

EXACT MICROCANONICAL FORMULATION
AND THERMODYNAMICS OF EQUISPACED
FINITE-LEVEL SYSTEMSJ. RICARDO DE SOUSA Universidade Federal do Amazonas, Departamento de Física
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We develop a unified microcanonical formulation for N noninteracting particles with an arbitrary number p of equally spaced local energy levels, $0, \varepsilon, 2\varepsilon, \dots, (p-1)\varepsilon$. The exact microcanonical multiplicity is expressed as a coefficient of a local generating function, providing a finite- N representation that can also be systematically analyzed in the thermodynamic limit. By converting this coefficient extraction into a contour integral and applying saddle-point analysis, we obtain analytical expressions for the entropy per particle $s(u, p)$ and the inverse temperature $\beta(u, p)$, where $u = E/(N\varepsilon) \in [0, p-1]$. The main result is a single arbitrary- p solution that contains the two-level and three-level systems as special cases and connects them continuously to the unbounded limit $p \rightarrow \infty$. For every finite p , the upper spectral bound produces an entropy maximum at $u_c = (p-1)/2$, where the inverse temperature vanishes and the positive-temperature branch changes into a negative-temperature branch. As p increases, this characteristic energy is displaced toward higher values, and in the limit $p \rightarrow \infty$ the upper bound is removed, the finite-energy entropy maximum disappears, and only positive temperatures remain. The formulation therefore provides an analytically controlled description of the crossover between bounded- and unbounded-spectrum thermodynamics. Beyond the present model, the method supplies an exact microcanonical template for discrete additive systems whose local spectra can be encoded by generating functions, including finite-level quantum and spin systems.

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1. Introduction

The microcanonical thermodynamics of finite-level systems provides a simple and useful route for discussing one of the distinctive features of bounded spectra: the possible emergence of negative absolute temperatures. In an unbounded spectrum, increasing the energy generally opens access to

a larger number of states, and the entropy is expected to grow monotonically. In a finite spectrum, by contrast, the number of accessible configurations may increase only up to an intermediate energy and then decrease as the upper spectral edge is approached. The entropy can therefore attain a maximum at a finite energy. At this point, the inverse temperature vanishes, and beyond this point, the microcanonical temperature becomes negative [1, 2]. This mechanism is well known in principle and has long been associated with bounded-spectrum systems, including the classic nuclear-spin experiment of Purcell and Pound [3].

Despite this general understanding, it is useful to have explicit microcanonical solutions in which the whole entropy curve and the corresponding inverse temperature can be obtained analytically. Such examples make it possible to follow, in a controlled way, how the entropy maximum appears, how the point $\beta = 0$ depends on the number of accessible levels, and how the negative-temperature branch disappears when the upper bound of the spectrum is removed. This is also relevant more broadly because the definition and interpretation of entropy and temperature in isolated finite or bounded systems remain topics of continuing interest in statistical mechanics [4, 5].

In this work, we consider a minimal model in which the microcanonical thermodynamics of bounded spectra can be treated exactly in the thermodynamic limit. The system consists of N noninteracting particles, each of which can occupy one of p equally spaced local energy levels,

$$\{0, \varepsilon, 2\varepsilon, \dots, (p-1)\varepsilon\}, \quad (1)$$

where $\varepsilon > 0$. Equivalently, the Hamiltonian can be written as

$$\mathcal{H} = \varepsilon \sum_{i=1}^N \sigma_i, \quad \sigma_i = 0, 1, 2, \dots, p-1. \quad (2)$$

For every finite value of p , the many-body spectrum is bounded from above. Consequently, the microcanonical entropy may attain a maximum at a finite energy, separating positive- and negative-temperature branches. In the limit $p \rightarrow \infty$, the upper spectral bound is removed and the negative-temperature branch disappears. The model therefore provides an analytically controlled setting in which the passage from bounded-spectrum to unbounded-spectrum thermodynamics can be followed continuously through the local dimension p .

