CONFORMAL SYMMETRY OF THE MASSLESS STARUSZKIEWICZ MODEL AND THE PHOTON-LIKE CHARGE PROBLEM

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Within classical electrodynamics, the problem of the existence or non-existence of massless charges is commonly associated with the problem of radiation reaction. Here, the relativistic system of two massless charged particles is considered within the Staruszkiewicz model, in which radiation reaction is neglected. The model is formulated in the Hamiltonian form with constraints. The system is invariant with respect to 15-parameter conformal group. The corresponding conserved canonical generators and the relativistic Laplace–Runge–Lenz vector imply the superintegrability of the system. The trajectory of the unbounded relative motion is hyperbolic, as in the non-relativistic Kepler problem. Surprisingly, particle world lines turn out to be lightlike straight rays, as if charges are switched off.

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1. Introduction

The Poincaré group of invariance enlarges to the conformal symmetry [1, 2] when considering massless fields and particles. Well-known examples are the Maxwell equations [1, 3, 4], and the relativistic mechanics of a free massless particle [2]. The latter is, perhaps, the simplest but not apt example of conformally invariant system: its dynamics is integrable due to the Poincaré invariance, and additional conserved quantities corresponding to the conformal transformations do not add anything essential in this regard.

The dynamical meaning of the conformal symmetry may become essential in the relativistic systems of interacting massless particles, such as proposed in [5]. Actually, this manifestly covariant two-particle Lagrangian system (and its N-particle generalization) is constructed solely by means of the formal requirement of conformal invariance. Remarkably, the model appears physically meaningful inasmuch as it is related to the bilocal field theory. The authors study symmetry aspects of their classical and quantized model, but the question of integrability is left outside the scope of that work.

It seems plausible that the electrodynamics of pointlike massless charges should be conformally invariant. But massless charges are not observed in nature, and they are theoretically problematic. We leave aside quantum aspects of this subject. Within the classical electrodynamics, divergences of the retarded field generated by a pointlike massless charge are stronger than those of the massive charge, and their renormalization has not yet led to a generally accepted expression for a radiation reaction force. One of them includes higher-than-third-order time derivatives, in contrast to the Abraham–Lorentz–Dirac formula [6]. In Ref. [7], it is concluded that the accelerating massless charge does not radiate and thus its reaction is absent. On the contrary, the authors of [8] and [9, Ch. 9] found the radiation reaction infinitely large thus, no finite force can accelerate such a particle, and it moves as free.

The action-at-a-distance electrodynamics [10] is based on the time-symmetric Fokker action integrals [11], and the radiation reaction is incorporated into the theory by Wheeler and Feynman as the response of the entire Universe [12, 13]. As far as a closed few-particle system is concerned, the radiation reaction does not arise. Thus, the Fokker description of a system of few massless charges may appear consistent. The conformal invariance of the action-at-a-distance electrodynamics has been discussed in the literature [14, 15], but its dynamical meaning is obscured. The reason is that even the two-body Fokker action variational problem is very complicated, its dynamics is governed by difference-differential equations which cannot be treated as a conventional dynamical system of finite degrees of freedom [16].

In order to avoid these difficulties, Staruszkiewicz [17] and Rudd and Hill [18] proposed the model describing the following time-asymmetric interaction of two pointlike charged particles: the advanced field of the first particle acts on the second particle, the retarded field of the second particle acts on the first particle, and a radiation reaction is not taken into account. A causally conjugated model was also proposed, in which the retarded and advanced interactions swap. This Fokker-type model was reduced by Staruszkiewicz to the Lagrangian form and then to the Hamiltonian form [19] which was shown integrable, due to the exact Poincaré invariance, with rather complicated dynamics [20, 21]. The Staruszkiewicz model was generalized for a variety of non-electromagnetic time-asymmetric interactions (scalar, gravitational, confining, *etc.*) [22–26], and corresponding quantum versions [27, 28] revealed their physical adequacy, despite an artificially broken causality of interactions.

Here, the Staruszkiewicz model is adopted for the description of two massless charged particles. The massless limit is taken in the framework of the Hamiltonian formalism with constraints [23]. The system is shown to be invariant with respect to 15-parameter conformal group acting in the 4D Minkowski space. The corresponding conserved canonical generators and the relativistic Laplace–Runge–Lenz vector provide the solution of the problem without resorting to quadratures. The trajectory of the unbounded relative motion is represented by the hyperbolic conic section, similarly to the case of the non-relativistic two-body system with Coulomb interaction. Surprisingly, however, the world lines of particles themselves are shown to be null rectilinear rays, *i.e.*, the particles behave as if non-interacting or neutral electrically. This result is consonant with the aforementioned works [8, 9].

