VACUUM CORRELATIONS OF THE STRESS-ENERGY-MOMENTUM TENSOR WITH CONSTITUENT QUARKS*

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The two-point correlation function of the stress-energy-momentum tensor describes the propagation of a space-time "micro-earthquake" in the vacuum. In the framework of the path integral formulation of field theory in curved space-time, we discuss the Ward–Takashi identity for two-point Green's function of the stress-energy-momentum tensor for a general case of a non-conformal theory. The identity constrains the longitudinal part of the correlator, with the vacuum expectation value of the stress-energy-momentum, non-zero in a non-conformal theory. The obtained formula is demonstrated on the free massive Dirac fermion theory, treated at the one-loop level. This example befits a class of phenomenological chiral quarks models which have been used successfully in numerous applications in the soft non-perturbative regime of strong interactions. We stress the role of the constraints following from the Ward–Takahashi identity for the correlation functions in these models and point out the differences with the low-energy theorems (sum rules), not complying to the symmetry requirements. We also show how the temporal representation of the two-point correlators is an object amenable to lattice QCD simulations.

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1. Introduction

The stress-energy-momentum tensor (SEM) is a conserved quantity in relativistic quantum field theory and, consequently, has a long history. The canonical Noether construction has been amended by Belinfante and Rosenfeld [1, 2], allowing one to obtain a symmetric form while respecting the conservation laws. With torsion-free connections, this prescription is equivalent to the Hilbert construction, where SEM is the response of the action to perturbations of the metric tensor in curved space-time about its flat-space value (for a recent discussion of these issues see, *e.g.*, [3, 4]). Further, within this framework, multiple vacuum SEM correlations (Green's functions) arise as multilinear responses to these perturbations and, in that regard, they provide basic information on the gravitational space-time fluctuations (see, *e.g.*, [5]). For the rest of the paper, we will embrace the physically motivated Hilbert interpretation of SEM as the response of the system to the presence of a space-time fluctuation in the metric caused either by gravity or other external sources.

The theoretical and practical phenomenological implications involving the SEM are vast, so we briefly mention only some of them to provide a motivating framework for our study. Since the early days of the deep inelastic scattering, the momentum sum rule, corresponding to the expectation value of SEM in a hadron, has played a major role, as it provides a measure of the relevance of the valence quarks, sea quarks, and gluons in a hadron state (see, *e.g.*, [6] for a comprehensive discussion). This connection to the parton distribution functions (PDFs), the generalized parton distribution functions (GPDs), and the corresponding gravitational form factors has been explored in lattice QCD simulations of hadron structure (for a recent lattice and theoretical progress on the gravitational form factors of the pion and nucleon, see [7–9] and numerous references therein).

Two-point vacuum correlators of SEM have been studied in the context of the operator product expansion (OPE) in the QCD sum rules for gluonium states by Novikov, Shifman, Vainshtein, and Zakharov [10] in phenomenological studies of the $J^{PC} = 2^{++}$ states. A dispersive analysis of the glueball masses was carried out within a method based on OPE in [11]. The mixed scalar–SEM correlator was explored within Chiral Perturbation Theory by Donoghue and Leutwyler [12].

At finite temperatures, the two-point correlators of SEM are related to viscosities via the Kubo formulas. These correlators are basic objects in the lattice QCD quest to find the transport coefficients of hot QCD medium [13, 14]. The shear viscosity of a hot medium in a chiral quark model was calculated in [15].

On a more fundamental though less practical setup, the n-point SEM Green's functions describe a multi-graviton scattering process, but the smallness of the gravitational constant and no access to gravitons, obviously, precludes a direct experimental information. Nevertheless, the case of *conformal* field theories has been quite thoroughly studied, starting with Osborn and collaborators' pioneering works on the two- and three-point functions in the coordinate space [16, 17]. The extension to the momentum space followed, first via the Fourier transform [18], and later through an approach contained entirely in the momentum space [19, 20]. The case of the scalar field has been considered by one of us (M.S.) in the momentum space for the four-point SEM correlation function [21, 22].