Several special or limiting cases are already known. The two-level system, corresponding to $p = 2$, is a standard example of negative-temperature thermodynamics. The three-level case, $p = 3$, was analyzed microcanonically in Ref. [6], while the limit $p \rightarrow \infty$ is related to the familiar unbounded equally

spaced spectrum. These cases, however, do not by themselves provide a general description of how the microcanonical thermodynamics depends on the number of accessible local levels. To the best of our knowledge, an exact thermodynamic-limit microcanonical solution valid for arbitrary finite p , together with a unified analysis of its bounded and unbounded limits, has not previously been presented.

The principal contribution of the present work is therefore the derivation of a single arbitrary- p formulation from which the entropy, inverse temperature, infinite-temperature energy, and positive- and negative-temperature branches follow systematically. This general solution does more than collect previously known cases. It makes explicit how the thermodynamic structure changes as the local state space is enlarged and identifies the precise manner in which the entropy maximum and the negative-temperature sector are displaced toward progressively higher energies before disappearing in the unbounded limit.

The three-level system plays an important methodological role in the development of the formulation. It is not introduced merely as an elementary illustration. Rather, it is the simplest nontrivial finite-level case in which the energy constraint leaves an internal occupation variable to be summed over, and it therefore exposes explicitly the combinatorial structure that underlies the general calculation. Its direct occupation-number representation provides a transparent bridge between multinomial microstate counting and the generating-function formulation used for arbitrary p . In addition, the available exact result for $p = 3$ supplies a nontrivial benchmark against which the general saddle-point construction can be tested.

A solution for arbitrary p is scientifically useful for two related reasons. First, it embeds the known cases $p = 2$, $p = 3$, and $p \rightarrow \infty$ within a common microcanonical framework, thereby separating model-independent features from properties specific to a particular number of levels. Second, it permits a quantitative analysis of the evolution of the thermodynamic functions with p . In particular, one can determine how the entropy profile broadens, how its maximum shifts with the upper spectral bound, and how the inverse-temperature curve evolves from a finite-spectrum structure with both signs of temperature to the purely positive-temperature behavior of an unbounded spectrum. An additional advantage of the generating-function formulation is that the exact coefficient representation remains valid for finite N , providing a natural basis for the analysis of finite-size microcanonical thermodynamics.

Finite local spectra are not restricted to abstract counting models. They arise naturally in quantum-spin systems, Zeeman multiplets, truncated local Hilbert spaces, and effective finite-state descriptions of discrete statistical-mechanical systems. Accordingly, the arbitrary- p solution derived here pro-

vides an exact reference model for bounded-spectrum thermodynamics and a transferable generating-function framework for more general discrete systems with additive energies.

The paper is organized as follows. In Section 2, we develop the microcanonical formulation and derive the exact multiplicity of a system with p equally spaced levels by means of a generating-function representation. While the model is analytically simple, the resulting formulation is not merely pedagogical: it provides a flexible microcanonical framework that can be adapted to other discrete systems whose multiplicities are encoded by local generating functions. In Section 3, we analyze the thermodynamic consequences of the solution, focusing on the role of the number of accessible levels in shaping the entropy, locating the infinite-temperature point, and determining the negative-temperature branch. Finally, Section 4 summarizes the main results and outlines possible extensions of the method.

2. Exact finite-system formulation and thermodynamic-limit analysis

This section develops the microcanonical formulation in four successive steps. We first examine the three-level system using occupation numbers and direct combinatorial counting. The purpose of this construction is not merely illustrative: the case $p = 3$ is the simplest nontrivial finite-level system in which the fixed-energy constraint leaves an internal occupation variable to be summed over, thereby making the underlying combinatorial structure explicit. We then reformulate the counting problem in terms of a generating function and extend the construction to an arbitrary number p of local energy levels. Next, we discuss the finite-system content of the resulting exact coefficient and contour-integral representations, emphasizing that no large- N approximation is required at this stage. Finally, starting from the same exact representation, we apply the saddle-point method to obtain the entropy and inverse temperature in the thermodynamic limit. This organization distinguishes the exact finite-system formulation from its asymptotic analysis and clarifies the methodological role played by the three-level case in constructing the arbitrary- p solution.

