# THE QUADRUPOLE–DIPOLE COLLECTIVITY IN HEAVY NUCLEI IN THE FRAMEWORK OF THE U(10) AND U(20) ALGEBRAS

A. Mekentichi[†], A. Bouldjedri [‡]

Laboratoire de physique des rayonnements et de leurs intéractions avec la matière
Department of Physics, Batna 1 University
Route de Biskra, 05000 Batna, Algeria

*Received 24 February 2025, accepted 27 August 2025, published online 9 September 2025*

We investigate a manifestation of low-energy dipole collectivity in heavy nuclei, known as clustering, using algebraic techniques. In the first step, the connection of the nuclear vibron model to the U(10) algebra (spanned by four types of bosons, within the interacting boson model I (IBM-I)) is extended to other possible models through a detailed study of the U(10) subalgebras. Subsequently, the ability of the vibron model to reproduce the experimental data is extended to a wider region of heavy nuclei belonging to the rare-earth and actinide regions. A more realistic irrep labelling has been introduced to take into account the Wildermuth condition. In a second step, we upgrade the model to the IBM-II level involving the U(20) algebra, where a new $G$-spin and hybrid limits have been introduced.

DOI:10.5506/APhysPolB.56.9-A3

## 1. Introduction

The algebraic methods, which invoke symmetries of the nucleus and use group theoretical approaches to handle the problem of nuclear structure, have been widely investigated. In this way, the interacting boson model has been introduced [1–5] by Iachello and Arima in terms of U(6) algebra. In its simplest version, the basic idea of the IBM model is the assumption that excited states in even–even nuclei can be described by a system of $s$ and $d$ interacting bosons with angular momenta (and parity), respectively, $J^\pi = 0^+$ and $J^\pi = 2^+$. This model had a great success in describing low-lying positive parity states in medium and heavy even–even nuclei. However, in this version, the model does not account for all observed structure features, and subsequent extensions were necessary.

[†] ahlam.mekentichi@univ-batna.dz
[‡] abdelhamid.bouldjedri@univ-batna.dz

In a peculiar approach to the nuclear many-body problem, the nucleus is reduced to a limited number of (interacting) clusters of nucleons. This mapping to a few-body system is based on early data about alpha radioactivity and the properties of light nuclei with even $Z$ and $A = 2\,Z$ (like $^{12}$C or $^{16}$O). Indeed, the mean binding energy per nucleon shows local maxima for such nuclei indicating the importance of $\alpha$-clustering in these nuclei. In 1928, Gamow developed the theory of alpha decay with the assumption that the alpha particle preexists in the nucleus. Subsequently, a model where the nucleus is treated as a set of alpha particles has been introduced [6–8].

In the cluster models, one is usually left with a system of two or three bodies, a situation similar to the one that occurs in molecular spectroscopy. Indeed, an algebraic approach was proposed to describe the roto-vibrational structure of diatomic molecules [9–11], in terms of U(4) algebra using $\sigma$ and $p$ bosons (molecular vibron model), subsequently extended to linear tri- and four-atomic molecules and certain non-linear triatomic molecules.

The previous analogy has been used to develop an algebraic-interacting bosons-nuclear cluster model. The model was proposed to treat the motion of $\alpha$ particles (clusters) in heavy nuclei, called the nuclear vibron model [12, 13]. It is used to describe the nucleus as a few-body system [14–18], consisting of a heavy core on which oscillate alpha particles. The appropriate algebraic structure of this model is $\mathrm{U}(6) \otimes \mathrm{U}(4)$, its SU(3) limit describes the harmonic vibration of the cluster in axially-deformed nuclei. It leads, at low energy, to a spectrum which is a mixture of quadrupole and cluster states.

The aim of this paper is twofold. First, we reconsider the works on dipole collectivity in heavy nuclei, either from the mathematical side or from the extension of the comparison of the calculations with the experimental data. Then we upgrade the model to a larger algebraic framework, taking into account the distinction between the proton and the neutron bosons.

In Section 2, we discuss the possibility of the occurrence of clustering in heavy nuclei. In Section 3, we introduce the U(10) as a symmetry algebra of a variety of algebraic models related to the dipole–quadrupole excitations. The approach to clustering based on $\mathrm{U}(6) \otimes \mathrm{U}(4)$ subalgebra is described in Section 4 in addition to the results of its application to heavy nuclei (energy levels and transitions). In Section 5, we introduce an IBM-II version of the nuclear vibron model. A summary and conclusions are given in Section 6.

## 2. Clustering phenomenon in heavy nuclei

Clustering is a known feature of light nuclei. There exists a rich literature on the cluster models describing the spectral properties of these nuclei [19–21].

Whether clustering may manifest itself in the structure of heavy nuclei has been a challenge. Alpha radioactivity is a well-known phenomenon in heavy nuclei. Moreover, cluster radioactivity was predicted in 1980 [22] and discovered experimentally [23] (emission of $^{12}$C and $^{14}$C in the decay of $^{223}$Ra) four years later. In 1973, the theoretical work of Brink and Castro [24] showed that $\alpha$-clustering effects are important at a density of approximately one third of the central nuclear density, suggesting the importance of such correlation in the nuclear surface of heavy nuclei. Subsequently, it has been suggested by Iachello and Jackson [12] that alpha clustering may play an important role in heavy nuclei. This assumption was based on experimental data related to the energy levels and alpha decay. A corresponding model has been developed by Daley and Iachello [13] and provides a complete description of data in the actinides.

In a series of papers, a successful description of spectra and electromagnetic decay properties of positive and negative parity rotational bands in the actinide nuclei has been achieved by Buck, Merchant, and Perez [25–30]. The approach deals with a cluster–core system, whose energies are obtained in a semi-classical way.

The hypothesis regarding clustering occurrence in heavy nuclei was confirmed experimentally in 2010. Indeed, cluster states have been observed in $^{212}$Po by postulating $\alpha$ transfer using the $^{208}$Pb($^{16}$O,$^{12}$C)$^{212}$Po reaction. The results are discussed in terms of an $\alpha$-cluster structure ($\alpha$+$^{208}$Pb) [31]. One may assume that adding more $\alpha$s to the $^{208}$Pb core may exhibit similar physics. Hence, complex structures are expected if the $\alpha$s move independently. In a recent paper on $\alpha$-like quasi-molecules in heavy nuclei, Delion *et al.* [32] confirmed that $\alpha$ clusters are born in the nuclear surface at low density. Moreover, for nuclei of $N > 126$, the probability of clusterization, including the picture of three cluster structures, is suggested to study the properties of the nucleus [33].

Clustering as a reflection symmetry-breaking mode leads to collective odd-parity states. Low-energy collective negative parity states have been first observed in Ra and Th isotopes by the Berkeley group [34]. They have been interpreted as a manifestation of a reflection asymmetric (pear) shape of the nucleus [35].

These collective negative parity states have been described in the framework of an extension of the original IBM model by naturally adding $f$ ($J^\pi = 3^-$) bosons (the *sdf*-IBM) [4, 36]. The $p$ boson ($J^\pi = 1^-$) has been introduced by Morrison and Weise [37] to describe dipole states in nuclei. The obtained model is called the *spdf*-IBM model [36, 38–46], where the $f$ boson is the leading ingredient and the $p$ boson has been shown to be necessary from a mathematical point of view and due to the fact that it plays

a crucial role in the calculation of E1 transitions [41, 42]. This $p$ boson has also been used to introduce the second (fundamentally different) mechanism of reflection symmetry breaking: clustering [12].

## 3. The U(10) Lie algebra

In addition to the quadrupole degrees of freedom, described by the $s$ and $d$ bosons (U(6) algebra), reproducing negative parity states (within the clustering model) is achieved by the introduction of negative parity $l = 1$ boson ($p$) and a second $l = 0$ boson ($\sigma$). Hence, the U(10) is the minimal symmetry algebra of the problem. It will be restricted to a subalgebra (the dynamical symmetries) relevant to our problem.

The algebraic realisation of the U(10) algebra is obtained by means of the following operators:

$$a^\dagger_{lm} = \left\{ s^\dagger, d^\dagger_m \right\}, \qquad l = 0, 2, \qquad m = 0, \pm 1, \pm 2, \tag{1}$$

$$b^\dagger_{lm} = \left\{ \sigma^\dagger, p^\dagger_m \right\}, \qquad l = 0, 1, \qquad m = 0, \pm 1. \tag{2}$$

The commutation relations are given by

$$\left[ a_{lm}, a^\dagger_{l'm'} \right] = \delta_{ll'}\delta_{mm'}, \tag{3}$$

$$[a_{lm}, a_{l'm'}] = \left[ a^\dagger_{lm}, a^\dagger_{l'm'} \right] = 0, \tag{4}$$

$$\left[ b_{lm}, b^\dagger_{l'm'} \right] = \delta_{ll'}\delta_{mm'}, \tag{5}$$

and

$$[b_{lm}, b_{l'm'}] = \left[ b^\dagger_{lm}, b^\dagger_{l'm'} \right] = 0. \tag{6}$$

All $a$ bosons commute with the $b$ bosons.

### 3.1. The U(10) generators

The unitary algebra U(10) has 100 generators that can be written in coupled form (of rank $k$ and $2k + 1$ components) as follows:

$$\begin{aligned} G^0(ss) &= \left[ s^\dagger \otimes \tilde{s} \right]^0, \\ G^k(dd) &= \left[ d^\dagger \otimes \tilde{d} \right]^k, \qquad k = 0, 1, 2, 3, 4, \\ G^2(ds) &= \left[ d^\dagger \otimes \tilde{s} \right]^2, \qquad G^2(sd) = \left[ s^\dagger \otimes \tilde{d} \right]^2, \end{aligned} \tag{7}$$

$$
\begin{aligned}
G^0(\sigma\sigma) &= \left[\sigma^\dagger \otimes \tilde{\sigma}\right]^0 , \\
G^k(pp) &= \left[p^\dagger \otimes \tilde{p}\right]^k , \qquad k = 0, 1, 2 , \\
G^1(\sigma p) &= \left[\sigma^\dagger \otimes \tilde{p}\right]^1 , \qquad G^1(p\sigma) = \left[p^\dagger \otimes \tilde{\sigma}\right]^1 ,
\end{aligned} \tag{8}
$$

and

$$
\begin{aligned}
G^0(s\sigma) &= \left[s^\dagger \otimes \tilde{\sigma}\right]^0 , \qquad G^0(\sigma s) = \left[\sigma^\dagger \otimes \tilde{s}\right]^0 , \\
G^1(sp) &= \left[s^\dagger \otimes \tilde{p}\right]^1 , \qquad G^1(ps) = \left[p^\dagger \otimes \tilde{s}\right]^1 , \\
G^2(\sigma d) &= \left[\sigma^\dagger \otimes \tilde{d}\right]^2 , \qquad G^2(d\sigma) = \left[d^\dagger \otimes \tilde{\sigma}\right]^2 , \\
G^k(dp) &= \left[d^\dagger \otimes \tilde{p}\right]^k , \qquad k = 1, 2, 3 , \\
G^k(pd) &= \left[p^\dagger \otimes \tilde{d}\right]^k , \qquad k = 1, 2, 3 .
\end{aligned} \tag{9}
$$

### *3.2. The U(10) subalgebras*

There are five ways to combine the four bosons to obtain subalgebras of U(10) (rows 1 and 2 of Table 1). If the *sd* or $p\sigma$ bosons are completely dropped from (7), then two other subalgebras appear: $\mathrm{U}(10) \supset \mathrm{U}_{sd}(6)$ and $\mathrm{U}(10) \supset \mathrm{U}_{p\sigma}(4)$ [47]. The former does not generate negative parity states, while the latter does not include the quadrupole degree of freedom.

