \partial_\mu\varphi\partial^\mu\varphi^+ - a|\varphi|^2 - \frac{b}{2}|\varphi|^4 = \frac{a^2}{b}\left[\xi^2\partial_\mu\widetilde{\varphi}\partial^\mu\widetilde{\varphi}^+ - |\widetilde{\varphi}|^2 - \frac{1}{2}|\widetilde{\varphi}|^4\right], \quad (97)$$

where the length $\xi \equiv \frac{1}{\sqrt{|a|}}$ determines the spatial scale of variations of the scalar field φ — the "coherence length". On the other hand, we could see that the mass of H-boson is $m_H = \sqrt{2|a|}$. Then we have

$$m_H = \frac{\sqrt{2}}{\xi}. \quad (98)$$

It is noteworthy that the mass of the Higgs boson and the mass of the gauge boson are related as

$$\frac{m_H}{m_A} = \sqrt{2}\kappa, \quad (99)$$

where $\kappa = \lambda/\xi$ is the Ginzburg–Landau parameter. Accordingly, the three-band system is characterized with the three coherence lengths $\xi_1 = \frac{\sqrt{2}}{m_{H1}}$, $\xi_2 = \frac{\sqrt{2}}{m_{H2}}$, $\xi_3 = \frac{\sqrt{2}}{m_{H3}}$, hence with three Ginzburg–Landau parameters $\kappa_1 \equiv \frac{\lambda}{\xi_1} = \frac{m_{H1}}{\sqrt{2}m_A}$, $\kappa_2 \equiv \frac{\lambda}{\xi_2} = \frac{m_{H2}}{\sqrt{2}m_A}$, $\kappa_3 \equiv \frac{\lambda}{\xi_3} = \frac{m_{H3}}{\sqrt{2}m_A}$.

Let us consider the term of interaction of modulus of the scalar fields $|\varphi_1|, |\varphi_2|, |\varphi_3|$ with the gauge field A_μ in Lagrangian (94). Using the small deviations $|\phi_i| \ll \varphi_{0i}$ from the corresponding equilibrium values $|\varphi_i|^2 \approx \varphi_{0i}^2 + 2\varphi_{0i}\phi_i(t,\boldsymbol{r})$, we obtain

$$\begin{aligned}\mathcal{U}_{\varphi A} &= e^2\left(|\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2\right)A_\mu A^\mu \\ &\approx e^2\left(\varphi_{01}^2 + \varphi_{02}^2 + \varphi_{03}^2\right)A_\mu A^\mu \\ &\quad + e^2\left(2\varphi_{01}\phi_1(t,\boldsymbol{r}) + 2\varphi_{02}\phi_2(t,\boldsymbol{r}) + 2\varphi_{03}\phi_3(t,\boldsymbol{r})\right)A_\mu A^\mu \\ &\equiv \frac{m_A^2}{8\pi}A_\mu A^\mu + e^2\left(2\varphi_{01}\phi_1(t,\boldsymbol{r}) + 2\varphi_{02}\phi_2(t,\boldsymbol{r}) + 2\varphi_{03}\phi_3(t,\boldsymbol{r})\right)A_\mu A^\mu \\ &\approx \frac{m_A^2}{8\pi}A_\mu A^\mu + e^2\left(2\varphi_{01}\phi_e(t,\boldsymbol{r}) + 2\varphi_{02}\phi_\mu(t,\boldsymbol{r}) + 2\varphi_{03}\phi_\tau(t,\boldsymbol{r})\right)A_\mu A^\mu,\end{aligned} \quad (100)$$

where we have used Eq. (96) and we have taken advantage of the strong band asymmetry and the weakness of interband coupling discussed in Subsection 2.3, where we could see that each collective mode is approximately oscillations of a single band according to the following correspondence $\phi_e \approx \phi_1, \phi_\mu \approx \phi_2, \phi_\tau \approx \phi_3$. As will be demonstrated below $\varphi_{01} : \varphi_{02} : \varphi_{03} = m_e : m_\mu : m_\tau = 0.00028 : 0.059 : 1$. Thus, the gauge boson A_μ interacts with τ-Higgs boson ϕ_τ predominantly, at the same time, the interaction with μ, e-Higgs bosons ϕ_μ, ϕ_e is very weak.

4. The band states and the flavor states of Dirac fields

We can consider three Dirac spinor fields ψ_1, ψ_2, ψ_3 as we have considered three scalar fields (32). The fields are massless, but each field interacts with the corresponding scalar field (*i.e.* in own band). Then, the Lagrangian will have the form

$$\begin{aligned}\mathcal{L} = & i\bar{\psi}_{\mathrm{L1}}\gamma^{\mu}\overset{\leftrightarrow}{\partial}_{\mu}\psi_{\mathrm{L1}} + i\bar{\psi}_{\mathrm{R1}}\gamma^{\mu}\overset{\leftrightarrow}{\partial}_{\mu}\psi_{\mathrm{R1}} - \chi\left(\bar{\psi}_{\mathrm{L1}}\varphi_1\psi_{\mathrm{R1}} + \bar{\psi}_{\mathrm{R1}}\varphi_1^{+}\psi_{\mathrm{L1}}\right) \\ & + i\bar{\psi}_{\mathrm{L2}}\gamma^{\mu}\overset{\leftrightarrow}{\partial}_{\mu}\psi_{\mathrm{L2}} + i\bar{\psi}_{\mathrm{R2}}\gamma^{\mu}\overset{\leftrightarrow}{\partial}_{\mu}\psi_{\mathrm{R2}} - \chi\left(\bar{\psi}_{\mathrm{L2}}\varphi_2\psi_{\mathrm{R2}} + \bar{\psi}_{\mathrm{R2}}\varphi_2^{+}\psi_{\mathrm{L2}}\right) \\ & + i\bar{\psi}_{\mathrm{L3}}\gamma^{\mu}\overset{\leftrightarrow}{\partial}_{\mu}\psi_{\mathrm{L3}} + i\bar{\psi}_{\mathrm{R3}}\gamma^{\mu}\overset{\leftrightarrow}{\partial}_{\mu}\psi_{\mathrm{R3}} - \chi\left(\bar{\psi}_{\mathrm{L3}}\varphi_3\psi_{\mathrm{R3}} + \bar{\psi}_{\mathrm{R3}}\varphi_3^{+}\psi_{\mathrm{L3}}\right), \end{aligned} \quad (101)$$

where γ^{μ} are Dirac matrices, $\bar{\psi}\gamma^{\mu}\overset{\leftrightarrow}{\partial}_{\mu}\psi \equiv \frac{1}{2}[\bar{\psi}\gamma^{\mu}(\partial_{\mu}\psi) - (\partial_{\mu}\bar{\psi})\gamma^{\mu}\psi]$ is a differential operator, $\bar{\psi} = \psi^{+}\gamma^{0}$ is the Dirac conjugated bispinor; $\psi_{\mathrm{R}} = \frac{1}{2}(1+\gamma^{5})\psi$ and $\psi_{\mathrm{L}} = \frac{1}{2}(1-\gamma^{5})\psi$ are the right-handed and the left-handed fields, respectively, so that $\psi = \psi_{\mathrm{L}} + \psi_{\mathrm{R}}$; χ is the dimensionless coupling constant between the corresponding Dirac field ψ_j and scalar field φ_j (Yukawa constant). Thus, by analogy with the Higgs modes, we will call the states ψ_1, ψ_2, ψ_3 as *the band states*.

Due to the presence of the scalar fieldcondensate $\langle 0|\varphi|0\rangle = \varphi_0\,\mathrm{e}^{i\theta^0}$, the Dirac fermion takes mass as follows. Let us consider a single-band case, then the term of the interaction of the scalar field φ with the Dirac field ψ has the following form:

$$\begin{aligned}\mathcal{U}_{\mathrm{D}} &= \chi\left(\bar{\psi}_{\mathrm{L}}\varphi\psi_{\mathrm{R}} + \bar{\psi}_{\mathrm{R}}\varphi^{+}\psi_{\mathrm{L}}\right) = \chi|\varphi|\left(\bar{\psi}_{\mathrm{L}}\psi_{\mathrm{R}} + \bar{\psi}_{\mathrm{R}}\psi_{\mathrm{L}}\right)\cos\theta \\ & \quad + i\chi|\varphi|\left(\bar{\psi}_{\mathrm{L}}\psi_{\mathrm{R}} - \bar{\psi}_{\mathrm{R}}\psi_{\mathrm{L}}\right)\sin\theta\,. \end{aligned} \quad (102)$$

Here, $\bar{\psi}_{\mathrm{L}}\psi_{\mathrm{R}} + \bar{\psi}_{\mathrm{R}}\psi_{\mathrm{L}}$ is a *scalar*, but $\bar{\psi}_{\mathrm{L}}\psi_{\mathrm{R}} - \bar{\psi}_{\mathrm{R}}\psi_{\mathrm{L}}$ is a *pseudoscalar*. Hence, in order to obtain the Dirac mass of a fermion, we must choose the vacuum so that $\theta^0 = 0$, that is $m_{\mathrm{D}} = \chi\varphi_0\cos\theta^0 = \chi\varphi_0$. It should be noted that, in the single-band system, this choice of phase is not principal, because, due to the U(1)-symmetry, the phase θ can always be set as $\theta = 0$. Then, the Dirac term takes the form

$$\mathcal{U}_{\mathrm{D}} = \chi|\varphi|\left(\bar{\psi}_{\mathrm{R}}\psi_{\mathrm{L}} + \bar{\psi}_{\mathrm{R}}\psi_{\mathrm{L}}\right) = m_{\mathrm{D}}\left(\bar{\psi}_{\mathrm{R}}\psi_{\mathrm{L}} + \bar{\psi}_{\mathrm{R}}\psi_{\mathrm{L}}\right) + \chi\phi\left(\bar{\psi}_{\mathrm{R}}\psi_{\mathrm{L}} + \bar{\psi}_{\mathrm{R}}\psi_{\mathrm{L}}\right). \quad (103)$$

Thus, the initially massless fermion obtains the mass $m_{\mathrm{D}} = \chi\varphi_0$, due to interaction with the condensate $\varphi_0 = \sqrt{\frac{-a}{b}}$ of the scalar field φ. The coupling $\chi\phi$ is the interaction of the Dirac field ψ with small variations of the modulus of the scalar field from its equilibrium value: $|\varphi| = \varphi_0 + \phi$, where $|\phi| \ll \varphi_0$, *i.e.* the interaction with the H-boson.

However, in the three-band system (multi-band system), there are many scalar fields: $|\varphi_1|\,\mathrm{e}^{i\theta_1}$, $|\varphi_2|\,\mathrm{e}^{i\theta_2}$, $|\varphi_3|\,\mathrm{e}^{i\theta_3}$, where the equilibrium phase differences $\theta^0_{12},\theta^0_{13},\theta^0_{23}$ are determined by Eq. (40). In the case of repulsive interband coupling $\epsilon>0$, we can have different phases: $\theta^0_1\neq\theta^0_2\neq\theta^0_3$, see Fig. 5, for example, $\theta^0_{12}=\theta^0_{23}=\frac{2\pi}{3},\theta^0_{13}=\frac{4\pi}{3}$ for symmetrical bands $\varphi_{01}=\varphi_{02}=\varphi_{03}$. This means that even if we set $\theta^0_1=0$, the other phases will be $\theta^0_2=\theta^0_{12}\neq 0$, $\theta^0_3=\theta^0_{13}\neq 0$. Hence, the coupling terms (101) cannot be reduced to the Dirac mass term (103) due to the pseudoscalar contribution. On the contrary, in the case of attractive interband coupling $\epsilon<0$, we have the same phases: $\theta^0_1=\theta^0_2=\theta^0_3$, see Fig. 5. This means that we should assume $\theta^0_1=\theta^0_2=\theta^0_3=0$, then the coupling terms in Eq. (101) can be reduced to the Dirac mass term (103)

$$
\begin{aligned}
\mathcal{L}=i\bar{\psi}_{\mathrm{L}1}\gamma^{\mu}\overset{\leftrightarrow}{\partial}_{\mu}\psi_{\mathrm{L}1}+i\bar{\psi}_{\mathrm{R}1}\gamma^{\mu}\overset{\leftrightarrow}{\partial}_{\mu}\psi_{\mathrm{R}1}-\chi|\varphi_1|\left(\bar{\psi}_{\mathrm{L}1}\psi_{\mathrm{R}1}+\bar{\psi}_{\mathrm{R}1}\psi_{\mathrm{L}1}\right)\\
+i\bar{\psi}_{\mathrm{L}2}\gamma^{\mu}\overset{\leftrightarrow}{\partial}_{\mu}\psi_{\mathrm{L}2}+i\bar{\psi}_{\mathrm{R}2}\gamma^{\mu}\overset{\leftrightarrow}{\partial}_{\mu}\psi_{\mathrm{R}2}-\chi|\varphi_2|\left(\bar{\psi}_{\mathrm{L}2}\psi_{\mathrm{R}2}+\bar{\psi}_{\mathrm{R}2}\psi_{\mathrm{L}2}\right)\\
+i\bar{\psi}_{\mathrm{L}3}\gamma^{\mu}\overset{\leftrightarrow}{\partial}_{\mu}\psi_{\mathrm{L}3}+i\bar{\psi}_{\mathrm{R}3}\gamma^{\mu}\overset{\leftrightarrow}{\partial}_{\mu}\psi_{\mathrm{R}3}-\chi|\varphi_3|\left(\bar{\psi}_{\mathrm{L}3}\psi_{\mathrm{R}3}+\bar{\psi}_{\mathrm{R}3}\psi_{\mathrm{L}3}\right).
\end{aligned}
\tag{104}
$$

Therefore, the masses of the Dirac fields ψ_1,ψ_2,ψ_3 are determined by coupling with the *equilibrium* module of the corresponding scalar fields φ_{01}, φ_{02}, φ_{03}

$$
m_{\mathrm{D}1}=\chi\varphi_{01}\,,\qquad m_{\mathrm{D}2}=\chi\varphi_{02}\,,\qquad m_{\mathrm{D}3}=\chi\varphi_{03}\,.
\tag{105}
$$

Thus, only the attractive interband coupling

$$
\epsilon<0
\tag{106}
$$

has a physical sense, unlike multi-band superconductivity, where the analog of the interaction of Dirac spinors and scalar field (the superconducting order parameter) is absent, therefore any interband couplings ϵ_{ik} are allowed [74, 75]. From Fig. 5 we can see that at $\varphi_{01,02}\ll\varphi_{03}$ and $\epsilon>0$, and we can assume that $\theta^0_3=0$, $\theta^0_2=\theta^0_2=\pi$ (then we should change signs of two Yukawa constants $\chi_1=\chi_2=-\chi_3$). However, such a system will have larger ground-state energy compared to the $\epsilon<0$ case.

Let us consider the Dirac terms in Eq. (104)

$$
\begin{aligned}
\mathcal{U}_{\mathrm{D}} &= \chi\varphi_{01}\left(\bar{\psi}_{\mathrm{L}1}\psi_{\mathrm{R}1}+\bar{\psi}_{\mathrm{R}1}\psi_{\mathrm{L}1}\right)+\chi\phi_1\left(\bar{\psi}_{\mathrm{L}1}\psi_{\mathrm{R}1}+\bar{\psi}_{\mathrm{R}1}\psi_{\mathrm{L}1}\right)\\
&+ \chi\varphi_{02}\left(\bar{\psi}_{\mathrm{L}2}\psi_{\mathrm{R}2}+\bar{\psi}_{\mathrm{R}2}\psi_{\mathrm{L}2}\right)+\chi\phi_2\left(\bar{\psi}_{\mathrm{L}2}\psi_{\mathrm{R}2}+\bar{\psi}_{\mathrm{R}2}\psi_{\mathrm{L}2}\right)\\
&+ \chi\varphi_{03}\left(\bar{\psi}_{\mathrm{L}3}\psi_{\mathrm{R}3}+\bar{\psi}_{\mathrm{R}3}\psi_{\mathrm{L}3}\right)+\chi\phi_3\left(\bar{\psi}_{\mathrm{L}3}\psi_{\mathrm{R}3}+\bar{\psi}_{\mathrm{R}3}\psi_{\mathrm{L}3}\right).
\end{aligned}
\tag{107}
$$

However, as we could see in Section 2, the fields ϕ_1,ϕ_2,ϕ_3 are not normal oscillations of the coupled scalar fields $|\varphi_1|,|\varphi_2|,|\varphi_3|$. Eigen-frequencies (the masses of H-bosons) have been found in Eq. (70), each normal oscillation mode involves all three scalar fields, see Eq. (71) and Fig. 7. Thus, we can introduce *the flavor states*: each flavor state of the Dirac fields interacts only with the corresponding normal mode $\phi_e, \phi_\mu, \phi_\tau$ of the scalar fields. At the same time, in Section 2.3, we have seen that due to the weakness of interband coupling $|\epsilon| \ll \alpha_{1,2,3}$ and the strong band asymmetry (69), the effective diagonalization (73) can be realized. As a result, we obtain the flavor states of condensates (77), in the sense that each normal mode $\phi_e, \phi_\mu, \phi_\tau$ is an oscillation of the corresponding effective band (flavor) $\varphi_e, \varphi_\mu, \varphi_\tau$. Then we can write

$$\begin{aligned}\mathcal{U}_{\mathrm{De}\mu\tau} = {} & \chi\varphi_{0e}\left(\bar{\psi}_{\mathrm{L}e}\psi_{\mathrm{R}e} + \bar{\psi}_{\mathrm{R}e}\psi_{\mathrm{L}e}\right) + \chi\phi_e\left(\bar{\psi}_{\mathrm{L}e}\psi_{\mathrm{R}e} + \bar{\psi}_{\mathrm{R}e}\psi_{\mathrm{L}e}\right) \\ & + \chi\varphi_{0\mu}\left(\bar{\psi}_{\mathrm{L}\mu}\psi_{\mathrm{R}\mu} + \bar{\psi}_{\mathrm{R}\mu}\psi_{\mathrm{L}\mu}\right) + \chi\phi_\mu\left(\bar{\psi}_{\mathrm{L}\mu}\psi_{\mathrm{R}\mu} + \bar{\psi}_{\mathrm{R}\mu}\psi_{\mathrm{L}\mu}\right) \\ & + \chi\varphi_{0\tau}\left(\bar{\psi}_{\mathrm{L}\tau}\psi_{\mathrm{R}\tau} + \bar{\psi}_{\mathrm{R}\tau}\psi_{\mathrm{L}\tau}\right) + \chi\phi_\tau\left(\bar{\psi}_{\mathrm{L}\tau}\psi_{\mathrm{R}\tau} + \bar{\psi}_{\mathrm{R}\tau}\psi_{\mathrm{L}\tau}\right) . \end{aligned} \tag{108}$$

Thus, the masses of the Dirac fields $\psi_e, \psi_\mu, \psi_\tau$ are determined by coupling with the equilibrium modules of the corresponding scalar fields $\varphi_{0e}, \varphi_{0\mu}, \varphi_{0\tau}$

$$m_{\mathrm{D}e} = \chi\varphi_{0e} , \qquad m_{\mathrm{D}\mu} = \chi\varphi_{0\mu} , \qquad m_{\mathrm{D}\tau} = \chi\varphi_{0\tau} . \tag{109}$$

Since $\varphi_{0e} \approx \varphi_{01}, \varphi_{0\mu} \approx \varphi_{02}, \varphi_{0\tau} \approx \varphi_{03}$ and $\phi_e \approx \phi_1, \phi_\mu \approx \phi_2, \phi_\tau \approx \phi_3$, we can present the Yukawa couplings in Table 2, similarly to Table 1, for 2HDM or 3HDM models. Thus, unlike in 2HDM or 3HDM models, in the three-band model, the Yukawa interactions with scalar fields are distributed over generations of fermions, not over leptons and quarks apart.

Table 2. Yukawa interactions for three generations of fermions (charged leptons, upper and bottom quarks).

e, u, d	μ, c, s	τ, t, b
φ_1	φ_2	φ_3

However, unlike the exact diagonalization (78) for potential energy of the excitations $\phi_{1,2,3} \to \phi_{e,\mu,\tau}$ of the coupled condensates (because it is a quadratic form), the diagonalization (73) is approximate. Hence, the flavor states must enter into the Lagrangian with some interflavor mixings, which compensate the inaccuracy of diagonalization (73). Then, the potential energy term for the flavor states $\psi_e, \psi_\mu, \psi_\tau$ takes the following form:

$$\begin{aligned}\mathcal{U}_{\mathrm{De}\mu\tau} + \mathcal{U}_{\mathrm{mix}} = {} & \begin{pmatrix} \bar{\psi}_{\mathrm{L}e}, & \bar{\psi}_{\mathrm{L}\mu}, & \bar{\psi}_{\mathrm{L}\tau} \end{pmatrix} \begin{pmatrix} m_{\mathrm{D}e} & \zeta_{e\mu} & \zeta_{e\tau} \\ \zeta_{e\mu} & m_{\mathrm{D}\mu} & \zeta_{\mu\tau} \\ \zeta_{e\tau} & \zeta_{\mu\tau} & m_{\mathrm{D}\tau} \end{pmatrix} \begin{pmatrix} \psi_{\mathrm{R}e} \\ \psi_{\mathrm{R}\mu} \\ \psi_{\mathrm{R}\tau} \end{pmatrix} \\ & + \mathrm{h.c.} \equiv \langle \psi_{e\mu\tau} | M_{e\mu\tau} | \psi_{e\mu\tau} \rangle , \end{aligned} \tag{110}$$

where ζ_{ik} are the mixing parameters analogous to the interband coupling ϵ. Thus, the coupling of "L" and "R" components $\bar{\psi}_{\mathrm{L}i}\psi_{\mathrm{R}i}+\bar{\psi}_{\mathrm{R}i}\psi_{\mathrm{L}i}$ gives the Dirac masses $m_{\mathrm{D}e}, m_{\mathrm{D}\mu}, m_{\mathrm{D}\tau}$, at the same time, the "L" and "R" components are mixed with the corresponding "R" and "L" components of the other flavors $\bar{\psi}_{\mathrm{L}i}\psi_{\mathrm{R}k}+\bar{\psi}_{\mathrm{R}k}\psi_{\mathrm{L}i}$. As a result of diagonalization of the matrix $M_{e\mu\tau}$: $M_{123}=\mathrm{diag}(M_{e\mu\tau})$, we obtain the potential energy term in Lagrangian (107) for the band states

$$\mathcal{U}_{\mathrm{D}e\mu\tau}+\mathcal{U}_{\mathrm{mix}}=\mathcal{U}_{\mathrm{D}}=\begin{pmatrix}\bar{\psi}_{\mathrm{L}1}, & \bar{\psi}_{\mathrm{L}2}, & \bar{\psi}_{\mathrm{L}3}\end{pmatrix}\begin{pmatrix} m_{\mathrm{D}1} & 0 & 0\\ 0 & m_{\mathrm{D}2} & 0\\ 0 & 0 & m_{\mathrm{D}3}\end{pmatrix}\begin{pmatrix}\psi_{\mathrm{R}1}\\ \psi_{\mathrm{R}2}\\ \psi_{\mathrm{R}3}\end{pmatrix} +\mathrm{h.c.}\equiv\langle\psi_{123}|M_{123}|\psi_{123}\rangle\,. \tag{111}$$

Thus, we have the system of equations for the mixing parameters $\zeta_{e\mu},\zeta_{e\tau},\zeta_{\mu\tau}$

$$\begin{vmatrix} m_{\mathrm{D}e}-m_{\mathrm{D}i} & \zeta_{e\mu} & \zeta_{e\tau}\\ \zeta_{e\mu} & m_{\mathrm{D}\mu}-m_{\mathrm{D}i} & \zeta_{\mu\tau}\\ \zeta_{e\tau} & \zeta_{\mu\tau} & m_{\mathrm{D}\tau}-m_{\mathrm{D}i}\end{vmatrix}=0\,,\quad \text{where}\quad i=1,2,3\,. \tag{112}$$

Obviously, $m_{\mathrm{D}e}-m_{\mathrm{D}1}\sim m_{\mathrm{D}\mu}-m_{\mathrm{D}2}\sim m_{\mathrm{D}\tau}-m_{\mathrm{D}3}\sim\zeta_{e\mu},\zeta_{e\tau},\zeta_{\mu\tau}$. Using Eq. (70), we obtain

$$\zeta_{\alpha\beta}\sim m_{\mathrm{D}i}\frac{\epsilon^2}{m_{Hi}^2\Delta m_{Hij}^2}\,, \tag{113}$$

where $\Delta m_{Hij}^2=m_{Hi}^2-m_{Hj}^2$. Thus, the mixing parameters $\zeta_{\alpha\beta}$ are determined by the interband coupling ϵ. If the interband coupling is weak $|\epsilon|\lll m_H^2, m_{\mathrm{D}}^2$, then the mixing parameter $|\zeta|\lll m_{\mathrm{D}}$.

