

MODULAR CLASSES AND SUPERSYMMETRIC BEREZIN VOLUMES

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*Received 22 December 2025, accepted 24 September 2026,
published online 5 October 2026*

We argue that modular classes of Q -manifolds provide an efficient method for addressing the existence of supersymmetric Berezin volumes in the supergeometric representation theory of the $\mathcal{N} = 2$ $d = 1$ supertranslation algebra. We establish a cohomological coherence criterion for the existence of a Berezin volume that is invariant under both of the supercharges.

DOI:10.5506/APhysPolB.57.11-A2

In this short note, we highlight the application of the cohomology and modular classes of Q -manifolds to the construction of supersymmetric Berezin volumes. Superspace methods provide a powerful geometric framework for constructing supersymmetric theories by generalising the notion of an action. A salient point is the behaviour of the Berezin volume/integration measure under supersymmetry transformations. The standard constructions in superspace require the Berezin volume to be invariant under the action of the supercharges; checking this may be non-trivial. For simplicity, we concentrate on the $\mathcal{N} = 2$ $d = 1$ *supertranslation algebra* (see Nicolai [1, Section 2] and Witten [2, Section 3])

$$Q_1^2 = Q_2^2 = 0, \quad \{Q_1, Q_2\} = P, \quad [P, Q_1] = 0, \quad [P, Q_2] = 0. \quad (1)$$

Note that via defining $Q_{\pm} := Q_1 \pm Q_2$, algebra (1) can be cast in the form

$$Q_{\pm}^2 = \pm P, \quad \{Q_+, Q_-\} = 0, \quad [P, Q_{\pm}] = 0, \quad (2)$$

which is the $\mathcal{N} = (1, 1)$ $d = 1$ supertranslation algebra. These superalgebras are essential in supersymmetric classical and quantum mechanics; for an overview, the reader may consult Junker [3]. For more on the irreducible representations of $d = 1$ extended superalgebras, the reader can consult [4, 5]. We will concentrate on (1) in this note and comment on (2) in due course.

We study real and finite-dimensional supermanifolds $M = (|M|, \mathcal{O}_M)$, equipped with a triple of vector fields $Q_1, Q_2 \in \text{Vect}(M)$ odd and $P \in \text{Vect}(M)$ even, that under the standard $\mathbb{Z}_2$ -graded Lie bracket, satisfy the Lie superalgebra (1). We set $C^\infty(M) := \mathcal{O}_M(|M|)$ and speak of functions. As we have a pair of homological vector fields, Q_i ($i = 1, 2$), *i.e.*, they ‘square to zero’, we have two homological structures on M (see Schwarz [6, Section 3]). As there are two supercharges, there are two modular classes that control the existence of an invariant Berezin volume. As far as the author is aware, modular classes have not been applied to the supergeometric representation theory of supersymmetry before now.

The first systematic studies of characteristic classes of Q -manifolds, including the modular class, were by Lyakhovich and Sharapov [7] and Lyakhovich, Mosman, and Sharapov [8, 9]. The author reviewed modular classes of Q -manifolds [10], which we will draw heavily from. Modular classes were first defined for Poisson manifolds and were subsequently extended to Lie algebroids; see the review article of Kosmann-Schwarzbach [11] for more details.

Recall that the divergence of a vector field X is given defined by $\rho \text{Div}_\rho X := L_X \rho$, where ρ is a chosen Berezin volume, and satisfies

$$\text{Div}_\rho(f X) = f \text{Div}_\rho X + (-1)^{\tilde{f}\tilde{X}} X(f), \quad (3a)$$

$$\text{Div}_{\rho'} X = \text{Div}_\rho X + X(g), \quad (3b)$$

$$\text{Div}_\rho[X, Y] = X \text{Div}_\rho Y - (-1)^{\tilde{X}\tilde{Y}} Y \text{Div}_\rho X. \quad (3c)$$

To clarify the notation used, for all $X, Y \in \text{Vect}(M)$, $f \in C^\infty(M)$, and $\rho' = e^g \rho$ with $g \in C^\infty(M)$ even. For local expressions, the reader may consult [10].