The following levels of reduction include subalgebras that are either already widely known or given in [43].

Rows 1–3 of Table! 1 show the subalgebras resulting from the addition of a second $l = 0$ boson ($\sigma$ boson). This means that the $\mathrm{U}_{sdp\sigma}(10)$ model is significantly richer than $\mathrm{U}_{sdp}(9)$, which justifies the introduction of the $\sigma$ boson. In order to build a model describing quadruole–dipole activity using a chain of subalgebras of U(10), it is necessary to make a compilation of the different chains of subalgebras involved.

There are two kinds of maximal subalgebras; they are either regular or irregular. The former (simple or non-simple) corresponds to the different ways of grouping the different bosons: there are 5 if we consider *s* and $\sigma$ bosons exchangeable, or 7 if not (rows 1 and 2 of Table 1). Among these subalgebras the subalgebras of $\mathrm{U}_p(3) \otimes \mathrm{U}_{sd\sigma}(7)$ generate states with well-defined parity.

We identified four dynamical symmetries (Figs. 1–4): the $\mathrm{SU}_{pd}(3)$ limit (Ia), the $\mathrm{SU}_{spd}(3)$ limit (Ib), the $\mathrm{U}_{pd}(5)$ limit (II), and the $\mathrm{SO}_{pd}(5)$ limit (III).

Table 1. The different subalgebras of U(10).

| Algebra | Subalgebras | | | |
|---|---|---|---|---|
| $U(10)$ | $U_{spd}(9) \otimes U_{\sigma}(1)$ | $U_{pd}(8) \otimes U_{s\sigma}(2)$ | $U_{sd\sigma}(7) \otimes U_{p}(3)$ | $U_{sd}(6) \otimes U_{p\sigma}(4)$ |
| | $U_{sp\sigma}(5) \otimes U_{d}(5)$ | | | |
| | $SO_{spd\sigma}(10)$ | $SU(2) \otimes SU(5)$ | | |
| $U_{spd}(9)$ | $U_{pd}(8) \otimes U_{s}(1)$ | $U_{sd}(6) \otimes U_{p}(3)$ | $U_{d}(5) \otimes U_{sp}(4)$ | |
| | $SO_{spd}(9)$ | $SU(3) \otimes SU(3)$ | | |
| $U_{pd}(8)$ | $U_{p}(3) \otimes U_{d}(5)$ | $SO_{pd}(8)$ | $SU(2) \otimes SU(4)$ | |
| $U_{sd\sigma}(7)$ | $U_{sd}(6) \otimes U_{\sigma}(1)$ | $U_{d}(5) \otimes U_{s\sigma}(2)$ | $SO_{sd\sigma}(7)$ | |
| $U_{sd}(6)$ | $U_{d}(5)$ | $SU_{sd}(3)$ | $SO_{sd}(6)$ | |
| $U_{d}(5)$ | $SO_{d}(5)$ | | | |
| $U_{s\sigma p}(5)$ | $U_{\sigma p}(4) \otimes U_{s}(1)$ | $U_{p}(3) \otimes U_{s\sigma}(2)$ | $SO_{\sigma ps}(5)$ | |
| $U_{\sigma p}(4)$ | $U_{p}(3)$ | $SO_{\sigma p}(4)$ | | |
| $U_{p}(3)$ | $SO_{p}(3)$ | | | |
| $SO_{spd\sigma}(10)$ | $SO_{p}(3) \otimes SO_{sd\sigma}(7)$ | $SO_{d}(5) \otimes SO_{sp\sigma}(5)$ | $SO_{p\sigma}(4) \otimes SO_{sd}(6)$ | $SO_{s\sigma}(2) \otimes SO_{pd}(8)$ |
| $SO_{spd}(9)$ | $SO_{\sigma p}(4) \otimes SO_{d}(3)$ | $SO_{p}(3) \otimes SO_{sd}(6)$ | $SO_{pd}(8)$ | |
| $SO_{pd}(8)$ | $SU_{pd}(3)$ | $SO_{p}(3) \otimes SO_{d}(5)$ | | |
| $SO_{sd\sigma}(7)$ | $SO_{s\sigma}(2) \otimes SO_{d}(5)$ | $SO_{sd}(6)$ | | |
| $SO_{sd}(6)$ | $SO_{d}(5)$ | | | |
| $SO_{ps\sigma}(5)$ | $SO_{s\sigma}(2) \otimes SO_{p}(3)$ | $SO_{p\sigma}(4) \otimes SO_{s}(1)$ | | |
| $SO_{d}(5)$ | $SO_{d}(3)$ | | | |
| $SO_{p\sigma}(4)$ | $SO_{p}(3)$ | | | |

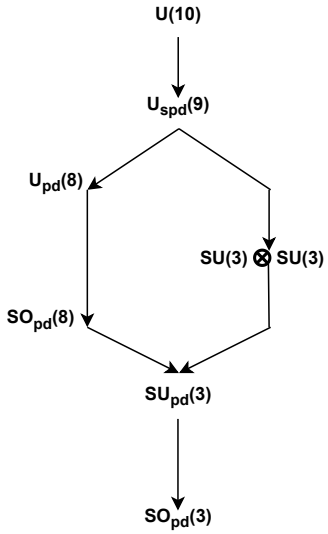


Fig. 1. The $SU_{pd}(3)$ limit (Ia).

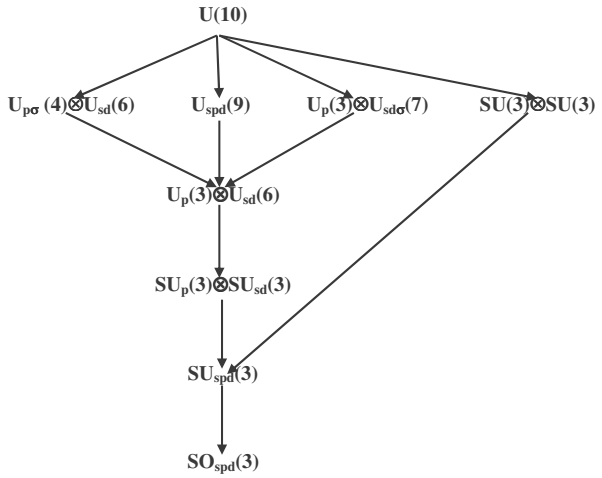


Fig. 2. The SU$_{spd}$(3) limit (Ib).

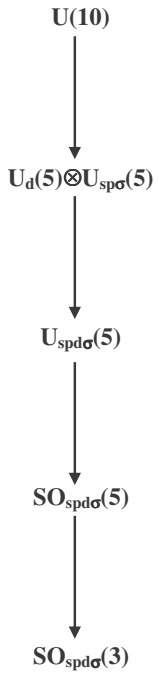


Fig. 3. The U$_{pd}$(5) limit (II).

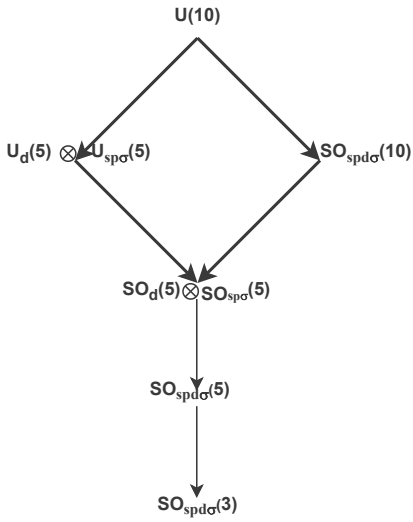


Fig. 4. The $\mathrm{SO}_{pd}(5)$ limit (III).

## 4. Clustering in the framework of IBM-I

### *4.1. U(10)⊃ U(6) ⊗ U(4): coexistence of cluster configurations*

Consider a heavy-deformed nucleus that can cluster into a core and an alpha particle. The internal structure of the core is described by the U(6) algebra (quadrupole deformation), and the cluster is assumed to be spherical with excited levels occurring at very large energies in comparison to those of interest.

The dynamics of the system is described by $\mathrm{U}(6) \otimes \mathrm{U}(4)$. The U(6) algebra has three possible chains of subalgebras [1]: the U(5) limit describing vibrational nuclei [3], the SU(3) limit describing axially-deformed ones [4], and the O(6) limit for $\gamma$-unstable nuclei [5]. The U(4) algebra has two dynamical symmetries [11], the U(3) limit associated with non-rigid vibrator (harmonic motion) and SO(4) describes rigid vibrator.

This model was introduced by Daley and Iachello (1986) [18] based on the Ikeda diagram presented in 1968 [48] for $N = Z$ nuclei. In the Ikeda diagram, a heavy nucleus may exhibit states described as a one cluster–core system, where the cluster is made of zero, one or more alpha particles, which we call horizontal subdivision (see [16], Fig. 2). One describes such a situation as the coexistence of configurations.

The Hamiltonian in this $\mathrm{U}(6)\otimes\mathrm{U}(4)$ model describing both dipole collectivities is given by the sum of two separate terms

$$\begin{aligned}\hat{H}_{sd} = {} & \varepsilon_s\left[s^\dagger\otimes\tilde{s}\right]^0+\varepsilon_d\left[d^\dagger\otimes\tilde{d}\right]^0\\ & +\sum_{L=0,2,4}\frac{1}{2}\sqrt{2L+1}c_L\left[\left[d^\dagger\otimes d^\dagger\right]^L\otimes\left[\tilde{d}\otimes\tilde{d}\right]^L\right]^0\\ & +\frac{1}{\sqrt{2}}\tilde{\nu}_2\left[\left[d^\dagger\otimes\ d^\dagger\right]^2\otimes\left[\tilde{d}\otimes\tilde{s}\right]^2+\left[d^\dagger\otimes s^\dagger\right]^2\otimes\left[\tilde{d}\otimes\tilde{d}\right]^2\right]^0\\ & +\frac{1}{2}\tilde{\nu}_0\left[\left[d^\dagger\otimes\ d^\dagger\right]^2\otimes[\tilde{s}\otimes\tilde{s}]^2+\left[d^\dagger\otimes\ d^\dagger\right]^2\otimes\left[\tilde{d}\otimes\tilde{d}\right]^2\right]^0\\ & +u_2\left[\left[d^\dagger\otimes\ s^\dagger\right]^2\otimes\left[\tilde{d}\otimes\tilde{s}\right]^2\right]^0+\frac{1}{2}u_0\left[\left[s^\dagger\otimes\ s^\dagger\right]^0\otimes[\tilde{s}\otimes\tilde{s}]^0\right]^0\end{aligned}\tag{10}$$

and

$$\begin{aligned}\hat{H}_{\sigma p} = {} & \varepsilon_\sigma\left[\sigma^\dagger\otimes\tilde{\sigma}\right]^0+\varepsilon_p\left[p^\dagger\otimes\tilde{p}\right]^0\\ & +\sum_{L=0,2}\frac{1}{2}\sqrt{2L+1}c'_L\left[\left[p^\dagger\otimes p^\dagger\right]^L\otimes[\tilde{p}\otimes\tilde{p}]^L\right]^0\\ & +\frac{1}{2}\tilde{\nu'}_0\left[\left[p^\dagger\otimes\ p^\dagger\right]^0\otimes[\tilde{\sigma}\otimes\tilde{\sigma}]^0+\left[\sigma^\dagger\otimes\ \sigma^\dagger\right]^0\otimes[\tilde{p}\otimes\tilde{p}]^0\right]^0\\ & +u'_1\left[\left[p^\dagger\otimes\ \sigma^\dagger\right]^1\otimes[\tilde{p}\otimes\tilde{\sigma}]^1\right]^0+\frac{1}{2}u'_0\left[\left[\sigma^\dagger\otimes\ \sigma^\dagger\right]^0\otimes[\tilde{\sigma}\otimes\tilde{\sigma}]^0\right]^0.\end{aligned}\tag{11}$$

The Hamiltonian in (10) and (11) can be written as a combination of the Casimir operator of the chain of subalgebra of $\mathrm{U}(6)\otimes\mathrm{U}(4)$. However, such a Hamiltonian is still too general and a numerical treatment is necessary. This work will be achieved mainly in the framework of the SU(3) dynamical symmetry.