2. Staruszkiewicz model in Hamiltonian form

The dynamics of the Staruszkiewicz model is determined by the two-particle Fokker action

$$I = I_{\mathrm{free}} + I_{\mathrm{int}}\,, \tag{2.1}$$

where

$$I_{\mathrm{free}} = -\sum_{a=1}^{2} m_a \int \mathrm{d}\tau_a \sqrt{\dot{x}_a^2} \tag{2.2}$$

is a free-particle term, and

$$I_{\mathrm{int}} = -q_1 q_2 \iint \mathrm{d}\tau_1\, \mathrm{d}\tau_2\, (\dot{x}_1 \cdot \dot{x}_2) G_\eta(x)\,, \qquad \eta = \pm 1 \tag{2.3}$$

describes the time-asymmetric electromagnetic interaction. Here, m_a and q_a $(a = 1, 2)$ are the rest mass and the charge of a^{th} particle; $x_a^\mu(\tau_a)$ $(\mu = 0, \dots, 3)$ are the covariant coordinates of a^{th} particle in the Minkowski space $\mathbb{M}_4$; τ_a is an arbitrary evolution parameter on the a^{th} world line; $\dot{x}_a^\mu(\tau_a) = \mathrm{d}x_a^\mu / \mathrm{d}\tau_a$; $G_\eta(x) = 2\Theta(\eta x^0)\delta(x^2)$ is the retarded (for $\eta = 1$) or advanced $(\eta = -1)$ Green function of the d'Alembert equation depending on the relative position vector $x^\mu = x_1^\mu - x_2^\mu$. Particle world lines are implied timelike, *i.e.*, $\dot{x}_a^2 > 0$ in terms of the timelike Minkowski metrics $\|\eta_{\mu\nu}\| = \mathrm{diag}(1, -1, -1, -1)$ chosen. This metrics defines the scalar product $a \cdot b = \eta_{\mu\nu} a^\mu b^\nu$ of 4-vectors a, b, the vector squared $a^2 = a \cdot a$, *etc.* Light speed $c = 1$.

It is possible to reformulate the model into the single-time manifestly covariant Lagrangian form [23], and then into the Hamiltonian formalism on the 16-dimensional phase space $\mathrm{T}^*\mathbb{M}_4^2$ parameterized by the particle positions $x_a^\mu(\tau)$ and the conjugated momenta $p_{a\mu}(\tau)$ which satisfy the standard Poisson-bracket (PB) relations: $\{x_a^\mu, p_{b\nu}\} = \delta_{ab}\delta_\nu^\mu$.

Due to the invariance under an arbitrary reparametrization $\tau \mapsto f(\tau)$, the canonical Hamiltonian vanishes while there are two constraints. One of them, the *light-cone constraint*, is holonomic

$$x^2 = 0\,, \qquad x^0 > 0\,, \qquad i.e., \qquad \eta x^0 = |\boldsymbol{x}|\,, \tag{2.4}$$

where $\boldsymbol{x} = (x^i \,|\, i = 1,2,3)$ and $\eta = 1$ or -1 specifies the choice of a complementary model. Another, the *dynamical constraint*, has the form

$$\varPhi\left(P^2,\ \upsilon^2,\ P\cdot x,\ \upsilon\cdot x\right) = \varPhi_{\text{free}} + \varPhi_{\text{int}} = 0\,, \tag{2.5}$$

$$\varPhi_{\text{free}} = \frac{1}{4}P^2 - \frac{1}{2}\left(m_1^2 + m_2^2\right) + \left(m_1^2 - m_2^2\right)\frac{\upsilon\cdot x}{P\cdot x} + \upsilon^2\,, \tag{2.6}$$

$$\varPhi_{\text{int}} = -\frac{\bar{\alpha}}{P\cdot x}\left[P^2 - \sum_{a=1}^{2} m_a^2\left(1 + \frac{\bar{\alpha}}{\frac{1}{2}P\cdot x - (-)^a\upsilon\cdot x - \bar{\alpha}}\right)\right] \tag{2.7}$$

with $\bar{\alpha} = \eta\alpha$, where $\alpha = q_1 q_2$ is the coupling constant, and

$$\upsilon_\mu = w_\mu - x_\mu\, P\cdot w/P\cdot x\,, \qquad w_\mu = (p_{1\mu} - p_{2\mu})/2 \quad \Longrightarrow \quad \upsilon\cdot P = 0\,. \tag{2.8}$$

The constraints (2.4) and (2.5) are of the 1st class, *i.e.*, $\{x^2, \varPhi\} = 0$ (see Appendix), thus they reduce effectively the dimension of the phase space $\mathrm{T}^*\mathbb{M}_4^2$ to 12, the same dimension as in the non-relativistic two-particle dynamics. Thus, the holonomic constraint (2.4) restricts only redundant nonphysical degree of freedom, and sets the simultaneity relation between points of different particle world lines[1]. The left-hand side function $\varPhi$ in (2.5) and x^2 constitute, together with the Lagrangian factors λ and λ_0, the Dirac's Hamiltonian $H_{\mathrm{D}} = \lambda\varPhi + \lambda_0 x^2$ determining the dynamics of the model.