It should be noted that in SM, we can write the mass matrix $M_{e\mu\tau}^{\mathrm{SM}}$ as

$$M_{e\mu\tau}^{\mathrm{SM}}=\begin{pmatrix} m_{\mathrm{D}e} & \zeta_{e\mu} & \zeta_{e\tau}\\ \zeta_{e\mu} & m_{\mathrm{D}\mu} & \zeta_{\mu\tau}\\ \zeta_{e\tau} & \zeta_{\mu\tau} & m_{\mathrm{D}\tau}\end{pmatrix}=\varphi_0\begin{pmatrix}\chi_{ee} & \chi_{e\mu} & \chi_{e\tau}\\ \chi_{e\mu} & \chi_{\mu\mu} & \chi_{\mu\tau}\\ \chi_{e\tau} & \chi_{\mu\tau} & \chi_{\tau\tau}\end{pmatrix}\,. \tag{114}$$

That is, both diagonal elements and off-diagonal elements are just Yukawa constants χ_{ij} due to the presence of the single-scalar field φ_0. However, in the three-band model, we have three scalar fields $\varphi_{0e},\varphi_{0\mu},\varphi_{0\tau}$ and Eq. (109), hence we cannot write the mass matrix $M_{e\mu\tau}$ in the form of Eq. (114). This means that the mixing coefficients ζ_{ij} are not off-diagonal Yukawa interaction. The mixing coefficients ζ_{ij} are the fermionic analog of the interband Josephson coupling. This mixing takes place due to the interband Josephson coupling of the scalar fields $\varphi_1,\varphi_2,\varphi_3$; from Eq. (113), we can see that $\zeta_{\alpha\beta}\propto\epsilon^2$.

The band states and the flavor states are connected by an unitary transformation $|\psi_{e\mu\tau}\rangle = U|\psi_{123}\rangle$, $|\psi_{123}\rangle = U^{\mathrm{T}}|\psi_{e\mu\tau}\rangle$, where U is an unitary matrix $U^{-1} = U^{\mathrm{T}}$, which can be written via the mixing angles $\alpha_{12}, \alpha_{13}, \alpha_{23}$, see Eqs. (79) and (80). The mixing angles can be found from Eq. (81). Assuming the independent mixing for each pair of bands $e \leftrightarrow \mu, e \leftrightarrow \tau, \mu \leftrightarrow \tau$, we obtain (for example, for $e \leftrightarrow \mu$ via the mixing of the band states 1 and 2)

$$\tan 2\alpha_{12} = \frac{2\zeta_{e\mu}}{m_{\mathrm{D}e} - m_{\mathrm{D}\mu}}, \tag{115}$$

and moreover, the band masses $m_{\mathrm{D1}}, m_{\mathrm{D2}}$ and the flavor masses $m_{\mathrm{D}e}, m_{\mathrm{D}\mu}$ are connected by the following way:

$$\begin{aligned} (m_{\mathrm{D1}} - m_{\mathrm{D2}})^2 &= (m_{\mathrm{D}e} - m_{\mathrm{D}\mu})^2 + 4\zeta_{e\mu}^2, \\ m_{\mathrm{D}e} + m_{\mathrm{D}\mu} &= m_{\mathrm{D1}} + m_{\mathrm{D2}}. \end{aligned} \tag{116}$$

In the case of weak interband coupling $|\epsilon| \ll |a_1|, |a_2|, |a_3|$ (hence, $|\zeta_{ik}| \ll m_{\mathrm{D1}}, m_{\mathrm{D2}}, m_{\mathrm{D3}}$) and strongly asymmetrical bands $\varphi_{01} \ll \varphi_{02} \ll \varphi_{03}$ (hence, $m_{\mathrm{D1}} \ll m_{\mathrm{D2}} \ll m_{\mathrm{D3}}$), the mixing angles are very small $|\tan\alpha_{ik}| \lll 1$, hence the oscillations charged leptons (*i.e.* electron–muon–tauon) are negligible and experimentally unobservable, unlike the neutrino oscillations.

Now, let us return to Eq. (102) again. Despite the fact that equilibrium phases are $\theta_1^0 = \theta_2^0 = \theta_3^0 = 0$, phase oscillations (50) can take place. The full interaction term has the form

$$\begin{aligned} \mathcal{U}_{\mathrm{D}} = {} & \chi|\varphi_1|\left(\bar{\psi}_{\mathrm{L1}}\psi_{\mathrm{R1}} + \bar{\psi}_{\mathrm{R1}}\psi_{\mathrm{L1}}\right) + \chi|\varphi_2|\left(\bar{\psi}_{\mathrm{L2}}\psi_{\mathrm{R2}} + \bar{\psi}_{\mathrm{R2}}\psi_{\mathrm{L2}}\right) \\ & + \chi|\varphi_3|\left(\bar{\psi}_{\mathrm{L3}}\psi_{\mathrm{R3}} + \bar{\psi}_{\mathrm{R3}}\psi_{\mathrm{L3}}\right) \\ & + i\chi\varphi_{01}\left(\bar{\psi}_{\mathrm{L1}}\psi_{\mathrm{R1}} - \bar{\psi}_{\mathrm{R1}}\psi_{\mathrm{L1}}\right)\theta_1 + i\chi\varphi_{02}\left(\bar{\psi}_{\mathrm{L2}}\psi_{\mathrm{R2}} - \bar{\psi}_{\mathrm{R2}}\psi_{\mathrm{L2}}\right)\theta_2 \\ & + i\chi\varphi_{03}\left(\bar{\psi}_{\mathrm{L3}}\psi_{\mathrm{R3}} - \bar{\psi}_{\mathrm{R3}}\psi_{\mathrm{L3}}\right)\theta_3, \end{aligned} \tag{117}$$

where $\theta = \theta(t, \boldsymbol{r})$ is small phase oscillations $|\theta| \ll 1$. Unlike the interaction with the amplitudes of the scalar fields $|\varphi_i|$, the interaction with the phase oscillations would have to violate the P-invariance. However, we can see that the Dirac field $\psi_i = \psi_{\mathrm{L}i} + \psi_{\mathrm{R}i}$ of each band interacts with the corresponding phase of the scalar field θ_i. As have been demonstrated in Section 3, the phase oscillations $\theta_i(t, \boldsymbol{r})$ are absorbed by the gauge fields A_μ due to the Higgs mechanism, hence in Eq. (117), the phases are equal to their equilibrium value $\theta_i = \theta_i^0 = 0$.

5. Spontaneous breaking of the SU(2)$_I$ gauge symmetry in the three-band system with the Josephson couplings

Let the fields Ψ_1, Ψ_2, Ψ_3 be isospinors, each of which has two complex (four real) scalar components

$$\Psi = \begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix}, \qquad \Psi^+ = \begin{pmatrix} \varphi^{(1)*}, & \varphi^{(2)*} \end{pmatrix} \tag{118}$$

being transformed during the rotation in the isospace as

$$\Psi = S\Psi', \quad S = \mathrm{e}^{i\frac{\vec{\tau}}{2}\vec{\vartheta}} = \left(\tau_0 \cos\frac{\vartheta}{2} + i(\vec{n}\vec{\tau})\sin\frac{\vartheta}{2}\right), \tag{119}$$

where $\vec{\tau} = (\tau_x, \tau_y, \tau_z)$ is a vector consisting of Pauli matrices, τ_0 is an identity matrix, $\vec{\vartheta} = \vec{n}\vartheta$, where $\vec{n}$ is an unit vector in the direction of the axis around which the rotation is made in the isospace. Thus, the isospinor fields, corresponding to each band, can be represented in the following form:

$$\begin{aligned} \Psi_1(x) &= \mathrm{e}^{i\frac{\vec{\tau}}{2}\vec{\vartheta}_1(x)} \begin{pmatrix} 0 \\ \varphi_1(x) \end{pmatrix}, \qquad \Psi_2(x) = \mathrm{e}^{i\frac{\vec{\tau}}{2}\vec{\vartheta}_2(x)} \begin{pmatrix} 0 \\ \varphi_2(x) \end{pmatrix}, \\ \Psi_3(x) &= \mathrm{e}^{i\frac{\vec{\tau}}{2}\vec{\vartheta}_3(x)} \begin{pmatrix} 0 \\ \varphi_3(x) \end{pmatrix}, \end{aligned} \tag{120}$$

where $\varphi_1, \varphi_2, \varphi_3$ are real and $\varphi_1, \varphi_2, \varphi_3 > 0$. Thus, we assign the third projection of the isospin as $I_z = -\frac{1}{2}$ to the scalar fields φ_i, then hypercharge $Y = 1$ and the electrical charge $Q = I_z + \frac{Y}{2} = 0$. At the same time, the phases $\vec{\vartheta}_i$ are characterized with zero charges $I_z = Y = Q = 0$. The corresponding Lagrangian $\mathcal{L}$ is a sum of the gauge-invariant part (relative to the SU(2) gauge symmetry) and the Josephson terms

$$\begin{aligned} \mathcal{L} = &\partial_\mu\Psi_1\partial^\mu\Psi_1^+ + \partial_\mu\Psi_2\partial^\mu\Psi_2^+ + \partial_\mu\Psi_3\partial^\mu\Psi_3^+ \\ &- a_1\Psi_1\Psi_1^+ - a_2\Psi_2\Psi_2^+ - a_3\Psi_3\Psi_3^+ \\ &- \frac{b_1}{2}\left(\Psi_1\Psi_1^+\right)^2 - \frac{b_2}{2}\left(\Psi_2\Psi_2^+\right)^2 - \frac{b_3}{2}\left(\Psi_3\Psi_3^+\right)^2 \\ &- \epsilon\left(\Psi_1^+\Psi_2 + \Psi_1\Psi_2^+\right) - \epsilon\left(\Psi_1^+\Psi_3 + \Psi_1\Psi_3^+\right) - \epsilon\left(\Psi_2^+\Psi_3 + \Psi_2\Psi_3^+\right). \end{aligned} \tag{121}$$

The Josephson terms $\Psi_i^+\Psi_j + \Psi_j\Psi_i^+$ are not invariant relatively to the SU(2) gauge symmetry, however, these terms should depend on the phase differences $\vartheta_i - \vartheta_j$ only: $\Psi_i\Psi_j^+ + \Psi_i^+\Psi_j = 2\varphi_i\varphi_j\cos\frac{\vartheta_i - \vartheta_j}{2}$, in order to have a physical sense as interference between condensates Ψ_1, Ψ_2, Ψ_3. To ensure such a property, it is necessary

$$\vec{n}_1 = \vec{n}_2 = \vec{n}_3, \tag{122}$$

that is the isospinors (120) must rotate around a common axis. Moreover, it is not difficult to see that

$$\Psi = \mathrm{e}^{i\frac{\vec{\tau}}{2}\vec{\vartheta}}\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} [i\varphi n_x + \varphi n_y]\sin\frac{\vartheta}{2} \\ \varphi\cos\frac{\vartheta}{2} - in_z\varphi\sin\frac{\vartheta}{2} \end{pmatrix}, \tag{123}$$

then

$$\begin{aligned}
\Psi_k\Psi_j^+ + \Psi_j^+\Psi_k \; = \; & \begin{pmatrix} [-i\varphi_j n_x + \varphi_j n_y]\sin\frac{\vartheta_j}{2}, & \varphi_j\cos\frac{\vartheta_j}{2} + in_z\varphi_j\sin\frac{\vartheta_j}{2} \end{pmatrix} \\
& \times\begin{pmatrix} [i\varphi_k n_x + \varphi_k n_y]\sin\frac{\vartheta_k}{2} \\ \varphi_k\cos\frac{\vartheta_k}{2} - in_z\varphi_k\sin\frac{\vartheta_k}{2} \end{pmatrix} \\
& + \begin{pmatrix} [-i\varphi_k n_x + \varphi_k n_y]\sin\frac{\vartheta_k}{2}, & \varphi_k\cos\frac{\vartheta_k}{2} + in_z\varphi_k\sin\frac{\vartheta_k}{2} \end{pmatrix} \\
& \times\begin{pmatrix} [i\varphi_j n_x + \varphi_j n_y]\sin\frac{\vartheta_j}{2} \\ \varphi_j\cos\frac{\vartheta_j}{2} - in_z\varphi_j\sin\frac{\vartheta_j}{2} \end{pmatrix} \\
= \; & 2\varphi_j\varphi_k\left[\cos\frac{\vartheta_j}{2}\cos\frac{\vartheta_k}{2} + n_z^2\sin\frac{\vartheta_j}{2}\sin\frac{\vartheta_k}{2}\right] \\
& +2\varphi_j\varphi_k\left[n_x^2 + n_y^2\right]\sin\frac{\vartheta_j}{2}\sin\frac{\vartheta k}{2}\,.
\end{aligned} \tag{124}$$

Therefore, it must be

$$n_x = n_y = 0 \Rightarrow n_z = \pm 1\,. \tag{125}$$

Then, substituting representation (120) into Lagrangian (121), we obtain

$$\begin{aligned}
\mathcal{L} = \; & \partial_\mu\varphi_1\partial^\mu\varphi_1 + \partial_\mu\varphi_2\partial^\mu\varphi_2 + \partial_\mu\varphi_3\partial^\mu\varphi_3 \\
& +\varphi_1^2\partial_\mu\frac{\vartheta_1}{2}\partial^\mu\frac{\vartheta_1}{2} + \varphi_2^2\partial_\mu\frac{\vartheta_2}{2}\partial^\mu\frac{\vartheta_2}{2} + \varphi_3^2\partial_\mu\frac{\vartheta_3}{2}\partial^\mu\frac{\vartheta_3}{2} \\
& -a_1\varphi_1^2 - \frac{b_1}{2}\varphi_1^4 - a_2\varphi_2^2 - \frac{b_2}{2}\varphi_2^4 - a_3\varphi_3^2 - \frac{b_3}{2}\varphi_3^4 \\
& -2\epsilon\varphi_1\varphi_2\cos\frac{\vartheta_1-\vartheta_2}{2} - 2\epsilon\varphi_1\varphi_3\cos\frac{\vartheta_1-\vartheta_3}{2} - 2\epsilon\varphi_2\varphi_3\cos\frac{\vartheta_2-\vartheta_3}{2}\,.
\end{aligned} \tag{126}$$

Let us consider stationary and spatially homogeneous case, *i.e.* $\partial_t\varphi = 0$, $\nabla\varphi = 0$, $\partial_t\vartheta = 0$, $\nabla\vartheta = 0$. Then we obtain equations for the equilibrium values of the fields φ_{0i} and $\vartheta_i^0 - \vartheta_j^0$

$$\left\{\begin{array}{c}
a_1\varphi_{01} + \epsilon\varphi_{02}\cos\frac{\vartheta_2^0-\vartheta_1^0}{2} + \epsilon\varphi_{03}\cos\frac{\vartheta_3^0-\vartheta_1^0}{2} + b_1\varphi_{01}^3 = 0 \\
a_2\varphi_{02} + \epsilon\varphi_{01}\cos\frac{\vartheta_1^0-\vartheta_2^0}{2} + \epsilon\varphi_{03}\cos\frac{\vartheta_3^0-\vartheta_2^0}{2} + b_1\varphi_{02}^3 = 0 \\
a_3\varphi_{03} + \epsilon\varphi_{01}\cos\frac{\vartheta_1^0-\vartheta_3^0}{2} + \epsilon\varphi_{02}\cos\frac{\vartheta_2^0-\vartheta_3^0}{2} + b_1\varphi_{03}^3 = 0 \\
\varphi_{02}\sin\frac{\vartheta_2^0-\vartheta_1^0}{2} + \varphi_{03}\sin\frac{\vartheta_3^0-\vartheta_1^0}{2} = 0 \\
\varphi_{01}\sin\frac{\vartheta_1^0-\vartheta_2^0}{2} + \varphi_{03}\sin\frac{\vartheta_3^0-\vartheta_2^0}{2} = 0 \\
\varphi_{01}\sin\frac{\vartheta_1^0-\vartheta_3^0}{2} + \varphi_{02}\sin\frac{\vartheta_2^0-\vartheta_3^0}{2} = 0
\end{array}\right\}. \tag{127}$$

If the interband coupling is weak, *i.e.* $|\epsilon| \ll |a_1|, |a_2|, |a_3|$, then we can assume $\varphi_{0i} = \sqrt{\frac{a_i}{b_i}}$.

On the other hand, let us consider three Dirac spinor fields ψ_1, ψ_2, ψ_3 as we have considered them in Section 4. However, now Lagrangian (101) has the form

$$\mathcal{L} = \sum_{i=1}^{3}\left[i\bar{\psi}_{\mathrm{L}i}\gamma^{\mu}\overleftrightarrow{\partial}_{\mu}\psi_{\mathrm{L}i} + i\bar{\psi}_{\mathrm{R}i}\gamma^{\mu}\overleftrightarrow{\partial}_{\mu}\psi_{\mathrm{R}i} - \chi\left(\bar{\psi}_{\mathrm{L}i}\Psi_i\psi_{\mathrm{R}i} + \bar{\psi}_{\mathrm{R}i}\Psi_i^{+}\psi_{\mathrm{L}i}\right)\right]. \tag{128}$$

For the terms of interaction of Dirac fields with isospinor fields $\mathcal{U}_{\mathrm{D}} = \chi(\bar{\psi}_{\mathrm{L}}\Psi\psi_{\mathrm{R}} + \bar{\psi}_{\mathrm{R}}\Psi^{+}\psi_{\mathrm{L}})$ to take the form of the mass term of the Dirac type $\mathcal{U}_{\mathrm{D}} = m_{\mathrm{D}}(\bar{\psi}_{\mathrm{L}}\psi_{\mathrm{R}} + \bar{\psi}_{\mathrm{R}}\psi_{\mathrm{L}})$, the following conditions must be satisfied:

1. Since $\mathcal{L}$ must be a scalar and Ψ is a doublet (118), then the spinor ψ_{L} must be a dublet (singlet) and ψ_{R} must be a singlet (dublet). That is, for example, $\psi_{\mathrm{L}} = \begin{pmatrix} \nu_{\mathrm{L}} \\ l_{\mathrm{L}} \end{pmatrix}$ and $\psi_{\mathrm{R}} = l_{\mathrm{R}}$. This means the violation of the spatial parity symmetry. Here, $l_{\mathrm{L,R}}$ are electrically charged leptons: $Q_{\mathrm{L}} = Q_{\mathrm{R}} = -1$, $I_{z\mathrm{L}} = -\frac{1}{2}, Y_{\mathrm{L}} = -1, I_{z\mathrm{R}} = 0, Y_{\mathrm{R}} = -2$, at the same time, ν_{L} are electrically neutral leptons (neutrinos): $Q = 0$, $I_{z\mathrm{L}} = \frac{1}{2}$, $Y_{\mathrm{L}} = -1$.

2. As in Section 4, the coupling terms $\mathcal{U}_{\mathrm{D}}$ in Eq. (128) can be reduced to the Dirac mass terms, only when three condensates (118) have the same equilibrium phases, which are assumed to be $\vartheta_1^0 = \vartheta_2^0 = \vartheta_3^0 = 0$. This is possible only in the case of the attractive interband coupling $\epsilon < 0$.

3. If we use an isospinor $\Psi = \begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix}$, the coupling terms $\mathcal{U}_{\mathrm{D}} = \chi(\bar{\psi}_{\mathrm{L}}\Psi\psi_{\mathrm{R}} + \bar{\psi}_{\mathrm{R}}\Psi^{+}\psi_{\mathrm{L}})$ take the form of $\mathcal{U}_{\mathrm{D}} = \chi(\bar{\nu}_{\mathrm{L}}\varphi^{(1)}l_{\mathrm{R}} + \bar{l}_{\mathrm{R}}\varphi^{(1)+}\nu_{\mathrm{L}}) + \chi(\bar{l}_{\mathrm{L}}\varphi^{(2)}l_{\mathrm{R}} + \bar{l}_{\mathrm{R}}\varphi^{(2)+}l_{\mathrm{L}})$. We can see that we must suppose $\varphi^{(1)} = 0$ to avoid mixing of neutrinos with the charged leptons, and $\varphi^{(2)+} = \varphi^{(2)} > 0$. This corresponds to our selection of the vacuum as (125).

4. Neutrino masses are assumed to be zero $m_{\nu 1} = m_{\nu 2} = m_{\nu 3} = 0$. We postpone discussion of this issue until Section 7.

At the same time, except for the band states ψ_1, ψ_2, ψ_3 (*i.e.* the states, which interact with the corresponding isospinor fields), the flavor states $\psi_e, \psi_\mu, \psi_\tau$ (*i.e.* the states, which interact with normal oscillation modes of the coupled isospinor fields) must exist

$$\begin{aligned}\psi_{\mathrm{L}e} &= \begin{pmatrix} \nu_{\mathrm{L}e} \\ e_{\mathrm{L}} \end{pmatrix}, \qquad \psi_{\mathrm{R}e} = e_{\mathrm{R}}, \qquad \psi_{\mathrm{L}\mu} = \begin{pmatrix} \nu_{\mathrm{L}\mu} \\ \mu_{\mathrm{L}} \end{pmatrix}, \qquad \psi_{\mathrm{R}\mu} = \mu_{\mathrm{R}}, \\ \psi_{\mathrm{L}\tau} &= \begin{pmatrix} \nu_{\mathrm{L}\tau} \\ \tau_{\mathrm{L}} \end{pmatrix}, \qquad \psi_{\mathrm{R}\tau} = \tau_{\mathrm{R}}.\end{aligned} \tag{129}$$

The relationship between the band states and the flavor states (*i.e.* the lepton oscillations) has been discussed in Section 4 and will be considered again in Section 7.