The SU(3) limit of $\mathrm{U}(6)\otimes\mathrm{U}(4)$ corresponds to the chain of subalgebra

$$\begin{aligned}&\mathrm{U}(10)\supset\mathrm{U}(6)\otimes\mathrm{U}(4)\supset\mathrm{SU}_{sd}(3)\otimes\mathrm{U}_p(3)\\ &\supset\mathrm{SU}_{sd}(3)\otimes\mathrm{SU}_p(3)\supset\mathrm{SU}_{spd}(3)\supset\mathrm{SO}_{spd}(3).\end{aligned}\tag{12}$$

The Hamiltonian expression in the SU(3) limit is

$$\begin{aligned}\hat{H}_0 \ = \ & a\hat{C}_2[\mathrm{U}(6)]+b\hat{C}_2[\mathrm{U}(4)]+k_d\hat{C}_2[\mathrm{SU}_{sd}(3)]\\ & +\alpha_p\hat{C}_2[\mathrm{SU}_p(3)]+k\hat{C}_2[\mathrm{SU}_{spd}(3)]+k_L\hat{C}_2[\mathrm{SO}_{spd}(3)].\end{aligned}\tag{13}$$

The boson number $N_{\mathrm{T}}$ of the nucleus is half the number of its valence nucleons. If we denote the number of $\alpha$ particles in the cluster by $n_\alpha$, then the number of bosons in the remaining nucleus (called core) is $N = N_{\mathrm{T}} - 2n_\alpha$. These variable numbers are defined as in [13, 16–18]. Two classes of states will be considered: $0\alpha$ states and $1\alpha$ states. The number of alpha particles in the cluster is supposed to increase with the excitation energy [18] so that $N_{\mathrm{T}} = N + 2n_\alpha$ is constant. The U(4) irreps are labelled $M$. In the work of Daley [16], the total number of bosons is $N_{\mathrm{T}}$, while the adopted expression for $M$ is $M = 2n_\alpha$.

The Hamiltonian (13) is diagonal in the basis

$$|(N_{\mathrm{T}}), N, M, (\lambda,\mu)_{sd}, (N_p, 0), (\lambda,\mu), \chi, L\rangle\,. \tag{14}$$

The values $(\lambda,\mu)$ of $\mathrm{SU}_{sd}(3)$ contained in a symmetric representation $[N, 0]$ of U(6) are

$$(\lambda,\mu) = [2N, 0] \oplus [2N-4, 2] \oplus [2N-8, 4] \oplus \ldots \tag{15}$$

The product of two SU(3) representations $(\lambda,\mu)_{sd} \otimes (N_p, 0)$ is given by O'Reilly [49]

$$(\lambda_1,\mu_1) \oplus (\lambda_2,\mu_2) = \sum_{k=0}^{\mu_2,\lambda_1+\mu_1} \sum_{j=0}^{\mu_1,\lambda_2,\lambda_1+\mu_1-k} \sum_{i=0,j=\mu_1+k}^{\lambda_2-j+k,\lambda_1} (\lambda_1+\lambda_2-j-2i, \mu_1+\mu_2+i-j-2k)\,. \tag{16}$$

The U(4) to SU(3) reduction gives $N_p = 0, 1, \ldots M$. The step from SU(3) to SO(3) is well known [1].

The eigenvalues of (13) are given by

$$E_0 = C + k_d C_2(\lambda,\mu)_{sd} + \alpha_p N_p(N_p+3) + kC_2(\lambda,\mu) + k_L L(L+1)\,. \tag{17}$$

The energy spectrum in the SU(3) limit is given in Table 2, where the labels of Daley are adopted [16]. It can be summarised as follows:

*(i)* The case $n_\alpha = 0$ is trivial, since it corresponds to the spectrum of a no clustered nucleus ($0\alpha$ cluster configuration) and can be described using only the *sd*-IBM.

*(ii)* The value $n_\alpha = 1$ generates negative parity states and hence, it is associated with a $1\alpha$ cluster, since every alpha particle is considered as a set of two bosons as it was proposed by Daley *et al.* Using Daley's labelling, the $n_\alpha = 1 (M = 2)$ case includes the previous spectrum (for

$N_p = 0$) and, at higher energy ($N_p = 1$), a set of negative parity states belonging to the bands: $K^\pi = 0^-$, $1^-$, $2^-$ and $3^-$. Moreover, for $N_p = 2$, it generates extra positive parity states. The exciting case is the one with $K^\pi = 1^+$ and $L = 1, 2, \ldots$

The $M = 0$ and 2 configurations are the leading ones since they generate the low-lying collective states. This situation is different from the case of molecular spectroscopy, where $M$ is large.

Table 2. Classification of zero- and one-alpha cluster bands, using Daley's labels [16].

| $(\lambda,\mu)_{sd}$ | $(N_p, 0)$ | $(\lambda,\mu)$ | $K^\pi$ | $L^\pi$ |
|---|---|---|---|---|
| Zero-alpha configurations $M = 0$ | | | | |
| $(2N_\mathrm{T}, 0)$ | $(0,0)$ | $(2N_\mathrm{T}, 0)$ | $0^+$ | $0^+,\ 2^+,$ |
| $(2N_\mathrm{T}-4, 2)$ | $(0,0)$ | $(2N_\mathrm{T}-4, 2)$ | $2^+$ | $2^+,\ 3^+,$ |
| | | | $0^+$ | $0^+,\ 2^+,$ |
| One-alpha configurations $M = 2$ | | | | |
| $(2(N_\mathrm{T}-2), 0)$ | $(0,0)$ | $(2(N_\mathrm{T}-2), 0)$ | $0^+$ | $0^+,\ 2^+,$ |
| $(2(N_\mathrm{T}-2)-4, 2)$ | $(0,0)$ | $(2(N_\mathrm{T}-2)-4, 2)$ | $2^+$ | $2^+,\ 3^+,$ |
| | | | $0^+$ | $0^+,\ 2^+,$ |
| $(2(N_\mathrm{T}-2), 0)$ | $(1,0)$ | $(2(N_\mathrm{T}-2)+1, 0)$ | $0^-$ | $1^-,\ 3^-,$ |
| | | $(2(N_\mathrm{T}-2)-1, 1)$ | $1^-$ | $1^-,\ 2^-,$ |
| $(2(N_\mathrm{T}-2)-4, 2)$ | $(1,0)$ | $(2(N_\mathrm{T}-2)-4+1, 2)$ | $2^-$ | $2^-,\ 3^-,$ |
| | | $(2(N_\mathrm{T}-2)-4-1, 3)$ | $3^-$ | $3^-,\ 4^-,$ |
| | | $(2(N_\mathrm{T}-2)-4-1, 1)$ | $1^-$ | $1^-,\ 2^-,$ |
| $(2(N_\mathrm{T}-2), 0)$ | $(2,0)$ | $(2(N_\mathrm{T}-2)+2, 0)$ | $0^+$ | $0^+,\ 2^+,$ |
| | | $(2(N_\mathrm{T}-2), 1)$ | $1^+$ | $1^+,\ 2^+,$ |
| | | $(2(N_\mathrm{T}-2)-2, 2)$ | $2^+$ | $2^+,\ 3^+,$ |
| | | | $0^+$ | $0^+,\ 2^+,$ |

### *4.2. Mixing of cluster configurations and symmetry breaking*

Up till now, the treatment is carried out in the framework of the subalgebra $\mathrm{U}(6) \otimes \mathrm{U}(4)$. It leads to configurations with a well-definite $M$ (and $N$) boson numbers separably, the ground band, for example, is associated with $M = 0$. However, the alpha decay shows that the ground configuration is actually contaminated by the $1\alpha$ cluster configuration. Even more, the cluster decay indicates the effect of $m\alpha$ configurations. This means that the actual configurations are mixtures of the $\mathrm{U}(6) \otimes \mathrm{U}(4)$ ones. This mixing can be taken into account by terms beyond $\mathrm{U}(6) \otimes \mathrm{U}(4)$. Indeed, the total Hamiltonian having U(10) as a symmetry algebra includes a third expression

(called the mixing Hamiltonian) describing a mixing between the $sd$ and $\sigma p$ bosons. Such expression will mix the previous configurations, and hence will break the U(6) ⊗ U(4) symmetry

$$\begin{aligned}\hat{H}_{\mathrm{C}} = \omega_0 &\left[\left[s^\dagger \otimes \sigma^\dagger\right]^0 \otimes [\tilde{s} \otimes \tilde{\sigma}]^0 + \left[\sigma^\dagger \otimes s^\dagger\right]^0 \otimes [\tilde{\sigma} \otimes \tilde{s}\,]^0\right]^0 \\ &+ \sum_{L=1,2,3} \omega_L \sqrt{2L+1} \left[\left[d^\dagger \otimes p^\dagger\right]^L \otimes \left[\tilde{d} \otimes \tilde{p}\right]^L + \left[p^\dagger \otimes d^\dagger\right]^L \otimes \left[\tilde{p} \otimes \tilde{d}\right]^L\right]^0 \\ &+ \omega_4 \left[\left[s^\dagger \otimes p^\dagger\right]^1 \otimes \left[\tilde{d} \otimes \tilde{p}\right]^1 + \left[p^\dagger \otimes d^\dagger\right]^1 \otimes [\tilde{p} \otimes \tilde{s}\,]^1\right]^0 \\ &+ \omega_5 \left[\left[s^\dagger \otimes p^\dagger\right]^1 \otimes [\tilde{s} \otimes \tilde{p}]^1 + \left[p^\dagger \otimes \ s^\dagger\right]^1 \otimes [\tilde{p} \otimes \tilde{s}\,]^1\right]^0 \\ &+ \omega_6 \left[\left[d^\dagger \otimes \sigma^\dagger\right]^2 \otimes \left[\tilde{d} \otimes \tilde{\sigma}\right]^2 + \left[\sigma^\dagger \otimes d^\dagger\right]^2 \otimes \left[\tilde{\sigma} \otimes \tilde{d}\right]^2\right]^0 . \end{aligned} \tag{18}$$

In his work [16], Daley considered only the first term in (18) that can be written as

$$V_{0\alpha-1\alpha} = \gamma \left[\left[\sigma^\dagger \otimes \sigma^\dagger\right]^0 \otimes [\tilde{s} \otimes \tilde{s}]^0 + \left[s^\dagger \otimes s^\dagger\right]^0 \otimes [\tilde{\sigma} \otimes \tilde{\sigma}\,]^0\right]^0 . \tag{19}$$

The calculated effect is given by Daley and Iachello [16]

$$\Delta E = \left[\left[\Delta k_L L(L+1) - \Delta'_\alpha\right]^2 + \frac{2}{9}\gamma^2 \varphi^2\right]^{1/2} , \tag{20}$$

where $\varphi = [\frac{(2N_{\mathrm{T}}+L+1)(2N_{\mathrm{T}}+L-1)(2N_{\mathrm{T}}-L)(2N_{\mathrm{T}}+L-2)}{(2N_{\mathrm{T}}-3)(2N_{\mathrm{T}}-1)}]^{1/2}$, and $\Delta'_\alpha$ is the difference in energy between the two configurations.