The Poincaré invariance of these constraints and thus of the Dirac's Hamiltonian guarantees that the generators

$$P_\mu = \sum_{a=1}^{2} p_{a\mu}\,, \qquad J_{\mu\nu} = \sum_{a=1}^{2}\left(x_{a\mu}p_{a\nu} - x_{a\nu}p_{a\mu}\right) \tag{2.9}$$

[1] Indeed, in Dirac's forms of relativistic dynamics (such as *instant*, *point*, or *front* forms) the simultaneity is defined by spacelike or null hypersurfaces (*hyperplanes*, *hyperboloids*, or *light fronts*) foliating the Minkowski space. In the present case, also known as the *isotropic* form of dynamics [25] or the mechanics on the *light cone* [23], the foliation is defined by a set of past (future) light-cones strung with their vertices on the world line of one of the particles.

of the Poincaré group are conserved: $\{P_\mu, H_{\rm D}\} \approx \lambda\{P_\mu, \varPhi\} + \lambda_0\{P_\mu, x^2\} = 0$, $\{J_{\mu\nu}, H_{\rm D}\} \approx 0$, where the sign "$\approx$" denotes the "weak equality", *i.e.*, the equality by virtue of the constraints.

It may seem that not α, but the complementary coupling constant $\bar\alpha$ determines the repulsive or attractive nature of the interaction. However, $\bar\alpha$ in the interaction part (2.7) of the dynamical constraint (2.5) is accompanied by the expression $P\cdot x$, the $\mathrm{sign}(P\cdot x) = \eta$ of which in the physically admissible domain $P^2 > 0$ adjusts an actual coupling constant to α.

3. Symmetries of the massless Staruszkiewicz model

Let us consider the case of the massless charged particles. The dynamical constraint (2.5)–(2.7) possesses a regular limit and simplifies considerably at $m_a \to 0$

$$\varPhi = \frac{1}{4}P^2 + v^2 - \frac{\bar\alpha P^2}{P\cdot x} = 0\,. \tag{3.1}$$

As expected, the generators of dilatation and special conformal transformations, additional to those in (2.9), arise

$$D = \sum_{a=1}^{2} x_a\cdot p_a\,, \tag{3.2}$$

$$K_\mu = \sum_{a=1}^{2}\left[2\left(x_a\cdot p_a\right)x_{a\mu} - x_a^2 p_{a\mu}\right] - 2\bar\alpha x_\mu\,, \tag{3.3}$$

which are in weak involution with both constraints (2.4) and (3.1), and are thus conserved (see Eqs. (A.2) and (A.3) in Appendix A). They, together with the Poincaré generators (2.9), satisfy the PB-relations of the Lie algebra $\mathcal{A}$C(1,3) (see Eqs. (2.4) in Ref. [2]). Thus, the symmetry of the problem expands to the 15-parameter conformal group C(1,3) $\simeq$ O(2,4).

Besides, there exists the relativistic analogue of the Laplace–Runge–Lenz vector

$$A_\mu = \varPi_\mu^\nu B_\nu\,, \qquad \text{where} \quad B_\nu = v^\lambda J_{\lambda\nu} + \frac{\bar\alpha P^2}{2P\cdot x}x_\nu \tag{3.4}$$

and $\varPi_\mu^\nu = \delta_\mu^\nu - P_\mu P^\nu/P^2$ is a projector: $\varPi_\mu^\nu P_\nu = 0$. The A_μ vector is in strong involution with both constraints (2.4) and (3.1), *i.e.*, it is conserved; see (A.4). This vector is not yet another generator of space-time symmetry transformations because of non-linear PB-relations (A.6). Rather, it is related intricately to the dynamical symmetry group of the system, SO(1,3) in the present case; see [29, 30], where properties of the relativistic Laplace–Runge–Lenz vector are discussed in more detail.

Not all components of the conserved quantities P_μ, $J_{\mu\nu}$, D, K_μ, and A_μ are independent due to the following relations between them:

$$P \cdot K \ = \ D^2 - \left(V^2 + W^2\right)/P^2\,, \tag{3.5}$$
$$A^2 \ = \ \left(W^2 - \alpha^2 P^2\right)/4\,, \tag{3.6}$$

where $V_\mu = J_{\mu\nu}P^\nu$; $W_\mu = *J_{\mu\nu}P^\nu$ is the Pauli–Lubanski vector, and $*J_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\varkappa\lambda}J^{\varkappa\lambda}$, where $\varepsilon_{\mu\nu\varkappa\lambda}$ is the Levi-Civita symbol. The amount of independent integrals of motion is sufficient to provide a superintegrability of the problem.