Definition 1. An $\mathcal{N} = 2$ supermanifold with a volume is a quintuple (M, Q_i, P, ρ) , where M is a supermanifold, (Q_1, Q_2, P) is a set of vector fields that satisfy the $\mathcal{N} = 2$ $d = 1$ supertranslation algebra (1), and ρ is a Berezin volume.

On any $\mathcal{N} = 2$ supermanifold with a volume, there is a pair of standard cohomologies on functions (see [8, 9])

$$\text{Hom}_{\text{st}}(M, Q_i) := \frac{\{Q_i - \text{closed functions}\}}{\{Q_i - \text{exact functions}\}},$$

and so a pair of modular classes

$$\text{Mod}(M, Q_i) := [\text{Div}_\rho Q_i]_{\text{st}} \in \text{Hom}_{\text{st}}(M, Q_i).$$

As $\{Q_1, Q_2\} = P \neq 0$, we do not have a bicomplex; or in supergeometric terms, we do not have a double Q -manifold¹. Furthermore, the modular classes $\text{Mod}(M, Q_i)$ can easily be shown to be independent of the choice of Berezin volume using (3a). This directly implies that one can work locally using coordinates when evaluating the modular classes. It is essential to note that the modular class of a Q -manifold is a global invariant that can be meaningfully captured locally; this fact makes the modular class particularly convenient.

The vanishing of the modular class $\text{Mod}(M, Q_i)$ is a necessary and sufficient condition for the existence of a Berezin volume that is invariant under the action of the supercharge Q_i . In particular, the divergence of the supercharge Q_i vanishes for such a Berezin volume. However, the vanishing of both modular classes is a necessary, but not a sufficient condition for the existence of a Berezin volume that is invariant under both supercharges.

Theorem 1 (Cohomological Coherence Criterion). *Let (M, Q_i, P, ρ) be an $\mathcal{N} = 2$ supermanifold with a volume. Assume that both modular classes $\text{Mod}(M, Q_i)$ vanish, i.e., there exists Berezin volumes ρ_i , such that $\text{Div}_{\rho_i} Q_i = 0$. Then there exists a Berezin volume ρ' such that $\text{Div}_{\rho'} Q_i = 0$ ($i = 1, 2$) if and only if there exist an even function $\psi \in C^\infty(M)$ such that*

$$\text{Div}_{\rho_1} Q_2 = Q_2(\psi), \quad \text{and} \quad Q_1(\psi) = 0.$$

Proof.

If Assuming that $\text{Div}_{\rho_1} Q_2 = Q_2(\psi)$ and $Q_1(\psi) = 0$ for some even function $\psi \in C^\infty(M)$. We then claim $\rho' := e^{-\psi} \rho_1$ is the required Berezin volume. By construction, we have

$$\text{Div}_{\rho'} Q_2 = \text{Div}_{\rho_1} Q_2 - Q_2(\psi) = 0.$$

Similarly, as $\text{Div}_{\rho_1} Q_1 = 0$, we have

$$\text{Div}_{\rho'} Q_1 = \text{Div}_{\rho_1} Q_1 - Q_1(\psi) = -Q_1(\psi) = 0.$$

Only if Assuming that such a Berezin volume ρ' exists, $\text{Div}_{\rho'} Q_1 = 0$ implies that $\rho_1 = e^\psi \rho'$ for some even $\psi \in C^\infty(M)$. As $\text{Div}_{\rho_1} Q_1 = 0$,

$$\text{Div}_{\rho'} Q_1 = \text{Div}_{\rho_1} Q_1 - Q_1(\psi) = -Q_1(\psi) = 0,$$

and so $Q_1(\psi) = 0$. As $\text{Div}_{\rho'} Q_2 = 0$, we have

$$\text{Div}_{\rho_1} Q_2 = \text{Div}_{e^\psi \rho'} Q_2 = \text{Div}_{\rho'} Q_2 + Q_2(\psi) = Q_2(\psi).$$

□

¹ By definition, a double Q -manifold is a supermanifold equipped with two homological vector fields that commute.