The mixing occurs only between states of the same parity. Since the $0\alpha$ states have a positive parity, the negative parity states (belonging to the $1\alpha$ cluster configuration) are not mixed if higher configurations are neglected. The main effect on the levels is a differential shift of the $1\alpha$ configuration states with respect to $0\alpha$ ones.

### *4.3. Electromagnetic transitions*

Electromagnetic transitions are a strong test of any nuclear model. Due to the dipole feature of the nuclear cluster, one expects strong electric dipole transitions ($B$(E1)). Indeed, an electric dipole moment appears because the center of charge and the center of mass are separated. In the present work,

we focused on the dipole transition in order to discuss the possible occurrence of clustering in heavy nuclei

$$T(\mathrm{E1}) = a_1^- \left[\sigma^\dagger \otimes \tilde{p} + p^\dagger \otimes \tilde{\sigma}\right]^{(1)} . \tag{21}$$

The dipole reduced transition probabilities have been calculated, and analytical expressions are available and already used in the actinide region [18]

$$B(\mathrm{E1}; K = 0^-, L \to K = 0^+, L+1) = B(\mathrm{E1}; K = 1^-, L \to K = 0^+, L-1)$$
$$= (a_1^-)^2 n_\sigma \left(\frac{2(L+1)}{2L+1}\right)\left(\frac{\lambda - L + 1}{\lambda + 1}\right) \tag{22}$$
$$B(\mathrm{E1}; K = 0^-, L \to K = 0^+, L-1) = B(\mathrm{E1}; K = 1^-, L \to K = 0^+, L+1)$$
$$= (a_1^-)^2 n_\sigma \left(\frac{2L}{2L+1}\right)\left(\frac{\lambda + L + 2}{\lambda + 1}\right) . \tag{23}$$

Ratios such as

$$R_{L^-} = \frac{B(E1; K = 0^-, L \to K = 0^+, L+1)}{B(E1; K = 0^-, L \to K = 0^+, L-1)} = \frac{(L+1)}{L}\frac{(\lambda - L + 1)}{(\lambda + L + 2)} \tag{24}$$

and

$$R'_{L^-} = \frac{B(E1; K = 1^-, L \to K = 0^+, L+1)}{B(E1; K = 1^-, L \to K = 0^+, L-1)} = \frac{L}{L+1}\frac{(\lambda + L + 2)}{(\lambda - L + 1)} \tag{25}$$

are used instead of absolute E1 transition rates which are very difficult to measure. $R_{L^-}$ and $R'_{L^-}$ expressions involve no parameters to be fitted. Configuration mixing can be taken into account. However, as shown by Daley [18], the effect on the electric transitions is small. Thus, it will not be taken into account in our calculations.

### *4.4. Alternative models*

The U(10) algebra allows us to propose alternative formulations derived from the different chains of subalgebras. The motivation is to extend the description of clustering into different regions of the isotopic chart. Indeed, the $\mathrm{U}(6) \otimes \mathrm{U}(4)$ model describes well clustering in deformed nuclei, while the first experimental evidence occurred in a vibrational (spherical) nucleus. It is then necessary to extend the study to other algebras. We will exclude any case where the coupling occurs only at the level of SO(3).

*Class 1: deformed nuclei, SU(3) limits*

There are two chains of subalgebras with an SU(3) limit. The difference occurs in the reductions from U(10) to the product SU(3) ⊗ SU(3) (Fig. 2)

$$\mathrm{U}(10) \supset \mathrm{U}_{spd}(9) \supset \mathrm{U}_{sd}(6) \otimes \mathrm{U}_p(3) \supset \mathrm{SU}_{sd}(3) \otimes \mathrm{SU}_p(3) \supset \mathrm{SU}_{spd}(3) \supset \mathrm{SO}_{spd}(3)\,, \tag{26}$$

$$|[N_{\mathrm{T}}]; [N_{spd}][N_{sd}][N_p](\lambda,\mu)_p(\lambda,\mu)_{sd}(\lambda,\mu)kL\rangle\,. \tag{27}$$

The U(10) to U(9) reduction is $N_{spd} = N_{\mathrm{T}}, N_{\mathrm{T}}-1, N_{\mathrm{T}}-2, \ldots 0$. The reduction U(9) to U(6) ⊗ U(3) is $N_p + N_{sd} = N_{spd}$

$$\mathrm{U}(10) \supset \mathrm{U}_{sd\sigma}(7) \otimes \mathrm{U}_p(3) \supset \mathrm{U}_{sd}(6) \otimes \mathrm{U}_p(3) \supset \mathrm{SU}_{sd}(3) \otimes \mathrm{SU}_p(3) \supset \mathrm{SU}_{spd}(3) \supset \mathrm{SO}_{spd}(3)\,, \tag{28}$$

$$|[N_{\mathrm{T}}]; [N_{sd\sigma}][N_{sd}][N_p](\lambda,\mu)_{sd}(\lambda,\mu)_p(\lambda,\mu)kL\rangle\,. \tag{29}$$

Here, one deals with only one reduction $\mathrm{U}(10) \supset \mathrm{U}_{sd\sigma}(7) \otimes \mathrm{U}_p(3)$ with the rule $N_{\mathrm{T}} = N_p + N_{sd\sigma}$.

*Class 2: vibrational nuclei, U(5) and SO(5) limits*

The first model (Fig. 3) is based on algebra coupling at the level of U(5)

$$\mathrm{U}(10) \supset \mathrm{U}_d(5) \otimes \mathrm{U}_{sp\sigma}(5) \supset \mathrm{U}_{spd\sigma}(5) \supset \mathrm{SO}_{spd\sigma}(5) \supset \mathrm{SO}_{spd\sigma}(3)\,. \tag{30}$$

The obtained spectrum is the dipole–quadrupole states in the vibrational spectrum, since the core is described by $\mathrm{U}_d(5)$. The irreps can be written as

$$|[N][N_d][N_{sp\sigma}], [n_1,n_2], (v_1,v_2), KL\rangle\,. \tag{31}$$

The eigenvalues are given by

$$\begin{aligned} E \;=\; & A + \xi N_d(N_d+4) + \delta N_{sd\sigma}(N_{sd\sigma}+4) + \varrho[n_1(n_1+4) + n_2(n_2+3)] \\ & + \upsilon[v_1(v_1+3) + v_2(v_2+1)] + \omega L(L+1)\,. \end{aligned} \tag{32}$$

The second possibility results from the coupling at the level of SO(5) (Fig. 4)

$$\mathrm{U}(10) \supset \mathrm{U}_d(5) \otimes \mathrm{U}_{sp\sigma}(5) \supset \mathrm{SO}_d(5) \otimes \mathrm{SO}_{sp\sigma}(5) \supset \mathrm{SO}_{spd\sigma}(5) \supset \mathrm{SO}_{spd\sigma}(3)\,, \tag{33}$$

$$|[N][N_d][N_{sp\sigma}], [n_1,0][n_2,0], (v_1,v_2), KL\rangle\,. \tag{34}$$

The eigenvalues are given here by

$$\begin{aligned} E \;=\; & B + \xi N_d(N_d+4) + \delta N_{sd\sigma}(N_{sd\sigma}+4) + \phi n_1(n_1+3) + \varphi n_2(n_2+3) \\ & + \omega[v_1(v_1+3) + v_2(v_2+1)] + \nu L(L+1)\,. \end{aligned} \tag{35}$$

### *4.5. Result of calculation*

The vibron model has been tested by Daley [18] in the actinide region. In order to assess the accuracy of the model in other regions, a larger list of nuclei was necessary. This led us to study the following: $^{156,158,160}$Gd, and $^{234,236,238}$U, $^{240}$Pu,$^{248}$Cm,$^{250}$Cf. Their $E_4^+/E_2^+$ ratios range from 3.23 ($^{156}$Gd) to 3.321 ($^{250}$Cf).

In his work [18], Daley considered the two leading bands $K^\pi = 0^-$ and $K^\pi = 1^-$ in $^{224-228}$Ra, $^{228-230}$Th, $^{232-238}$U, and $^{238-240}$Pu (the latter band being missing in $^{228}$Ra and $^{240}$Pu).

In the present work, we investigated additional nuclei in the actinide region and extended the calculations to the rare-earth regions. Moreover, our calculations include the $K^\pi = 2^-$ band for the first time. Additional bands are indeed useful to test the coherence of Daley's labelling. The fit has been achieved using the least square technique, taking into account $0\alpha$ and $1\alpha$ configuration mixing. Tables 3 and 4 show the optimised free parameters and Figs. 5–8 give the corresponding results in comparison to the experimental data [50] of the studied nuclei.

Table 3. The empirically determined parameters of the mixing energy (in keV).

| Nucleus | $k_{0\alpha}$ | $k_{1\alpha}$ | $\Delta k$ | $\Delta'_\alpha$ | $\gamma$ |
|---|---|---|---|---|---|
| $^{156}$Gd | 14.828 | 10.541 | 4.287 | 860.579 | 176.063 |
| $^{158}$Gd | 13.252 | 13.205 | 0.047 | 980.85 | 193.48 |
| $^{160}$Gd | 12.543 | 7.684 | 4.859 | 1131.294 | 215.63 |
| $^{234}$U | 7.249 | 6.118 | 1.131 | 664.124 | 131.001 |
| $^{236}$U | 7.54 | 6.308 | 1.232 | 753.72 | 143.70 |
| $^{238}$U | 7.486 | 5.555 | 1.93 | 760.312 | 140.398 |
| $^{238}$Pu | 7.344 | 5.629 | 1.715 | 772 | 142.55 |
| $^{240}$Pu | 7.137 | 5.538 | 1.599 | 705.78 | 126.47 |
| $^{246}$Cm | 7.14 | 5.058 | 2.082 | 963.27 | 159.238 |
| $^{250}$Cf | 7.12 | 7.369 | 0.249 | 946.476 | 149.22 |

Table 4. Free parameters (in keV) of the U(6) ⊗ U(4) Hamiltonian.

| Nucleus | $\alpha_p$ | $k_d$ | $k$ | $k_L$ | r.m.s. |
|---|---|---|---|---|---|
| $^{156}$Gd | 482.396 | −3.995 | 2.519 | 10.541 | 32.2 |
| $^{158}$Gd | 799.219 | −8.125 | 4.199 | 13.204 | 4.4 |
| $^{160}$Gd | 761.070 | −0.989 | −1.713 | 7.684 | 7.2 |
| $^{234}$U | 438.154 | 8.431 | −9.390 | 6.118 | 1.2 |
| $^{236}$U | 660.277 | 1.026 | −3.761 | 6.308 | 2.9 |
| $^{238}$U | 729.451 | 0.331 | −3.087 | 5.555 | 2.1 |
| $^{238}$Pu | 473.964 | 3.016 | −4.415 | 5.629 | 0.1 |
| $^{240}$Pu | 1013.231 | 0.198 | −3.923 | 5.538 | 2.4 |
| $^{246}$Cm | 807.349 | −3.287 | 1.614 | 5.058 | 1.9 |
| $^{250}$Cf | 685.648 | 4.026 | −5.177 | 7.369 | 21.5 |

We have calculated relative E1 transition probabilities, using equations of Section 4. As already mentioned there, generally only the relative data are available. The comparison of the theoretical and experimental data could be achieved in $^{156}$Gd,$^{158}$Gd,$^{160}$Gd, and $^{238}$U. Table 5 shows a good theory–experiment agreement.