4. Relative trajectory in center-of-inertia reference frame

Let us consider in this section the case of $P^2 > 0$. Then, without loss of generality, one can choose the reference frame and the origin in it to provide the equalities

$$P_i = 0\,, \qquad J_{0i} = 0\,, \qquad i = 1, 2, 3\,. \tag{4.1}$$

We will refer to as the *center-of-inertia* (CI) reference frame (since the term *center-of-mass* is inappropriate in the massless case), and use in this frame the notation: $x = (\eta r, \boldsymbol{r})$, where $\boldsymbol{r} = (r^i | i = 1, 2, 3)$ and $r = |\boldsymbol{r}|$ by (2.4), $\upsilon = (0, \boldsymbol{k})$ since $\upsilon^0 = 0$ by (2.8) and (4.1). In these terms, the dynamical constraint (3.1)

$$\frac{1}{4}M^2 - \boldsymbol{k}^2 - \alpha\frac{M}{r} = 0\,, \tag{4.2}$$

defines implicitly the total mass $M = M(\boldsymbol{r}, \boldsymbol{k})$ as a function of the relative position vector $\boldsymbol{r}$ and the momentum-type variable $\boldsymbol{k}$. Note that on the effective 6-dimensional phase space reduced by the constraints (2.4), (3.1), and the conditions (4.1), the variables $\boldsymbol{r}$, $\boldsymbol{k}$ are canonical, while the total mass $M(\boldsymbol{r}, \boldsymbol{k})$ generates the evolution; see [23]. These facts, however, are not important in the present work.

Other conserved quantities of interest take the form: $V_\mu = 0$ and

$$W_0 \ = \ 0\,, \qquad \boldsymbol{W} = M\boldsymbol{S} = M\boldsymbol{r}\times\boldsymbol{k}\,, \tag{4.3}$$
$$D \ = \ tM - \boldsymbol{r}\cdot\boldsymbol{k}\,, \tag{4.4}$$
$$K_0 \ = \ 2Dt - M\left[t^2 - \left(1/2 - k^2/M^2\right)r^2\right] - 2\alpha r\,, \qquad \boldsymbol{K} = 0\,, \tag{4.5}$$
$$A_0 \ = \ 0\,, \qquad \boldsymbol{A} = \boldsymbol{k}\times\boldsymbol{S} + \alpha\frac{M\boldsymbol{r}}{2r}\,, \tag{4.6}$$

where $\boldsymbol{S} = \boldsymbol{r}\times\boldsymbol{k}$ is the angular momentum of the system, "$\times$" denotes the vector product symbol, and $t = (x_1^0 + x_2^0)/2$ is the CI average time. These integrals are related by (3.5) and (3.6) as follows:

$$MK_0 \ = \ D^2 + \boldsymbol{S}^2\,, \tag{4.7}$$
$$\boldsymbol{A}^2 \ = \ M^2\left(\boldsymbol{S}^2 + \alpha^2\right)/4\,. \tag{4.8}$$

Eliminating $\boldsymbol{k}$ from (4.5) by (4.2) and using (4.7)–(4.8) yields the relation

$$\frac{1}{4}\left(r-2\frac{\alpha}{M}\right)^2-\left(t-\frac{D}{M}\right)^2=\frac{4A^2}{M^4}=\frac{S^2+\alpha^2}{M^2} \tag{4.9}$$

which defines implicitly the relative distance r as a function of the average time t; here $S=|\boldsymbol{S}|$ and $A=|\boldsymbol{A}|$. Putting $D=0$ which is equivalent to specifying the beginning on the timeline t yields the function $r(t)$ explicitly

$$\frac{r}{2}=\frac{\alpha}{M}+\sqrt{t^2+\frac{4A^2}{M^4}}\,. \tag{4.10}$$

In order to build the trajectory of a relative motion, let us consider the scalar products

$$\boldsymbol{A}\cdot\boldsymbol{r}=S^2+\alpha M r/2\,. \tag{4.11}$$

Since $\boldsymbol{r}\perp\boldsymbol{S}$ and $\boldsymbol{A}\perp\boldsymbol{S}$, the trajectory of $\boldsymbol{r}$ lies in the plane which is orthogonal to $\boldsymbol{S}$. Introducing in this plane the polar angle ϕ through the relation $\boldsymbol{A}\cdot\boldsymbol{r}=Ar\cos\phi$ and using (4.11), one arrives at the relative trajectory $r(\phi)$ equation

$$e\cos\phi-\operatorname{sgn}\alpha=p/r\,, \tag{4.12}$$

which actually is the conic section equation with the following eccentricity e, the semi-latus rectum p, and the semi-major axis a parameters:

$$e=\sqrt{1+S^2/\alpha^2}>1\,,\qquad p=\frac{2S^2}{M|\alpha|}\,,\qquad a=\frac{p}{e^2-1}\,. \tag{4.13}$$