Let us consider the movement of the phases $\vartheta_{1,2,3}$. The corresponding Lagrange equations for Lagrangian (126) are

$$\begin{aligned}
&\varphi_{01}^2\partial_\mu\partial^\mu\vartheta_1 - 2\varphi_{01}\varphi_{02}\epsilon\sin\frac{\vartheta_1-\vartheta_2}{2} - 2\varphi_{01}\varphi_{03}\epsilon\sin\frac{\vartheta_1-\vartheta_3}{2} = 0\,,\\
&\varphi_{02}^2\partial_\mu\partial^\mu\vartheta_2 + 2\varphi_{01}\varphi_{02}\epsilon\sin\frac{\vartheta_1-\vartheta_2}{2} - 2\varphi_{02}\varphi_{03}\epsilon\sin\frac{\vartheta_2-\vartheta_3}{2} = 0\,,\\
&\varphi_{03}^2\partial_\mu\partial^\mu\vartheta_3 + 2\varphi_{01}\varphi_{03}\epsilon\sin\frac{\vartheta_1-\vartheta_3}{2} + 2\varphi_{02}\varphi_{03}\epsilon\sin\frac{\vartheta_2-\vartheta_3}{2} = 0\,.
\end{aligned}\tag{130}$$

As we have seen above, the coupling terms $\mathcal{U}_\mathrm{D}$ in Eq. (128) can be reduced to the Dirac mass terms, only when three condensates $\langle\Psi_1\rangle, \langle\Psi_2\rangle, \langle\Psi_3\rangle$ have the same equilibrium phases $\vartheta_1^0 = \vartheta_2^0 = \vartheta_3^0 = 0$. This is only possible in the case of the attractive interband coupling $\epsilon < 0$. Considering small variations, *i.e.* $|\vartheta| \ll \pi$, we can linearize Eq. (130)

$$\begin{aligned}
&\varphi_{01}^2\partial_\mu\partial^\mu\vartheta_1 - \varphi_{01}\varphi_{02}\epsilon(\vartheta_1-\vartheta_2) - \varphi_{01}\varphi_{03}\epsilon(\vartheta_1-\vartheta_3) = 0\,,\\
&\varphi_{02}^2\partial_\mu\partial^\mu\vartheta_2 + \varphi_{01}\varphi_{02}\epsilon(\vartheta_1-\vartheta_2) - \varphi_{02}\varphi_{03}\epsilon(\vartheta_2-\vartheta_3) = 0\,,\\
&\varphi_{03}^2\partial_\mu\partial^\mu\vartheta_3 + \varphi_{01}\varphi_{03}\epsilon(\vartheta_1-\vartheta_3) + \varphi_{02}\varphi_{03}\epsilon(\vartheta_2-\vartheta_3) = 0\,,
\end{aligned}\tag{131}$$

which coincides with Eq. (51) for the phases $\theta_1, \theta_2, \theta_3$ when $\epsilon < 0$, *i.e.* all equilibrium phase differences are $\theta_{ij}^0 = 0 \Rightarrow \cos\theta_{ij}^0 = 1$. Therefore, the spectrum of Goldstone modes due to the spontaneous breaking of the SU(2) gauge symmetry in the three-band system with the interband coupling coincides with the spectrum (53) of Goldstone modes resulting from the spontaneous breaking of the U(1) gauge symmetry in the three-band system with the interband coupling.

Let us consider oscillations of φ_i only. Then, at $\epsilon < 0$ (*i.e.* equilibrium phase differences are $\vartheta_{ij}^0 = 0$), Lagrangian (126) takes the form

$$\begin{aligned}
\mathcal{L} \;=\; &\partial_\mu\varphi_1\partial^\mu\varphi_1 + \partial_\mu\varphi_2\partial^\mu\varphi_2 + \partial_\mu\varphi_3\partial^\mu\varphi_3\\
&-a_1\varphi_1^2 - \frac{b_1}{2}\varphi_1^4 - a_2\varphi_2^2 - \frac{b_2}{2}\varphi_2^4 - a_3\varphi_3^2 - \frac{b_3}{2}\varphi_3^4\\
&-2\epsilon\varphi_1\varphi_2 - 2\epsilon\varphi_1\varphi_3 - 2\epsilon\varphi_2\varphi_3\,,
\end{aligned}\tag{132}$$

which coincides with Lagrangian (41) for the fields $|\varphi_1|, |\varphi_2|, |\varphi_3|$ when $\epsilon < 0$. Therefore, the spectrum of Higgs modes due to the spontaneous breaking of the SU(2) gauge symmetry in the three-band system with the interband coupling coincides with the spectrum (66) of Higgs modes resulting from the spontaneous breaking of the U(1) gauge symmetry in the three-band system with the interband couplings.

Let us consider the interaction of the isospinor fields $\Psi_{1,2,3}$ (120), breaking the gauge SU(2) symmetry each, with the gauge Yang–Mills field $\vec{A}_\mu$. The corresponding gauge-invariant Lagrangian has the form

$$\begin{aligned}\mathcal{L} \;=\; & D_\mu\Psi_1\left(D^\mu\Psi_1\right)^+ + D_\mu\Psi_2\left(D^\mu\Psi_2\right)^+ + D_\mu\Psi_3\left(D^\mu\Psi_3\right)^+ \\ & -a_1\Psi_1\Psi_1^+ - a_2\Psi_2\Psi_2^+ - a_3\Psi_3\Psi_3^+ \\ & -\frac{b_1}{2}\left(\Psi_1\Psi_1^+\right)^2 - \frac{b_2}{2}\left(\Psi_2\Psi_2^+\right)^2 - \frac{b_3}{2}\left(\Psi_3\Psi_3^+\right)^2 \\ & -\epsilon\left(\Psi_1^+\Psi_2 + \Psi_1\Psi_2^+\right) - \epsilon\left(\Psi_1^+\Psi_3 + \Psi_1\Psi_3^+\right) - \epsilon\left(\Psi_2^+\Psi_3 + \Psi_2\Psi_3^+\right) \\ & -\frac{1}{16\pi}\vec{F}_{\mu\nu}\vec{F}^{\mu\nu}\,,\end{aligned} \tag{133}$$

where

$$D_\mu \equiv \tau_0\partial_\mu - ig\frac{\vec{\tau}}{2}\vec{A}_\mu \tag{134}$$

is the covariant derivation,

$$\vec{F}_{\mu\nu} = \partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu + g\left[\vec{A}_\mu \times \vec{A}_\nu\right] \tag{135}$$

is the tensor of the Yang–Mills field. Using Eq. (120), from which we can assume that $|\vartheta| \ll 1 \Rightarrow S = \tau_0 + i\frac{\vec{\tau}}{2}\vec{\vartheta}$, and using a property of the Pauli matrixes $-i\left[\frac{\vec{\tau}}{2}\cdot\vec{\vartheta}^\mu, \frac{\vec{\tau}}{2}\cdot\vec{A}^\mu\right] = \left[\vec{\vartheta}\times\vec{A}^\mu\right]\cdot\frac{\vec{\tau}}{2}$ [1], Lagrangian (133) can be rewritten in the following form:

$$\begin{aligned}\mathcal{L} \;=\; & \sum_{i=1}^{3}\begin{pmatrix} 0, & \varphi_i \end{pmatrix}\left(\tau_0\partial^\mu - i\frac{\vec{\tau}}{2}\partial^\mu\vec{\vartheta}_i + ig\frac{\vec{\tau}}{2}\vec{A}^\mu + ig\left[\frac{\vec{\tau}}{2}\times\vec{\vartheta}_i\right]\vec{A}^\mu\right) \\ & \times\left(\tau_0\partial_\mu + i\frac{\vec{\tau}}{2}\partial_\mu\vec{\vartheta}_i - ig\frac{\vec{\tau}}{2}\vec{A}_\mu - ig\left[\frac{\vec{\tau}}{2}\times\vec{\vartheta}_i\right]\vec{A}_\mu\right)\begin{pmatrix} 0 \\ \varphi_i \end{pmatrix} \\ & -a_1\varphi_1^2 - a_2\varphi_2^2 - a_3\varphi_3^2 - \frac{b_1}{2}\varphi_1^4 - \frac{b_2}{2}\varphi_2^4 - \frac{b_3}{2}\varphi_3^4 \\ & -2\epsilon\varphi_1\varphi_2\cos\frac{\vartheta_1-\vartheta_2}{2} - 2\epsilon\varphi_1\varphi_3\cos\frac{\vartheta_1-\vartheta_3}{2} - 2\epsilon\varphi_2\varphi_3\cos\frac{\vartheta_2-\vartheta_3}{2} \\ & -\frac{1}{16\pi}\vec{F}_{\mu\nu}\vec{F}^{\mu\nu}\,.\end{aligned} \tag{136}$$

The corresponding Lagrange equation

$$\partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\vec{A}_\mu)} - \frac{\partial\mathcal{L}}{\partial\vec{A}_\mu} = 0 \tag{137}$$

with the Yang–Mills equation $\partial_\nu\vec{F}^{\mu\nu} + g\left[\vec{A}_\nu\times\vec{F}^{\mu\nu}\right] = 4\pi\vec{J}^\mu$ gives the current

$$\vec{J}^\mu \;=\; \frac{g}{2}\sum_{i=1}^{3}\varphi_{0i}^2\left(\partial^\mu\vec{\vartheta}_i - g\vec{A}^\mu - g\left[\vec{\vartheta}_i\times\vec{A}^\mu\right]\right)\,. \tag{138}$$

The gauge field can be transformed as

$$\vec{A}'_\mu = \vec{A}_\mu - \alpha\left(\frac{1}{g}\partial_\mu\vec{\vartheta}_1 - \left[\vec{\vartheta}_1 \times \vec{A}_\mu\right]\right) - \beta\left(\frac{1}{g}\partial_\mu\vec{\vartheta}_2 - \left[\vec{\vartheta}_2 \times \vec{A}_\mu\right]\right) - \gamma\left(\frac{1}{g}\partial_\mu\vec{\vartheta}_3 - \left[\vec{\vartheta}_3 \times \vec{A}_\mu\right]\right), \tag{139}$$

where

$$\alpha = \frac{\varphi_{01}^2}{\varphi_{01}^2+\varphi_{02}^2+\varphi_{03}^2}, \qquad \beta = \frac{\varphi_{02}^2}{\varphi_{01}^2+\varphi_{02}^2+\varphi_{03}^2}, \qquad \gamma = \frac{\varphi_{03}^2}{\varphi_{01}^2+\varphi_{02}^2+\varphi_{03}^2}, \tag{140}$$

which are analogous to Eqs. (88) and (89). Then, neglecting the second order of smallness in the phase $\vartheta\partial_\mu\vartheta$, Eq. (138) can be reduced to the "London law"

$$\vec{J}^\mu = -\frac{g^2}{2}\left(\varphi_{01}^2+\varphi_{02}^2+\varphi_{03}^2\right)\vec{A}^\mu \equiv -\frac{1}{4\pi\lambda^2}\vec{A}^\mu, \tag{141}$$

where

$$\lambda = \frac{1}{\sqrt{2\pi g^2\left(\varphi_{01}^2+\varphi_{02}^2+\varphi_{03}^2\right)}} \tag{142}$$

is the "penetration depth" — the length of interaction mediated by the gauge bosons $\vec{A}_\mu$.

Applying the transformation (139), we can transform Lagrangian (133) to the following form:

$$\begin{aligned}
\mathcal{L} = {} & \partial_\mu\varphi_1\partial^\mu\varphi_1 + \partial_\mu\varphi_2\partial^\mu\varphi_2 + \partial_\mu\varphi_3\partial^\mu\varphi_3 + \frac{g^2}{4}\left(\varphi_1^2+\varphi_2^2+\varphi_3^2\right)\vec{A}_\mu\vec{A}^\mu \\
& -2\epsilon\varphi_1\varphi_2\cos\frac{\vartheta_1-\vartheta_2}{2} - 2\epsilon\varphi_1\varphi_3\cos\frac{\vartheta_1-\vartheta_3}{2} - 2\epsilon\varphi_2\varphi_3\cos\frac{\vartheta_2-\vartheta_3}{2} \\
& +\left(\varphi_1^2\beta^2+\varphi_2^2\alpha^2\right)\partial_\mu\frac{\vartheta_1-\vartheta_2}{2}\partial^\mu\frac{\vartheta_1-\vartheta_2}{2} \\
& +\left(\varphi_1^2\gamma^2+\varphi_3^2\alpha^2\right)\partial_\mu\frac{\vartheta_1-\vartheta_3}{2}\partial^\mu\frac{\vartheta_1-\vartheta_3}{2} \\
& +\left(\varphi_2^2\gamma^2+\varphi_3^2\beta^2\right)\partial_\mu\frac{\vartheta_2-\vartheta_3}{2}\partial^\mu\frac{\vartheta_2-\vartheta_3}{2} \\
& -\varphi_1^2 2\gamma\beta\partial_\mu\frac{\vartheta_1-\vartheta_2}{2}\partial^\mu\frac{\vartheta_1-\vartheta_3}{2} - \varphi_2^2 2\alpha\gamma\partial_\mu\frac{\vartheta_1-\vartheta_2}{2}\partial^\mu\frac{\vartheta_2-\vartheta_3}{2} \\
& -\varphi_3^2 2\alpha\beta\partial_\mu\frac{\vartheta_1-\vartheta_3}{2}\partial^\mu\frac{\vartheta_2-\vartheta_3}{2} \\
& -a_1\varphi_1^2 - a_2\varphi_2^2 - a_3\varphi_3^2 - \frac{b_1}{2}\varphi_1^4 - \frac{b_2}{2}\varphi_2^4 - \frac{b_3}{2}\varphi_3^4 - \frac{1}{16\pi}\vec{F}_{\mu\nu}\vec{F}^{\mu\nu},
\end{aligned} \tag{143}$$

which is analogous to the calibrated Lagrangian (94) with the spontaneous breaking of the U(1) symmetry. We can see that the phases $\vartheta_1, \vartheta_2, \vartheta_3$ have been excluded from the Lagrangian individually leaving only their differences: $\vartheta_1 - \vartheta_2, \vartheta_1 - \vartheta_3, \vartheta_2 - \vartheta_3$. Thus, the gauge field $\vec{A}_\mu$ absorbs the Goldstone boson (*i.e.* the common mode oscillations, where $\nabla\vartheta_1 = \nabla\vartheta_2 = \nabla\vartheta_3$) with the acoustic spectrum (55). At the same time, the Leggett bosons (*i.e.* the oscillations of the relative phases $\vartheta_i - \vartheta_j$) with massive spectrum (56), (57) "survive".

Substituting the calibrated Lagrangian (143) into Eq. (137), we obtain the equation for the field $\vec{A}_\mu$

$$\partial_\nu \vec{F}^{\nu\mu} + g\left[\vec{A}_\nu \times \vec{F}^{\nu\mu}\right] + \frac{1}{\lambda^2}\vec{A}^\mu = 0\,, \tag{144}$$

where

$$\frac{1}{\lambda^2} = 2\pi g^2\left(\varphi_{01}^2 + \varphi_{02}^2 + \varphi_{03}^2\right) \equiv m_A^2 \tag{145}$$

is the squared mass of the gauge boson $\vec{A}^\mu$, which is the squared reciprocal "penetration depth" (142) in the "London law" (141).

6. Spontaneous breaking of the $SU(2)_I \otimes U(1)_Y$ gauge symmetry in the three-band system with the Josephson couplings

It is not difficult to notice that the scalar product of the isospinors (118) $\Psi\Psi^+$ is invariant under both the SU(2) transition and the U(1) transition. Thus, we can write by analogy with Eq. (120)

$$\Psi_1(x) = \mathrm{e}^{i\theta_1(x)}\mathrm{e}^{i\frac{\vec{\tau}}{2}\vec{\vartheta}_1(x)}\begin{pmatrix} 0 \\ \varphi_1(x)\end{pmatrix}, \qquad \Psi_2(x) = \mathrm{e}^{i\theta_2(x)}\mathrm{e}^{i\frac{\vec{\tau}}{2}\vec{\vartheta}_2(x)}\begin{pmatrix} 0 \\ \varphi_2(x)\end{pmatrix},$$
$$\Psi_3(x) = \mathrm{e}^{i\theta_3(x)}\mathrm{e}^{i\frac{\vec{\tau}}{2}\vec{\vartheta}_3(x)}\begin{pmatrix} 0 \\ \varphi_3(x)\end{pmatrix}. \tag{146}$$

Lagrangian (121) is a sum of the gauge-invariant part (relative to the SU(2)⊗ U(1) gauge symmetry) and the Josephson terms. As in the previous case, the Josephson terms are not invariant relatively to this gauge symmetry due to the terms $\Psi_i^+\Psi_j + \Psi_j\Psi_i^+$, however these terms depend on the phase differences $\vartheta_i - \vartheta_j$ and $\theta_i - \theta_j$ only, but not on the single phases ϑ_i, θ_i, if the conditions (122), (125) are satisfied. Then, substituting representation (146) into Lagrangian (121) and considering (125), we obtain

$$
\begin{aligned}
\mathcal{L} = \sum_{i=1}^{3} \Big[& \partial_\mu \varphi_i \partial^\mu \varphi_i + \varphi_i^2 \left(\partial_\mu \frac{\vartheta_i}{2} \partial^\mu \frac{\vartheta_i}{2} + \partial_\mu \theta_i \partial^\mu \theta_i \right) - a_i \varphi_i^2 - \frac{b_i}{2} \varphi_i^4 \Big] \\
& -2\epsilon \varphi_1 \varphi_2 \left[\cos \frac{\vartheta_1 - \vartheta_2}{2} \cos(\theta_1 - \theta_2) + n_z \sin \frac{\vartheta_1 - \vartheta_2}{2} \sin(\theta_1 - \theta_2) \right] \\
& -2\epsilon \varphi_1 \varphi_3 \left[\cos \frac{\vartheta_1 - \vartheta_3}{2} \cos(\theta_1 - \theta_3) + n_z \sin \frac{\vartheta_1 - \vartheta_3}{2} \sin(\theta_1 - \theta_3) \right] \\
& -2\epsilon \varphi_2 \varphi_3 \left[\cos \frac{\vartheta_2 - \vartheta_3}{2} \cos(\theta_2 - \theta_3) + n_z \sin \frac{\vartheta_2 - \vartheta_3}{2} \sin(\theta_2 - \theta_3) \right] . \quad (147)
\end{aligned}
$$

Considering small variations of the phases from their equilibrium values $\vartheta_{0i} = \theta_{0i} = 0$, we can rewrite this Lagrangian in the form

$$
\begin{aligned}
\mathcal{L} = \sum_{i=1}^{3} \Big[& \partial_\mu \varphi_i \partial^\mu \varphi_i + \varphi_i^2 \left(\partial_\mu \frac{\vartheta_i}{2} \partial^\mu \frac{\vartheta_i}{2} + \partial_\mu \theta_i \partial^\mu \theta_i \right) - a_i \varphi_i^2 - \frac{b_i}{2} \varphi_i^4 \Big] \\
& -2\epsilon \varphi_1 \varphi_2 \left[1 - \frac{(\vartheta_1 - \vartheta_2)^2}{8} - \frac{(\theta_1 - \theta_2)^2}{2} + n_z \frac{\vartheta_1 - \vartheta_2}{2} (\theta_1 - \theta_2) \right] \\
& -2\epsilon \varphi_1 \varphi_3 \left[1 - \frac{(\vartheta_1 - \vartheta_3)^2}{8} - \frac{(\theta_1 - \theta_3)^2}{2} + n_z \frac{\vartheta_1 - \vartheta_3}{2} (\theta_1 - \theta_3) \right] \\
& -2\epsilon \varphi_2 \varphi_3 \left[1 - \frac{(\vartheta_2 - \vartheta_3)^2}{8} - \frac{(\theta_2 - \theta_3)^2}{2} + n_z \frac{\vartheta_2 - \vartheta_3}{2} (\theta_2 - \theta_3) \right] . \quad (148)
\end{aligned}
$$

We can see that the Goldstone modes corresponding to the U(1) gauge symmetry (oscillations of the phases $\theta_1, \theta_2, \theta_3$) are coupled with the Goldstone modes corresponding to the SU(2) symmetry (oscillations of the phases $\vartheta_1, \vartheta_2, \vartheta_3$) by the component $n_z = \pm 1$ of the unit vector $\vec{n} = \boldsymbol{k} n_z$ in the direction of the axis around which the rotation is made in isospace $\vec{\vartheta} = \vec{n}\vartheta$.

In the presence of the Abelian field B_μ, corresponding to the local gauge U(1) symmetry, and the non-Abelian field $\vec{A}_\mu$, corresponding to the local gauge SU(2) symmetry, we must apply the covariant derivative

$$D_\mu \equiv \tau_0 \partial_\mu - i\tau_0 \frac{f}{2} B_\mu - ig \frac{\vec{\tau}}{2} \vec{A}_\mu \,, \quad (149)$$

where f and g are corresponding coupling constants. Using the gauge transformations (88) and (139), Lagrangian (133) can be presented in the form

$$
\begin{aligned}
\mathcal{L} \;=\;& \partial_\mu\varphi_1\partial^\mu\varphi_1+\partial_\mu\varphi_2\partial^\mu\varphi_2+\partial_\mu\varphi_3\partial^\mu\varphi_3 \\
&+\left(\varphi_1^2\beta^2+\varphi_2^2\alpha^2\right)\left[\partial_\mu\frac{\vartheta_1-\vartheta_2}{2}\partial^\mu\frac{\vartheta_1-\vartheta_2}{2}+\partial_\mu\left(\theta_1-\theta_2\right)\partial^\mu\left(\theta_1-\theta_2\right)\right] \\
&+\left(\varphi_1^2\gamma^2+\varphi_3^2\alpha^2\right)\left[\partial_\mu\frac{\vartheta_1-\vartheta_3}{2}\partial^\mu\frac{\vartheta_1-\vartheta_3}{2}+\partial_\mu\left(\theta_1-\theta_3\right)\partial^\mu\left(\theta_1-\theta_3\right)\right] \\
&+\left(\varphi_2^2\gamma^2+\varphi_3^2\beta^2\right)\left[\partial_\mu\frac{\vartheta_2-\vartheta_3}{2}\partial^\mu\frac{\vartheta_3-\vartheta_3}{2}+\partial_\mu\left(\theta_2-\theta_3\right)\partial^\mu\left(\theta_2-\theta_3\right)\right] \\
&-\varphi_1^2 2\gamma\beta\left[\partial_\mu\frac{\vartheta_1-\vartheta_2}{2}\partial^\mu\frac{\vartheta_1-\vartheta_3}{2}+\partial_\mu(\theta_1-\theta_2)\partial^\mu(\theta_1-\theta_3)\right] \\
&-\varphi_2^2 2\alpha\gamma\left[\partial_\mu\frac{\vartheta_1-\vartheta_2}{2}\partial^\mu\frac{\vartheta_2-\vartheta_3}{2}+\partial_\mu(\theta_1-\theta_2)\partial^\mu(\theta_2-\theta_3)\right] \\
&-\varphi_3^2 2\alpha\beta\left[\partial_\mu\frac{\vartheta_1-\vartheta_3}{2}\partial^\mu\frac{\vartheta_2-\vartheta_3}{2}+\partial_\mu(\theta_1-\theta_3)\partial^\mu(\theta_2-\theta_3)\right] \\
&-2\epsilon\varphi_1\varphi_2\left[1-\frac{(\vartheta_1-\vartheta_2)^2}{8}-\frac{(\theta_1-\theta_2)^2}{2}+n_z\frac{\vartheta_1-\vartheta_2}{2}(\theta_1-\theta_2)\right] \\
&-2\epsilon\varphi_1\varphi_3\left[1-\frac{(\vartheta_1-\vartheta_3)^2}{8}-\frac{(\theta_1-\theta_3)^2}{2}+n_z\frac{\vartheta_1-\vartheta_3}{2}(\theta_1-\theta_3)\right] \\
&-2\epsilon\varphi_2\varphi_3\left[1-\frac{(\vartheta_2-\vartheta_3)^2}{8}-\frac{(\theta_2-\theta_3)^2}{2}+n_z\frac{\vartheta_2-\vartheta_3}{2}(\theta_2-\theta_3)\right] \\
&-a_1\varphi_1^2-a_2\varphi_2^2-a_3\varphi_3^2-\frac{b_1}{2}\varphi_1^4-\frac{b_2}{2}\varphi_2^4-\frac{b_3}{2}\varphi_3^4 \\
&+\frac{g^2}{4}\left(\varphi_1^2+\varphi_2^2+\varphi_3^2\right)\left(A_{x\mu}A_x^\mu+A_{y\mu}A_y^\mu\right) \\
&+\frac{1}{4}\left(\varphi_1^2+\varphi_2^2+\varphi_3^2\right)\left(g^2A_{z\mu}A_z^\mu+f^2B_\mu B^\mu\right) \\
&-\frac{1}{16\pi}\vec{F}_{\mu\nu}\vec{F}^{\mu\nu}-\frac{1}{16\pi}G_{\mu\nu}G^{\mu\nu},
\end{aligned}
\tag{150}
$$

where

$$G_{\mu\nu}=\partial_\mu B_\nu-\partial_\nu B_\mu \tag{151}$$

is the field tensor for the Abelian gauge field B_μ. In the GWS theory, the linear combinations

$$W_\mu \;=\; \frac{1}{\sqrt{2}}\left(A_{x\mu}+iA_{y\mu}\right), \tag{152}$$

$$
\begin{array}{l} Z_\mu=A_{z\mu}\cos\alpha-B_\mu\sin\alpha \\ A_\mu=A_{z\mu}\sin\alpha+B_\mu\cos\alpha \end{array}
\;\Rightarrow\;
\begin{array}{l} A_{z\mu}=Z_\mu\cos\alpha+A_\mu\sin\alpha \\ B_\mu=-Z_\mu\sin\alpha+A_\mu\cos\alpha \end{array}, \tag{153}
$$

where

$$\cos\alpha = \frac{g}{\widetilde{g}}\,, \qquad \sin\alpha = \frac{f}{\widetilde{g}}\,, \qquad \widetilde{g} = \sqrt{g^2+f^2}\,, \tag{154}$$

allow us to make the transformation

$$\frac{g^2}{4}\left(A_{x\mu}A_x^{\mu}+A_{y\mu}A_y^{\mu}\right)+\frac{1}{4}\left(g^2A_{z\mu}A_z^{\mu}+f^2B_{\mu}B^{\mu}\right)=\frac{g^2}{2}W_{\mu}W^{*\mu}+\frac{\widetilde{g}^2}{4}Z_{\mu}Z^{\mu}\,. \tag{155}$$