Unlike other conserved quantities such as the electromagnetic or flavor currents, SEM carries additional complications from higher-order space-time derivatives of the fields, which requires introducing the T^* time-ordering [23]. The path-integral framework, as used in this paper, is equivalent to choosing a certain T^* time-ordered product. Also, the correlations of SEM induce a highly divergent high-energy behavior. In a pioneering paper, Callan, Coleman, and Jackiw [24] showed how to alter the construction in such a way that a new improved tensor not only has a better ultraviolet behavior, but its source is the gravitational field itself, in harmony with the equivalence principle.

We believe that a more phenomenological and practical discussion in the case of strong interactions seems worthwhile and insightful, as it corresponds to the fluctuating mass distribution properties of matter and the vacuum subjected to proper symmetry constraints. In this paper, we first derive, with path integrals in curved space-time, the Ward–Takashi identity (WTI) for two-point Green's function of SEM for a general case of a *non-conformal* field theory. We then analyze the case of free massive Dirac fermions. With appropriate Feynman rules, we obtain an explicit one-quark-loop formula for the two-point function which complies to the general symmetry requirements, in particular WTI. Such fermionic systems with quarks have been used extensively to model low-energy dynamics of strong interactions, where quarks acquire a large (constituent) mass due to the dynamical chiral symmetry breaking induced in QCD (for a review, see [25]). The model has no gluons at the so-called quark-model scale [25], denoted by $\mu_{\rm QM} \simeq 320$ MeV, where massive quarks are the only degrees of freedom (for early studies on the interplay of the chiral quark models and SEM within chiral effective Lagrangians, see, *e.g.*, [26–28]). Finally, with possible lattice QCD simulations [29–31] in mind, we discuss the quark-model SEM correlators in the temporal representation, where only their absorptive parts contribute.

2. Correlators and the gravitational Ward–Takahashi identities

Our notation follows closely Chapter 2 of [21], adapted to the Minkowski space-time convention. The standard Hilbert definition of SEM in a classical field theory described by an action $\mathcal{S}$ of the matter fields is given in terms of a functional derivative with respect to the metric tensor $g_{\mu\nu}(x)$, once the theory has been embedded in a curved space-time,

$$\theta^{\mu\nu}(x) = -\frac{2}{\sqrt{g_x}} \frac{\delta \mathcal{S}}{\delta g_{\mu\nu}(x)}\,, \tag{1}$$

where $g_x \equiv -\det g_{\mu\nu}(x)$. The generating functional is given by the path integral

$$\mathcal{Z} = e^{i\mathcal{W}} = \frac{1}{\mathcal{N}} \int \mathcal{D}\Phi\, \mathrm{e}^{i\mathcal{S}}\,, \tag{2}$$

where $\mathcal{W}$ generates the connected diagrams, $\mathcal{N}$ is a normalization, and Φ denotes all the quantum fields of the theory.

The n-point connected correlators of SEM are obtained as

$$\begin{aligned} &\langle \theta^{\mu_1\nu_1}(x_1) \dots \theta^{\mu_n\nu_n}(x_n) \rangle \\ &= \frac{(2i)^n}{\sqrt{g_{x_1}} \cdots \sqrt{g_{x_n}}} \frac{i\,\delta^n \mathcal{W}}{\delta g_{\mu_1\nu_1}((x_1)) \dots \delta g_{\mu_n\nu_n}((x_n))}\,, \end{aligned} \tag{3}$$

where the brackets $\langle . \rangle$ denote the path-integral expectation value in the curved space-time. Note the symmetry property with the indices interchange, $(\mu_i, \nu_i) \leftrightarrow (\nu_i, \mu_i)$, following from the definition. As already mentioned, in this formulation $\langle . \rangle$ defines a particular T^* time-ordered product. The first two connected n-point functions are given explicitly in terms of the action $\mathcal{S}$ as

$$\langle \theta^{\mu\nu}(x) \rangle = \int \mathcal{D}\Phi\, \frac{2i}{\sqrt{g_x}} \frac{\delta(i\mathcal{S})}{\delta g_{\mu\nu}(x)}\, \mathrm{e}^{i\mathcal{S}} \tag{4}$$

and

$$\begin{aligned} \left\langle \theta^{\mu\nu}(x_1) \theta^{\alpha\beta}(x_2) \right\rangle &= \int \mathcal{D}\Phi\, \mathrm{e}^{i\mathcal{S}} \frac{(2i)^2}{\sqrt{g_{x_1}}\sqrt{g_{x_2}}} \\ &\times \left[\left\langle \frac{\delta(i\mathcal{S})}{\delta g_{\mu\nu}(x_1)} \frac{\delta(i\mathcal{S})}{\delta g_{\alpha\beta}(x_2)} \right\rangle + \left\langle \frac{\delta^2(i\mathcal{S})}{\delta g_{\mu\nu}(x_1) \delta g_{\alpha\beta}(x_2)} \right\rangle \right]. \end{aligned} \tag{5}$$

Note the presence of the tadpole piece (the second term).