In both cases $\alpha\gtrless 0$, and thus $\operatorname{sgn}\alpha=\alpha/|\alpha|=\pm 1$, equation (4.12) describes the hyperbolic trajectories of an unbounded relative motion depicted

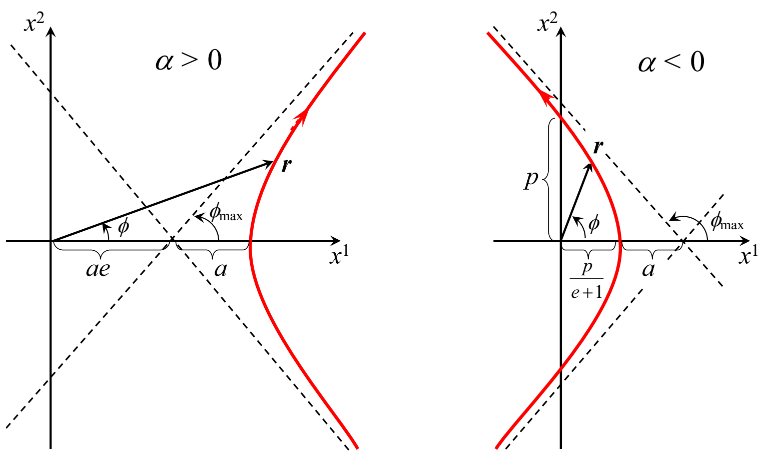

Fig. 1. Hyperbolic trajectories of relative motion.

in Fig. 1. The asymptotic value of the angle ϕ at $r \to \infty$ is determined by the equality

$$\cos\phi_{\max} = \frac{\operatorname{sgn}\alpha}{e}\,. \tag{4.14}$$

It is noteworthy that both functions $r(t)$ and $r(\phi)$ defined by Eqs. (4.10) and (4.12), respectively, are derived from the integrals of motion, without resorting to quadratures. Thus, the reduced system is maximally superintegrable.

5. Particle world lines

In the CI reference frame, the particle positions in $\mathbb{M}_4$ are the following functions of $t, \boldsymbol{r}, \boldsymbol{k}$:

$$x_a^0 = t + \frac{1}{2}(-)^{\bar a}\eta r\,, \qquad a = 1,2\,, \tag{5.1}$$

$$\boldsymbol{x}_a = \frac{1}{2}(-)^{\bar a}\boldsymbol{r} + \eta r\boldsymbol{k}/M\,, \qquad \bar a = 3 - a\,. \tag{5.2}$$

It follows that $\boldsymbol{x}_a \perp \boldsymbol{S}$, *i.e.*, particle positions $\boldsymbol{x}_a$ lie in the plane which is orthogonal to $\boldsymbol{S}$.

Defining the shifted particle positions

$$\boldsymbol{r}_a = \boldsymbol{x}_a + 2(-)^a\boldsymbol{A}/M^2\,, \qquad a = 1,2 \tag{5.3}$$

and introducing the angles ϕ_a between $\boldsymbol{A}$ and $\boldsymbol{r}_a$ as follows: $\boldsymbol{A}\cdot\boldsymbol{r}_a = Ar_a\cos\phi_a$, one arrives at the equations for particle trajectories

$$e\cos\phi_a + (-)^a\operatorname{sgn}\alpha = 0\,, \qquad a = 1,2\,, \tag{5.4}$$

which, actually, are the equations of degenerated conic sections with zero *semi-latus recta*. It follows from (5.4) and (5.2) that values of ϕ_a are constant

$$\phi_1 = \eta\phi_{\max}\,, \qquad \phi_2 = \pi - \eta\phi_{\max}\,, \qquad \forall\ r_a > 0\,, \tag{5.5}$$

where $\phi_{\max}$ is defined by (4.14). Surprisingly, the trajectories of $\boldsymbol{r}_a$ and thus of $\boldsymbol{x}_a$ turn out to be rectilinear, in contrast to the relative trajectory; see Fig. 2.

Using (5.1) and other results of this section, one can derive the functions $x_a^0(t)$, $\boldsymbol{x}_a(t)$ and thus recover the particle world lines in $\mathbb{M}_4$ without integration. We have

$$x_a^0 = (-)^{\bar a}\eta\left(r_a + \alpha/M\right)\,, \qquad a = 1,2\,, \tag{5.6}$$

$$\boldsymbol{x}_a = r_a\boldsymbol{\epsilon}(\phi_a) + 2(-)^{\bar a}\boldsymbol{A}/M^2\,, \qquad \bar a = 3 - a\,, \tag{5.7}$$