Thus, the masses of charged W-boson and neutral Z-boson are

$$m_W = g\sqrt{2\pi\left(\varphi_{01}^2+\varphi_{02}^2+\varphi_{03}^2\right)}\,, \quad m_Z=\widetilde{g}\sqrt{2\pi\left(\varphi_{01}^2+\varphi_{02}^2+\varphi_{03}^2\right)}=\frac{m_W}{\cos\alpha}\,, \tag{156}$$

but the field A_μ (photon) remains massless (with the interaction constant — electrical charge $e = g\sin\alpha$). However, separation of the components $A_{x\mu}, A_{y\mu}$ from the component $A_{z\mu}$ (which is mixed with the Abelian field B_μ) takes place only in the London gauge, where we exclude the single phases θ, ϑ from Lagrangian, see Eq. (150). Then, let us consider the gauge transformation (139)

$$\vec{A}'_\mu = \vec{A}_\mu - \frac{1}{g}\partial_\mu\vec{\vartheta}+\left[\vec{\vartheta}\times\vec{A}_\mu\right] = \vec{A}_\mu - \frac{1}{g}\partial_\mu\vec{\vartheta}+\vartheta\cdot\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ n_x & n_y & n_z \\ A_{x\mu} & A_{y\mu} & A_{z\mu}\end{vmatrix}\,. \tag{157}$$

We can see that the gauge transformation mixes the components $A_{x\mu}$ and $A_{y\mu}$ with the component $A_{z\mu}$. On the other hand, separating the field W_μ (152) from the component $A_{z\mu}$ makes physical sense if and only if the fields $W_\mu = (A_{\mu 1} + iA_{\mu 2})/\sqrt{2}$ and $A_{z\mu}$ are transformed by themselves. From Eq. (157), we can see that it is possible only when $n_x = n_y = 0$

$$\begin{array}{l} A'_{x\mu} = A_{x\mu} + \vartheta n_z A_{y\mu} \\ A'_{y\mu} = A_{y\mu} - \vartheta n_z A_{x\mu} \end{array}\,, \tag{158}$$

$$A'_{z\mu} = A_{z\mu} - n_z\tfrac{1}{g}\partial_\mu\vartheta\,. \tag{159}$$

Therefore, $n_z = \pm 1$ which coincides with Eq. (125) as the condition for interference of condensates Ψ_1, Ψ_2, Ψ_3. Thus, separating of the field W_μ also selects a direction in isospace. The spectrum of excitations depends only on n_z^2, n_z^4, so the sign of n_z is not important.

As before, we must assume that $\epsilon < 0$, hence the equilibrium phases are such that $\cos\theta_{ij} = \cos\vartheta_{ij} = 1$. Then, from Lagrangian (147), we can see that the spectrum of Higgs oscillations coincides with the spectrum (66). At the same time, the spectrum of the Goldstone modes takes the form

$$(q_\mu q^\mu)^4\left((q_\mu q^\mu)^2+(q_\mu q^\mu)\,b+c\right)=0\,, \tag{160}$$

where

$$
\begin{aligned}
b &= 2\epsilon\frac{\varphi_{01}^2(\varphi_{02}+\varphi_{03})+\varphi_{02}^2(\varphi_{01}+\varphi_{03})+\varphi_{03}^2(\varphi_{01}+\varphi_{02})}{\varphi_{01}\varphi_{02}\varphi_{03}}\,,\\
c &= 4\epsilon^2\frac{\varphi_{01}^3+\varphi_{02}^3+\varphi_{03}^3+\varphi_{01}^2(\varphi_{02}+\varphi_{03})+\varphi_{02}^2(\varphi_{01}+\varphi_{03})+\varphi_{03}^2(\varphi_{01}+\varphi_{02})}{\varphi_{01}\varphi_{02}\varphi_{03}}\,.
\end{aligned}
\tag{161}
$$

From Eq. (160) we can see that one of the dispersion relations is $q_\mu q^\mu = 0$. This relation corresponds to the twofold degenerated common mode oscillations, which are absorbed by the gauge fields W_μ, W_μ^*, Z_μ and to the twofold degenerated massless Leggett mode. The remaining quadratic equation determines two Leggett modes with massive spectra: $m_{\mathrm{L}1,2}^2 = q_\mu q^\mu = \frac{1}{2}(-b \mp \sqrt{b^2-4c})$. As we could see above, the L-bosons are not absorbed by the gauge fields. Thus, if all bands are independent, *i.e.* $\epsilon = 0$, then we have two massless Goldstone modes per band (independent oscillations of the phase ϑ and θ), a total of six independent Goldstone modes. Due to the internal proximity effect, *i.e.* $\epsilon \neq 0$, the Goldstone modes from each band transform to the following normal oscillations for all bands: twofold degenerated common mode oscillations with the acoustic spectrum, the twofold degenerated massless Leggett mode, and two Leggett modes with the energy gaps. Squared masses of the L-bosons are proportional to the interband coupling $m_{\mathrm{L}1,2}^2 \sim |\epsilon|$. For the symmetrical three-band system, *i.e.* $\varphi_{01} = \varphi_{02} = \varphi_{03}$, masses of both massive L-bosons are equal

$$
m_{\mathrm{L}1} = m_{\mathrm{L}2} = \sqrt{6|\epsilon|}\,. \tag{162}
$$

Thus, we can see that, unlike the cases of U(1) and SU(2) symmetries, for the case of the SU(2) ⊗ U(1) symmetry, we have two massless L-bosons and two massive L-bosons. However, the massless bosons, like relic photons, lose their energy in the process of space expansion. Hence, the role of these bosons can be neglected. In contrast to them, the massive L-bosons are able to form stable gravitationally bound structures (clusters, halo). Moreover, the L-bosons are sterile. Therefore, the massive L-bosons are a suitable candidate for Dark Matter.

It should be noted that if we suppose the nonsymmetrical Josephson coupling $\epsilon_{12} \neq \epsilon_{13} \neq \epsilon_{23}$ instead of the uniform coefficient ϵ, then the twofold degenerated massless Leggett mode splits into one massless mode and one massive mode. However, in what follows, we will consider only the minimal model with the uniform coefficient ϵ.

7. Lepton mixing and the mass states of neutrinos

From Eq. (128) we can see that the band states of the Dirac fields, *i.e.* ψ_1, ψ_2, ψ_3, are determined by the coupling between the corresponding Dirac field ψ_j and the scalar field φ_j (isospinor field $\varPsi_j$). Then, the gauge-invariant Dirac Lagrangian for the lepton fields has the form

$$L_{\rm D} = \sum_{j=1}^{3} i\bar{\psi}_{{\rm L}j}\gamma^{\mu}\overset{\leftrightarrow \vec{A},B}{D_{\mu}}\psi_{{\rm L}j} + i\bar{\psi}_{{\rm R}j}\gamma^{\mu}\overset{\leftrightarrow B}{D_{\mu}}\psi_{{\rm R}j} - \chi\left[\bar{\psi}_{{\rm L}j}\varPsi_j\psi_{{\rm R}j} + \bar{\psi}_{{\rm R}j}\varPsi_j^{+}\psi_{{\rm L}j}\right], \tag{163}$$

where

$$D_{\mu}^{\vec{A},B} \equiv \tau_0\partial_{\mu} - ig\frac{\vec{\tau}}{2}\vec{A}_{\mu} + i\tau_0\frac{f}{2}B_{\mu}, \qquad D_{\mu}^{B} \equiv \tau_0\partial_{\mu} + i\tau_0 f B_{\mu} \tag{164}$$

are covariant derivations. Thus, each band state ψ_1, ψ_2, ψ_3, emitting or absorbing the gauge bosons $\vec{A}_{\mu}, B_{\mu}$, transforms only to itself, *i.e.* $1 \leftrightarrow 1$, $2 \leftrightarrow 2$, $3 \leftrightarrow 3$. Analogously, for the flavor states $\psi_e, \psi_{\mu}, \psi_{\tau}$ (129),

$$L_{\rm D} = i\bar{\psi}_{{\rm L}e}\gamma^{\mu}\overset{\leftrightarrow \vec{A},B}{D_{\mu}}\psi_{{\rm L}e} + i\bar{e}_{\rm R}\gamma^{\mu}\overset{\leftrightarrow B}{D_{\mu}}e_{\rm R} - \chi\left[\bar{\psi}_{\rm L}\varPsi_e e_{\rm R} + \bar{e}_{\rm R}\varPsi_e^{+}\psi_{\rm L}\right] + L_{\mu} + L_{\tau}. \tag{165}$$

We can conditionally use e for the electron e and the electron neutrino ν_e, μ for the muon μ and the muon neutrino ν_{μ}, τ for the tauon τ and the tauon neutrino ν_{τ}. If $m_e \ll m_{\mu} \ll m_{\tau}$, then the bands should be strongly asymmetrical: $\varphi_{01} \ll \varphi_{02} \ll \varphi_{03}$. As for the band states, each flavor state $\psi_e, \psi_{\mu}, \psi_{\tau}$ emitting or absorbing the gauge bosons $\vec{A}_{\mu}, B_{\mu}$, transforms only to itself, *i.e.* $e \leftrightarrow e$, $\mu \leftrightarrow \mu$, $\tau \leftrightarrow \tau$.

As we could see in Section 4 each of "L" and "R" components should mix with the corresponding "R" and "L" components of other flavors $\bar{\psi}_{{\rm L}i}\psi_{{\rm R}k} + \bar{\psi}_{{\rm R}k}\psi_{{\rm L}i}$, which is the fermionic analog of the interband Josephson coupling, unlike SM, where the mixing coefficients are off-diagonal Yukawa interactions. Thus, we can take the SU(2)-symmetric mixing term for the Dirac fields in the following form:

$$\mathcal{U}_{\rm mix} = \bar{\psi}_{{\rm L}e}\begin{pmatrix} 0 \\ \zeta_{e\mu} \end{pmatrix}\psi_{{\rm R}\mu} + \bar{\psi}_{{\rm L}e}\begin{pmatrix} 0 \\ \zeta_{e\tau} \end{pmatrix}\psi_{{\rm R}\tau} + \bar{\psi}_{{\rm L}\mu}\begin{pmatrix} 0 \\ \zeta_{\mu\tau} \end{pmatrix}\psi_{{\rm R}\tau} + {\rm h.c.}, \tag{166}$$

where ζ_{ik} are mixing parameters determined by the interband coupling ϵ of the scalar fields, see Eq. (113), $\psi_{{\rm L}e} = \begin{pmatrix} \nu_{e_{\rm L}} \\ e_{\rm L} \end{pmatrix}$ is the left-handed bispinor, $\psi_{{\rm R}e} = e_{\rm R}$ is the right-handed spinor. The band masses $m_{\rm D1}, m_{\rm D2}, m_{\rm D3}$ are determined by the Yukawa interaction of the Dirac fields with the corresponding band states of scalar fields $\varphi_{01}, \varphi_{02}, \varphi_{03}$. In turn, the flavor masses

$m_{\mathrm{D}e}, m_{\mathrm{D}\mu}, m_{\mathrm{D}\tau}$ are determined by the Yukawa interaction of the Dirac fields with the corresponding flavor states of scalar fields $\varphi_{0e}, \varphi_{0\mu}, \varphi_{0\tau}$ which are result of the diagonalization (73). The mixing R $\leftrightarrow$ L results in the transition from the flavor masses to the band masses via diagonalization of the matrix $M_{e\mu\tau}$ as demonstrated in Section 4

$$\mathcal{U}_{\mathrm{D}e\mu\tau}+\mathcal{U}_{\mathrm{mix}} = \left(\,\bar{e}_{\mathrm{L}},\quad \bar{\mu}_{\mathrm{L}},\quad \bar{\tau}_{\mathrm{L}}\,\right)\begin{pmatrix} m_{\mathrm{D}e} & \zeta_{e\mu} & \zeta_{e\tau} \\ \zeta_{e\mu} & m_{\mathrm{D}\mu} & \zeta_{\mu\tau} \\ \zeta_{e\tau} & \zeta_{\mu\tau} & m_{\mathrm{D}\tau} \end{pmatrix}\begin{pmatrix} e_{\mathrm{R}} \\ \mu_{\mathrm{R}} \\ \tau_{\mathrm{R}} \end{pmatrix} + \mathrm{h.c.}$$
$$= \left(\,\bar{l}_{1\mathrm{L}},\quad \bar{l}_{2\mathrm{L}},\quad \bar{l}_{3\mathrm{L}}\,\right)\begin{pmatrix} m_{\mathrm{D}1} & 0 & 0 \\ 0 & m_{\mathrm{D}2} & 0 \\ 0 & 0 & m_{\mathrm{D}3} \end{pmatrix}\begin{pmatrix} l_{1\mathrm{R}} \\ l_{2\mathrm{R}} \\ l_{2\mathrm{R}} \end{pmatrix} + \mathrm{h.c.} \tag{167}$$

The mixing takes place due to the interband Josephson coupling of the scalar fields $\varphi_1, \varphi_2, \varphi_3$: from Eq. (113) we can see that $\zeta_{\alpha\beta} \propto \epsilon^2$. As will be demonstrated in Section 9, the interband coupling is extremely small $\epsilon \sim 10^{-40}$ eV^2. Taking masses of H-bosons as $\Delta m_H^2 \sim 10^2$ GeV^2 (see Section 8), we can see that the mixing angles, determined by Eqs. (113), (115), are extremely small: $\alpha_{ij} \sim 10^{-100}$. Probability of interflavor transition is $P_{ik} \sim \sin^2(2\alpha_{ik})$ [36–41], hence, for the massive leptons (electrons, muons, tauons), the effect of mixing is negligible $P \sim 10^{-200}$. Thus, the mixing of charged leptons is negligible and it lies beyond the sensitivity of any experiment.

In SM, masses of neutrinos are zero. However, observation of the neutrino oscillations in vacuum means the presence of mass of neutrinos [36–41], and the differences in the squares of the masses have been measured: $|\Delta m_{23}^2| \equiv |m_3^2 - m_2^2| \approx 2.51 \times 10^{-3}$ eV^2, $|\Delta m_{12}^2| \approx 7.41 \times 10^{-5}$ eV^2 [42, 43]. Formally, we can write the Dirac mass term (Yukawa interaction) for both the charged lepton and the neutrino in the form (1), assigning neutrinos a small but non-zero Yukawa constant χ_ν and introducing the sterile right-handed neutrino. Thus, the neutrino mass becomes similar to the mass of charged leptons. However, in the proposed three-band model, the problem of mass is fundamental. As we have seen, the interaction of Dirac fields with the corresponding scalar fields leads to lepton oscillations as a consequence of the Josephson coupling between scalar fields. Hence, the mixing angles are extremely small: $\alpha_{ij} \sim 10^{-100}$. At the same time, the experimental mixing angles for neutrinos are large: $\alpha_{12} = 33.4°$, $\alpha_{23} = 42.2 \ldots 49.5°$, $\alpha_{13} = 8.6°$ [42, 43].

However, within the framework of the three-band model, the presence of mixing alone, without interaction with scalar fields, can lead to mass generation. Let us suppose the existence of massless *sterile* right-handed neutrinos $\nu_{\mathrm{R}e}, \nu_{\mathrm{R}\mu}, \nu_{\mathrm{R}\tau}$, *i.e.* which are characterized by zero isospin and hypercharge: $I_z = 0, Y = 0$, unlike the *active* left-handed neutrinos $\nu_{\mathrm{L}e}, \nu_{\mathrm{L}\mu}, \nu_{\mathrm{L}\tau}$ which are characterized by $I_z = \frac{1}{2}, Y = -1$. Then, the SU(2)-symmetric mixing term for neutrinos should have the following form:

$$\mathcal{U}^{\nu}_{\rm mix} = \bar{\psi}_{\rm Le} \begin{pmatrix} \varsigma_{e\mu} \\ 0 \end{pmatrix} \nu_{{\rm R}\mu} + \bar{\psi}_{\rm Le} \begin{pmatrix} \varsigma_{e\tau} \\ 0 \end{pmatrix} \nu_{{\rm R}\tau} + \bar{\psi}_{{\rm L}\mu} \begin{pmatrix} \varsigma_{\mu\tau} \\ 0 \end{pmatrix} \nu_{{\rm R}\tau} + \text{h.c.}\,, \quad (168)$$

where $\varsigma_{ik} \neq \zeta_{ik}$ are mixing parameters specially for neutrinos, $\psi_{\rm Le} = \begin{pmatrix} \nu_{e_{\rm L}} \\ e_{\rm L} \end{pmatrix}$ is the left-handed bispinor, $\nu_{\rm Re}$ is the right-handed neutrino. Then, the corresponding neutrino Lagrangian has the form

$$\begin{aligned}\mathcal{L}_\nu = i &\left(\bar{\nu}_{\rm Le}\gamma^\sigma \overset{\leftrightarrow}{\partial}_\sigma \nu_{\rm Le} + \bar{\nu}_{{\rm L}\mu}\gamma^\sigma \overset{\leftrightarrow}{\partial}_\sigma \nu_{{\rm L}\mu} + \bar{\nu}_{{\rm L}\tau}\gamma^\sigma \overset{\leftrightarrow}{\partial}_\sigma \nu_{{\rm L}\tau} \right) \\ &- \varsigma_{e\mu}(\bar{\nu}_{\rm Le}\nu_{{\rm R}\mu} + \bar{\nu}_{{\rm L}\mu}\nu_{\rm Re}) - \varsigma_{e\tau}(\bar{\nu}_{\rm Le}\nu_{{\rm R}\tau} + \bar{\nu}_{{\rm L}\tau}\nu_{\rm Re}) - \varsigma_{\mu\tau}(\bar{\nu}_{{\rm L}\mu}\nu_{{\rm R}\tau} + \bar{\nu}_{{\rm L}\tau}\nu_{{\rm R}\mu}) \\ &+ i \left(\bar{\nu}_{\rm Re}\gamma^\sigma \overset{\leftrightarrow}{\partial}_\sigma \nu_{\rm Re} + \bar{\nu}_{{\rm R}\mu}\gamma^\sigma \overset{\leftrightarrow}{\partial}_\sigma \nu_{{\rm R}\mu} + \bar{\nu}_{{\rm R}\tau}\gamma^\sigma \overset{\leftrightarrow}{\partial}_\sigma \nu_{{\rm R}\tau} \right) \\ &- \varsigma_{e\mu}(\bar{\nu}_{\rm Re}\nu_{{\rm L}\mu} + \bar{\nu}_{{\rm R}\mu}\nu_{\rm Le}) - \varsigma_{e\tau}(\bar{\nu}_{\rm Re}\nu_{{\rm L}\tau} + \bar{\nu}_{{\rm R}\tau}\nu_{\rm Le}) - \varsigma_{\mu\tau}(\bar{\nu}_{{\rm R}\mu}\nu_{{\rm L}\tau} + \bar{\nu}_{{\rm R}\tau}\nu_{{\rm L}\mu})\,.\end{aligned} \quad (169)$$

We can diagonalize the matrix $M_{e\mu\tau}$ as

$$\begin{aligned}&\begin{pmatrix} \bar{\nu}_{\rm Le}, & \bar{\nu}_{{\rm L}\mu}, & \bar{\nu}_{{\rm L}\tau} \end{pmatrix} \begin{pmatrix} 0 & \varsigma_{e\mu} & \varsigma_{e\tau} \\ \varsigma_{e\mu} & 0 & \varsigma_{\mu\tau} \\ \varsigma_{e\tau} & \varsigma_{\mu\tau} & 0 \end{pmatrix} \begin{pmatrix} \nu_{\rm Re} \\ \nu_{{\rm R}\mu} \\ \nu_{{\rm R}\tau} \end{pmatrix} + \text{h.c.} \\ &= \begin{pmatrix} \bar{\nu}_{\rm 1L}, & \bar{\nu}_{\rm 2L}, & \bar{\nu}_{\rm 3L} \end{pmatrix} \begin{pmatrix} m_{\nu 1} & 0 & 0 \\ 0 & m_{\nu 2} & 0 \\ 0 & 0 & m_{\nu 3} \end{pmatrix} \begin{pmatrix} \nu_{\rm 1R} \\ \nu_{\rm 2R} \\ \nu_{\rm 2R} \end{pmatrix} + \text{h.c.}\end{aligned} \quad (170)$$

The corresponding characteristic equation is

$$m_\nu^3 - \left(\varsigma_{e\mu}^2 + \varsigma_{e\tau}^2 + \varsigma_{\mu\tau}^2\right) m_\nu - 2\varsigma_{e\mu}\varsigma_{e\tau}\varsigma_{\mu\tau} = 0\,. \quad (171)$$

For the symmetrical interband mixing $\varsigma_{e\mu} = \varsigma_{e\tau} = \varsigma_{\mu\tau} \equiv \varsigma$, we obtain the following solutions of Eq. (171):

$$m_{\nu 1} = m_{\nu 2} = -\varsigma\,, \qquad m_{\nu 3} = 2\varsigma\,. \quad (172)$$

Obviously, the right-handed (sterile) neutrinos and left-handed (active) neutrinos have exactly the same masses: $m_{\nu {\rm R}i} = m_{\nu {\rm L}i}$. We can see that neutrino masses can take both the positive and the negative magnitudes. This means that the mass states of neutrino ν_1, ν_2, ν_3 are quasiparticles (unlike the band state of charged leptons l_1, l_2, l_3, which are determined by the Yukawa coupling with the scalar fields of the corresponding bands). Respectively, the masses (172) are effective masses of the quasiparticles. Only the square root of the squares of masses $\sqrt{m_{\nu 1}^2}, \sqrt{m_{\nu 2}^2}, \sqrt{m_{\nu 3}^2}$ makes physical sense since only the differences $|\Delta m_{23}^2| \equiv |m_3^2 - m_2^2|$, $|\Delta m_{12}^2| \equiv |m_2^2 - m_1^2|$ are measured in experiments regarding neutrino oscillations, and the upper limits of the masses $\sqrt{m_{\nu e}^2}, \sqrt{m_{\nu\mu}^2}, \sqrt{m_{\nu\tau}^2}$ have been determined experimentally from the β-decay of tritium, pion decay, τ-decays into multi-pion final states, respectively, where the spectral distribution of leptons is determined by m_ν^2, but not by m_ν. Moreover, $m_{\nu\alpha} = \sqrt{\sum_{i=1}^3 |U_{\alpha i}|^2 m_{\nu i}^2}$, where $U_{\alpha i}$ is the PMNS matrix [38, 44, 45].