Remark 1. Note that the proof of Theorem 1 does not use the Lie superalgebra (1). Thus, the Cohomological Coherence Criterion holds for double Q -manifolds. As far as the author is aware, the criterion has not appeared in the literature before in this context.

Geometrically, the Cohomological Coherence Criterion is understood as follows. A jointly invariant Berezin volume exists if and only if the change in a chosen Q_1 -invariant volume ρ_1 along the odd flow generated by Q_2 , is governed by a Q_2 -exact term $Q_2(\psi)$, and moreover the even function ψ must be constant along the odd flow generated by Q_1 , *i.e.*, it is Q_1 -closed.

Theorem 1 is symmetric under the exchange of $1 \leftrightarrow 2$. In particular, if we have

$$\text{Div}_{\rho} Q_1 = Q_1(\psi_1), \quad Q_2(\psi_1) = 0,$$

and

$$\text{Div}_{\rho} Q_2 = Q_2(\psi_2), \quad Q_1(\psi_2) = 0,$$

then $\rho' = e^{-(\psi_1 + \psi_2)} \rho$ provides the (generally non-unique) Berezin volume of Theorem 1. Clearly, $\text{Div}_{\rho'} Q_i = 0$ directly implies $L_{Q_i} \rho' = 0$ and so the Berezin volume ρ' is ‘supersymmetric’. Locally, using coordinates and assuming the conditions of the theorem, we can construct ρ' by using the coordinate Berezin volume and explicitly find suitable ψ_1 and ψ_2 .

If the conditions of Theorem 1 hold, then

$$\text{Div}_{\rho'} Q_{\pm} = \text{Div}_{\rho'} Q_1 \pm \text{Div}_{\rho'} Q_2 = 0.$$

In summary and recalling algebra (2), we have the following corollary.

Corollary 1. *Under the conditions of Theorem 1, there exists a Berezin volume ρ' that is invariant under the action of the supercharges $Q_{\pm}$, *i.e.*, $L_{Q_{\pm}} \rho' = 0$.*

The converse holds, in that if there is a Berezin volume that is invariant under the action of both $Q_{\pm}$, then the Berezin volume is also invariant under both Q_1 and Q_2 .

From the properties of the divergence operator (3c), we have

$$\text{Div}_{\rho} P = \text{Div}_{\rho} \{Q_1, Q_2\} = Q_1(\text{Div}_{\rho} Q_2) + Q_2(\text{Div}_{\rho} Q_1).$$

Thus, if there exists a Berezin volume ρ' that is invariant under both Q_1 and Q_2 , then $\text{Div}_{\rho'} P = 0$, and so $L_P \rho' = 0$.

In general, the volume is not preserved along P ; however, the modular classes are well-behaved with respect to the action of P . More carefully, from (1), we have $[P, Q_i] = 0$. Thus,

$$\text{Div}_{\rho} [P, Q_i] = P (\text{Div}_{\rho} Q_i) - Q_i (\text{Div}_{\rho} P) = 0.$$

Hence, $P(\text{Div}_{\boldsymbol{\rho}} Q_i) = Q_i(\text{Div}_{\boldsymbol{\rho}} P)$, which is Q_i -exact and so by definition, is zero in the respective standard cohomology. This implies that $P([\text{Div}_{\boldsymbol{\rho}} Q_i]_{\text{st}})$ is zero in the standard cohomology of Q_i , respectively. Thus, the modular classes are invariant under the action of the even generator P .