We remark that in the presence of fermions, one usually uses the formulation with the Vierbein (tetrad) fields $V_A^\mu(x)$, which encode the transformation properties of the Dirac gamma matrices and the spin connection,

allowing for a more efficient treatment of their variation with $g^{\mu\nu}$. Then the equivalent of Eq. (1) is [5]

$$\theta^{\mu\nu} = \frac{1}{2V(x)}\left[V_A^{\mu}(x)\frac{\delta S}{\delta V_A^{\nu}(x)} + V_A^{\nu}(x)\frac{\delta S}{\delta V_A^{\mu}(x)}\right], \tag{6}$$

where symmetrization is being imposed. Using Eq. (6) yields the same structure of WTI identities as Eq. (1) [18]. A detailed derivation of SEM in the Vierbein formalism can be found, *e.g.*, in [32].

The derivation of the gravitational WTI for the two-point function follows exactly [18, 21, 22]. The general covariance in a curved space implies the master equation

$$\nabla_{\nu}\langle\theta^{\mu\nu}(x_1)\rangle = \nabla_{\nu}\left(\frac{2i}{\sqrt{g_{x_1}}}\frac{i\,\delta\mathcal{W}}{\delta g_{\mu\nu}(x_1)}\right) = 0\,. \tag{7}$$

By expanding the covariant derivative and using the symmetry properties of the Christoffel symbols $\Gamma^{\mu}_{\kappa\nu}$, one gets

$$\partial^{x_1}_{\nu}\frac{\delta\mathcal{W}}{\delta g_{\mu\nu}(x_1)} + \Gamma^{\mu}_{\kappa\nu}(x_1)\frac{\delta\mathcal{W}}{\delta g_{\kappa\nu}(x_1)} = 0\,. \tag{8}$$

Carrying out a functional derivative of this relation with respect to $\delta g_{\alpha\beta}$ and taking the flat space-time limit (where $\Gamma^{\mu}_{\kappa\nu} = 0$ but $\delta\Gamma^{\mu}_{\kappa\nu}/\delta g_{\alpha\beta}$ is, in general, non-zero) yields (*cf.* Eq. (30) of [18])

$$\begin{aligned}&\nabla_{\nu}\left\langle\theta^{\mu\nu}(x_1)\theta^{\alpha\beta}(x_2)\right\rangle_c \\ &= \left[\partial^{x_1}_{\nu}\frac{\delta^2\mathcal{W}}{\delta g_{\mu\nu}(x_1)\delta g_{\alpha\beta}(x_2)} + \frac{\delta\Gamma^{\mu}_{\kappa\nu}(x_1)}{\delta g_{\alpha\beta}(x_2)}\frac{\delta\mathcal{W}}{\delta g_{\kappa\nu}(x_1)}\right]_{g=\eta} = 0\,.\end{aligned} \tag{9}$$

Next, we perform the Fourier transform to the momentum space (all the momenta are conventionally taken to be incoming), which together with definition (3) yields the WTI in the form (from now on, taking the $g = \eta$ limit is understood, with $\eta = \mathrm{diag}(1, -1, -1, -1)$)

$$q_{\nu}\left\langle\theta^{\mu\nu}(q)\theta^{\alpha\beta}(-q)\right\rangle + 2\frac{\delta\Gamma^{\mu}_{\kappa\lambda}}{\delta g_{\alpha\beta}}(q)\left\langle\theta^{\kappa\lambda}\right\rangle = 0\,. \tag{10}$$