In view of the above, we should consider equations that include only the squares of the effective masses. The Lagrange equations for Lagrangian (169) are

$$
\begin{aligned}
i\gamma^\sigma \partial_\sigma \nu_{\mathrm{L}e} - \varsigma_{e\mu}\nu_{\mathrm{R}\mu} - \varsigma_{e\tau}\nu_{\mathrm{R}\tau} &= 0\,,\\
i\gamma^\sigma \partial_\sigma \nu_{\mathrm{L}\mu} - \varsigma_{e\mu}\nu_{\mathrm{R}e} - \varsigma_{\mu\tau}\nu_{\mathrm{R}\tau} &= 0\,,\\
i\gamma^\sigma \partial_\sigma \nu_{\mathrm{L}\tau} - \varsigma_{e\tau}\nu_{\mathrm{R}e} - \varsigma_{\mu\tau}\nu_{\mathrm{R}\mu} &= 0\,,\\
i\gamma^\sigma \partial_\sigma \nu_{\mathrm{R}e} - \varsigma_{e\mu}\nu_{\mathrm{L}\mu} - \varsigma_{e\tau}\nu_{\mathrm{L}\tau} &= 0\,,\\
i\gamma^\sigma \partial_\sigma \nu_{\mathrm{R}\mu} - \varsigma_{e\mu}\nu_{\mathrm{L}e} - \varsigma_{\mu\tau}\nu_{\mathrm{L}\tau} &= 0\,,\\
i\gamma^\sigma \partial_\sigma \nu_{\mathrm{R}\tau} - \varsigma_{e\tau}\nu_{\mathrm{L}e} - \varsigma_{\mu\tau}\nu_{\mathrm{L}\mu} &= 0\,.
\end{aligned}
\tag{173}
$$

Then Eq. (173) can be transformed to the system of the Klein–Gordon-like equations for the left-handed fields separately

$$
\begin{aligned}
\partial^\sigma\partial_\sigma \nu_{\mathrm{L}e} + \left(\varsigma_{e\mu}^2 + \varsigma_{e\tau}^2\right)\nu_{\mathrm{L}e} + \varsigma_{e\tau}\varsigma_{\mu\tau}\nu_{\mathrm{L}\mu} + \varsigma_{e\mu}\varsigma_{\mu\tau}\nu_{\mathrm{L}\tau} &= 0\,,\\
\partial^\sigma\partial_\sigma \nu_{\mathrm{L}\mu} + \varsigma_{e\tau}\varsigma_{\mu\tau}\nu_{\mathrm{L}e} + \left(\varsigma_{e\mu}^2 + \varsigma_{\mu\tau}^2\right)\nu_{\mathrm{L}\mu} + \varsigma_{e\mu}\varsigma_{e\tau}\nu_{\mathrm{L}\tau} &= 0\,,\\
\partial^\sigma\partial_\sigma \nu_{\mathrm{L}\tau} + \varsigma_{e\mu}\varsigma_{\mu\tau}\nu_{\mathrm{L}e} + \varsigma_{e\tau}\varsigma_{e\mu}\nu_{\mathrm{L}\mu} + \left(\varsigma_{e\tau}^2 + \varsigma_{\mu\tau}^2\right)\nu_{\mathrm{L}\tau} &= 0\,,
\end{aligned}
\tag{174}
$$

where we have used $(\gamma^\nu\partial_\nu)(\gamma^\mu\partial_\mu) = \frac{1}{2}\partial_\mu\partial_\nu(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = \frac{1}{2}\partial_\mu\partial_\nu 2g^{\mu\nu} = \partial_\mu\partial^\mu$. Thus, we obtain Lorentz-covariant equations of motion only for the left-handed neutrinos $\nu_{\mathrm{L}e}, \nu_{\mathrm{L}\mu}, \nu_{\mathrm{L}\tau}$. Analogously, we can obtain such equations for the right-handed fields.

Let us consider the spinors $\nu_{\mathrm{L}e,\mu,\tau}$ in the form of plane waves $\nu_{e,\mu,\tau} = u_{e,\mu,\tau}\,\mathrm{e}^{-ip_\sigma x^\sigma}$, where $u_{e,\mu,\tau}$ are the corresponding spinor amplitudes. Then Eq. (174) takes the form

$$
\begin{aligned}
\left(\varsigma_{e\mu}^2 + \varsigma_{e\tau}^2 - p^\sigma p_\sigma\right)u_e + \varsigma_{e\tau}\varsigma_{\mu\tau}u_\mu + \varsigma_{e\mu}\varsigma_{\mu\tau}u_\tau &= 0\,,\\
\varsigma_{e\tau}\varsigma_{\mu\tau}u_e + \left(\varsigma_{e\mu}^2 + \varsigma_{\mu\tau}^2 - p^\sigma p_\sigma\right)u_\mu + \varsigma_{e\mu}\varsigma_{e\tau}u_\tau &= 0\,,\\
\varsigma_{e\mu}\varsigma_{\mu\tau}u_e + \varsigma_{e\tau}\varsigma_{e\mu}u_\mu + \left(\varsigma_{e\tau}^2 + \varsigma_{\mu\tau}^2 - p^\sigma p_\sigma\right)u_\tau &= 0\,.
\end{aligned}
\tag{175}
$$

The corresponding characteristic equation is

$$\begin{aligned}&(p^\sigma p_\sigma)^3 - \left(2\varsigma_{e\mu}^2 + 2\varsigma_{e\tau}^2 + 2\varsigma_{\mu\tau}^2\right)(p^\sigma p_\sigma)^2 \\ &+ \left(\varsigma_{e\mu}^4 + \varsigma_{e\tau}^4 + \varsigma_{\mu\tau}^4 + 2\varsigma_{e\mu}^2\varsigma_{e\tau}^2 + 2\varsigma_{e\mu}^2\varsigma_{\mu\tau}^2 + 2\varsigma_{e\tau}^2\varsigma_{\mu\tau}^2\right)(p^\sigma p_\sigma) - 4\varsigma_{e\mu}^2\varsigma_{e\tau}^2\varsigma_{\mu\tau}^2 = 0\,.\end{aligned} \tag{176}$$

This equation has three positive real solutions: $m_{\nu 1}^2 = (p^\sigma p_\sigma)_1$, $m_{\nu 2}^2 = (p^\sigma p_\sigma)_2$, $m_{\nu 3}^2 = (p^\sigma p_\sigma)_3$, which can be associated with the mass states of neutrinos $\nu_{\mathrm{L}1}, \nu_{\mathrm{L}2}, \nu_{\mathrm{L}3}$

$$\begin{aligned}\partial^\sigma\partial_\sigma\nu_{\mathrm{L}1} + m_{\nu 1}^2\nu_{\mathrm{L}1} &= 0\,,\\ \partial^\sigma\partial_\sigma\nu_{\mathrm{L}2} + m_{\nu 2}^2\nu_{\mathrm{L}2} &= 0\,,\\ \partial^\sigma\partial_\sigma\nu_{\mathrm{L}3} + m_{\nu 3}^2\nu_{\mathrm{L}3} &= 0\,,\end{aligned} \tag{177}$$

and we can assume the hierarchy of the masses as $m_{\nu 1}^2 \le m_{\nu 2}^2 \le m_{\nu 3}^2$. Thus, for the symmetrical interband mixing $\varsigma_{e\mu} = \varsigma_{e\tau} = \varsigma_{\mu\tau} \equiv \varsigma$, we obtain the following solutions of Eq. (176):

$$m_{\nu 1}^2 = m_{\nu 2}^2 = \varsigma^2\,, \qquad m_{\nu 3}^2 = 4\varsigma^2\,. \tag{178}$$

Thus, the effective masses of neutrinos are of order of the interband mixing parameters. It should be noted that the masses $m_{\nu 1}, m_{\nu 2}, m_{\nu 3}$ are the result of interband mixing, unlike the electron–muon–tauon masses, which are the result of coupling with the corresponding scalar fields $\varphi_{e,\mu,\tau}$. Obviously, the flavor states $\nu_{\mathrm{L}e}, \nu_{\mathrm{L}\mu}, \nu_{\mathrm{L}\tau}$ must be linear combinations of the mass states $\nu_{\mathrm{L}1}, \nu_{\mathrm{L}2}, \nu_{\mathrm{L}3}$ and *vice versa*, that can be written in the following way:

$$\begin{pmatrix}\nu_{\mathrm{L}e}\\ \nu_{\mathrm{L}\mu}\\ \nu_{\mathrm{L}\tau}\end{pmatrix} = U\cdot\begin{pmatrix}\nu_{\mathrm{L}1}\\ \nu_{\mathrm{L}2}\\ \nu_{\mathrm{L}3}\end{pmatrix}, \qquad \begin{pmatrix}\nu_{\mathrm{L}1}\\ \nu_{\mathrm{L}2}\\ \nu_{\mathrm{L}3}\end{pmatrix} = U^{\mathrm{T}}\cdot\begin{pmatrix}\nu_{\mathrm{L}e}\\ \nu_{\mathrm{L}\mu}\\ \nu_{\mathrm{L}\tau}\end{pmatrix}, \tag{179}$$

where U and U^{T} are mixing matrices (79), (80). Let us find a relation that the angles $\alpha_{12}, \alpha_{13}, \alpha_{23}$ must satisfy. At first, let us introduce the designations in Eq. (174)

$$\begin{aligned}\partial^\sigma\partial_\sigma\nu_{\mathrm{L}e} + A\nu_{\mathrm{L}e} + B\nu_{\mathrm{L}\mu} + C\nu_{\mathrm{L}\tau} &= 0\,,\\ \partial^\sigma\partial_\sigma\nu_{\mathrm{L}\mu} + B\nu_{\mathrm{L}e} + E\nu_{\mathrm{L}\mu} + D\nu_{\mathrm{L}\tau} &= 0\,,\\ \partial^\sigma\partial_\sigma\nu_{\mathrm{L}\tau} + C\nu_{\mathrm{L}e} + D\nu_{\mathrm{L}\mu} + F\nu_{\mathrm{L}\tau} &= 0\,.\end{aligned} \tag{180}$$

Using Eqs. (177), (179), we can write Eq. (180) as

$$
\begin{aligned}
&c_{12}c_{13}m_1^2\nu_{\mathrm{L}1} + c_{13}s_{12}m_2^2\nu_{\mathrm{L}2} + s_{13}m_3^2\nu_{\mathrm{L}3} = A\nu_{\mathrm{L}e} + B\nu_{\mathrm{L}\mu} + C\nu_{\mathrm{L}\tau}\,,\\
&(-s_{13}s_{23}c_{12} - c_{23}s_{12})m_1^2\nu_{\mathrm{L}1} + (-s_{12}s_{13}s_{23} + c_{23}c_{12})m_2^2\nu_{\mathrm{L}2} + c_{13}s_{23}m_3^2\nu_{\mathrm{L}3}\\
&= B\nu_{\mathrm{L}e} + E\nu_{\mathrm{L}\mu} + D\nu_{\mathrm{L}\tau}\,,\\
&(-c_{23}c_{12}s_{13} + s_{23}s_{12})m_1^2\nu_{\mathrm{L}1} + (-c_{23}s_{12}s_{13} - s_{23}c_{12})m_2^2\nu_{\mathrm{L}2} + c_{13}c_{23}m_3^2\nu_{\mathrm{L}3}\\
&= C\nu_{\mathrm{L}e} + D\nu_{\mathrm{L}\mu} + F\nu_{\mathrm{L}\tau}\,.
\end{aligned}
\tag{181}
$$

The right-hand side can be transformed like this

$$
\begin{aligned}
&\begin{pmatrix} A\nu_{\mathrm{L}e} + B\nu_{\mathrm{L}\mu} + C\nu_{\mathrm{L}\tau}\\ B\nu_{\mathrm{L}e} + E\nu_{\mathrm{L}\mu} + D\nu_{\mathrm{L}\tau}\\ C\nu_{\mathrm{L}e} + D\nu_{\mathrm{L}\mu} + F\nu_{\mathrm{L}\tau}\end{pmatrix} \equiv \begin{pmatrix} A & B & C\\ B & E & D\\ C & D & F\end{pmatrix}\begin{pmatrix}\nu_{\mathrm{L}e}\\ \nu_{\mathrm{L}\mu}\\ \nu_{\mathrm{L}\tau}\end{pmatrix}\\
&= \begin{pmatrix} A & B & C\\ B & E & D\\ C & D & F\end{pmatrix}\cdot\widehat{U}\cdot\begin{pmatrix}\nu_{\mathrm{L}1}\\ \nu_{\mathrm{L}2}\\ \nu_{\mathrm{L}3}\end{pmatrix}\\
&= \begin{pmatrix}
[Ac_{13}c_{12} - B(s_{23}s_{13}c_{12} + c_{23}s_{12}) - C(c_{23}s_{13}c_{12} - s_{23}s_{12})]\nu_{\mathrm{L}1}\\
+[Ac_{13}s_{12} - B(s_{23}s_{13}s_{12} - c_{23}c_{12}) - C(c_{23}s_{13}s_{12} + s_{23}c_{12})]\nu_{\mathrm{L}2}\\
+[As_{13} + Bs_{23}c_{13} + Cc_{23}c_{13}]\nu_{\mathrm{L}3}\\
\\
[Bc_{13}c_{12} - E(s_{23}s_{13}c_{12} + c_{23}s_{12}) - D(c_{23}s_{13}c_{12} - s_{23}s_{12})]\nu_{\mathrm{L}1}\\
+[Bc_{13}s_{12} - E(s_{23}s_{13}s_{12} - c_{23}c_{12}) - D(c_{23}s_{13}s_{12} + s_{23}c_{12})]\nu_{\mathrm{L}2}\\
+[Bs_{13} + Es_{23}c_{13} + Dc_{23}c_{13}]\nu_{\mathrm{L}3}\\
\\
[Cc_{13}c_{12} - D(s_{23}s_{13}c_{12} + c_{23}s_{12}) - F(c_{23}s_{13}c_{12} - s_{23}s_{12})]\nu_{\mathrm{L}1}\\
+[Cc_{13}s_{12} - D(s_{23}s_{13}s_{12} - c_{23}c_{12}) - F(c_{23}s_{13}s_{12} + s_{23}c_{12})]\nu_{\mathrm{L}2}\\
+[Cs_{13} + Ds_{23}c_{13} + Fc_{23}c_{13}]\nu_{\mathrm{L}3}
\end{pmatrix}.
\end{aligned}
\tag{182}
$$

Then, the angles $\alpha_{12}, \alpha_{13}, \alpha_{23}$ satisfy the following equation:

$$
\varDelta \equiv \begin{vmatrix} \varDelta_{11} & \varDelta_{12} & \varDelta_{13}\\ \varDelta_{21} & \varDelta_{22} & \varDelta_{23}\\ \varDelta_{31} & \varDelta_{32} & \varDelta_{33}\end{vmatrix} = 0\,, \tag{183}
$$

where

$$\begin{aligned}
\varDelta_{11} &= \left(A - m_1^2\right) c_{13}c_{12} - B(s_{23}s_{13}c_{12} + c_{23}s_{12}) - C(c_{23}s_{13}c_{12} - s_{23}s_{12})\,,\\
\varDelta_{12} &= \left(A - m_2^2\right) c_{13}s_{12} - B(s_{23}s_{13}s_{12} - c_{23}c_{12}) - C(c_{23}s_{13}s_{12} + s_{23}c_{12})\,,\\
\varDelta_{13} &= \left(A - m_3^2\right) s_{13} + Bs_{23}c_{13} + Cc_{23}c_{13}\,,\\
\varDelta_{21} &= Bc_{13}c_{12} - \left(E - m_1^2\right)(s_{23}s_{13}c_{12} + c_{23}s_{12}) - D(c_{23}s_{13}c_{12} - s_{23}s_{12})\,,\\
\varDelta_{22} &= Bc_{13}s_{12} - \left(E - m_2^2\right)(s_{23}s_{13}s_{12} - c_{23}c_{12}) - D(c_{23}s_{13}s_{12} + s_{23}c_{12})\,,\\
\varDelta_{23} &= Bs_{13} + \left(E - m_3^2\right) s_{23}c_{13} + Dc_{23}c_{13}\,,\\
\varDelta_{31} &= Cc_{13}c_{12} - D(s_{23}s_{13}c_{12} + c_{23}s_{12}) - \left(F - m_1^2\right)(c_{23}s_{13}c_{12} - s_{23}s_{12})\,,\\
\varDelta_{32} &= Cc_{13}s_{12} - D(s_{23}s_{13}s_{12} - c_{23}c_{12}) - \left(F - m_2^2\right)(c_{23}s_{13}s_{12} + s_{23}c_{12})\,,\\
\varDelta_{33} &= Cs_{13} + Ds_{23}c_{13} + \left(F - m_3^2\right) c_{23}c_{13}\,. \qquad (184)
\end{aligned}$$

It is noteworthy that for the case of two-band system, we obtain

$$\begin{array}{l}
i\gamma^\sigma\partial_\sigma\nu_{\mathrm{L}e} - \varsigma_{e\mu}\nu_{\mathrm{R}\mu} = 0\\
i\gamma^\sigma\partial_\sigma\nu_{\mathrm{L}\mu} - \varsigma_{e\mu}\nu_{\mathrm{R}e} = 0\\
i\gamma^\sigma\partial_\sigma\nu_{\mathrm{R}e} - \varsigma_{e\mu}\nu_{\mathrm{L}\mu} = 0\\
i\gamma^\sigma\partial_\sigma\nu_{\mathrm{R}\mu} - \varsigma_{e\mu}\nu_{\mathrm{L}e} = 0
\end{array} \Rightarrow \begin{array}{l}
\partial^\sigma\partial_\sigma\nu_{\mathrm{L}e} + \varsigma_{e\mu}^2\nu_{\mathrm{L}e} = 0\\
\partial^\sigma\partial_\sigma\nu_{\mathrm{L}\mu} + \varsigma_{e\mu}^2\nu_{\mathrm{L}\mu} = 0
\end{array}. \qquad (185)$$

Thus we can see that, unlike the three-band system, in the two-band system the flavor states coincide with the mass states. This means that neutrino oscillations in the two-band system are impossible.

We can see that the mixing of massive (charged) leptons and the mixing of neutrinos have completely different nature. From Eq. (113), we can see that the lepton mixing parameters $\zeta_{\alpha\beta}$ are determined by the interband coupling ϵ, since the masses of the electron, muon, and tauon $m_{\mathrm{D}i}$ are determined by coupling the with scalar fields $\varphi_1, \varphi_2, \varphi_3$, respectively, see Eq. (105), and in turn, these scalar fields are mixed by the interband coupling ϵ, see Eq. (73). Thus, if we turn off the interband interaction, *i.e.* $\epsilon = 0$ is assumed, then the lepton mixing will be absent. On the contrary, neutrinos do not interact with the scalar fields, therefore the neutrino mixing parameters $\varsigma_{\alpha\beta}$ are not determined by the interband coupling ϵ. Thus, the neutrino mixing parameters $\varsigma_{\alpha\beta}$ remain free parameters of the theory. Cosmological data (anisotropy of cosmic microwave background radiation, formation of structures, *etc.*) impose restrictions on the masses: $\sum_\nu m_\nu < 0.19$ eV [46], $\sum_\nu m_\nu < 0.28$ eV [47]. Since $m_\nu \sim |\varsigma|$, then $|\varsigma| \sim 0.1$ eV.

The matrix U (79), which is determined by the three mixing angles $\alpha_{12}, \alpha_{13}, \alpha_{23}$, is unitary. The unitarity property is preserved in the presence of one more parameter, the phase δ, so that

$$\begin{pmatrix}\nu_{\mathrm{L}e}\\ \nu_{\mathrm{L}\mu}\\ \nu_{\mathrm{L}\tau}\end{pmatrix}=\begin{pmatrix}1&0&0\\0&c_{23}&s_{23}\\0&-s_{23}&c_{23}\end{pmatrix}\begin{pmatrix}c_{13}&0&s_{13}\mathrm{e}^{-i\delta}\\0&1&0\\-s_{13}\mathrm{e}^{i\delta}&0&c_{13}\end{pmatrix}\begin{pmatrix}c_{12}&s_{12}&0\\-s_{12}&c_{12}&0\\0&0&1\end{pmatrix}\begin{pmatrix}\nu_{\mathrm{L}1}\\ \nu_{\mathrm{L}2}\\ \nu_{\mathrm{L}3}\end{pmatrix},\tag{186}$$

$$\begin{pmatrix}\nu_{\mathrm{L}1}\\ \nu_{\mathrm{L}2}\\ \nu_{\mathrm{L}3}\end{pmatrix}=\begin{pmatrix}c_{12}&-s_{12}&0\\s_{12}&c_{12}&0\\0&0&1\end{pmatrix}\begin{pmatrix}c_{13}&0&-s_{13}\mathrm{e}^{-i\delta}\\0&1&0\\s_{13}\mathrm{e}^{i\delta}&0&c_{13}\end{pmatrix}\begin{pmatrix}1&0&0\\0&c_{23}&-s_{23}\\0&s_{23}&c_{23}\end{pmatrix}\begin{pmatrix}\nu_{\mathrm{L}e}\\ \nu_{\mathrm{L}\mu}\\ \nu_{\mathrm{L}\tau}\end{pmatrix},\tag{187}$$

As is well known, the complex multipliers $\mathrm{e}^{i\delta},\mathrm{e}^{-i\delta}$ produce the violation of CP-invariance [5, 8, 39–41]. Then, instead of Eqs. (183), (184), we obtain

$$\mathit{\Delta}\equiv\begin{vmatrix}\mathit{\Delta}_{11}&\mathit{\Delta}_{12}&\mathit{\Delta}_{13}\\ \mathit{\Delta}_{21}&\mathit{\Delta}_{22}&\mathit{\Delta}_{23}\\ \mathit{\Delta}_{31}&\mathit{\Delta}_{32}&\mathit{\Delta}_{33}\end{vmatrix}=\mathrm{Re}(\mathit{\Delta})+i\,\mathrm{Im}(\mathit{\Delta})=0\Longrightarrow\left\{\begin{array}{c}\mathrm{Re}(\mathit{\Delta})=0\\ \mathrm{Im}(\mathit{\Delta})=0\end{array}\right\},\tag{188}$$

$$\begin{aligned}
\mathit{\Delta}_{11}&=\left(A-m_1^2\right)c_{13}c_{12}-B(s_{23}s_{13}c_{12}\mathrm{e}^{i\delta}+c_{23}s_{12})-C(c_{23}s_{13}c_{12}\mathrm{e}^{i\delta}-s_{23}s_{12})\,,\\
\mathit{\Delta}_{12}&=\left(A-m_2^2\right)c_{13}s_{12}-B(s_{23}s_{13}s_{12}\mathrm{e}^{i\delta}-c_{23}c_{12})-C(c_{23}s_{13}s_{12}\mathrm{e}^{i\delta}+s_{23}c_{12})\,,\\
\mathit{\Delta}_{13}&=\left(A-m_3^2\right)s_{13}\mathrm{e}^{-i\delta}+Bs_{23}c_{13}+Cc_{23}c_{13}\,,\\
\mathit{\Delta}_{21}&=Bc_{13}c_{12}-\left(E-m_1^2\right)(s_{23}s_{13}c_{12}\mathrm{e}^{i\delta}+c_{23}s_{12})-D(c_{23}s_{13}c_{12}\mathrm{e}^{i\delta}-s_{23}s_{12})\,,\\
\mathit{\Delta}_{22}&=Bc_{13}s_{12}-\left(E-m_2^2\right)(s_{23}s_{13}s_{12}\mathrm{e}^{i\delta}-c_{23}c_{12})-D(c_{23}s_{13}s_{12}\mathrm{e}^{i\delta}+s_{23}c_{12})\,,\\
\mathit{\Delta}_{23}&=Bs_{13}\mathrm{e}^{-i\delta}+\left(E-m_3^2\right)s_{23}c_{13}+Dc_{23}c_{13}\,,\\
\mathit{\Delta}_{31}&=Cc_{13}c_{12}-D\left(s_{23}s_{13}c_{12}\mathrm{e}^{i\delta}+c_{23}s_{12}\right)-\left(F-m_1^2\right)\left(c_{23}s_{13}c_{12}\mathrm{e}^{i\delta}-s_{23}s_{12}\right)\,,\\
\mathit{\Delta}_{32}&=Cc_{13}s_{12}-D(s_{23}s_{13}s_{12}\mathrm{e}^{i\delta}-c_{23}c_{12})-\left(F-m_2^2\right)\left(c_{23}s_{13}s_{12}\mathrm{e}^{i\delta}+s_{23}c_{12}\right)\,,\\
\mathit{\Delta}_{33}&=Cs_{13}\mathrm{e}^{-i\delta}+Ds_{23}c_{13}+\left(F-m_3^2\right)c_{23}c_{13}\,.
\end{aligned}\tag{189}$$

In the case of symmetrical interband mixing $\varsigma_{e\mu}=\varsigma_{e\tau}=\varsigma_{\mu\tau}\equiv\varsigma$, it is not difficult to see that any magnitudes of the mixing angles $\alpha_{12},\alpha_{13},\alpha_{23}$ and the CP-violation phase δ satisfy Eq. (188). Thus, the asymmetry of interband neutrino mixing selects values of the mixing angles α_{ik} and the CP-violation phase δ.