2.1. Three-level system and combinatorial structure

To introduce the method in its simplest nontrivial finite setting, and to provide a direct benchmark against the available three-level solution, we first consider the case $p = 3$, with single-particle spectrum

$$\{0, \varepsilon, 2\varepsilon\}. \quad (3)$$

Let N_1 , N_2 , and N_3 denote the occupation numbers of the three levels. For a macrostate with N particles and total energy E , these variables satisfy

$$N = N_1 + N_2 + N_3, \quad E = \varepsilon (N_2 + 2N_3). \quad (4)$$

Writing $q = E/\varepsilon$, one may take N_3 as the independent variable, so that

$$N_2 = q - 2N_3, \quad N_1 = N - q + N_3. \quad (5)$$

For each allowed value of N_3 , the corresponding number of configurations is given by the multinomial factor

$$\frac{N!}{N_1!N_2!N_3!}. \quad (6)$$

Summing over all admissible values of N_3 , one obtains

$$\Omega_3(E, N) = \sum_{r=r_0}^{[q/2]} \frac{N!}{r! (N - q + r)! (q - 2r)!}, \quad (7)$$

where $r = N_3$ and $r_0 = \max(0, q - N)$.

Although the thermodynamics of the three-level system has been discussed previously [6], the present treatment serves a distinct methodological purpose. The occupation-number calculation displays explicitly how the fixed-energy constraint organizes the admissible configurations and leads to the sum of multinomial contributions. This structure can be expressed more compactly through the generating function

$$\Omega_3(E, N) = [x^q] (1 + x + x^2)^N, \quad (8)$$

where $q = E/\varepsilon = 0, 1, 2, \dots, 2N$. Here, $[x^q] f(x)$ denotes the coefficient of x^q in the expansion of $f(x)$, so that the coefficient of x^q in $(1 + x + x^2)^N$ gives the number of configurations with total energy $E = q\varepsilon$.

The importance of Eq. (8) is that it replaces the explicit summation over occupation numbers by a coefficient-extraction problem. The local energy spectrum is encoded in a single polynomial, and the fixed-energy condition is imposed by selecting the appropriate power of x . This observation provides the structural link between the direct combinatorial calculation for $p = 3$ and the general finite-level formulation developed below.

2.2. Exact formulation for arbitrary p

The generating-function representation obtained for the three-level system does not rely on any property specific to $p = 3$, apart from the form of

the corresponding local polynomial. For a particle with the p equally spaced energy levels defined in Eq. (1), the single-particle generating polynomial is

$$g_p(x) = 1 + x + x^2 + \cdots + x^{p-1}. \quad (9)$$

Because the particles are independent and their energies are additive, the generating function for N particles is $[g_p(x)]^N$. The exact number of microstates with total energy $E = q\varepsilon$ is therefore

$$\Omega_p(E, N) = [x^q] (1 + x + x^2 + \cdots + x^{p-1})^N, \quad (10)$$

with $q = 0, 1, 2, \dots, (p-1)N$.

Since the single-particle polynomial in Eq. (10) is a finite geometric series, the exact microcanonical multiplicity can be written as

$$\Omega_p(E, N) = [x^q] \left(\frac{1 - x^p}{1 - x} \right)^N. \quad (11)$$

Equation (11) is a central result of the present formulation. It gives the complete microstate counting for arbitrary finite N and p , without requiring an explicit summation over the occupation numbers of the individual levels. All information concerning the local spectrum is contained in the polynomial $(1 - x^p)/(1 - x)$, while the coefficient extraction imposes the fixed total-energy condition $E = q\varepsilon$. In particular, the numerator $(1 - x^p)$ encodes the upper bound of the local spectrum and therefore distinguishes the finite- p problem from the unbounded limit. When $p \rightarrow \infty$, this truncation factor disappears on the physical positive-temperature branch, providing a direct algebraic representation of the crossover from bounded- to unbounded-spectrum thermodynamics.