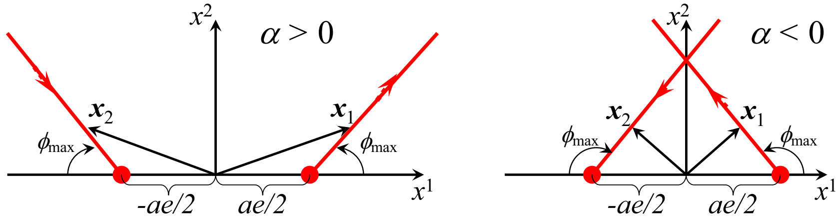

Fig. 2. Rectilinear particle trajectories for $\eta = 1$.

where $\boldsymbol{\epsilon}(\phi)$ is the unit radial orth in the ϕ-direction, and the functions $r_a(t)$,

$$r_a = |\boldsymbol{r}_a| = (-)^{\bar{a}}\eta t + \sqrt{t^2 + 4A^2/M^4}\,, \tag{5.8}$$

reveal the following asymptotical behaviour:

$$r_a \to 0 \qquad \text{if} \qquad t \to (-)^a \eta \cdot \infty\,, \qquad a = 1, 2\,, \tag{5.9}$$

$$r_a \to \infty \qquad \text{if} \qquad t \to (-)^{\bar{a}} \eta \cdot \infty\,, \qquad \bar{a} = 3 - a\,. \tag{5.10}$$

Thus, the particle world lines are lightlike straight rays which begin or end at finite points with coordinates $X_a^0 = (-)^{\bar{a}}\eta\alpha/M$ and $\boldsymbol{X}_a = 2(-)^{\bar{a}}\boldsymbol{A}/M^2$. This implies an incomplete description of particle evolution.

It turns out that complementary models corresponding to the opposite signs of η and α but the same $\bar{\alpha} = \eta\alpha$ can be used to complete smoothly particle world lines corresponding to the same values of integrals of motion; see Fig. 3.

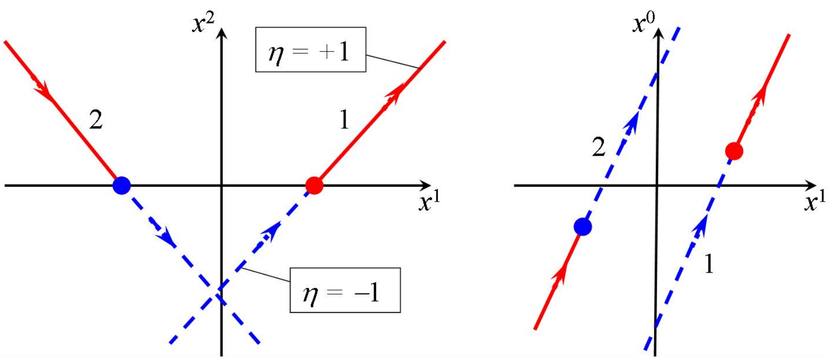

Fig. 3. Particle world lines corresponding to the same values of integrals of motion and $\bar{\alpha} = \eta\alpha > 0$ but different $\eta = \pm 1$.

Above, the restriction $P^2 > 0$ was imposed to find the solution of the problem in the CI reference frame. Farther, the manifestly covariant description is used to derive particle world lines in a general reference frame.

6. Manifestly covariant equations of motion

Despite the problem is maximally superintegrable, it is convenient to begin from the manifestly covariant equations of motion for the relative position $x^\mu = x_1^\mu - x_2^\mu$ and the average position $Y^\mu = \frac{1}{2}(x_1^\mu + x_2^\mu)$

$$\dot{x}^\mu = \{x^\mu, H_{\rm D}\} \approx \lambda\{x^\mu, \varPhi\} \approx \frac{2\lambda}{P\cdot x}\left[P^2 Y^\mu - V^\mu - DP^\mu\right], \tag{6.1}$$

$$\dot{Y}^\mu = \{Y^\mu, H_{\rm D}\} \approx \lambda\{Y^\mu, \varPhi\} \approx \frac{\lambda}{P\cdot x}\left[\frac{1}{2}P^2 x^\mu - 2A^\mu - \bar{\alpha}P^\mu\right]. \tag{6.2}$$

The inverse relations $x_a^\mu = Y^\mu - \frac{1}{2}(-)^a x^\mu$ $(a = 1, 2)$ yield immediately the equations of particle motion

$$\dot{x}_a = (-)^{\bar{a}}\tilde{\lambda}\,(x_a^\mu - X_a^\mu)\,, \qquad a = 1,2\,, \qquad \bar{a} = 3 - a\,, \tag{6.3}$$

where $\tilde{\lambda} = \lambda P^2/P\cdot x$, and

$$X_a^\mu = \left[V^\mu + DP^\mu - (-)^a\,(2A^\mu + \bar{\alpha}P^\mu)\right]/P^2\,, \qquad a = 1,2 \tag{6.4}$$

are conserved 4-vectors.

One can show by direct calculations with the use of (3.2)–(3.6) that $(x_a - X_a)^2 = 0$. Thus, the world lines are lightlike, $\dot{x}_a^2 = 0$, as it expected of massless particles.