At present, it is known from experiments that $|\Delta m_{23}^2|\equiv|m_3^2-m_2^2|\approx 2.51\times10^{-3}\ \mathrm{eV}^2$, $|\Delta m_{12}^2|\approx7.41\times10^{-5}\ \mathrm{eV}^2$, $\alpha_{12}=33.4^\circ$, $\alpha_{23}=42.2\ldots49.5^\circ$, $\alpha_{13}=8.6^\circ$, $\delta/^\circ=195^{+51}_{-25}$ [42, 43]. Thus, to find masses of neutrinos, we should solve an inverse problem: knowing the mixing angles α_{ik}, the CP-violation phase δ, and the mass differences $|\Delta m_{ik}^2|$, we can find the mixing parameters $\varsigma_{e\mu},\varsigma_{e\tau},\varsigma_{\mu\tau}$. However, such a problem is very difficult to calculate. At the same time, we can see that the two angles α_{12},α_{23} are close to

$\pi/4$, *i.e.* this mixing is close to full mixing. On the other hand, the mass differences are strongly asymmetric $\Delta m_{12}^2 \ll \Delta m_{23}^2$. In the case of symmetrical mixing, *i.e.* $\varsigma_{e\mu} = \varsigma_{e\tau} = \varsigma_{\mu\tau} \equiv \varsigma$, we have effective masses (178). We can see that there is a tendency $\Delta m_{23}^2 \gg \Delta m_{12}^2 \to 0$. Thus, we can estimate the mixing parameters as

$$\varsigma^2 \sim \frac{1}{3}\Delta m_{23}^2 \sim 8.369 \times 10^{-4}\ \mathrm{eV}^2\,. \tag{190}$$

Hence, the band masses of neutrinos can be estimated as

$$\sqrt{m_{\nu 1}^2} \approx \sqrt{m_{\nu 2}^2} \approx |\varsigma| = 0.0289\ \mathrm{eV}\,, \qquad \sqrt{m_{\nu 3}^2} \approx 2|\varsigma| = 0.0579\ \mathrm{eV}\,. \tag{191}$$

Magnitudes of the band masses (191) are the result of a very rough approximation of the symmetric mixing $\varsigma_{e\mu} = \varsigma_{e\tau} = \varsigma_{\mu\tau}$, in reality, $m_{\nu 1}^2 \neq m_{\nu 2}^2$, although $|\Delta m_{12}^2| \ll |\Delta m_{23}^2|$. Then, we can choose the mixing parameters $\varsigma_{e\mu}, \varsigma_{e\tau}, \varsigma_{\mu\tau}$ to obtain the experimentally observed difference in squared masses $\Delta m_{12}^2, \Delta m_{23}^2$ by slightly changing the parameter ς from Eq. (190) (by module)

$$\varsigma_{e\mu} = 2.988 \times 10^{-2}\ \mathrm{eV}\,, \qquad \varsigma_{e\tau} = \varsigma_{\mu\tau} = 2.893 \times 10^{-2}\ \mathrm{eV}\,. \tag{192}$$

Then, using Eq. (176), the band masses of neutrinos can be estimated as

$$\sqrt{m_{\nu 1}^2} = 0.0286\ \mathrm{eV}\,, \quad \sqrt{m_{\nu 2}^2} = 0.0299\ \mathrm{eV}\,, \quad \sqrt{m_{\nu 3}^2} = 0.0585\ \mathrm{eV}\,. \tag{193}$$

Unfortunately, Eq. (188) is extremely sensitive to the parameters $m_{\nu i}, c_{ik}$, so we can only make some estimations. Let us suppose $\alpha_{13} \to 0$ in Eqs. (188) and (189), then we should take

$$\alpha_{13} \to 0 \Rightarrow \alpha_{12} \approx \alpha_{23} \approx 38^\circ \tag{194}$$

that is close to the *tribimaximal mixing* $\alpha_{12} = 35.3^\circ, \alpha_{23} = 45^\circ, \alpha_{13} = 0$. Then $\sqrt{m_{\nu 1}^2} + \sqrt{m_{\nu 2}^2} + \sqrt{m_{\nu 3}^2} \approx 0.12$ eV, that is consistent with current cosmological data $\sum_\nu m_\nu < 0.19$ eV [46, 47] (where all $m_\nu > 0$).

8. Systematics of elementary particles, masses of Higgs bosons, and Dark Matter

Summarizing the results of previous sections, we can make Table 3 of elementary particles in the three-band GWS theory (excluding quarks). We can see that, unlike the single-band theory, in the three-band case, we have

Table 3. Elementary particles in the three-band GWS theory: leptons, Higgs bosons (scalar), Leggett bosons (scalar), gauge bosons (vector). Each flavor of leptons can interact with the Higgs field of only corresponding flavor. The Leggett bosons are sterile particles, therefore the massive modes form the so-called "ultra-light Dark Matter". The sterile right-handed neutrinos have exactly the same effective masses as the corresponding active left-handed neutrinos. Each charged lepton can be both left- and right-handed.

	Electron flavor	Muon flavor	Tauon flavor
Higgs bosons	H_e	H_μ	H_τ
Charged leptons	$e_{\mathrm{L,R}}$	$\mu_{\mathrm{L,R}}$	$\tau_{\mathrm{L,R}}$
Active neutrinos	$\nu_{\mathrm{L}e}$	$\nu_{\mathrm{L}\mu}$	$\nu_{\mathrm{L}\tau}$
Sterile neutrinos	$\nu_{\mathrm{R}e}$	$\nu_{\mathrm{R}\mu}$	$\nu_{\mathrm{R}\tau}$

Leggett bosons			Gauge bosons		
Massive	L_1	L_2	Massive	$W^\pm$	Z
Massless	$\mathrm{L}_3 \leftrightarrow \mathrm{L}_4$		Massless	γ	

three H-bosons with somewhat different masses. In the limit of weak inter-band coupling $|\epsilon| \ll |a_{1,2,3}|$, we can write their flavor masses via the band parameters

$$m_{He} = \sqrt{2|a_1|} < m_{H\mu} = \sqrt{2|a_2|} < m_{H\tau} = \sqrt{2|a_3|} \sim 100\ \mathrm{GeV}\,. \tag{195}$$

All H-bosons have zero electrical charge $Q = 0$, zero lepton charges l_e, l_μ, $l_\tau = 0$, hypercharge $Y = 1$, and the third projection of isospin $I_3 = -1/2$. At the same time, the bosons H_e, H_μ, H_τ interact only with the corresponding leptons e, μ, τ changing their chirality according to Eq. (108) as shown in Fig. 8 (a). The masses of leptons are

$$m_e = \chi\varphi_{01}\,, \qquad m_\mu = \chi\varphi_{02}\,, \qquad m_\tau = \chi\varphi_{03}\,, \tag{196}$$

where

$$\varphi_{01} = \sqrt{\frac{|a_1|}{b_1}} = \frac{m_{He}}{\sqrt{2b_1}}\,, \quad \varphi_{02} = \sqrt{\frac{|a_2|}{b_2}} = \frac{m_{H\mu}}{\sqrt{2b_2}}\,, \quad \varphi_{03} = \sqrt{\frac{|a_3|}{b_3}} = \frac{m_{H\tau}}{\sqrt{2b_3}} \tag{197}$$

are the equilibrium values of the scalar fields, χ is the dimensionless coupling constant between the corresponding Dirac fields and the scalar fields (Yukawa coupling).

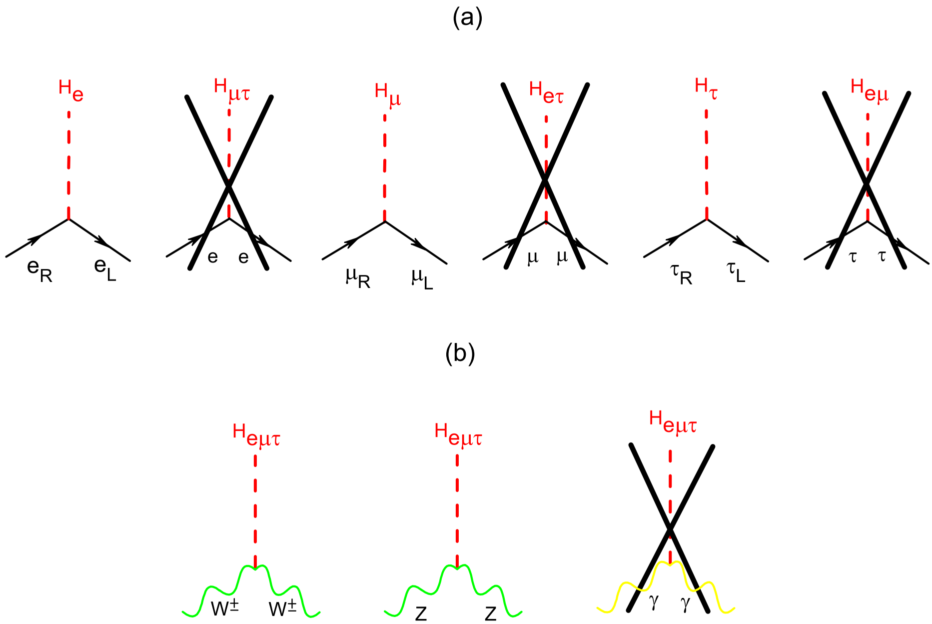

Fig. 8. The Higgs-lepton vertices (a), and the Higgs-gauge boson vertices (b). Leptons of each flavor can only interact with the H-bosons of corresponding flavor. $W^{\pm}$ and Z gauge bosons can interact with H-bosons of all flavors, but the photon γ does not interact with the Higgs fields.

According to Eqs. (150), (155), and (156), the gauge fields $W^{\pm}$ and Z interact with all scalar fields as shown in Fig. 8 (b). At the same time, photon γ does not interact with the scalar fields and remains massless. The masses of the charged W-boson and neutral Z-boson are

$$
\begin{aligned}
m_W &= \frac{e}{\sin\alpha}\sqrt{2\pi\left(\varphi_{01}^2+\varphi_{02}^2+\varphi_{03}^2\right)} = \frac{e}{\sin\alpha}\sqrt{2\pi\left(m_e^2+m_\mu^2+m_\tau^2\right)\frac{1}{\chi}}\,, \\
m_Z &= \frac{m_W}{\cos\alpha}\,,
\end{aligned}
\tag{198}
$$

where $\sin\alpha = 0.4721$ is the Weinberg angle, $e = 1/\sqrt{128}$ is the electromagnetic coupling constant at energy of ~ 100 GeV. Using masses of the gauge boson $m_W = 80.377$ GeV, lepton masses $m_e = 0.51\times10^{-3}$ GeV, $m_\mu = 0.1057$ GeV, $m_\tau = 1.7768$ GeV, we obtain the coupling constant χ

$$
\chi = 0.0104\,, \tag{199}
$$

and the amplitudes of the scalar fields $\varphi_{0i} = m_i/\chi$

$$\begin{aligned} \varphi_{0e} &\approx \varphi_{01} = 0.05 \text{ GeV}\,, \qquad \varphi_{0\mu} \approx \varphi_{02} = 10.17 \text{ GeV}\,, \\ \varphi_{0\tau} &\approx \varphi_{03} = 170.98 \text{ GeV}\,. \end{aligned} \tag{200}$$

In the standard representation of the isospinor field $\varPsi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \varphi \end{pmatrix}$, we have $\varphi_{0e} = 0.07$ GeV, $\varphi_{0\mu} = 14.38$ GeV, $\varphi_{0\tau} = 241.80$ GeV, so the effective amplitude of the scalar field is $\varphi_{\text{eff}} = \sqrt{(\varphi_{01}^2 + \varphi_{02}^2 + \varphi_{03}^2)} = 242$ GeV. If we take the electromagnetic coupling constant at energy of ~ 1 GeV: $e \approx 1/\sqrt{132}$, then we obtain $\varphi_{\text{eff}} = 246$ GeV.

Unfortunately, both the single-band GWS theory and the three-band GWS theory do not allow us to calculate the masses of H-bosons (195). We only know one H-boson with a mass of $m_H = 125.10$ GeV. Since the H-boson mediates interactions between leptons (as illustrated in Fig. 8 (a)), these interactions are interactions of a common nature, characterized by the same coupling constant (199) in our model. They should therefore have approximately the same effective interaction constants $\sim \frac{\chi^2}{m_{He}^2} \approx \frac{\chi^2}{m_{H\mu}^2} \approx \frac{\chi^2}{m_{H\tau}^2}$ and radii $\sim \frac{1}{m_{He}} \approx \frac{1}{m_{H\mu}} \approx \frac{1}{m_{H\tau}}$, similar to the weak interactions which have approximately equal interaction constants $\sim \frac{g^2}{m_W^2} \approx \frac{\widetilde{g}^2}{m_Z^2}$ and radii $\sim \frac{1}{m_W} \approx \frac{1}{m_Z}$, since the masses of mediators are of the same order: $m_W = 80.4$ GeV $\sim m_Z = 91.2$ GeV. Therefore, the masses of H-bosons should be of the same order too: $m_{He} \sim m_{H\mu} \sim m_{H\tau}$. At the same time, different Dirac masses of leptons $m_e \ll m_\mu \ll m_\tau$ are caused by different amplitudes of scalar fields $\varphi_{01} \ll \varphi_{02} \ll \varphi_{03}$. The amplitudes of scalar fields $\varphi_{01}, \varphi_{02}, \varphi_{03}$ from Eqs. (197), (200) differ from each other by orders, namely $\varphi_{01} : \varphi_{02} : \varphi_{03} = m_e : m_\mu : m_\tau$. Thus, the small changes in the mass of H-bosons $m_{He} < m_{H\mu} < m_{H\tau}$ should be accompanied by the significant changes of the scalar fields $\varphi_{01} \ll \varphi_{02} \ll \varphi_{03}$.

In a single-band case, the critical temperature is determined by the equilibrium magnitude of the scalar field at $T = 0$: $T_{\text{c}} = 2\varphi_0$, at the same time, at nonzero temperatures, we have $\varphi(T) = \varphi(0)\sqrt{1 - \frac{T^2}{T_{\text{c}}^2}}$ [77]. Let us write coefficients $a(T)$ and b in the following manner:

$$a = \mathcal{N}\left(\frac{T^2}{T_{\text{c}}^2} - 1\right)\,, \qquad b = \frac{4\mathcal{N}}{T_{\text{c}}^2}\,. \tag{201}$$

Then, the coefficient $\mathcal{N}$ does not take part in the condensate density $\varphi_0(T) = \sqrt{\frac{|a(T)|}{b}} = \frac{T_c}{2}\sqrt{1-\frac{T^2}{T_c^2}}$. Since $m_H = \sqrt{2|a|}$ at $T=0$, we have

$$\mathcal{N} = \frac{m_H^2}{2}\,. \tag{202}$$

For the three-band system, we can write the coefficients $a_{1,2,3}$ and $b_{1,2,3}$ as

$$a_1 = \mathcal{N}_1\left(\frac{T^2}{T_{c1}^2}-1\right), \qquad a_2 = \mathcal{N}_2\left(\frac{T^2}{T_{c2}^2}-1\right), \qquad a_3 = \mathcal{N}_3\left(\frac{T^2}{T_{c3}^2}-1\right), \tag{203}$$

$$b_1 = \frac{4\mathcal{N}_1}{T_{c1}^2}, \qquad b_2 = \frac{4\mathcal{N}_2}{T_{c2}^2}, \qquad b_3 = \frac{4\mathcal{N}_3}{T_{c3}^2}\,. \tag{204}$$

Here, T_{c1}, T_{c2}, T_{c3} are the critical temperatures of the corresponding bands, if the bands were independent, *i.e.* $\epsilon = 0$. In the presence of interband coupling $\epsilon \neq 0$, the system is characterized by the single critical temperature T_c, which can be calculated using the linearized Eq. (45) as the condition of the existence of nonzero solutions at T_c

$$\begin{vmatrix} a_1(T_c) & \epsilon & \epsilon \\ \epsilon & a_2(T_c) & \epsilon \\ \epsilon & \epsilon & a_3(T_c) \end{vmatrix} = 0 \;\Rightarrow\; a_1(T_c)a_2(T_c)a_3(T_c) + 2\epsilon^3 - \epsilon^2\left(a_1(T_c) + a_2(T_c) + a_3(T_c)\right) = 0\,. \tag{205}$$

It should be noted that the coefficient d in Eq. (67) is such that $d(T_c) = 0$ (here $\alpha_i(T_c) = a_i(T_c) > 0$ as follows from Eq. (63)). The solutions of Eq. (45) are illustrated in Fig. 9 for the case of strongly asymmetrical bands $T_{c1,c2} \ll T_{c3}$. The effect of interband coupling $\epsilon \neq 0$, even if the coupling is weak $|\epsilon| \ll |a_i(0)|$, is non-perturbative for the smaller scalar fields $\varphi_{1,2}$ — applying the interband coupling drags the smaller amplitudes up to a new critical temperature $T_c \gg T_{c1,c2}$. At the same time, the effect on the largest scalar fields φ_3 is not so significant — applying the interband coupling slightly increases only the critical temperature $T_c \gtrsim T_{c3}$. If the interband coupling is weak, then the magnitude of the scalar fields $\varphi_{01,02,03}$ at $T = 0$ changes very little [74, 75], for example,

$$\varphi_{01}(0) = \sqrt{\frac{|a_1(0)|}{b_1} + \frac{|\epsilon|}{|a_1(0)|}\left(\sqrt{\frac{|a_2(0)|}{b_2}} + \sqrt{\frac{|a_3(0)|}{b_3}}\right)} \approx \sqrt{\frac{|a_1(0)|}{b_1}}\,, \tag{206}$$

i.e. $\varphi_{0i}(0)$ is predominantly determined by the intraband coefficients $a_i(0), b_i$.

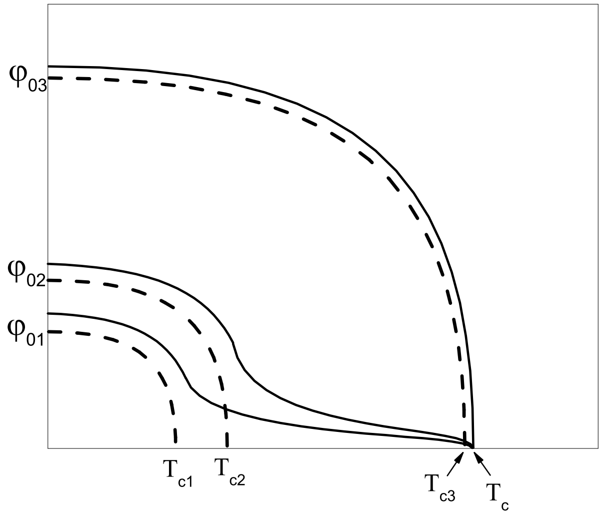

Fig. 9. The scalar fields $\varphi_{01}(T), \varphi_{02}(T), \varphi_{03}(T)$ as solutions of Eq. (45), if the interband coupling is absent, *i.e.* $\epsilon = 0$ (dashed lines), and if the weak interband interaction takes place, *i.e.* $|\epsilon| \ll |a_i(0)|$ (solid lines). Applying the weak interband coupling drags the smaller parameters $\varphi_{01,02}$ up to a new critical temperature $T_c \gg T_{c1,c2}$. The effect on the larger parameter φ_{03} is not so significant. The magnitudes of the scalar fields $\varphi_{01,02,03}$ at $T = 0$ change very little.

The coefficients $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$, *i.e.* Higgs masses as a generalization of Eq. (202) in a sense

$$m_{He}^2 = 2\mathcal{N}_1\,, \qquad m_{H\mu}^2 = 2\mathcal{N}_2\,, \qquad m_{H\tau}^2 = 2\mathcal{N}_3 \tag{207}$$

cannot be calculated at the present time. In SM, the mass of H-boson $m_H = 125.10$ GeV is taken from experiment as a parameter of the theory. In superconductors, the coefficient $\mathcal{N}$ plays role of the density of electron states on the Fermi surface. The critical temperature T_c depends exponentially on $\mathcal{N}$, $T_c \sim \varphi_0(0) \sim \Omega \exp\left(-\frac{1}{g\mathcal{N}}\right)$, at weak coupling, where Ω is the phonon frequency and g is the constant of the electron–phonon interaction. The larger the parameter $\mathcal{N}$, the higher the critical temperature T_c. In our model, $\mathcal{N}_1 < \mathcal{N}_2 < \mathcal{N}_3$ and $\varphi_{01} \ll \varphi_{02} \ll \varphi_{03}$ mean that small changes to the parameter $\mathcal{N}$ cause large (exponential) changes to the scalar field φ_0. Then, by analogy with the BCS theory, we can assume that the amplitudes of the scalar fields φ_{0i} at $T = 0$ are determined by the corresponding parameters $\mathcal{N}_i$

$$\varphi_{01} = \Omega \exp\left(-\frac{1}{g\mathcal{N}_1}\right), \quad \varphi_{02} = \Omega \exp\left(-\frac{1}{g\mathcal{N}_2}\right), \quad \varphi_{03} = \Omega \exp\left(-\frac{1}{g\mathcal{N}_3}\right), \tag{208}$$

where the parameters g, Ω are some common parameters for all three bands. Thus, the change in φ (T_c) is accompanied by the logarithmic change of $\mathcal{N}$. Moreover, if the "interaction constant" is zero, *i.e.* $g\mathcal{N} = 0$, then the "condensate" is absent $\varphi_0 = 0$. Thus, the scalar fields φ_i can be a result of the Cooper pairing of some more fundamental fermions as, for example, in models with the top-quark condensation [78, 79] or with the technicolor [80].

We can get rid of the parameter Ω

$$\begin{aligned} \ln\frac{\varphi_{02}}{\varphi_{01}} &= \frac{1}{g}\left(\frac{1}{\mathcal{N}_1} - \frac{1}{\mathcal{N}_2}\right), \qquad \ln\frac{\varphi_{03}}{\varphi_{01}} = \frac{1}{g}\left(\frac{1}{\mathcal{N}_1} - \frac{1}{\mathcal{N}_3}\right), \\ \ln\frac{\varphi_{03}}{\varphi_{02}} &= \frac{1}{g}\left(\frac{1}{\mathcal{N}_2} - \frac{1}{\mathcal{N}_3}\right). \end{aligned} \tag{209}$$

By eleminating the parameter g, we obtain an expression connecting the parameters $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$ between themselves

$$\frac{\mathcal{N}_2 - \mathcal{N}_1}{\mathcal{N}_3 - \mathcal{N}_1}\frac{\mathcal{N}_3}{\mathcal{N}_2} = \frac{A}{B} \Rightarrow \frac{m_{H\mu}^2 - m_{He}^2}{m_{H\tau}^2 - m_{He}^2}\frac{m_{H\tau}^2}{m_{H\mu}^2} = \frac{A}{B}, \tag{210}$$

where we have used Eq. (207), $m_{He} \neq m_{H\mu} \neq m_{H\tau}$, and we have denoted

$$A \equiv \frac{\ln\frac{\varphi_{02}}{\varphi_{01}}}{\ln\frac{\varphi_{03}}{\varphi_{02}}} = 1.89, \qquad B \equiv \frac{\ln\frac{\varphi_{03}}{\varphi_{01}}}{\ln\frac{\varphi_{03}}{\varphi_{02}}} = A + 1 = 2.89. \tag{211}$$

Thus, due to the three-band system, the magnitudes of the Higgs masses $m_{He} < m_{H\mu} < m_{H\tau}$ are related by Eq. (210). We assume (as will be demonstrated below) that the τ-Higgs boson coincides with the observed H-boson of mass $m_H = 125.10$ GeV, *i.e.* the masses of $m_{H\mu}$ and m_{He} are limited from the above by the mass 125.10 GeV. Using Eq. (210), we can find mass of the lightest H-boson m_{He} as a function of the boson of medium mass $m_{H\mu}$ at known mass of the heaviest H-boson $m_{H\tau} = 125.10$ GeV

$$m_{He} = m_{H\mu}\sqrt{\frac{Bm_{H\tau}^2 - Am_{H\tau}^2}{Bm_{H\tau}^2 - Am_{H\mu}^2}}, \tag{212}$$

as illustrated in Fig. 10.