The significance of Eq. (11) is not limited to its compact form. For finite N , it provides an exact starting point for coefficient extraction and for the analysis of finite-size microcanonical properties. In the thermodynamic limit, the same expression can be converted into a contour integral and evaluated systematically by complex-analytic and saddle-point methods. Thus, the finite-system formulation and the thermodynamic-limit solution are obtained from a single microcanonical representation rather than from separate calculations.

The local generating polynomial,

$$g_p(x) = \frac{1 - x^p}{1 - x}, \quad (12)$$

formally coincides with the one-particle canonical partition sum after the identification $x = \exp(-\beta\varepsilon)$. Its role in the present derivation, however, is

fundamentally different. In the canonical ensemble, x is fixed by the temperature and the partition function sums over all possible energies. Here, x is an auxiliary generating variable, and the operation $[x^q]$ selects exactly those configurations having the prescribed energy $E = q\varepsilon$. The fixed-energy constraint is therefore maintained throughout the derivation, and the canonical relation emerges only later, in the thermodynamic limit, through the saddle-point condition. More generally, this construction illustrates how local generating functions can be used to obtain exact microcanonical multiplicities for discrete systems with additive energies.

2.3. Finite-system content of the exact representation

Although the main objective of the present work is the thermodynamic-limit solution, the exact generating-function representation also provides direct access to systems with finite N . Indeed, Eq. (11) gives the microcanonical multiplicity for arbitrary finite values of N and p , without requiring any asymptotic approximation. The corresponding finite-system entropy is

$$S_p(E, N) = k_B \ln \Omega_p(E, N). \quad (13)$$

A detailed finite-size analysis lies outside the scope of the present paper. Nevertheless, it can be carried out within the same framework by extracting the coefficients of the generating polynomial exactly, either analytically for sufficiently small systems or numerically for larger values of N and p . The resulting entropy can then be compared with its thermodynamic-limit form in order to determine finite-size corrections and to investigate how the entropy maximum and the positive- and negative-temperature sectors develop as N increases.

At finite N , the allowed energies form a discrete set. Consequently, the microcanonical inverse temperature must be defined through an appropriate finite-difference prescription or through a suitable interpolation of the entropy, rather than by an ordinary derivative. These issues are part of the broader discussion of thermodynamics in finite isolated systems considered in Ref. [4]. Thus, although finite-system thermodynamics is not pursued here, Eq. (11) provides an exact and readily implementable starting point for such an analysis. In the following subsection, we restrict attention to the thermodynamic limit and derive the leading large- N behavior from the same exact representation.

2.4. Thermodynamic-limit solution

Having established the exact finite-system representation, we now turn to its thermodynamic-limit evaluation. The objective is to determine the

leading exponential behavior of the multiplicity when $N \rightarrow \infty$, with the reduced energy per particle held fixed. For this purpose, the coefficient-extraction formula is particularly suitable, since it can be converted into a contour integral whose asymptotic behavior is governed by a saddle point. This procedure provides a direct connection between the exact combinatorial multiplicity and the thermodynamic functions of the model, without abandoning the fixed-energy formulation.

The coefficient extraction in Eq. (11) can be expressed as a contour integral by means of Cauchy's coefficient formula. If

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad (14)$$

then

$$a_n = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{z^{n+1}} dz, \quad (15)$$

where the contour $\mathcal{C}$ encircles the origin counterclockwise and lies within a domain in which $f(z)$ is analytic.

Applying this result to Eq. (11), we obtain

$$\Omega_p(E, N) = \frac{1}{2\pi i} \oint \frac{dz}{z^{q+1}} \left(\frac{1 - z^p}{1 - z} \right)^N. \quad (16)$$

Equation (16) may therefore be regarded as the corresponding discrete inversion formula, recovering the microcanonical multiplicities at fixed energy from the generating function.