The equations of particle motion (6.3) can be easily solved

$$x_1^\mu(\tau) = [x_1^\mu(0) - X_1^\mu]\,f(\tau) + X_1^\mu\,, \qquad x_2^\mu(\tau) = [x_2^\mu(0) - X_2^\mu]\,/f(\tau) + X_2^\mu\,, \tag{6.5}$$

$$f(\tau) = \exp\left[\int\limits_0^\tau {\rm d}\tau\,\tilde{\lambda}(\tau)\right], \tag{6.6}$$

where the initial particle positions $x_a^\mu(0)$ are subjected to the light-cone constraint (2.4). These formulae describe lightlike straight lines.

Since the Lagrangian factor $\lambda(\tau)$ is arbitrary, the function $\tilde{\lambda}(\tau)$ can be chosen finite and positive for $\tau \in \mathbb{R}$. Thus, the properties of the function $f(\tau)$ follow from (6.6)

$$f(\tau > 0) > 1\,, \qquad 0 < f(\tau < 0) < 1\,, \tag{6.7}$$

$$f(+\infty) \to +\infty\,, \qquad f(-\infty) \to +0\,. \tag{6.8}$$

Consequently, the solution (6.5) behaves as follows:

$$x_1^\mu(-\infty) \to X_1^\mu\,, \qquad x_1^\mu(+\infty) \to [x_1^\mu(0) - X_1^\mu]\cdot\infty\,, \tag{6.9}$$

$$x_2^\mu(-\infty) \to [x_2^\mu(0) - X_2^\mu]\cdot\infty\,, \qquad x_2^\mu(+\infty) \to X_2^\mu\,. \tag{6.10}$$

Were the function $\tilde{\lambda}(\tau)$ negative, the asymptotes (6.8)–(6.10) at $\tau \to \pm\infty$ would swap, but particle world lines would remain the same.

Thus, the world lines are the lightlike straight rays. One of the particles starts at the point X_1 (or X_2) and passes through $x_1(0)$ (or $x_2(0)$) in future along the light-cone surface. Another particle comes from the past through $x_2(0)$ (or $x_1(0)$) and ends at the light-cone vertex X_2 (or X_1). Such a general property of the particle world lines coincides with results derived in the previous section for the case $P^2 > 0$: a description of particle evolution within the massless Staruszkiewicz model is not complete but it can be completed by means of the complementary model.

7. Conclusions

The Staruszkiewicz model represents a Poincaré-invariant integrable two-particle dynamics that accounts for the relativistic kinematics as well as certain aspects of the electromagnetic particle interaction, although the radiation reaction is neglected. Despite the causally conjugated time-asymmetric models yield different particle world lines, they were regarded as complementary sources in some approximation scheme for finding solutions to more physically acceptable problem within the Wheeler–Feynman electrodynamics [17]. In slow-motion domain, the systems with time-asymmetric, time-symmetric, and purely retarded interactions reveal very close particle dynamics; see [25] and references therein.

Here, a strongly relativistic system of two massless charged pointlike particles is considered in the framework of the Staruszkiewicz model. The manifestly covariant Hamiltonian description with constraints is used within which the massless limit is straightforward and regular. The alternative starting point could be the manifestly covariant Lagrangian description which must be modified, namely complemented by the additional einbein variables to have a meaningful massless limit [2, 5]. Both ways lead to the same Hamiltonian dynamics but the former way is simpler.

The system is invariant with respect to the 15-parameter conformal group, and possesses the corresponding conserved generators. A relativistic Laplace–Runge–Lentz vector is also found, components of which are conserved and imply, together with the conformal generators, a superintegrability of the system. Namely, within the center-of-inertia reference frame fixed by the conditions (4.1), the problem is reduced to the effective single-body Coulomb-like potential problem describing the relative motion of particles. The solution is derived by means of the aforementioned integrals of motion only, *i.e.*, without resorting to quadratures. The equation of the relative trajectory $r(\phi)$ yields, up to redefinition of parameters, the standard hyperbolic

Kepler orbit (4.12), see Fig. 1. The time dependence $r(t)$ is also hyperbolic (4.9), (4.10) — in contrast to the non-relativistic case, where this function is implicit and involves the eccentric anomaly.

Surprisingly, the particle trajectories, in contrast to the relative trajectory, appear recti-linear. Similarly, the world lines in the Minkowski space are the null straight rays, as if the particles are photon-like and free of one another, *i.e.*, not interacting or electrically neutral. Such a behaviour of massless charges is in agreement with the result of Refs. [8, 9] based strongly on the radiation reaction account. It is likely that the effective non-interaction of massless particles is due not so much to the inclusion of the radiation reaction as to the conformal symmetry of the electrodynamics of massless charges.