We continue this note with simple illustrative examples. Recall that in (local) coordinates x^a on a supermanifold, the divergence of a vector field is given by

$$\text{Div}_{\boldsymbol{\rho}} X = (-1)^{\tilde{a}} (\tilde{X}+1) \frac{1}{\rho} \frac{\partial (X^a \rho)}{\partial x^a},$$

where (locally) $\boldsymbol{\rho} = D[x]\rho(x)$. Here, $D[x]$ is the coordinate Berezin volume and ρ is even and nowhere vanishing. The Grassmann parity we denote as $\tilde{x}^a := \tilde{a} \in \mathbb{Z}_2$, and $\tilde{X} \in \mathbb{Z}_2$. For brevity, we will set $\text{Mod}(Q_i) := \text{Mod}(M, Q_i)$ as the underlying supermanifold will be clear.

Example 1. Consider $\mathbb{R}^{1|2}$ equipped with global coordinates (t, θ^1, θ^2) and vector fields

$$Q_1 = \frac{\partial}{\partial \theta^1} + \frac{1}{2} \theta^2 \frac{\partial}{\partial t}, \quad Q_2 = \frac{\partial}{\partial \theta^2} + \frac{1}{2} \theta^1 \frac{\partial}{\partial t}, \quad P = \frac{\partial}{\partial t}.$$

The reader can quickly confirm that these vector fields satisfy (1). As the modular classes are independent of the chosen Berezin volume, we can pick the coordinate Berezin volume $\boldsymbol{\rho} = D[t, \theta^2, \theta^1]$. Then

$$\begin{aligned} \text{Div}_{\boldsymbol{\rho}} Q_1 &= \frac{\partial}{\partial \theta^1}(1) + \frac{\partial}{\partial \theta^2}(0) + \frac{\partial}{\partial t}(\tfrac{1}{2} \theta^2) = 0, \\ \text{Div}_{\boldsymbol{\rho}} Q_2 &= \frac{\partial}{\partial \theta^1}(0) + \frac{\partial}{\partial \theta^2}(1) + \frac{\partial}{\partial t}(\tfrac{1}{2} \theta^1) = 0. \end{aligned}$$

Thus, both $\text{Mod}(Q_i)$ vanish and so there exists an invariant Berezin volume; in this case, the coordinate Berezin volume is invariant.

Example 2. Consider $\mathbb{R}^{1|2}$ equipped with global coordinates (t, θ^1, θ^2) and the vector fields

$$Q_1 = \frac{\partial}{\partial \theta^1} + \mu(t) \theta^2 \frac{\partial}{\partial t}, \quad Q_2 = \frac{\partial}{\partial \theta^2}, \quad P = \mu(t) \frac{\partial}{\partial t}.$$

As the modular classes are independent of the chosen Berezin volume, we can pick the coordinate volume $\boldsymbol{\rho} = D[t, \theta^1, \theta^2]$. Then

$$\begin{aligned} \text{Div}_{\boldsymbol{\rho}} Q_1 &= \frac{\partial}{\partial \theta^1}(1) + \frac{\partial}{\partial \theta^2}(0) + \frac{\partial}{\partial t}(\mu(t) \theta^2) = \mu'(t) \theta^2 = Q_1(\log(\mu(t))), \\ \text{Div}_{\boldsymbol{\rho}} Q_2 &= \frac{\partial}{\partial \theta^1}(0) + \frac{\partial}{\partial \theta^2}(1) + \frac{\partial}{\partial t}(0) = 0, \end{aligned}$$

where we have taken $\mu(t) > 0$ for all $t \in \mathbb{R}$. Thus, both the modular classes vanish, i.e., $\text{Mod}(Q_i) = 0$. Moreover, we observe that $Q_2(\log(\mu(t))) = 0$ and so we can find a Berezin volume that is invariant under both supercharges. Specifically, $\rho' = e^{-\log(\mu(t))} \rho$ is such a Berezin volume.