To extract the thermodynamic behavior, it is convenient to introduce the reduced energy per particle,

$$u = \frac{E}{N\varepsilon} = \frac{q}{N}, \quad (17)$$

so that $q = Nu$. Equation (16) can then be cast in the exponential form

$$\Omega_p(E, N) = \frac{1}{2\pi i} \oint \frac{dz}{z} e^{N\phi_p(z)}, \quad (18)$$

where

$$\phi_p(z) = \ln \left(\frac{1 - z^p}{1 - z} \right) - u \ln z. \quad (19)$$

Equation (18) is the central representation of the problem, since it gives direct access to the thermodynamic limit $N \rightarrow \infty$. Its contour-integral form

and asymptotic evaluation follow standard methods of coefficient extraction and saddle-point analysis [7, 8]. In this limit, the dominant contribution is determined by the saddle-point condition

$$\phi'_p(z_0) = 0, \quad (20)$$

which yields

$$u = \frac{z_0}{1 - z_0} - \frac{pz_0^p}{1 - z_0^p}. \quad (21)$$

If $x = x(u)$ denotes the physically relevant solution of Eq. (21), the Boltzmann entropy per particle in the thermodynamic limit is given by

$$s(u, p) = \lim_{N \rightarrow \infty} \left[\frac{S_p(E, N)}{N} \right] = k_B \left[\ln \left(\frac{1 - x^p}{1 - x} \right) - u \ln(x) \right]. \quad (22)$$

Equation (22) provides the fundamental thermodynamic relation of the model for arbitrary p in the limit $N \rightarrow \infty$. Thus, the generating-function approach combined with saddle-point analysis yields an exact microcanonical expression for the entropy, from which the remaining thermodynamic quantities follow directly. In particular, the inverse temperature is

$$\beta(u, p) = \frac{1}{\varepsilon k_B} \frac{ds(u, p)}{du} = -\frac{1}{\varepsilon} \ln x(u). \quad (23)$$

This relation also makes transparent the connection with the canonical treatment, under the formal identification $x = e^{-\beta\varepsilon}$. In this way, the present microcanonical formulation recovers the corresponding canonical thermodynamic structure in the thermodynamic limit.

The infinite-temperature point therefore corresponds to $x = 1$. Substituting this value into Eq. (21), one obtains

$$u_c = \frac{p - 1}{2}. \quad (24)$$

For arbitrary finite p , this is the energy at which the entropy reaches its maximum and the inverse temperature vanishes. At the same time, it identifies the change in the physical saddle-point branch: one has $0 < x(u) < 1$ for $u < u_c$, $x(u) = 1$ at $u = u_c$, and $x(u) > 1$ for $u > u_c$. Accordingly, u_c separates the positive- and negative-temperature sectors of the microcanonical description. The general solution therefore makes explicit both the location of the entropy maximum and the way in which the thermodynamic structure of the model depends on the number of accessible energy levels.

3. Thermodynamic consequences and limiting cases

An important consistency test of the formulation is that it reproduces the known limiting cases in a unified way. For $p = 2$, one recovers the standard two-level expressions

$$x(u) = \frac{u}{1-u}, \quad s(u, 2) = k_B [-u \ln(u) - (1-u) \ln(1-u)] \quad (25)$$

and

$$\beta(u, 2) = \frac{1}{\varepsilon} \ln \left(\frac{1-u}{u} \right). \quad (26)$$

In this case, $u_c = 1/2$, so that the temperature is positive for $0 \leq u < 1/2$, infinite at $u = 1/2$, and negative for $1/2 < u \leq 1$. Thus, the general arbitrary- p formalism correctly reduces to the textbook two-level result.

For $p = 3$, the saddle-point equation takes the form

$$u = \frac{x(1+2x)}{1+x+x^2}, \quad (27)$$

whose physically relevant solution is

$$x(u) = \frac{u-1+\sqrt{1+6u-3u^2}}{2(2-u)}. \quad (28)$$

It follows that

$$s(u, 3) = k_B [\ln(1+x+x^2) - u \ln x], \quad x = x(u), \quad (29)$$

and

$$\beta(u, 3) = -\frac{1}{\varepsilon} \ln \left[\frac{u-1+\sqrt{1+6u-3u^2}}{2(2-u)} \right]. \quad (30)$$

Here $u_c = 1$, so that the temperature is positive for $0 \leq u < 1$, infinite at $u = 1$, and negative for $1 < u \leq 2$. This case is especially relevant because, among finite nontrivial values of p , it is the one for which an explicit thermodynamic treatment appears to be available in the literature [6]. Our result coincides exactly with that case, but it is obtained here from a different and more general perspective: rather than treating $p = 3$ as an isolated problem, we derive it as a particular realization of a generating-function formulation that extends directly to arbitrary p . The agreement with the three-level result therefore provides a nontrivial validation of the present approach and, at the same time, highlights its broader scope.