The massless Staruszkiewicz model reveals an incomplete particle evolution, in contrast to the original model which yields entire timelike world lines of massive charges. The reason resides in the peculiarity of the light-cone condition (2.4): it cannot be satisfied by spacetime points on null asymptotes of different world lines both tending toward the infinite future (past).

On the other hand, a pair of causally conjugated models corresponding to the opposite signs of $\eta = \pm 1$, but the same complementary coupling constant $\bar{\alpha} = \eta\alpha$, provides a smooth continuation of the null world lines of both particles to infinity in both temporal directions. A physical interpretation of such dynamics leads beyond the framework of the Staruszkiewicz model as well as the scope of the present work.

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Appendix A

Here, PB relations between certain variables and functions used in this paper are calculated in more detail. Using the collective variables $Y^\mu = (x_1^\mu + x_2^\mu)/2$, $P_\mu = p_{1\mu} + p_{2\mu}$, $x^\mu = x_1^\mu - x_2^\mu$, and $w_\mu = (p_{1\mu} - p_{2\mu})/2$ which are canonical, *i.e.*, $\{Y^\mu, P_\nu\} = \delta^\mu_\nu$, $\{x^\mu, w_\nu\} = \delta^\mu_\nu$ (other PB are zero), and the definition (2.8) of υ_μ, one obtains

$$\begin{aligned}
\{Y^\mu, \upsilon_\nu\} &= -\frac{\upsilon^\mu x_\nu}{P{\cdot}x}\,, \qquad \{P_\mu, \upsilon_\nu\} = 0\,, \qquad \{x^\mu, \upsilon_\nu\} = \delta^\mu_\nu - \frac{P^\mu x_\nu}{P{\cdot}x}\,,\\
\{w_\mu, \upsilon_\nu\} &= \frac{P{\cdot}w}{P{\cdot}x}\left(\eta_{\mu\nu} - \frac{P_\mu x_\nu}{P{\cdot}x}\right), \qquad \{\upsilon_\mu, \upsilon_\nu\} = 0\,.
\end{aligned} \tag{A.1}$$

Then, an arbitrary function f of 15 independent components of 4-vectors Y, P, x, and υ referred to as *observables* [23] is identically in involution with x^2, *i.e.*, $\{f(Y,P,x,\upsilon),x^2\}=0$. Thus, the constraints (2.4) and (2.5) or (3.1) are identically of the 1st class.

The dilatation and special conformal generators (3.2) and (3.3) satisfy the following PB relations with the constraint functions (2.4) and (3.1):

$$\begin{aligned} \left\{D,x^2\right\} &= -2x^2\approx 0\,, \qquad \{D,\Phi\}=2\Phi\approx 0\,, && \text{(A.2)}\\ \left\{K^\mu,x^2\right\} &= -4Y^\mu x^2\approx 0\,, \\ \{K^\mu,\Phi\} &= 4\left(Y^\mu-\frac{w\cdot x}{P\cdot x}\,x^\mu\right)\Phi+\frac{p_1^2p_2^\mu-p_2^2p_1^\mu-2\Phi w^\mu}{P\cdot x}\,x^2\approx 0\,. && \text{(A.3)}\end{aligned}$$

Since the angular momentum (2.9), $J_{\mu\nu}=Y_\mu P_\nu+x_\mu w_\nu-(\mu\leftrightarrow\nu)=Y_\mu P_\nu+x_\mu \upsilon_\nu-(\mu\leftrightarrow\nu)$, is observable, the Laplace–Runge–Lenz vector (3.4) is observable. Besides, $\{B_\mu,\Phi\}\propto P_\mu$. Thus,

$$\left\{A_\mu,x^2\right\}=0\,,\qquad \{A_\mu,\Phi\}=\Pi_\mu^\nu\{B_\nu,\Phi\}\propto\Pi_\mu^\nu P_\nu=0\,. \tag{A.4}$$

BP relations of A_μ with other integrals of motion (2.9), (3.2), and (3.3) are

$$\begin{aligned} \{A_\mu,P_\nu\} &= 0\,,\qquad \{A_\mu,J_{\nu\lambda}\}=-\eta_{\mu\nu}A_\lambda+\eta_{\mu\lambda}A_\nu\,,\qquad \{A_\mu,D\}=-A_\mu\,, && \text{(A.5)}\\ \{A_\mu,K_\nu\} &\approx 2\left[P_\mu(J_{\nu\lambda}+\eta_{\nu\lambda}D)+V_\mu\eta_{\nu\lambda}-(\mu\leftrightarrow\nu)-\eta_{\mu\nu}V_\lambda\right]A^\lambda/P^2\,, \\ \{A_\mu,A_\nu\} &\approx \frac{1}{4}P^2\Pi_\mu^\lambda\Pi_\nu^\kappa J_{\lambda\kappa}\,, && \text{(A.6)}\end{aligned}$$

where V_μ and Π_ν^μ are defined below Eqs. (3.6) and (3.4), respectively.

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