Table 5. Experimental [50–52] and theoretical ratios of $B$(E1) reduced transition probabilities (for $^{156}$Gd, $^{158}$Gd, $^{160}$Gd, and $^{238}$U) using the SU(3) limit of the U(6) ⊗ U(4) model for ($K^\pi = 0^- \to K^\pi = 0^+$).

| | $^{156}$Gd | | $^{158}$Gd | | $^{160}$Gd | | $^{238}$U | |
|---|---|---|---|---|---|---|---|---|
| | exp. | th. | exp. | th. | exp. | th. | exp. | th. |
| $\frac{1^-\to 2^+}{1^-\to 0^+}$ | 2.26(115) | 1.78 | 1.83(87) | 1.76 | $1.80^{+40}_{-34}$ | 1.74 | 1.57(79) | 1.79 |
| $\frac{3^-\to 4^+}{3^-\to 2^+}$ | 1.63(64) | 1.01 | $1.37^{+65}_{-42}$ | 0.99 | $0.87^{+17}_{-14}$ | 0.96 | — | 1.03 |

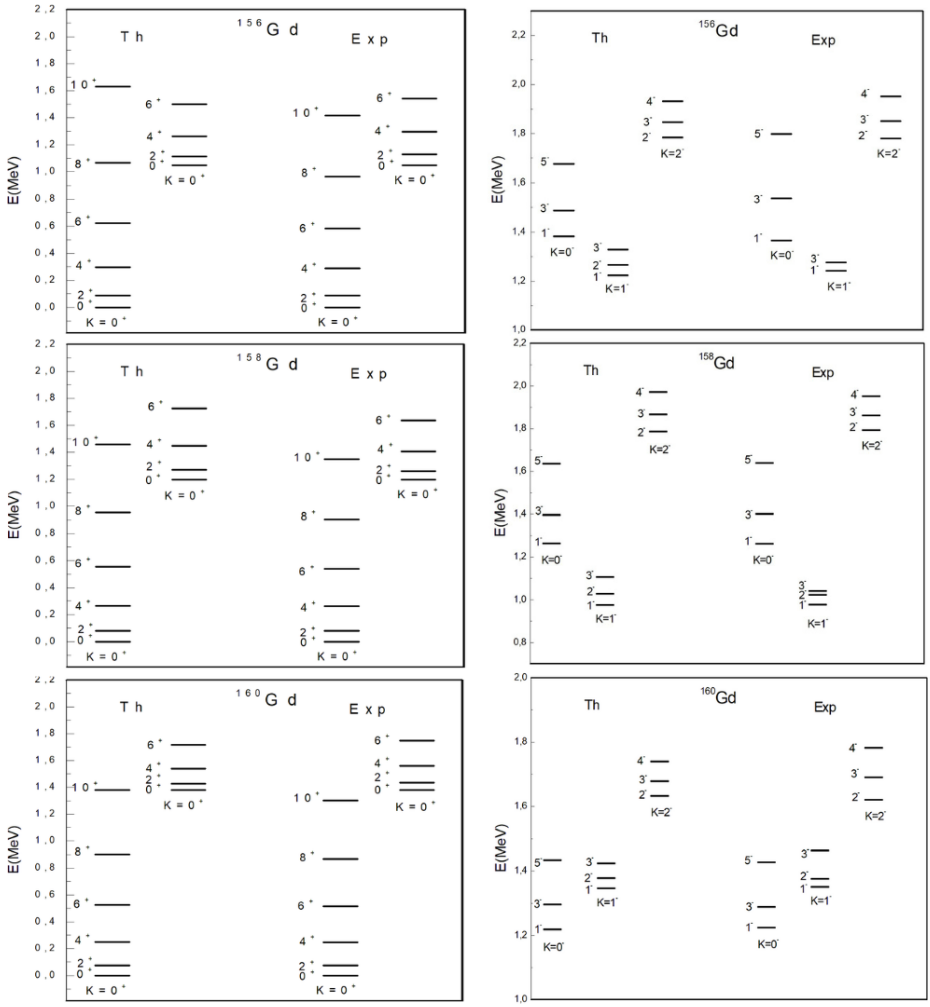


Fig. 5. Theoretical spectrum of $^{156-160}$Gd compared to the experimental data [50]. The left side includes the mixing, while the right side shows a pure $1\alpha$ configuration.

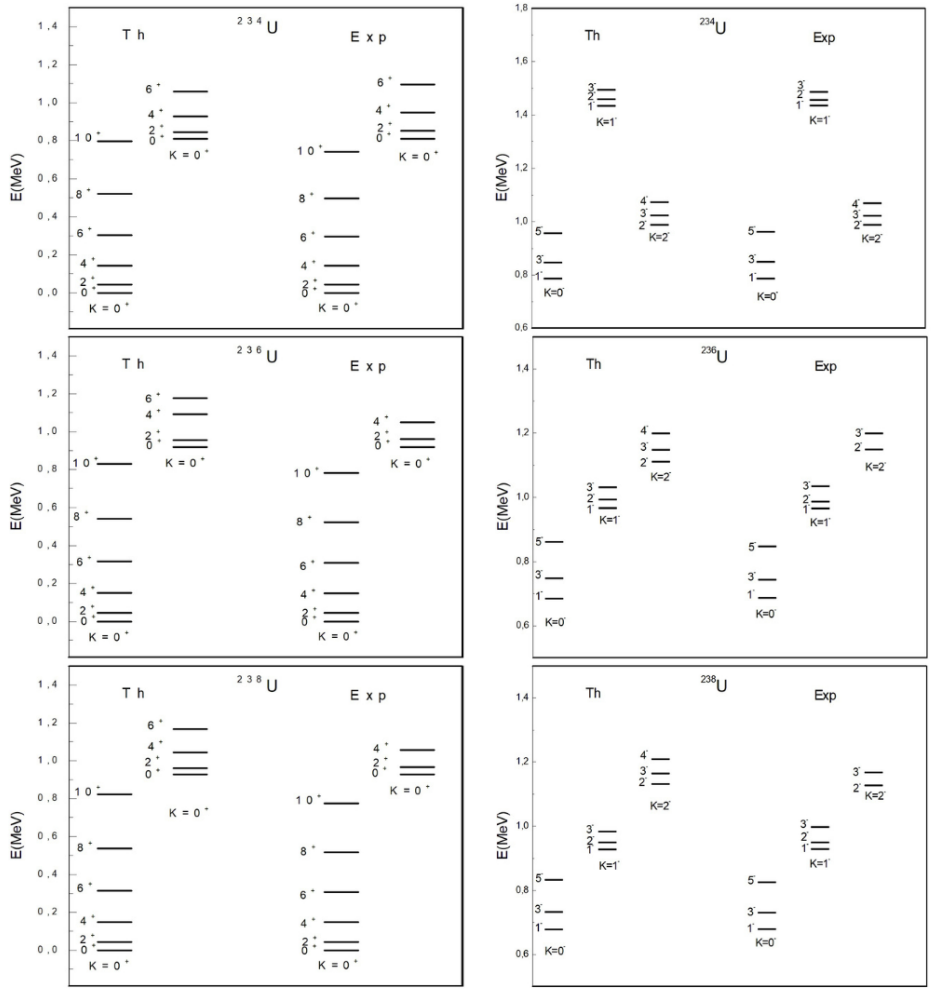


Fig. 6. The same as Fig. 5, but for $^{234-238}$U.

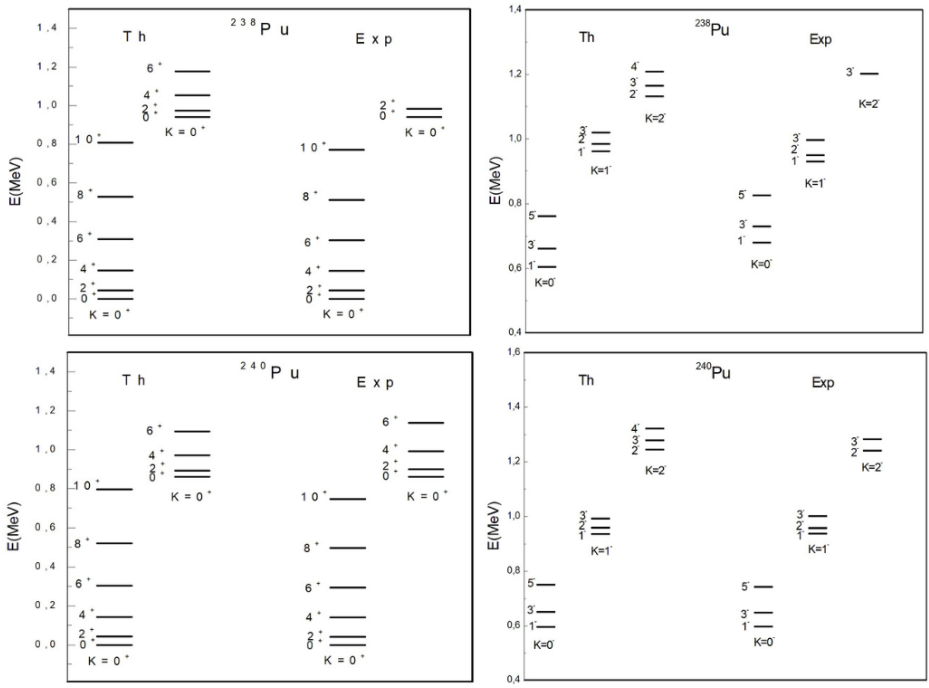


Fig. 7. The same as Fig. 5, but for $^{238-240}$Pu.

### *4.6. The problem of the algebra irrep labels and the Wildermuth condition*

In the work of Daley [16], the adopted expression for $M$ leads to values of $N_p$ starting from zero and independent of the studied nucleus. From the microscopic point of view, this feature represents a violation of the Pauli exclusion principle. Indeed, if the cluster and the core were sufficiently far apart, they could be regarded as two independent quantum systems, each with its own set of occupied states. However, when the cluster approaches the core closely enough to form a common system, the nucleon wave functions begin to overlap significantly. In such a situation, identical nucleons from the cluster and the core would, without the Pauli principle, be allowed to occupy the same quantum states, which is forbidden in fermionic systems. To heal this violation, it is then important to review the vibron model to take into account the Wildermuth condition [53]. This condition ensures that the relative motion between the cluster and the core contains a sufficient number of oscillator quanta to prevent forbidden overlaps of nucleon states, otherwise it is necessary the existence of a minimal value for $N_p$ describing the relative motion of the cluster and the core.