The limit $p \rightarrow \infty$ is particularly instructive, because it connects the present bounded-spectrum problem to the unbounded case. In this limit, the upper bound of the spectrum is removed, and Eq. (21) reduces to

$$u = \frac{x}{1-x},$$

since $x^p \rightarrow 0$ for the physically relevant solution $0 \leq x < 1$. Hence,

$$x(u) = \frac{u}{1+u}. \quad (31)$$

Substituting Eq. (31) into Eq. (22), one obtains

$$s(u, \infty) = k_B [(1+u) \ln(1+u) - u \ln u], \quad (32)$$

while Eq. (23) gives

$$\beta(u, \infty) = \frac{1}{\varepsilon} \ln \left(\frac{1+u}{u} \right). \quad (33)$$

Therefore, in the limit $p \rightarrow \infty$, the inverse temperature remains positive for all $u \geq 0$, and the negative-temperature branch disappears. This behavior is fully consistent with the loss of the upper bound in the energy spectrum. More importantly, it shows that the present formulation not only reproduces isolated special cases, but also captures the full crossover from finite bounded spectra, where the entropy has a maximum, to the infinite-level limit, where this maximum is lost. This limiting behavior reinforces the generality of the solution and clarifies the role played by the number of accessible levels in the microcanonical thermodynamics of equispaced spectra.

To illustrate the thermodynamic content of the general solution, Fig. 1 shows the dimensionless entropy per particle, $s(u, p)/k_B$ (upper panel), and the dimensionless inverse temperature, $\beta(u, p)\varepsilon$ (lower panel), as functions of the reduced energy $u = E/(N\varepsilon)$, for $p = 2, 4, 6, 8$, and ∞ . For every finite p , the entropy maximum occurs at $u_c = (p-1)/2$, where the inverse temperature vanishes. As p increases, the entropy maximum shifts to higher energies and the curve broadens, reflecting the enlargement of the accessible energy range. The figure thus makes explicit how the microcanonical thermodynamics evolves with the number of available levels.