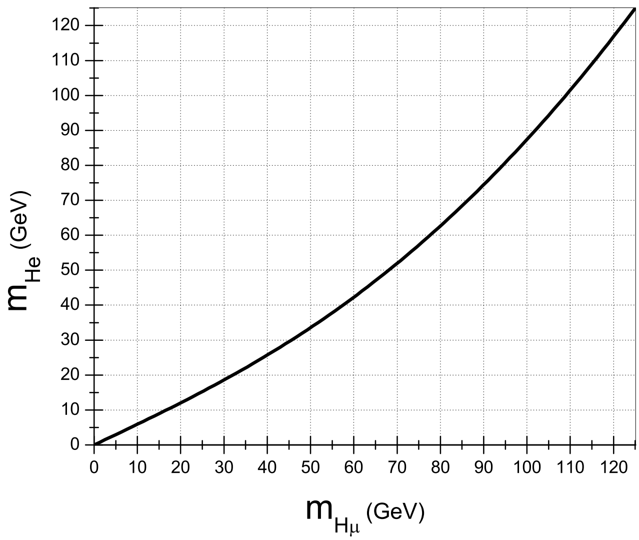

Fig. 10. The mass of the e-Higgs boson m_{He} as a function of the mass of μ-Higgs boson $m_{H\mu}$, which is limited above by the mass of the τ-Higgs boson $m_{H\tau} = 125.10$ GeV.

Thus, in the proposed model, we have H-bosons of three flavors (generations) H_e, H_μ, H_τ which should be characterized by quantum numbers similar to, for example, lepton numbers or quark flavors. However, at present, only one H-boson of mass 125 GeV is observed experimentally. Let us consider the processes of the H-boson production [48–50]. These processes can be categorized into two types: (a) production by the vector bosons — Fig. 11 (a) due to interaction (100), (b) production by the heaviest quarks (t and b) — Fig. 11 (b) due to the Yukawa interaction similar to Eq. (108). First, let us compare the constants for coupling between gauge bosons and H-bosons of each flavor (generation) from Eq. (100) using Eq. (200)

$$2e^2\varphi_{0e} : 2e^2\varphi_{0\mu} : 2e^2\varphi_{0\tau} = 0.00028 : 0.059 : 1\,. \tag{213}$$

Thus, gauge bosons $W^{\pm}, Z$ most efficiently radiate H_τ bosons. H_μ and H_e bosons must also be radiated, but they are extremely inefficient compared with H_τ bosons.

Now, let us consider the production of H-bosons by quarks (or leptons). We should calculate the constants of Yukawa coupling χ as $\chi = \frac{m_i}{\varphi_{0i}}$, where an index i means flavor. The amplitudes of the scalar fields φ_{0i} are taken from Eq. (200). The results of this calculation are presented in Table 4.

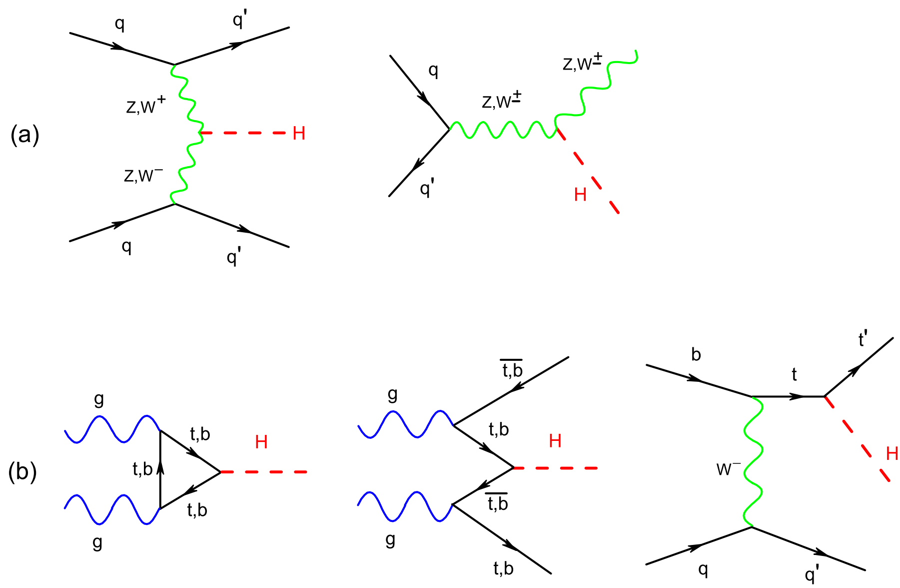

Fig. 11. Some processes of Higgs-boson production: (a) production by the vector bosons $W^{\pm}, Z$, (b) production by t and b quarks (here the blue lines g represent gluons).

Table 4. Masses (experimental) and Yukawa constants of elementary fermions χ calculated in the three-band GWS theory using amplitudes of the scalar fields φ_{0i} from Eq. (200) for the corresponding "flavors" (generations).

	Electron flavor	Muon flavor	Tauon flavor
Scalar fields	$\varphi_{0e} = 0.05$GeV	$\varphi_{0\mu} = 10.51$GeV	$\varphi_{0\tau} = 176.70$GeV
Charged leptons	$m_e = 0.0005$ GeV $\chi = 0.010$	$m_\mu = 0.1057$ GeV $\chi = 0.010$	$m_\tau = 1.7768$ GeV $\chi = 0.010$
Up quarks	$m_u = 0.0023$ GeV $\chi = 0.046$	$m_c = 1.275$ GeV $\chi = 0.121$	$m_t = 173.210$ GeV $\boldsymbol{\chi = 0.975}$
Down quarks	$m_d = 0.0048$ GeV $\chi = 0.096$	$m_s = 0.095$ GeV $\chi = 0.009$	$m_b = 4.180$ GeV $\chi = 0.024$

The probability of producing or decaying of the H-bosons is $\Gamma \propto \chi^2$. The squared Yukawa constants related to the t-quark coupling constant: χ^2/χ_t^2 are shown in Fig. 12. For comparison, the Yukawa constants for SM $\frac{\chi^2}{\chi_t^2} = \frac{m^2}{m_t^2}$ are shown in Fig. 13. In the single-band GWS theory (*i.e.* in SM), the masses of fermions are controlled by χ only, because the scalar field φ is single. In the multi-band GWS model, the masses of fermions are controlled by both χ and the corresponding (for each generation) amplitudes of the scalar

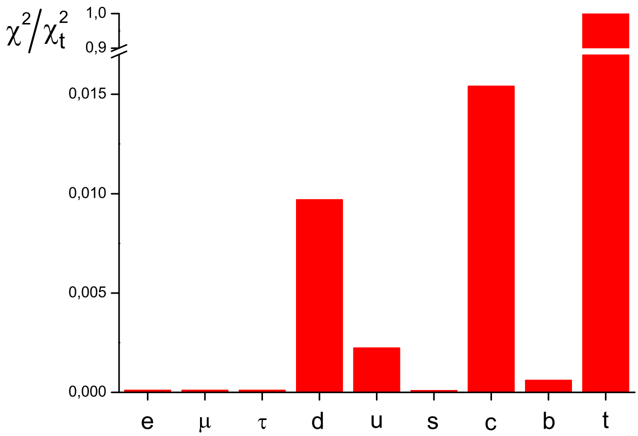

Fig. 12. Squared Yukawa constants related to the t-quark coupling constant: $\frac{\chi^2}{\chi_t^2}$ in the three-band GWS model.

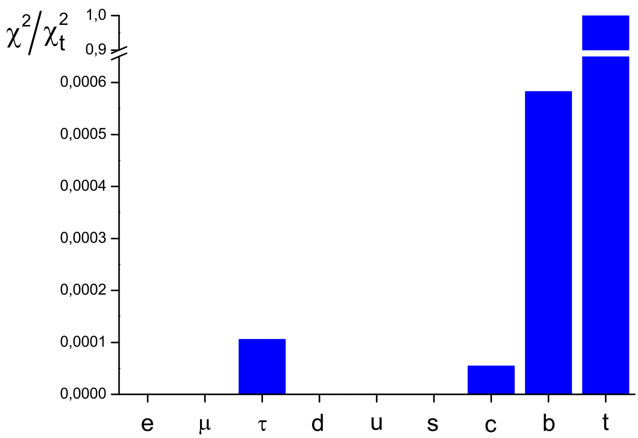

Fig. 13. Squared Yukawa constants related to the t-quark coupling constant: $\frac{\chi^2}{\chi_t^2} = \frac{m^2}{m_t^2}$ in the single-band GWS theory (Standard Model).

fields φ_{0i}. Thus, the differences between Yukawa constants for different flavors are somewhat smoothed out, as we saw earlier for leptons of all flavors $\chi = 0.01$. However, from Table 4 and Fig. 12 (and also Fig. 13), we can see that χ_t is giant, moreover $m_t > m_H$ (but $m_{c,b} \ll m_H$). This means that H_τ-bosons are produced in the vast majority of cases, as in the described above production, by the vector bosons $W^\pm, Z$.

Let us consider the decays of the H-boson into quarks or leptons shown in Fig. 2. According to SM, the H-boson should decay as $H \to b\bar{b}$ with a probability of 57.5%, $H \to \tau\bar{\tau}$ with a probability of 6.30%, $H \to c\bar{c}$ with a probability of 2.90%, and $H \to \mu\bar{\mu}$ with a probability of $\lesssim 0.022\%$ [50]. As an illustration, the squared Yukawa constants for fermions of the second and third generations (muon, tauon, s-quark, c-quark, b-quark) related to the b-quark coupling constant $\frac{\chi^2}{\chi_b^2} = \frac{m^2}{m_b^2}$ calculated in SM are shown in Fig. 14. Thus, in SM, the H-boson interacts most strongly with the third generation, which is the most massive, therefore the $H \to b\bar{b}$ and $H \to \tau\bar{\tau}$ decays are dominant. However, the decays into the second-generation fermions $H \to c\bar{c}$ should also be noticeable.

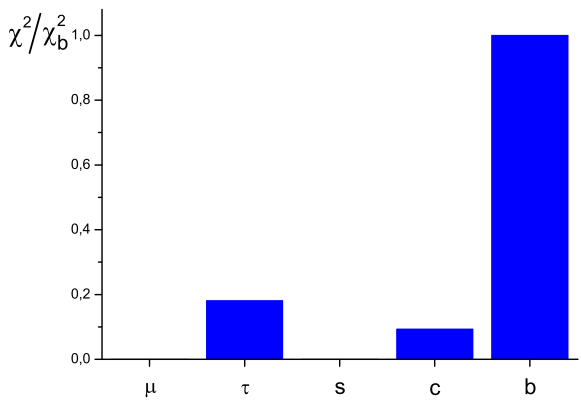

Fig. 14. Squared Yukawa constants for fermions of the second and third generations (muon, tauon, s-quark, c-quark, b-quark) related to the b-quark coupling constant: $\frac{\chi^2}{\chi_b^2} = \frac{m^2}{m_b^2}$ in the single-band GWS theory (Standard Model).

At the same time, there has been no experimental evidence found in direct searches by the ATLAS and CMS collaborations [51, 52] of the H-boson decaying into charm quark–antiquark, into strange quark–antiquark, and into electron–positron. The decay into muon–antimuon has been detected with a significance of 3σ [53], which is clearly not enough for an experimental fact (*i.e.* more than 5σ), moreover, there are similar decays such as $H \to \gamma\mu\bar{\mu}, \gamma e\bar{e}$, which occur through many intermediate channels due to various interactions (via virtual photon, Z-, W-bosons, quarks) with a significance of 3.2σ [54]. This fact (the absence of observations of decays of H-bosons into fermions of the first and second generations) is usually associated with the small Yukawa constants of the first and second generations. However, from Fig. 14, we can see that the decay rate into a pair of c-quarks is not much less than the decay rate into a pair of τ-leptons, *i.e.* $\chi_c^2 \lesssim \chi_\tau^2$ (the decay probabilities are 2.9% and 6.4%, respectively). On the other hand, such rare decays as two-photon decay $H \to \gamma\gamma$ with probabilities of $\approx 0.2\%$ have been detected.

If we turn to the three-band GWS model, then we have the squared Yukawa constants for fermions of the second and third generations (muon, tauon, s-quark, c-quark, b-quark) related to the b-quark coupling constant: $\frac{\chi^2}{\chi_b^2}$ shown in Fig. 15. Let us compare these relations with those in Fig. 14. It is not difficult to see that

$$\frac{\Gamma(H_\tau \to \tau\bar{\tau})}{\Gamma(H_\tau \to b\bar{b})} \approx \frac{\Gamma(H \to \tau\bar{\tau})}{\Gamma(H \to b\bar{b})} \approx 0.2\,. \tag{214}$$

This means that the probability of decay of the H_τ-boson into τ-leptons regarding the decay into b-quarks in the three-band GWS model and probability of decay of H-boson into τ-leptons regarding the decay into b-quarks in SM model are almost equal. However, in the three-band model, the $H_\tau \to c\bar{c}, H_\tau \to s\bar{s}, H_\tau \to \mu\bar{\mu}$ decays are prohibited. But the $H_\mu \to c\bar{c}, s\bar{s}, \mu\bar{\mu}$ decays are allowed, with the $H_\mu \to c\bar{c}$ decay dominating sharply. As have been demonstrated previously, the H_μ-boson is emitted extremely inefficiently. Thus, due to the inefficiency of production of H_e and H_μ by gauge

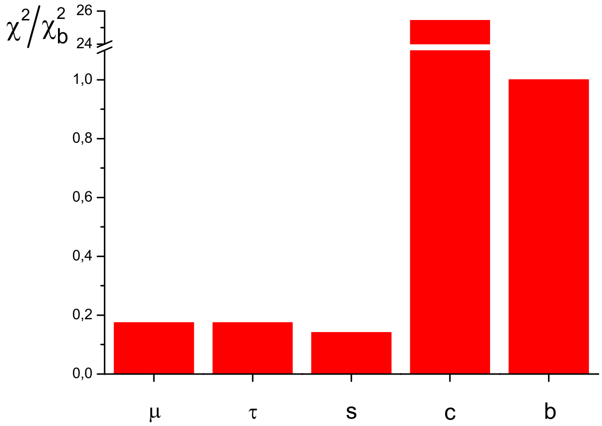

Fig. 15. Squared Yukawa constants for fermions of the second and third generations (muon, tauon, s-quark, c-quark, b-quark) related to the b-quark coupling constant: $\frac{\chi^2}{\chi_b^2}$ in the three-band GWS model.

bosons, the anomalously large Yukawa constant of the t-quark and the huge background from QCD, searching for H_e and H_μ by hadron–hadron collisions at the LHC requires to probe the Higgs decays to a deeper level with sufficient accuracy. At the same time, there are plans to build the Future Circular electron–positron Collider (FCC-ee) that would provide measurements with unprecedented precision and potentially point the way to physics beyond the SM [81]. It would allow us to look for H_e in direct e^-e^+ collisions at high energies, since then the background from QCD should be absent, electron–positron pairs can annihilate directly to H_e-bosons (similarly to how muon–antimuon pairs can annihilate to H_μ-bosons and tauon–antitauon pairs can annihilate to H_τ-bosons).

Proceeding from the aforesaid, we should identify H_τ-boson with experimentally observed H-boson

$$H_\tau \equiv H_{\text{observed}}\,. \tag{215}$$

Other generations (flavors) of H-bosons, H_μ and H_e, require detection. Thus, the H-boson of the electron flavor (the first generation) should decay as $H_e \to d\bar{d}, u\bar{u}, e\bar{e}$.

It should be noted that in recent years, the observation of so-called “multi-lepton anomalies” [82, 83] at the Large Hadron Collider is interpreted (with a local significance of $\lesssim 3\sigma$) as the existence of beyond the SM Higgs bosons: a new scalar particle S with a mass of $m_S = 151$ GeV, produced from the decay of a new heavier scalar particle H (with a mass of $m_H \geqslant 276$ GeV) into a lighter one S and the SM Higgs h: $H \to Sh, SS$ according to 2HDM + S model [84, 85]. However, the significance of this anomaly is debatable [86]. The CMS and ATLAS collaborations reported on the signal with the production cross section of the SM-like scalar ϕ with a mass of ~ 95 GeV which manifests itself as the diphoton decay $pp \to \phi \to \gamma\gamma$ [87–89] with a local significance of $1.7\sigma \ldots 2.9\sigma$. During the search for additional Higgs bosons ϕ and vector leptoquarks in $\tau\tau$ final states CMS found a 3.1σ excess of events for $pp \to \phi \to \tau\tau$ at an invariant mass $m_\phi \simeq 100$ GeV and a 2.6σ at an invariant mass $m_\phi \simeq 95$ GeV. Thus, the low significance of these anomalies ($\lesssim 3\sigma$) does not make it possible to interpret them as unambiguous confirmation of multi-HDM models. It is possible that these recorded signals correspond to some very heavy meson resonances or tetraquark resonances.

As we could see in Sections 2, 3, 5, 6, due to the interband coupling, the Goldstone modes from each band (oscillations of θ and ϑ phases) transform into the following normal modes of the system. The twofold degenerated acoustic mode $q_\mu q^\mu = 0$ is a common mode oscillations of the phases of the isospinor fields $\Psi_{1,2,3}$. The propagation of this mode is accompanied by the current $J^\mu \neq 0$, hence this mode is absorbed by the gauge fields W_μ, W^*_μ, Z_μ. Other modes are the Leggett modes, which are antiphase oscillations of the phases of the isospinor fields. The propagation of the L-modes is not accompanied by the current $J^{\mu}=0$, hence they do not interact with the gauge fields. Moreover, the Leggett modes do not interact with the Higgs modes in the linear approximation if attractive interband coupling takes place $\epsilon<0$. Furthermore, these modes do not interact with the Dirac fields $\psi_{1,2,3}$, unlike the Higgs modes. Thus, the Leggett modes do not interact with any particles, *i.e.* they are sterile. These modes can only manifest themselves through gravity on astrophysical scales. One of the L-modes is the twofold degenerated acoustic mode $q_{\mu}q^{\mu}=0$, which we labeled $L_3\leftrightarrow L_4$ in Table 3. However, the massless bosons lose their energy in the process of space expansion, similarly to the relic photons. Moreover, ultrarelativistic particles cannot be assembled into a self-gravitating halo. Therefore, such particles do not contribute to DM. However, other two modes L_1 and L_2 are massive with masses determined by the coefficient of the interband coupling as $q_{\mu}q^{\mu}\sim\epsilon$. The masses of L-bosons can be calculated using Eqs. (160), (161), and (200)

$$m_{\mathrm{L1}}=5.83\sqrt{|\epsilon|}\,,\qquad m_{\mathrm{L2}}=85.98\sqrt{|\epsilon|}\,. \tag{216}$$

Since the L-bosons do not take part in the electro-weak interaction, they cannot decay, for example, into two photons, therefore the L-bosons are stable particle. Obviously, the massive L-bosons are able to form stable gravitationally bound structures (halo, clusters, *etc.*). Therefore, the massive L-bosons are suitable candidates for DM.

9. The masses of Leggett bosons and the cuspy halo problem

In Section 8, we found that the Leggett modes do not interact with any particles, *i.e.* these modes are sterile and they can only manifest themselves through gravity on astrophysical scales. Therefore, the massive L-bosons are particles of so-called Dark Matter (massless, *i.e.* ultrarelativistic, L-bosons cannot be accumulated in self-gravitating clusters). Masses of L-bosons are determined by the coefficient of the interband coupling ϵ, see Eq. (216). This coefficient can be arbitrary small because the effect of interband coupling is nonperturbative.

Observation of DM density distributions (halo around a galaxy) seems to prefer a central density as $\rho\sim r^{0}$. For example, the empirical core profiles can be described by the following function with two parameters: a scale radius r_0 and a scale density ρ_0 [25]:

$$\rho(r)=\frac{\rho_0}{1+(\frac{r}{r_0})^2}\,. \tag{217}$$

However, in the large-scale simulations using the collisionless cold Dark Matter model, the inner region of the halo shows a density distribution described by a power law $\rho \sim r^{\alpha}$, where $\alpha = -1$. Such behavior is now called a "cusp". One example is the Navarro–Frenk–White profile

$$\rho(r) = \frac{\rho_0}{\frac{r}{r_0}\left(1 + \frac{r}{r_0}\right)^2} . \tag{218}$$

Since the mass of L-bosons can be extremely small and the critical temperature of BEC can be very high (because the coefficient $|\epsilon|$ can be arbitrarily small), then the Bose–Einstein condensate Dark Matter (BEC DM or Scalar Field Dark Matter, Fuzzy, Wave, Ultra-light Dark Matter) can form [25, 29, 31]. This means that the halo is described by the macroscopic wave function

$$\sqrt{M}\psi(r) = \sqrt{\rho(\boldsymbol{r}, t)}\, \mathrm{e}^{iS(\boldsymbol{r},t)} , \tag{219}$$

where $M = mN$ is the total mass of the DM halo, m is the DM particle mass (mass of L-bosons, see Eq. (216)), N is the number of the particles in the halo. Then the quantum Euler–Madelung equation for the stationary case $\frac{D(\nabla S(\boldsymbol{r},t))}{Dt} = 0$ is

$$\boldsymbol{g} - \frac{\nabla p}{\rho} + \sigma \nabla T + \frac{\nabla Q}{m} = 0 , \tag{220}$$

where $\boldsymbol{g}$ is the gravitational field strength

$$\boldsymbol{g} = -\frac{4\pi G \boldsymbol{r}}{r^3} \int_0^r \rho(r') r'^2 \mathrm{d}r' . \tag{221}$$

L-bosons do not interact with anything except through gravity, so we can assume "dust" matter $p = 0$. In BEC at $T \to 0$, the entropy can be supposed $\sigma = 0$ or the profile can be suppose isothermal $\nabla T = 0$. Q is a quantum potential

$$Q = \frac{\hbar^2}{2m} \frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} , \tag{222}$$

i.e. $\frac{1}{m}\nabla Q$ is the quantum pressure term.

Let us consider the central cusp of profile (218) in a form of $\rho(r) = \rho_0 \frac{r_0}{r}$. Then the gravitational field strength is

$$\boldsymbol{g} = -4\pi G \rho_0 r_0 \frac{\boldsymbol{r}}{r} , \tag{223}$$

and the quantum pressure term takes the form

$$\frac{\nabla Q}{m} = \frac{\hbar^2}{2m^2} \frac{1}{r^3} \frac{\boldsymbol{r}}{r} . \tag{224}$$

We can see that, while the field strength is finite, the quantum pressure is singular at $r = 0$. Thus, the equilibrium cannot be achieved. Such a cusp should be blurred by itself.