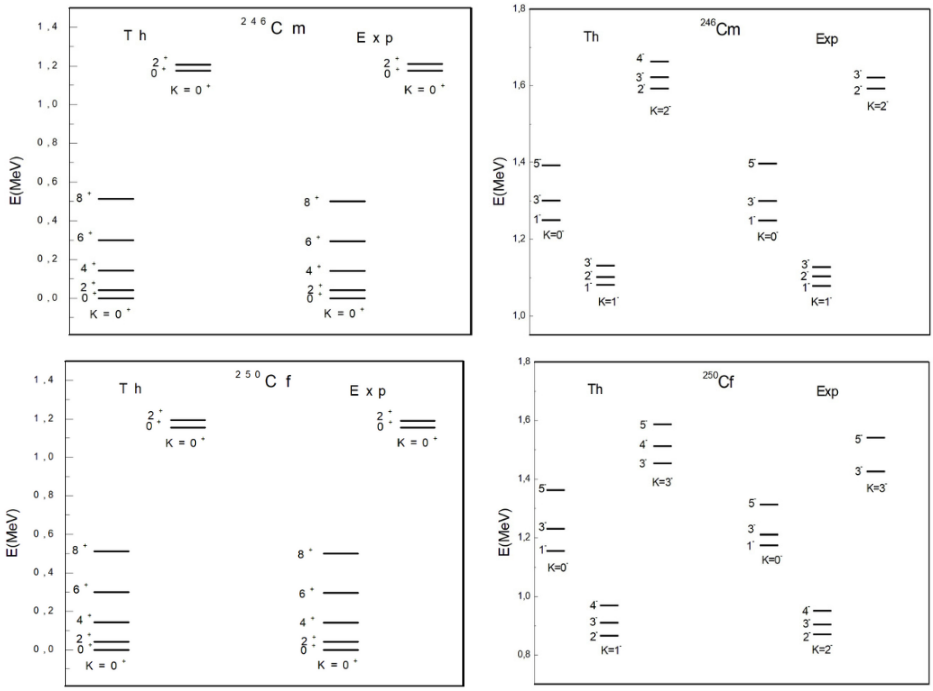


Fig. 8. The same as Fig. 5, but for $^{246}$Cm and $^{250}$Cf.

A solution to this problem has been introduced by Cseh in the framework of the semimicroscopic algebraic clustering model (SACM) [54, 55]. In such an approach, the core is described by the SU(3) shell model, and the Wildermuth condition, related to the Pauli principle, results from the distribution of the nucleon over the harmonic oscillator major shells. The SACM model has been mainly limited to light nuclei. In heavy nuclei, the situation is more complicated, and the pseudo SU(3) symmetry is necessary [56].

In our work, investigating heavy nuclei, the core is described by a truncated version of the shell model (IBM). Hence, we can improve the traditional vibron model by taking into account the Wildermuth condition, as is done in the SACM model. If U(10) irreps are labelled $N_{10}$, the question is how to express $M$ and $N_{10}$ as a function of $n_\alpha$ and $N_{\mathrm{T}}$?

To solve this problem, we adopted a generalisation of Daley's labelling as follows:

$$\begin{aligned} M &= n_\alpha(N_p + k)\,, \qquad k = 0, 1, \ldots, \\ N_{10} &= N_{\mathrm{T}} + n_\alpha R\,. \end{aligned} \tag{36}$$

The embedding in U(10) guarantees that $N_{10} = N + M$ is constant. As a result of Eq. (36), for $n_\alpha = 0$, one gets $M = 0$ and $N = N_{\rm T} = N_{10}$. This situation corresponds to the absence of clustering. On the other hand, for $n_\alpha = 1$, one gets $M = N_p + k$, $N = N_{\rm T} - 2$, and $N_{10} = N_{\rm T} + R$. The Wildermuth condition is then fulfilled, providing that $R \geq N_{p,\,\rm min}$ and hence, $N_{10} \geq N_{\rm T} + N_{p,\,\rm min}$.

## 5. IBM-II formulation of clustering

Up till now, the algebraic treatment of clustering has been formulated in IBM-I, where no distinction is made between the proton and the neutron bosons. However, it is known that IBM-II is richer and more predictive. In 2000, an *sdf*-IBM-II model was introduced [57]. Later, in 2013, a work on the IBM-II vibron model was published by Zouioueche and Bouldjedri [58], where for $p$ bosons, only one kind of bosons is considered. It has been shown that without configuration mixing a non-zero $f$ spin leads to both positive and negative mixed symmetry states. More recently, an *spdf*-IBM-II model was introduced by Vallejos and Barea [59] and applied to $^{214-226}$Rn, but without consideration of the corresponding dynamical symmetries.

In the present work, we will formulate the IBM-II version of the U(6) $\otimes$ U(4) model by generalising the work of [58] to $\mathrm{U}_{sd,\pi}(6) \otimes \mathrm{U}_{sd,\nu}(6) \otimes \mathrm{U}_{\sigma p,\pi}(4) \otimes \mathrm{U}_{\sigma p,\nu}(4)$.

In addition to the standard algebras, we also consider their conjugate algebras (denoted with a bar), which arise naturally in the context of particle–hole transformations. These conjugate algebras correspond to the same Lie algebra but are realized in the dual representation. When a standard algebra and a conjugate algebra are combined to form a single, coupled algebra, the resulting algebra is typically denoted with a star (see, for example, [60]).

Thus, for the U(6) algebra, we have

$$\mathrm{U}_\pi(6) \otimes \mathrm{U}_\nu(6) \supset \mathrm{U}_{\pi+\nu}(6) \supset \mathrm{SU}_{\pi+\nu}(3) \supset \mathrm{SO}_{\pi+\nu}(3)\,, \tag{37}$$

$$\bar{\mathrm{U}}_\pi(6) \otimes \mathrm{U}_\nu(6) \supset \mathrm{U}^*_{\pi+\nu}(6) \supset \mathrm{SU}_{\pi+\nu}(3) \supset \mathrm{SO}_{\pi+\nu}(3)\,, \tag{38}$$

$$\mathrm{U}_\pi(6) \otimes \mathrm{U}_\nu(6) \supset \mathrm{SU}_\pi(3) \otimes \mathrm{SU}_\nu(3) \supset \mathrm{SU}^*_{\pi+\nu}(3) \supset \mathrm{SO}_{\pi+\nu}(3)\,, \tag{39}$$

$$\bar{\mathrm{U}}_\pi(6) \otimes \mathrm{U}_\nu(6) \supset \bar{\mathrm{SU}}_\pi(3) \otimes \mathrm{SU}_\nu(3) \supset \mathrm{SU}^*_{\pi+\nu}(3) \supset \mathrm{SO}_{\pi+\nu}(3)\,, \tag{40}$$

while for the U(4) algebra,

$$\mathrm{U}_\pi(4) \otimes \mathrm{U}_\nu(4) \supset \mathrm{U}_{\pi+\nu}(4) \supset \mathrm{SU}_{\pi+\nu}(3) \supset \mathrm{SO}_{\pi+\nu}(3)\,, \tag{41}$$

$$\mathrm{U}_\pi(4) \otimes \mathrm{U}_\nu(4) \supset \mathrm{SU}_\pi(3) \otimes \mathrm{SU}_\nu(3) \supset \mathrm{SU}_{\pi+\nu}(3) \supset \mathrm{SO}_{\pi+\nu}(3)\,. \tag{42}$$

Thus, there are many possible couplings in comparison to the case where only one kind of $p$ boson is used.

We have already mentioned that the number of bosons $N$ in the core is variable; only the total number of bosons is constant. This number $N$ decreases with the number of bosons in the cluster. The U(10) algebra guarantees the constancy of the total number of bosons.

### *5.1. The U(20) algebra*

To formulate the IBM-II version, four boson numbers are necessary: $N_\pi$, $N_\nu$,$M_\pi$, and $M_\nu$. The boson number constancy is written now: $N_{\mathrm{T}} = N_\pi + N_\nu + M_\pi + M_\nu = N + M$. There are two ways of embedding: U(20) $\supset$ U(10) $\otimes$ U(10) or U(20) $\supset$ U(12) $\otimes$ U(8).

The U(20) algebra has 400 generators that can be written in coupled form as follows:

$$\begin{aligned}
G^0\left(s_\rho s_{\rho'}\right) &= \left[s^\dagger_\rho \otimes \tilde{s_{\rho'}}\right]^0, \\
G^k\left(d_\rho d_{\rho'}\right) &= \left[d^\dagger_\rho \otimes \tilde{d_{\rho'}}\right]^k, \qquad k = 0,1,2,3,4, \\
G^2\left(d_\rho s_{\rho'}\right) &= \left[d^\dagger_\rho \otimes \tilde{s_{\rho'}}\right]^2, \qquad G^2\left(s_\rho d_{\rho'}\right) = \left[s^\dagger_\rho \otimes \tilde{d_{\rho'}}\right]^2,
\end{aligned} \tag{43}$$

$$\begin{aligned}
G^0\left(\sigma_\rho \sigma_{\rho'}\right) &= \left[\sigma^\dagger_\rho \otimes \tilde{\sigma_{\rho'}}\right]^0 \\
G^k\left(p_\rho p_{\rho'}\right) &= \left[p^\dagger_\rho \otimes \tilde{p_{\rho'}}\right]^k, \qquad k = 0,1,2, \\
G^1\left(\sigma_\rho p_{\rho'}\right) &= \left[\sigma^\dagger_\rho \otimes \tilde{p_{\rho'}}\right]^1, \qquad G^1\left(p_\rho \sigma_{\rho'}\right) = \left[p^\dagger_\rho \otimes \tilde{\sigma_{\rho'}}\right]^1,
\end{aligned} \tag{44}$$

and

$$\begin{aligned}
G^0\left(s_\rho \sigma_{\rho'}\right) &= \left[s^\dagger_\rho \otimes \tilde{\sigma_{\rho'}}\right]^0, \qquad G^0\left(\sigma_\rho s_{\rho'}\right) = \left[\sigma^\dagger_\rho \otimes \tilde{s_{\rho'}}\right]^0, \\
G^1\left(s_\rho p_{\rho'}\right) &= \left[s^\dagger_\rho \otimes \tilde{p_{\rho'}}\right]^1, \qquad G^1\left(p_\rho s_{\rho'}\right) = \left[p^\dagger_\rho \otimes \tilde{s_{\rho'}}\right]^1, \\
G^2\left(\sigma_\rho d_{\rho'}\right) &= \left[\sigma^\dagger_\rho \otimes \tilde{d_{\rho'}}\right]^2, \qquad G^2\left(d_\rho \sigma_{\rho'}\right) = \left[d^\dagger_\rho \otimes \tilde{\sigma_{\rho'}}\right]^2, \\
G^k\left(d_\rho p_{\rho'}\right) &= \left[d^\dagger_\rho \otimes \tilde{p_{\rho'}}\right]^k, \qquad k = 1,2,3, \\
G^k\left(p_\rho d_{\rho'}\right) &= \left[p^\dagger_\rho \otimes \tilde{d_{\rho'}}\right]^k, \qquad k = 1,2,3
\end{aligned} \tag{45}$$

with $\rho, \rho' = \pi$ or $\nu$.

### *5.2. F–G spin symmetric limit*

Consider now the coupled algebras $\mathrm{U}_{\pi+\nu}(6)$ and $\mathrm{U}_{\pi+\nu}(4)$. For the *sd* sector, we define the $F$ spin as usual

$$F = \frac{N_{_T} - M}{2} - f\,, \qquad F_0 = \left|\frac{N_\pi - N_\nu}{2}\right| \tag{46}$$

with $N_{\mathrm{T}} = N+M$, $F = N/2, N/2-1, \ldots, F_0$, and $f = 0, 1, \ldots \min(N_\nu, N_\pi)$.