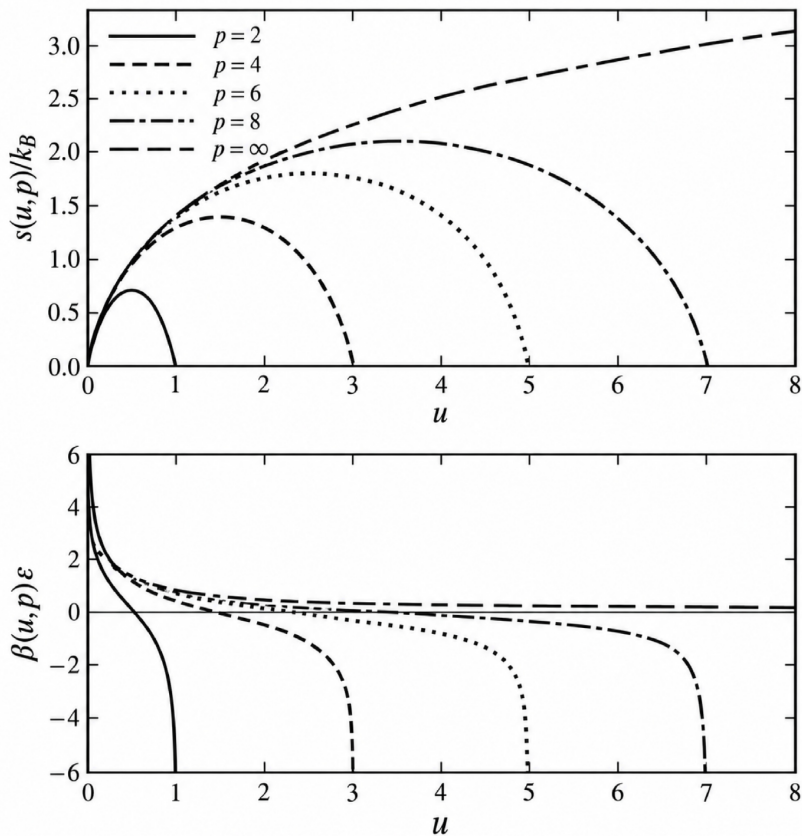


Fig. 1. Dimensionless entropy per particle, $s(u, p)/k_B$ (upper panel), and dimensionless inverse temperature, $\beta(u, p)\epsilon$ (lower panel), as functions of the reduced energy $u = E/(N\epsilon)$, for $p = 2, 4, 6, 8$, and ∞ .

4. Conclusions

The principal result of this work is an exact microcanonical construction that treats an arbitrary number p of equally spaced local energy levels within a single analytical framework. The generating-function representation retains the fixed-energy constraint explicitly and yields the complete multiplicity for finite N , while its contour-integral form permits a systematic thermodynamic-limit evaluation. The saddle-point solution then provides the entropy and inverse temperature for arbitrary p , rather than only for isolated low-dimensional cases.

This arbitrary- p solution identifies the upper spectral bound, rather than the equal level spacing alone, as the essential ingredient controlling the thermodynamic structure. For every finite p , the entropy reaches its maximum at

$$u_c = \frac{p-1}{2},$$

where the inverse temperature vanishes and the positive-temperature sector changes continuously into a negative-temperature sector. Increasing p shifts this characteristic energy toward the upper part of an increasingly broad spectrum. The limit $p \rightarrow \infty$ is therefore not simply another special case: it removes the truncation responsible for the entropy maximum and produces a qualitative change in the thermodynamics, leaving only the positive-temperature branch. The formulation thus gives an analytically controlled account of how bounded-spectrum thermodynamics is transformed into the unbounded-spectrum regime.

The known results for $p = 2$, $p = 3$, and $p \rightarrow \infty$ serve as consistency tests of the general construction, but they do not exhaust its content. The main advance is the explicit solution for arbitrary local dimension and the resulting ability to follow the dependence of the complete entropy and temperature curves on p . Moreover, because the coefficient representation remains exact before the large- N approximation is introduced, the same formalism can be used to investigate finite-size corrections, discrete definitions of microcanonical temperature, and the approach to the thermodynamic-limit entropy. These extensions are not developed here, but they follow directly from the exact multiplicity and constitute a natural continuation of the present analysis.

An immediate physical application is provided by a spin- S Zeeman multiplet [1]. The spectrum

$$E_m = -m\varepsilon, \quad m = -S, -S+1, \dots, S$$

is equivalent, up to an additive shift of the energy origin, to the p -level spectrum considered here. The replacements

$$p \longrightarrow 2S+1, \quad u \longrightarrow u+S$$

therefore yield an exact thermodynamic-limit microcanonical description for arbitrary S . To the best of our knowledge, the arbitrary-spin problem has not previously been expressed in this unified analytical microcanonical form. The present result consequently supplies a direct reference solution for bounded magnetic multiplets and for finite-dimensional local quantum systems.

More generally, the essential requirement of the method is not equal level spacing, but the existence of a local generating function whose powers encode the multiplicities of additive many-body observables. Replacing the polynomial used here by the generating function of another local spectrum leads to an analogous coefficient-extraction problem. This makes the

construction relevant to finite local quantum spectra [5], as well as to discrete statistical-mechanical models such as polymer and helix–coil systems [9, 10], for which complete analytical microcanonical treatments are often unavailable. The present solution should therefore be viewed not only as an exactly solvable bounded-spectrum model, but also as a general analytical template for constructing and studying the microcanonical thermodynamics of discrete finite-state systems.

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