Now, let us consider the observed profile (217). At $r \to 0$, we obtain

$$\boldsymbol{g} = -\frac{4\pi}{3}G\rho_0 r\frac{\boldsymbol{r}}{r}, \qquad \frac{\nabla Q}{m} = \frac{\hbar^2}{m^2}\frac{6r}{r_0^4}\frac{\boldsymbol{r}}{r}, \tag{225}$$

from which we can see that the Euler equation (220) can be satisfied when

$$\frac{4\pi}{3}G\rho_0 = \frac{\hbar^2}{m^2}\frac{6}{r_0^4}. \tag{226}$$

Obviously, if $\rho_0 r_0^3 \sim M$, then we have from Eq. (226)

$$\rho_0 \sim G^3\frac{m^6 M^4}{\hbar^6}, \qquad r_0 \sim \frac{\hbar^2}{Gm^2 M}. \tag{227}$$

Thus, due to the quantum pressure, the central density $\rho(r \to 0)$ is not singular. From Eq. (227), we can see that the extremely small mass $m = m_{\mathrm{L}} \sim 10^{-20}$ eV can ensure the small central density ρ_0 and the large profile width r_0. At the same time, we can see that the spatial distribution (217) does not give a finite mass of a self-gravitating Dark Matter halo: $\int_0^\infty \frac{r^2 \mathrm{d}r}{1+(r/r_0)^2} = \infty$. A good approximation would be the profile obtained in Ref. [91] for a self-gravitating system

$$\rho(r) = \frac{\rho_0}{\cosh^2\left(\frac{r}{r_0}\right)}. \tag{228}$$

This distribution becomes the profile (217) at $r \ll r_0$, at the same time, it gives a finite mass of the halo: $M = \frac{\pi^3}{3}\rho_0 r_0^3$. Unfortunately, we cannot verify (228) by direct substitution into Eq. (220) because the integral $\int_0^r \frac{r^2 \mathrm{d}r}{\cosh^2(r/r_0)}$ cannot be calculated analytically. However, we can verify it at another limit $r \gg r_0$. Then we have

$$\boldsymbol{g} = -G\frac{M}{r^2}\frac{\boldsymbol{r}}{r}, \qquad \frac{\nabla Q}{m} = \frac{\hbar^2}{m^2}\frac{1}{r^2 r_0}\frac{\boldsymbol{r}}{r}. \tag{229}$$

Obviously, Eq. (220) is satisfied at $r_0 = \frac{\hbar^2}{Gm^2 M}$, which corresponds to Eq. (227). Thus, the spatial distribution (228) can describe the DM halo.

Using Eq. (227), the mass and radius of the DM halo of the Milky Way $M \sim 10^{12} M_\odot$ and $r_0 \sim 120$ kpc [11], we can estimate the mass of an L-boson $m = m_{\rm L} \sim \sqrt{|\epsilon|}$, and using Eq. (216), we can then estimate the magnitude of the parameter of the interband coupling $|\epsilon|\, m_{\rm L} \sim 10^{-25}$ eV $\Rightarrow$ $|\epsilon| \sim 10^{-54}$ eV^2. However, numerical simulations demonstrate that the DM halo has some structure: a core from BEC of a size of ~ 1 kpc and the above-condensate Bose gas behaving as the cold DM, then a mass of $\sim 10^{-22} \ldots 10^{-20}$ eV [28–32] is assumed. Indeed, observations of the stellar kinematics of dwarf galaxies give the mass of just $\sim 10^{-22} \ldots 10^{-20}$ eV [33–35]. Then, we can suppose

$$m_{\rm L} \sim 10^{-20} \text{ eV} \Rightarrow |\epsilon| \sim 10^{-44} \text{ eV}^2 \,. \tag{230}$$

As mentioned above, the interband coupling is nonperturbative, therefore even such a small magnitude of ϵ determines the symmetry and spectrum of the system.

The L-bosons can appear due to vacuum decay after inflation and precipitate into the Bose-condensate. The temperature of the Bose condensation is $T_{\rm BEC} \sim n_{\rm cr}^{2/3} \frac{h^2}{m k_{\rm B}} = \rho_{\rm cr}^{2/3} \frac{h^2}{m^{5/3} k_{\rm B}} \sim 10^{31}$ K, where $\rho_{\rm cr}$ is the critical density of the universe. Thus, $T_{\rm BEC}$ is commensurable with the Plank temperature $T_{\rm Plank} \sim 10^{32}$ K. Thus, the L-bosons are so light that $T_{\rm BEC} \sim T_{\rm Plank}$ which means that the L-bosons should always be in BEC. This indicates a purely condensate nature of the DM halo and not a two-component structure with the condensate core of a size $\lesssim 1$ kps and a cloud of above-condensate Bose gas of a size ~ 120 kps. L-bosons could condense in BEC during the early years of the universe. Galaxies, galaxy clusters, and superclusters are immersed in the Bose-condensate clouds of sterile massive L-bosons that create the effect of Dark Matter.

Using Eq. (227), let us estimate the radius of the Dark Matter halo r_0 and the mass of the Milky Way (mass of Dark Matter) $M \sim 10^{12} M_\odot$, assuming $m_{\rm L} \sim 10^{-20}$ eV. Then we obtain $r_0 \sim 10^{-5}$ pc, which is in no way comparable to the radius of the DM halo being around $R \sim 120$ kpc. In connection with this fact, a hypothesis has been proposed [92, 93], regarding the formation of Bose stars, a large number of which can form the dark halo. However, we can propose another model. Let us compare the energy of the halos with sizes r_0 and R, respectively,

$$E_{r_0} \sim -G\frac{M^2}{r_0} \sim -10^{63} \text{ J} \,, \qquad E_R \sim -G\frac{M^2}{R} \sim -10^{53} \text{ J} \,. \tag{231}$$

These energies correspond to two different states of BEC — the ground state ψ_{r_0} with energy E_{r_0} and excited state ψ_R with energy E_R, which are solutions to the Gross–Pitaevskii equation

$$-\frac{\hbar^2}{2m}\Delta\psi(r) - m\frac{4\pi GM}{r}\int\limits_0^r |\psi(r')|^2 r'^2 \mathrm{d}r' \psi(r) = \mu\psi(r)\,, \qquad (232)$$

where $\mu = E/N = E\frac{m}{M}$. We can see that the Hamiltonian of the self-gravitating system is determined by its eigen-state ψ. Thus, different states (ground and excited) correspond to different Hamiltonians of one and the same system. This means that the states corresponding to different energies may not be orthogonal to each other, for example, $\int \psi_{r_0}\psi_R^+ \mathrm{d}^3 r \neq 0$. Suppose we excite the system from the ground state ψ_{r_0} to an excited state ψ_R. Such a transition stipulates the restructuring of the potential $U_{r_0} \to U_R$, so that our excited states ψ_R become the ground state of the new potential U_R as demonstrated in Fig. 16.

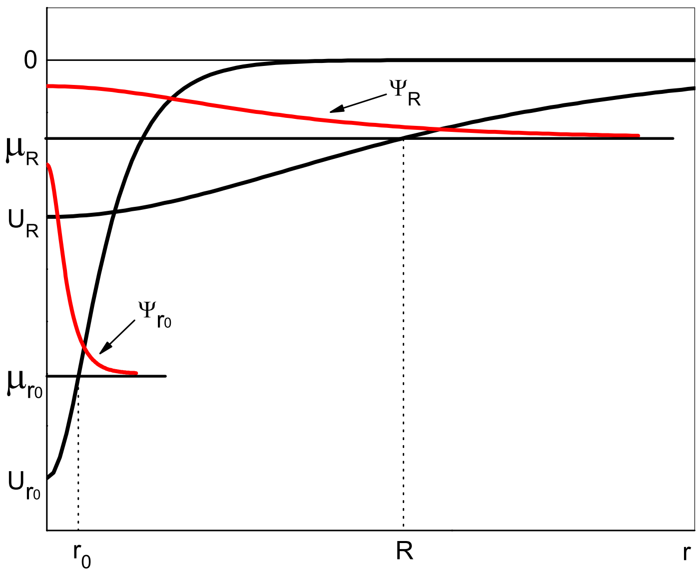

Fig. 16. Potentials U_{r_0} and U_R for the ground state Ψ_{r_0} with energy $E_{r_0} = \mu_{r_0}\frac{M}{m}$ and an excited state Ψ_R with energy $E_R = \mu_R\frac{M}{m}$, respectively. The corresponding radii of the DM halos are r_0 and R.

Let us enlarge the radius of the system by $n > 1$ times, *i.e.* $r_0 \to nr_0$, where r_0 is the "Bohr radius" for the self-gravitating system. The wave functions for the ground and excited states have the corresponding forms

$$\psi_{r_0}(r) \sim \frac{\sqrt{3/\pi^3}}{(r_0)^{3/2}} \frac{1}{\cosh\left(\frac{r}{r_0}\right)}, \tag{233}$$

$$\psi_R(r) \sim \frac{\sqrt{3/\pi^3}}{(nr_0)^{3/2}} \frac{1}{\cosh\left(\frac{r}{nr_0}\right)}, \quad n > 1. \tag{234}$$

Here, the value r_0 plays the role of the Bohr radius, $R \equiv r_0 n$. The average energy of a self-gravitating system in the state ψ is

$$E = \frac{M}{m}\frac{\hbar^2 4\pi}{2m}\int_0^\infty (\nabla\psi)^2 r^2 \mathrm{d}r - \frac{16\pi^2}{3} G M^2 \int_0^\infty \psi^4 r^4 \mathrm{d}r. \tag{235}$$

Substituting the ground-state wave function (233) into Eq. (235), we obtain

$$E = \frac{6\hbar^2 M}{\pi^2 m^2}\frac{0.607}{r_0^2} - \frac{48}{\pi^4} G M^2 \frac{0.249}{r_0} \equiv \frac{A}{r_0^2} - \frac{B}{r_0}. \tag{236}$$

Minimizing this energy by the radius r_0, we obtain

$$\begin{aligned} r_0 = \frac{2A}{B} = 2.44\frac{\pi^2}{4}\frac{\hbar^2}{Gm^2 M} \Rightarrow E_{r_0} &= -\frac{B^2}{4A} = -\frac{B}{2r_0} = -\frac{5.98}{\pi^4}\frac{GM^2}{r_0} \\ &= -\frac{9.80}{\pi^6}\frac{G^2 M^3 m^2}{\hbar^2}. \end{aligned} \tag{237}$$

Substituting the excited-state wave function (234) into Eq. (235), we obtain

$$R = nr_0 \Rightarrow E = \frac{A}{n^2 r_0^2} - \frac{B}{nr_0} \Rightarrow E_R = E_{r_0}\left(\frac{2}{n} - \frac{1}{n^2}\right). \tag{238}$$

For the highly excited states $n \gg 1$, we have $E_R = \frac{2E_{r_0}}{n}$, unlike the excited energies of a hydrogen atom: $E_n = \frac{E_1}{n^2}$.

Obviously, the self-gravitating system aspires to transition to an underlying state. To do so, the system must give somewhere the released energy $E(R_1) - E(R_2) > 0$, where $R_2 < R_1$. Let us consider transition of the system from a high "orbit" to a lower one. As a result, the cloud collapses and heats up. However, as we could see above, the L-bosons are sterile particles, that is, they do not scatter with each other or with baryonic matter. Hence, the above-mentioned mechanism of cloud collapse does not work.

Then there is only one way: in order to make a transition from the state ψ_R to the state ψ_{r_0}, it is necessary to radiate gravitation waves (since an atom making a quantum transition from an exited state to an underlying state radiates photons). The energy loss rate and the halo compression rate due to gravitational radiation can be estimated using the two-particle attraction problem according to Newton's law [94]

$$\frac{\mathrm{d}E}{\mathrm{d}t} \sim \frac{G^4M^5}{c^5R^5} \sim 10^{21}\ \mathrm{J/s}\,, \qquad \frac{\mathrm{d}R}{\mathrm{d}t} \sim \frac{G^3M^3}{c^5R^3} \sim 10^{-10}\ \mathrm{m/s}\,, \tag{239}$$

from where we obtain the relaxation time to the ground state ψ_{r_0}

$$\tau = \frac{\left(R^4 - r_0^4\right)c^5}{4G^3M^3} \sim 10^{32}\ \mathrm{s}\,, \tag{240}$$

which is incommensurably greater than the age of the universe 4×10^{17} s. Thus, the DM halo of a galaxy is similar to Rydberg atoms (instead of the Coulomb interaction — self-gravity, and instead of electro-magnetic radiation — gravitational radiation, however instead of electrons — L-bosons). A notable feature of Rydberg atoms is their very long lifetime compared to the lifetime of low-excited states, so for the hydrogen atom $\tau(n=2)\sim10^{-8}$ s against $\tau(n=1000)\sim1$ s. As we could see above, the analogous situation takes place for DM halos. Thus, the DM halo is a Rydberg, self-gravitating, many-boson atom. It should be noted that we have proposed the simplest model of the halo as an excited state of a self-gravitating many-boson system. However, the excited states can also be much more complex structures.

10. Higgs modes at $T = T_\mathrm{c}$

Let us consider a three-band system near the critical temperature $T_{\mathrm{c}1}$, $T_{\mathrm{c}2}$, $T_{\mathrm{c}3} < T < T_\mathrm{c}$. In this region, $\varphi_{0i}^2 \sim |\epsilon|/b_i$ [74]. Then we have from Eq. (63)

$$\alpha_i(T) = a_i(T) > 0\,. \tag{241}$$

Then coefficients a, b, d (67) in the dispersion equation (66) take the form

$$\begin{aligned} b(T) &= -a_1 - a_2 - a_3\,,\\ c(T) &= a_1a_2 + a_1a_3 + a_2a_3 - 3\epsilon^2\,,\\ d(T) &= -a_1a_2a_3 - 2\epsilon^3 + \epsilon^2(a_1+a_2+a_3)\,, \end{aligned} \tag{242}$$

where we have accounted $\cos\theta_{ik} = 1$. From the equation for critical temperature (205), we have $d(T_\mathrm{c}) = 0$. Then, from the dispersion equation (66), we obtain the corresponding dispersion relations at the critical temperature

$$q_\mu q^\mu(T_\mathrm{c}) = 0\,, \tag{243}$$

$$q_\mu q^\mu(T_\mathrm{c}) = \left(-b \pm \sqrt{b^2 - 4c}\right)/2 > 0\,. \tag{244}$$

For the first mode (243) (*i.e.* for the common-mode oscillations, see Fig. 7) the energy gap vanishes at the critical temperature, as it takes place in the single-band model. At the same time, the energy gaps of the second and third modes (244) (*i.e.* for the antiphase oscillations, see Fig. 7) do not vanish at the critical temperature. Thus, for symmetrical bands $\alpha_1 = \alpha_2 = \alpha_3 \equiv \alpha$, the massive modes have the same spectrum ($b^2 - 4c = 0$ taking into account the condition $d(T_c) = 0 \Rightarrow a(T_c) = 2|\epsilon|$)

$$q_\mu q^\mu(T_c) = 3|\epsilon| \Rightarrow m_{H1,2}(T_c) = \sqrt{3|\epsilon|}\,. \tag{245}$$

Thus, the energy gaps of the second and third Higgs modes are determined by the interband coupling ϵ. At the same time, at $T = T_c$, the second-order phase transition occurs: all equilibrium scalar fields become zero $\varphi_{01}(T_c) = \varphi_{02}(T_c) = \varphi_{03}(T_c) = 0$, see Fig. 9. Higgs bosons are oscillations of the modules $|\varphi_1|, |\varphi_2|, |\varphi_3|$ of the condensates. Since all $\varphi_{0i}(T_c) = 0$, then the nonzero energy gap $q_\mu q^\mu(T_c) \neq 0$ of the Higgs modes at $T = T_c$ is a nonphysical property. In other words, at $T \geq T_c$, there is nothing to oscillate, there are only fluctuations, where $\langle\varphi_i\rangle = 0$, $\langle\varphi_i^2\rangle \neq 0$ [95]. Therefore, nonzero masses of Higgs bosons at $T = T_c$ are incompatible with the second-order phase transition.

In order to solve this problem, in Refs. [70, 74, 75], the intergradient interaction in the form of $\eta_{ik}\left(\partial_\mu \Psi_i \partial^\mu \Psi_k^+ + \partial^\mu \Psi_i^+ \partial_\mu \Psi_k\right)$ has been proposed. With special choice of the coefficients η_{ik}, we obtain a single Higgs mode with the correct dispersion law $q_\mu q^\mu(T_c) = 0$ (but $q_\mu q^\mu(T < T_c) > 0$) and single Goldstone modes $q_\mu q^\mu = 0$ (*i.e.* the Leggett modes are absent). However, unlike in superconductivity, there is no restriction on the type of phase transition in the field theory. Thus, the second-order phase transition can be turned into the first-order phase transition by, for example, quantum corrections to the Lagrangian of the scalar field which interacts with the gauge fields [77]. Alternatively, we can use the effective potential [96] in the following form:

$$U(\varphi, T) = \frac{1}{2}\mathcal{N}\left(\frac{T^2}{T_-^2} - 1\right)\varphi^2 - \frac{1}{3}cT\varphi^3 + \frac{1}{4}bT\varphi^4\,, \tag{246}$$

where T_- is the lower spinodal temperature. Due to presence of the cubic term $cT\varphi^3$, the potential describes the first-order phase transition at the critical temperature $T_c = \frac{T_-}{\sqrt{1-2c^2T_-^2/9b\mathcal{N}}}$ and the jump of the density of the condensate $\frac{\Delta\varphi_0(T_c)}{T_c} = \frac{3}{2}\frac{c}{b}$. Thus, the presence of the jump, *i.e.* $\varphi_0(T_c) \neq 0$, allows for the existence of the nonzero energy gap $q_\mu q^\mu(T_c) \neq 0$ of the Higgs modes at $T = T_c$. Any other options can be considered.

11. Results

In this work, we proposed an extension of the Glashow–Weinberg–Salam model of the electro-weak interaction using analogy involving three-band superconductors with interband Josephson couplings. The proposed model describes important phenomena formulated in Section 1:

— There are two ultra-light sterile bosons — the Leggett bosons, the Bose–Einstein condensate of which plays the role of the Dark Matter halo. The halo is in an excited yet stable quantum state. There is no a central cusp due to the quantum pressure counteracting gravitational compression. In order to obtain the L-boson, at least two bands are required. In the case of the multi-band system, the attractive inter-band coupling $\epsilon < 0$ should take place in order for fermions to acquire Dirac masses.

— Dirac neutrinos receive effective masses which manifest themselves in the neutrino oscillations and β-decays. In order to obtain the neutrino oscillations and violation of CP-invariance, at least three bands are required. The mixing angles for charged leptons are negligibly small, so the flavor oscillations of electron–muon–tauon cannot be observed.

— There are neutral Higgs bosons of three flavors: H_e, H_μ, H_τ. Each interacts only with the corresponding generation of fermions, where the heaviest boson H_τ is associated with the observed H-boson. Therefore, decays of the H-boson into fermions of the second and first generations through the Yukawa interaction are prohibited. Another more light flavors H_e and H_μ require detection as an experimental test of the proposed model. At the same time, these two additional H-bosons interact very weakly with gauge and Dirac fields, which makes them difficult to detect.

— The masses of each generation of fermions are determined by the Yukawa couplings with the amplitudes of the corresponding condensates $\varphi_{0e}, \varphi_{0\mu}, \varphi_{0\tau}$ of the scalar fields. The slight mass asymmetry $m_{He} < m_{H\mu} < m_{H\tau}$ leads to the strong band asymmetry $\varphi_{0e} \ll \varphi_{0\mu} \ll \varphi_{0\tau}$. Therefore, the fermion masses differ by orders of magnitude $m_e \ll m_\mu \ll m_\tau$.

It should be noted that, unlike extensions of SM such as nHDM or nHDM+S, the proposed model does not generate a large number of other particles (for example, charged Higgs bosons), that can essentially interact with ordinary matter. In addition, the proposed particle candidates for DM — the Leggett bosons — are absolutely sterile, which means that they cannot even weakly interact with matter (as neutrinos).

This research was supported by a thematic grants of the Department of Physics and Astronomy of NAS of Ukraine: "Noise-induced dynamics and correlations in nonequilibrium systems" No. 0120U101347, "Stochastic processes in condensed media, biological systems and radiation fields" No. 0125U000031, and by grant of the Simons Foundation.

Appendix A

Some symmetric 3HDM potentials

Following [65], a scalar 3HDM potential symmetric under group G can be written as

$$V = V_0 + V_G \,, \tag{A.1}$$

where

$$\begin{aligned} V_0 \;=\; & \sum_{i=1}^{3} a_i \,|\varPsi_i|^2 + \frac{b_i}{2}\,|\varPsi_i|^4 + b_{12}|\varPsi_1|^2|\varPsi_2|^2 + b_{13}|\varPsi_1|^2|\varPsi_3|^2 + b_{23}|\varPsi_2|^2|\varPsi_3|^2 \\ & + b'_{12}\left(\varPsi_1^+\varPsi_2\right)\left(\varPsi_2^+\varPsi_1\right) + b'_{13}\left(\varPsi_1^+\varPsi_3\right)\left(\varPsi_3^+\varPsi_1\right) + b'_{23}\left(\varPsi_2^+\varPsi_3\right)\left(\varPsi_3^+\varPsi_2\right) \end{aligned} \tag{A.2}$$

is invariant under the most general $\mathrm{U}(1)\otimes\mathrm{U}(1)$ gauge transformation and V_G is a collection of extra terms ensuring the symmetry group G. The $\mathrm{U}(1)\otimes\mathrm{U}(1)$ group is generated by

$$\begin{pmatrix} \mathrm{e}^{-i\alpha} & 0 & 0 \\ 0 & \mathrm{e}^{i\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-2i\beta/3} & 0 & 0 \\ 0 & \mathrm{e}^{i\beta/3} & 0 \\ 0 & 0 & \mathrm{e}^{i\beta/3} \end{pmatrix} . \tag{A.3}$$

However, in the present work, we use the minimal model, where $b_{ik} = b'_{ik} = 0$. A potential symmetric under the $\mathrm{U}(1)$ group is

$$V_{\mathrm{U}(1)} = V_0 + \lambda_{123}\left[\left(\varPsi_1^+\varPsi_3\right)\left(\varPsi_2^+\varPsi_3\right) + \left(\varPsi_1\varPsi_3^+\right)\left(\varPsi_2\varPsi_3^+\right)\right] . \tag{A.4}$$

The $\mathrm{U}(1)$ group is generated by

$$\begin{pmatrix} \mathrm{e}^{-i\alpha} & 0 & 0 \\ 0 & \mathrm{e}^{i\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{A.5}$$

A potential symmetric under the $\mathrm{U}(1)\otimes Z_2$ group is

$$V_{\mathrm{U}(1)\otimes Z_2} = V_0 + \lambda_{23}\left[\left(\varPsi_2^+\varPsi_3\right)^2 + \left(\varPsi_2\varPsi_3^+\right)^2\right] . \tag{A.6}$$

The $\mathrm{U}(1)\otimes Z_2$ group is generated by

$$\begin{pmatrix} \mathrm{e}^{-2i\beta/3} & 0 & 0 \\ 0 & \mathrm{e}^{i\beta/3} & 0 \\ 0 & 0 & \mathrm{e}^{i\beta/3} \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.7}$$

A potential symmetric under the Z_2 group is

$$V_{Z_2} = V_0 + \epsilon_{12}\left[\Psi_1^+\Psi_2 + \Psi_1\Psi_2^+\right] + \lambda_{12}\left[\left(\Psi_1^+\Psi_2\right)^2 + \left(\Psi_1\Psi_2^+\right)^2\right] \\ +\lambda_{13}\left[\left(\Psi_1^+\Psi_3\right)^2 + (\Psi_1\Psi_3^+)^2\right] + \lambda_{23}\left[\left(\Psi_2^+\Psi_3\right)^2 + \left(\Psi_2\Psi_3^+\right)^2\right]. \tag{A.8}$$

The Z_2 group is generated by

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.9}$$

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