For the $p$ sector, we define a $G$ spin as

$$G = \frac{M}{2} - g\,, \qquad G_0 = 0 \tag{47}$$

with $M = 0$, it is clear that $g = G = 0$. In contrast, when $M = 2$, the possible values of $g$ are 0 and 1, which correspond, respectively, to $G = 1$ and $G = 0$.

For axisymmetric nuclei, the combined chain of subalgebras is

$$\begin{aligned}&\mathrm{U}_\pi(6) \otimes \mathrm{U}_\nu(6) \otimes \mathrm{U}_\pi(4) \otimes \mathrm{U}_\nu(4) \supset \mathrm{U}_{(\pi+\nu)sd}(6) \otimes \mathrm{U}_{(\pi+\nu)p\sigma}(4) \supset \\ &\mathrm{SU}_{(\pi+\nu)sd}(3) \otimes \mathrm{SU}_{(\pi+\nu)p\sigma}(3) \supset \mathrm{SU}_{(\pi+\nu)spd}(3) \supset \mathrm{SO}_{(\pi+\nu)spd}(3)\,.\end{aligned} \tag{48}$$

The states are then characterised by the quantum numbers

$$\begin{aligned}&|([N_{\mathrm{T}}]); [N_\pi][N_\nu][M_\pi][M_\nu]; [N_{\mathrm{T}} - M - f, f][M - g, g]; \\ &(\lambda,\mu)_{sd}(\lambda,\mu)_p; (\lambda,\mu); kL\rangle\,.\end{aligned} \tag{49}$$

We get four kinds of states:

1. $f$ and $g$ totally symmetric states ($f = 0, g = 0$): they are in one-to-one correspondence with the IBM-I states;
2. $f$ mixed symmetry states ($f \neq 0, g = 0$): this corresponds to the work in reference [58], since only the IBM-II formulation of the *sd* bosons is taken into account;
3. $g$ mixed symmetry states ($f = 0, g = 1$): this is the inverse of the case 2. They have not been discussed before;
4. mixed symmetry states ($f \neq 0$ and $g = 1$): this is the general case, that we will discuss here.

Another situation occurs when the proton bosons and the neutron bosons have a different nature (particle or hole). As a result, the obtained spectrum is significantly different from the first case and corresponds to a triaxial nucleus. In such a case, a different chain is obtained

$$\begin{aligned}&\bar{\mathrm{U}}_\pi(6) \otimes \mathrm{U}_\nu(6) \otimes \mathrm{U}_\pi(4) \otimes \mathrm{U}_\nu(4) \supset \mathrm{U}^*_{(\pi+\nu)sd}(6) \otimes \mathrm{U}_{(\pi+\nu)p}(4) \supset \\ &\mathrm{SU}^*_{(\pi+\nu)sd}(3) \otimes \mathrm{SU}_{(\pi+\nu)p}(3) \supset \mathrm{SU}_{(\pi+\nu)spd}(3) \supset \mathrm{SO}_{(\pi+\nu)spd}(3)\,.\end{aligned} \tag{50}$$

The hole bosons are described by conjugate SU(3) irreps.

### *5.3. The F–G spin non-symmetric limits*

If the proton bosons and neutron bosons are both particles or holes, one obtains the following chain of subalgebra:

$$\begin{aligned}&\mathrm{U}_\pi(6)\otimes\mathrm{U}_\nu(6)\otimes\mathrm{U}_\pi(4)\otimes\mathrm{U}_\nu(4)\supset\mathrm{SU}_{\pi sd}(3)\otimes\mathrm{SU}_{\nu sd}(3)\otimes\mathrm{SU}_{\pi p}(3)\otimes\mathrm{SU}_{\nu p}(3)\\&\supset\mathrm{SU}_{(\pi+\nu)sd}(3)\otimes\mathrm{SU}_{(\pi+\nu)p}(3)\supset\mathrm{SU}_{(\pi+\nu)spd}(3)\supset\mathrm{SO}_{(\pi+\nu)spd}(3)\,.\end{aligned}\tag{51}$$

The eigenvectors of the Hamiltonian built from this chain are given by

$$\begin{aligned}&|([N_{\mathrm{T}}]);[N_\pi][N_\nu][M_\pi][M_\nu];(\lambda,\mu)_{\pi sd}(\lambda,\mu)_{\nu sd}(\lambda,\mu)_{\pi p}(\lambda,\mu)_{\nu p};\\&(\lambda,\mu)_{sd}(\lambda,\mu)_p;(\lambda,\mu);KL\rangle\,.\end{aligned}\tag{52}$$

If we consider both particle bosons and hole bosons (triaxial nuclei), then another chain is possible

$$\begin{aligned}&\bar{\mathrm{U}}_\pi(6)\otimes\mathrm{U}_\nu(6)\otimes\mathrm{U}_\pi(4)\otimes\mathrm{U}_\nu(4)\supset\bar{\mathrm{SU}}_{\pi sd}(3)\otimes\mathrm{SU}_{\nu sd}(3)\otimes\mathrm{SU}_{\pi p}(3)\otimes\mathrm{SU}_{\nu p}(3)\\&\supset\mathrm{SU}^*_{(\pi+\nu)sd}(3)\otimes\mathrm{SU}_{(\pi+\nu)p}(3)\supset\mathrm{SU}_{(\pi+\nu)spd}(3)\supset\mathrm{SO}_{(\pi+\nu)spd}(3)\,.\end{aligned}\tag{53}$$

The corresponding eigenvectors are

$$\begin{aligned}&|([N_{\mathrm{T}}]);[N_\pi^5][N_\nu][M_\pi][M_\nu];(\mu,\lambda)_{\pi sd}(\lambda,\mu)_{\nu sd}(\lambda,\mu)_{\pi p}(\lambda,\mu)_{\nu p};\\&(\lambda,\mu)_{sd}(\lambda,\mu)_p;(\lambda,\mu);KL\rangle\,,\end{aligned}\tag{54}$$

where $[N^5]\equiv[N,N,N,N,N]$ and $(\mu,\lambda)$ is the conjugate representation of SU(3) [61].

### *5.4. The hybrid limits*

In this paper, we introduce the hybrid limits resulting from the combination of the previous chains; there are four possible cases. Either we form first $\mathrm{U}_{\pi+\nu}(6)$ and $\mathrm{U}_{\pi+\nu}(6)^*$, subsequently combined with $\mathrm{U}(4)\otimes\mathrm{U}(4)$, or we form first $\mathrm{U}_{\pi+\nu}(4)$ which is then combined with $\mathrm{U}(6)\otimes\mathrm{U}(6)$ or $\bar{\mathrm{U}}(6)\otimes\mathrm{U}(6)$. The coupling occurs, in each case, at the level of SU(3)

$$\begin{aligned}&\mathrm{U}_\pi(6)\otimes\mathrm{U}_\nu(6)\otimes\mathrm{U}_\pi(4)\otimes\mathrm{U}_\nu(4)\supset(\mathrm{U}_{\pi+\nu}(6)\supset\mathrm{SU}_{(\pi+\nu)sd}(3))\otimes\\&\mathrm{SU}_{\pi p}(3)\otimes\mathrm{SU}_{\nu p}(3)\supset\mathrm{SU}_{(\pi+\nu)sd}(3)\otimes\mathrm{SU}_{(\pi+\nu)p}(3)\supset\\&\mathrm{SU}_{(\pi+\nu)spd}(3)\supset\mathrm{SO}_{(\pi+\nu)spd}(3)\,,\end{aligned}\tag{55}$$

$$\begin{aligned}&\bar{\mathrm{U}}_\pi(6)\otimes\mathrm{U}_\nu(6)\otimes\mathrm{U}_\pi(4)\otimes\mathrm{U}_\nu(4)\supset(\mathrm{U}^*{}_{\pi+\nu}(6)\supset\mathrm{SU}^*_{(\pi+\nu)sd}(3))\otimes\\&\mathrm{SU}_{\pi p}(3)\otimes\mathrm{SU}_{\nu p}(3)\supset\mathrm{SU}_{(\pi+\nu)sd}(3)\otimes\mathrm{SU}_{(\pi+\nu)p}(3)\supset\\&\mathrm{SU}_{(\pi+\nu)spd}(3)\supset\mathrm{SO}_{(\pi+\nu)spd}(3)\,,\end{aligned}\tag{56}$$

$$\begin{aligned}&\mathrm{U}_\pi(6)\otimes\mathrm{U}_\nu(6)\otimes\mathrm{U}_\pi(4)\otimes\mathrm{U}_\nu(4)\supset\mathrm{SU}_{\pi sd}(3)\otimes\mathrm{SU}_{\nu sd}(3)\otimes\\&(\mathrm{U}_{\pi+\nu}(4)\supset\mathrm{SU}_{(\pi+\nu)p}(3))\supset\mathrm{SU}_{(\pi+\nu)sd}(3)\otimes\mathrm{SU}_{(\pi+\nu)p}(3)\supset\\&\mathrm{SU}_{(\pi+\nu)spd}(3)\supset\mathrm{SO}_{(\pi+\nu)spd}(3)\,,\end{aligned}\tag{57}$$

$$\begin{aligned}&\bar{\mathrm{U}}_\pi(6)\otimes\mathrm{U}_\nu(6)\otimes\mathrm{U}_\pi(4)\otimes\mathrm{U}_\nu(4)\supset\bar{\mathrm{SU}}_{\pi sd}(3)\otimes\mathrm{SU}_{\nu sd}(3)\otimes\\&(\mathrm{U}_{\pi+\nu}(4)\supset\mathrm{SU}_{(\pi+\nu)p}(3))\supset\mathrm{SU}^*_{(\pi+\nu)sd}(3)\otimes\mathrm{SU}_{(\pi+\nu)p}(3)\supset\\&\mathrm{SU}_{(\pi+\nu)spd}(3)\supset\mathrm{SO}_{(\pi+\nu)spd}(3)\,.\end{aligned}\tag{58}$$

## 6. Summary and conclusions

We investigated the problem of dipole–quadrupole low-energy collectivity in heavy nuclei. The mathematical framework in IBM-I, namely the U(10) algebra, has been analysed and shown to have a significantly richer structure than the U(9) algebra of the *spd* model. This large set of subalgebras is able to describe a variety of collective phenomena with a dipole nature. As an application, we studied the clustering in a set of heavy nuclei belonging to the rare-earth and actinide regions in the framework of the $\mathrm{U}(6)\otimes\mathrm{U}(4)$-based model.

In spite of the phenomenological nature of the nuclear vibron model, it has been successful in describing the energy spectrum and the electric dipole transition probabilities. Our study suggests a close connection between E1 strength and $\alpha$ clustering in heavy nuclei. Prior to our work, evidence of the formation of an $\alpha$ cluster in rare-earth nuclei was claimed by Spieker *et al.* [62] in studying the origin of low-lying E1 transitions. A possible extension of this work is the study of the $K^\pi = 3^-$ and $K^\pi = 1^+$ in these regions (see Table 2).

In comparison to the SACM model, the nuclear vibron model has the advantage of being able to describe the coexistence of different clustering configuration and their mixing. Furthermore, the U(10) embedding provides the possibility to describe the clustering in vibrational nuclei. With the proposed solution for the Wildermuth condition problem, the model is expected to be more realistic in describing the experimental data. This solution will be tested in a forthcoming paper.