Example 3. Consider $\mathbb{R}_{>0}^{2|2}$ equipped with global coordinates $(x, y, \theta^1, \theta^2)$, by definition we have $x, y > 0$. We equip this supermanifold with the vector fields

$$Q_1 = \frac{\partial}{\partial \theta^1} + \theta^2 x \frac{\partial}{\partial x}, \quad Q_2 = \frac{\partial}{\partial \theta^2} + \theta^1 y \frac{\partial}{\partial y}, \quad P = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}.$$

The reader can quickly confirm that these vector fields satisfy (1). As the modular classes are independent of the chosen Berezin volume, we can pick the coordinate Berezin volume $\rho = D[x, y, \theta^1, \theta^2]$. Then

$$\begin{aligned} \text{Div}_{\rho} Q_1 &= \frac{\partial}{\partial \theta^1}(1) + \frac{\partial}{\partial \theta^2}(0) + \frac{\partial}{\partial x}(\theta^2 x) + \frac{\partial}{\partial y}(0) = \theta^2, \\ \text{Div}_{\rho} Q_2 &= \frac{\partial}{\partial \theta^1}(0) + \frac{\partial}{\partial \theta^2}(1) + \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(\theta^1 y) = \theta^1. \end{aligned}$$

Note that

$$\begin{aligned} \theta^2 &= Q_1(a\theta^1\theta^2 + b\ln x + f(y)) \implies \text{Mod}(Q_1) = 0, \\ \theta^1 &= Q_2(c\theta^2\theta^1 + d\ln y + g(x)) \implies \text{Mod}(Q_2) = 0, \end{aligned}$$

with $a, b, c, d \in R$ such that $a + b = 1$, and $c + d = 1$. There exists invariant Berezin volumes ρ_i such that $\text{Div}_{\rho_i} Q_i = 0$ ($i = 1, 2$). We explicitly see that $\rho_1 = e^{-(a\theta^1\theta^2 + b\ln x + f(y))\theta^1\theta^2} \rho$, where ρ is the coordinate Berezin volume. Checking this, we have

$$\begin{aligned} \text{Div}_{\rho_1} Q_1 &= \theta^2 - Q_1(a\theta^1\theta^2 + b\ln x + f(y)) = \theta^2 - a\theta^2 - b\theta^2 = 0, \\ \text{Div}_{\rho_1} Q_2 &= \theta^1 + Q_2(a\theta^1\theta^2 + b\ln x + f(y)) = \theta^1 + \theta^1 = \theta^1 + a\theta^1 - \theta^1 y f'(y). \end{aligned}$$

Setting $\psi := (1 + a)\theta^2\theta^1 - f(y)$, we observe that $\text{Div}_{\rho_1} Q_2 = Q_2(\psi)$. However, $Q_1(\psi) = -(1 + a)\theta^2$, and so ψ is Q_1 -closed if and only if $a = -1$, which implies that $b = 2$. Thus, in light of Theorem 1, we know there exists a Berezin volume that is invariant under the action of both the supercharges. We modify ρ_1 using ψ , which gives

$$\rho' = e^{\theta^1\theta^2 - 2\ln x} \rho,$$

as an invariant Berezin volume. However, this is not unique and noting the symmetry of the vector fields under $x \leftrightarrow y$ and $\theta^1 \leftrightarrow \theta^2$ tells us that

$$\rho'' = e^{\theta^2\theta^1 - 2\ln y} \rho$$

is another invariant Berezin volume.