In a different direction, the vibron model has been extended to IBM-II by the introduction of the U(20) algebra. The resulting chains of subalgebras have been classified using two kinds of spins: the $F$ and $G$ spins (symmetric or non-symmetric limits). A new kind of subalgebra chains has been introduced: the hybrid limits. Hence, this extension triggers new motivation for an experimental investigation proving, once more, the predictive power of symmetry.

## REFERENCES


[1] A. Arima, F. Iachello, «The Interacting Boson Model», *Cambridge University Press*, Cambridge 1987.

[2] A. Arima, F. Iachello, «Nuclear States as Representations of a SU(6) Group», *Phys. Rev. Lett.* **35**, 1069 (1975).

[3] A. Arima, F. Iachello, «Interacting Boson Model of Collective States I. The Vibrational Limit», *Ann. Phys.* **99**, 253 (1976).

[4] A. Arima, F. Iachello, «Interacting boson model of collective nuclear states II. The rotational limit», *Ann. Phys.* **111**, 201 (1978).

[5] A. Arima, F. Iachello, «Interacting boson model of collective nuclear states IV. The O(6) limit», *Ann. Phys.* **123**, 468 (1979).

[6] J.A. Wheeler, «On the Mathematical Description of Light Nuclei by the Method of Resonating Group Structure», *Phys. Rev.* **52**, 1107 (1937).

[7] L.A. Hafstadt, E. Teller, «The Alpha-Particle Model of the Nucleus», *Phys. Rev. B* **54**, 681 (1938).

[8] C.F.V. Weizsacker, «Neuere Modellvorstellungen über den Bau der Atomkerne», *Naturwissenschaften* **26**, 209 (1938).

[9] F. Iachello, «Algebraic methods for molecular rotation-vibration spectra», *Chem. Phys. Lett.* **78**, 581 (1981).

[10] F. Iachello, R.D. Levine, «Algebraic approach to molecular rotation-vibration spectra. I. Diatomic molecules», *J. Chem. Phys.* **77**, 3046 (1982).

[11] F. Iachello, R. D.Levine, «Algebraic Theory of Molecules», *Oxford University Press*, 1995.

[12] F. Iachello, A.D. Jackson, «A phenomenological approch to $\alpha$-clustering in heavy nuclei», *Phys. Lett. B* **108**, 151 (1982).

[13] H. Daley, F. Iachello, «Alpha-clustering in heavy nuclei», *Phys. Lett. B* **131**, 281 (1983).

[14] F. Iachello, «Algebraic approach to nuclear structure», *Nucl. Phys. A* **396**, 233 (1983).

[15] F. Iachello, «Unification of nuclear structure models», *Nucl. Phys. A* **421**, 97 (1984).

[16] H.J. Daley, F. Iachello, «Nuclear vibron model. I. The SU(3) limit», *Ann. Phys.* **167**, 73 (1986).

[17] H.J. Daley, B.R. Barrett, «An interacting boson model of clustering in nuclei: $\alpha$-like clustering in the heavy actinides», *Nucl. Phys. A* **449**, 256 (1986).

[18] H.J. Daley, «An interacting boson model of clustering in nuclei: alpha-particle clustering in heavy nuclei», Ph.D. Thesis, Arizona University, 1984.

[19] Y. Funaki *et al.*, «$\alpha$-cluster states and 4$\alpha$-particle bose condensate in $^{16}$O», *J. Phys.: Conf. Ser.* **111**, 012012 (2008).

[20] M. Theeten *et al.*, «Three-cluster models for light nuclei», *J. Phys.: Conf. Ser.* **111**, 012046 (2008).

[21] C. Beck, «Present status of nuclear cluster physics and experimental perspectives», *J. Phys.: Conf. Ser.* **569**, 012002 (2014).

[22] A. Sandulescu, D.N. Poenaru, W. Greiner, «New type of decay of heavy nuclei intermediate between fission and cap alpha decay», *Sov. J. Part. Nucl.* **11**, 528 (1980).

[23] H.J. Rose, G.A. Jones, «A new kind of natural radioactivity», *Nature* **307**, 245 (1984).

[24] D.M. Brink, J.J. Castro, «Alpha clustering effects in nuclear matter», *Nucl. Phys. A* **216**, 109 (1973).

[25] B. Buck, A.C. Merchant, S.M. Perez, «$\alpha$ decay calculations with a realistic potential», *Phys. Rev. C* **45**, 2247 (1992).

[26] B. Buck, A.C. Merchant, S.M. Perez, «Exotic clustering in $^{222,224,226}$Ra», *Nucl. Phys. A* **617**, 195 (1997).

[27] B. Buck, A.C. Merchant, S.M. Perez, «Exotic clustering in $^{232,234,236,238}$U», *Nucl. Phys. A* **625**, 554 (1997).

[28] B. Buck, A.C. Merchant, S.M. Perez, «Alternative view of collective bands in actinide nuclei», *Phys. Rev. C* **57**, R2095 (1998).

[29] B. Buck, A.C. Merchant, S.M. Perez, «Exotic cluster model of band structure in actinides», *J. Phys. G: Nucl. Part. Phys.* **25**, 901 (1999).

[30] B. Buck, A.C. Merchant, S.M. Perez, «Octupole bands in even isotopes of Ra, Th, U and Pu», *J. Phys.: Conf. Ser.* **111**, 012041 (2008).

[31] A. Astier *et al.*, «Novel Manifestation of $\alpha$-Clustering Structures: New "$\alpha + ^{208}$Pb" States in $^{212}$Po Revealed by Their Enhanced E1 Decays», *Phys. Rev. Lett.* **104**, 042701 (2010).

[32] D.S. Delion, A. Dumitrescu, V.V. Baran, «Theoretical investigation of $\alpha$-like quasimolecules in heavy nuclei», *Phys. Rev. C* **97**, 064303 (2018).

[33] S.M. Ahmed, R. Yahaya, Sh. Radiman, M.S. Yasir, «Alpha-cluster preformation factors in alpha decay for even–even heavy nuclei using the cluster-formation model», *J. Phys. G: Nucl. Part. Phys.* **51**, 079601 (2013).

[34] F. Asaro, F. Stephens, I. Perlman, «Complex Alpha Spectra of Radiothorium ($Th^{228}$) and Thorium-X ($Ra^{224}$)», *Phys. Rev.* **92**, 1495 (1953).

[35] K. Lee, D.R. Inglis, «Stability of Pear-Shaped Nuclear Deformations», *Phys. Rev.* **108**, 774 (1957).

[36] A.F. Barfield, B.R. Barrett, J.L. Wood, O. Scholten, «An interacting boson model description of octupole states in nucleic», *Ann. Phys.* **182**, 344 (1988).

[37] I. Morrison, J. Weise, «Dipole states in the interacting boson model», *J. Phys. G: Nucl. Phys.* **8**, 687 (1982).

[38] J. Engel, F. Iachello, «Quantization of Asymmetric Shapes in Nuclei», *Phys. Rev. Lett.* **54**, 1126 (1985).

[39] J. Engel, F. Iachello, «Interacting boson model of collective octupole states: (I). The rotational limit», *Nucl. Phys. A* **472**, 61 (1987).

[40] T. Otsuka, M. Sugita,«Unified description of quadrupole–octupole collective states in nuclei», *Phys. Lett. B* **209**, 140 (1988).

[41] C.S. Han, D.S. Chun, S.T. Hsieh, H.C. Chiang, «Negative-parity states of $N = 88$ isotones in the interacting boson approximation», *Phys. Lett. B* **163**, 295 (1985).

[42] D. Kusnezov, F. Iachello, «A study of collective octupole states in barium in the interacting boson model», *Phys. Lett. B* **209**, 420 (1988).

[43] D.F. Kusnezov, «Nuclear Collective Quadrupole–Octupole Excitations in the U(16) *spdf* Interacting Boson Model», Ph.D. Thesis, Princeton University, 1988.

[44] D. Kusnezov, «The U(16) algebraic lattice. II. Analytic construction», *J. Phys. A: Math. Gen.* **23**, 5673 (1990).

[45] P. von Brentano, N.V. Zamfir, A. Zilges,«E1 operator in the sdf-interacting boson model from an Alaga rule constraint», *Phys. Lett. B* **278**, 221 (1992).

[46] Y. Liu, H. Sun, E. Zhao,«Dynamical symmetries of the spdf interacting boson model», *J. Phys. G: Nucl. Part. Phys.* **20**, 407 (1994).

[47] H.J. Daley, «Towards a unified interacting boson model: collective modes of arbitrary multipolarity», *J. Phys. G: Nucl. Phys.* **12**, L51 (1986).

[48] K. Ikeda, N. Tagikawa, H. Horiuchi, «The Systematic Structure-Change into the Molecule-like Structures in the Self-Conjugate 4$n$ Nuclei», *Prog. Theor. Phys. Suppl.* **E68**, 464 (1968).

[49] M.F. O'Reilly, «A closed formula for the product of irreducible representations of SU(3)», *J. Math. Phys.* **23**, 2022 (1982).

[50] https://www.nndc.bnl.gov/ensdf/

[51] R. Budaca, P. Buganu, A.I. Budaca «Nuclear collective motion of heavy nuclei with axial quadrupole and octupole deformation», *Phys. Rev. C* **106**, 014311 (2022).

[52] R. Budaca, A.I. Budaca, P. Buganu, «Quadrupole–octupole collective excitations in medium mass nuclei», *Phys. Scr.* **99**, 035309 (2024).

[53] K. Wildermuth, Y.C. Tang, «A unified Theory of the Nucleus», *Academic Press*, New York 1977.

[54] J. Cseh, «Semimicroscopic algebraic description of nuclear cluster states. Vibron model coupled to the SU(3) shell model», *Phys. Lett. B* **281**, 173 (1992).

[55] J. Cseh, G. Levai, «Semimicroscopic Algebraic Cluster Model of Light Nuclei. I. Two-Cluster-Systems with Spin–Isospin-Free Interactions», *Ann. Phys. N.Y.* **230**, 165 (1994).

[56] J. Cseh, «Semimicroscopic algebraic approach to clusterization in heavy nuclei. Application to the $^{210}$Pb+$^{14}$C system», *Phys. Lett. B* **299**, 205 (1993).

[57] N.A. Smirnova, N. Pietralla, T. Mizusaki, P.V. Isacker, «Interrelation between the isoscalar octupole phonon and the proton–neutron mixed-symmetry quadrupole phonon in near-spherical nuclei», *Nucl. Phys. A* **678**, 235 (2000).

[58] M. Zouioueche, A. Bouldjedri, «A proton–neutron nuclear vibron model», *J. Phys. G: Nucl. Part. Phys.* **40**, 075101 (2013).

[59] O. Vallejos, J. Barea, «Octupole and quadrupole modes in radon isotopes using the proton–neutron interacting boson model», *Phys. Rev. C* **104**, 014308 (2021).

[60] A. Frank, P. Van Isacker, «Symmetry Methods in Moleculs and Nuclei», *SyG Editors*, Mexico 2005.

[61] O. Scholten *et al.*, «Mixed-symmetry states in the neutron–proton interacting boson model», *Nucl. Phys. A* **438**, 41 (1985).

[62] M. Spieker, S. Pascu, A. Zilges, F. Iachello, «Origin of Low-Lying Enhanced E1 Strength in Rare-Earth Nuclei», *Phys. Rev. Lett.* **114**, 192504 (2015).