Example 4. Consider $\mathbb{R}^{2|2}$ equipped with global coordinates $(x, y, \theta^1, \theta^2)$ and vector fields

$$Q_1 = \frac{\partial}{\partial \theta^1} + \theta^2 x \frac{\partial}{\partial x}, \quad Q_2 = \frac{\partial}{\partial \theta^2} + \theta^1 y \frac{\partial}{\partial y}, \quad P = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}.$$

The reader can quickly confirm that these vector fields satisfy (1). As the modular classes are independent of the chosen Berezin volume, we can pick the coordinate Berezin volume $\rho = D[x, y, \theta^1, \theta^2]$. Then

$$\begin{aligned} \operatorname{Div}_{\rho} Q_1 &= \frac{\partial}{\partial \theta^1}(1) + \frac{\partial}{\partial \theta^2}(0) + \frac{\partial}{\partial x}(\theta^2 x) + \frac{\partial}{\partial y}(0) = \theta^2, \\ \operatorname{Div}_{\rho} Q_2 &= \frac{\partial}{\partial \theta^1}(0) + \frac{\partial}{\partial \theta^2}(1) + \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(\theta^1 y) = \theta^1. \end{aligned}$$

Note that

$$\begin{aligned} \theta^2 &= Q_1(\theta^1 \theta^2 + f(y)) \implies \operatorname{Mod}(Q_1) = 0, \\ \theta^1 &= Q_2(\theta^2 \theta^1 + g(x)) \implies \operatorname{Mod}(Q_2) = 0. \end{aligned}$$

Note that, unlike Example 3, we do not have the terms $\ln x$ and $\ln y$, as these are not smooth functions on $\mathbb{R}$. Thus, there exists invariant Berezin volumes ρ_i such that $\operatorname{Div}_{\rho_i} Q_i = 0$ ($i = 1, 2$). We explicitly see that $\rho_1 = e^{-(\theta^1 \theta^2 + f(y))} \rho$, where ρ is the coordinate Berezin volume. Checking this, we have

$$\begin{aligned} \operatorname{Div}_{\rho_1} Q_1 &= \theta^2 - Q_1(\theta^1 \theta^2 + f(y)) = \theta^2 - \theta^2 = 0, \\ \operatorname{Div}_{\rho_1} Q_2 &= \theta^1 - Q_2(\theta^1 \theta^2 + f(y)) = \theta^1 + \theta^1 = 2\theta^1 - \theta^1 y f'(y). \end{aligned}$$

Setting $\psi := 2\theta^2 \theta^1 - f(y)$, we observe that $\operatorname{Div}_{\rho_1} Q_2 = Q_2(\psi)$. However, $Q_1(\psi) = -2\theta^2 \neq 0$, and so ψ is not Q_1 -closed. Thus, in light of Theorem 1, we know that there cannot be a single Berezin volume invariant under both Q_1 and Q_2 .

We end this note with a few remarks.

The Cohomological Coherence Criterion (Theorem 1) directly generalises to the $\mathcal{N} = 2 \mid d > 1$ supertranslation algebra. One needs to take care with spinor conventions and indices, but the theorem itself is unchanged.

For $\mathcal{N} > 2$, we can use an inductive argument (which we omit) to show that a common invariant Berezin volume exists, assuming all $\operatorname{Mod}(M, Q_i) = 0$ for $i = 1, 2, \dots, \mathcal{N}$, if and only if there exists an even function $\psi \in C^\infty(M)$, such that $Q_1(\psi) = 0$ and $\operatorname{Div}_{\rho_1} Q_j = Q_j(\psi)$ for all $j = 2, 3, \dots, \mathcal{N}$. Thus, the Cohomological Coherence Criterion directly generalises to higher supersymmetry, and indeed to n -tuple Q -manifolds.

As for applications, the Cohomological Coherence Criterion as stated here can be used to construct actions for novel supersymmetric classical mechanics, including those on topologically non-trivial base manifolds. In common approaches to supergravity, one typically checks the divergence of the supercharges using local coordinates to investigate the existence of invariant volumes. Using modular classes provides a coordinate-free cohomological understanding of the existence of supersymmetric Berezin volumes that may prove useful in supergravity.

The author cordially thanks Francesco Toppan for comments on earlier drafts of this note. The author is grateful to the anonymous referee for helpful suggestions.

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