The derivative of the Christoffel symbol in the momentum space is

$$\frac{\delta\Gamma^{\mu}_{\kappa\nu}}{\delta g_{\alpha\beta}}(q) = \frac{1}{2}\,\eta^{\mu\lambda}\left(s^{\alpha\beta}{}_{\kappa\nu}\,q_{\lambda} - s^{\alpha\beta}{}_{\kappa\lambda}\,q_{\nu} - s^{\alpha\beta}{}_{\lambda\nu}\,q_{\kappa}\right), \tag{11}$$

with $s^{\mu\nu}{}_{\alpha\beta} \equiv \delta g_{\alpha\beta}/\delta g_{\mu\nu} = \frac{1}{2}\left(\delta^\mu_\alpha \delta^\nu_\beta + \delta^\nu_\alpha \delta^\mu_\beta\right)$. By covariance, the vacuum expectation value of SEM is proportional to the flat-space metric tensor,

$$\langle \theta^{\mu\nu} \rangle = B\, \eta^{\mu\nu}\,, \tag{12}$$

hence Eq. (10) becomes

$$q_\nu \left\langle \theta^{\mu\nu}(q)\theta^{\alpha\beta}(-q) \right\rangle = -B\left(q^\mu \eta^{\alpha\beta} - q^\alpha \eta^{\mu\beta} - q^\beta \eta^{\mu\alpha}\right)\,, \tag{13}$$

which is the key result. It can equivalently be written as

$$\begin{aligned} q_\nu \left\langle \theta^{\mu\nu}(q)\theta^{\alpha\beta}(-q) \right\rangle &= -q^\mu \left\langle \theta^{\alpha\beta} \right\rangle + q^\alpha \left\langle \theta^{\mu\beta} \right\rangle + q^\beta \left\langle \theta^{\mu\alpha} \right\rangle \\ &= \left(-q^\mu g^{\alpha\beta} + q^\alpha g^{\mu\beta} + q^\beta g^{\mu\alpha}\right) B\,. \end{aligned} \tag{14}$$

In non-conformal theories, such as in the case of massive fermions described in the following, $B \neq 0$ and the WTI has a non-trivial form, involving on the right-hand side the vacuum expectation values of SEM. We note that the tensorial structure of Eq. (13) agrees with Eq. (15) in [33], which was obtained via old-fashioned techniques involving an analysis of the Schwinger terms, as well as with Eq. (6) of [34].

We will later consider also the correlation of SEM with the quark condensate[1] (scalar field) in a theory with massive quarks. This correlator is defined as [12]

$$\langle \theta^{\mu\nu}(q)\, \bar{q}q(-q) \rangle = i \int \mathrm{d}^4x\, \mathrm{e}^{-iqx} \langle \theta^{\mu\nu}(x) \bar{q}q(0) \rangle\,. \tag{15}$$

Introducing a scalar source to the generating function [35] and repeating the steps outlined above, one obtains the corresponding WTI in the simple form

$$q_\mu \langle \theta^{\mu\nu}(q) \bar{q}q(-q) \rangle = -q^\nu \langle \bar{q}q \rangle\,. \tag{16}$$

3. Identity with the trace of SEM

In this paper, we use the short-hand notation for the trace operator,

$$\theta \equiv \theta^\mu_\mu\,. \tag{17}$$

In curved space, in general, one can write

$$g_{\mu\nu}(x) \langle \theta^{\mu\nu}(x) \rangle = \mathcal{A}[g, x] + \langle \theta(x) \rangle\,, \tag{18}$$

[1] In our notation, $\bar{q}q$ denotes the quark condensate of a single flavor.

where $\mathcal{A}$ denotes the anomalous terms which account for the difference between the trace operation taken before and after quantization (for their explicit form see, *e.g.*, [5]). Similarly as in the derivation of Section 2, we differentiate functionally Eq. (18) with respect to $\delta g_{\alpha\beta}(x')$, take the flat space-time limit, and pass to the momentum space, which yields

$$\eta_{\mu\nu} \left\langle \theta^{\mu\nu}(q)\theta^{\alpha\beta}(-q) \right\rangle = 2\,i\,\frac{\delta \mathcal{A}}{\delta g_{\alpha\beta}}[q] + \left\langle \theta(q)\theta^{\alpha\beta}(-q) \right\rangle + 2\left\langle \theta^{\alpha\beta} \right\rangle . \qquad (19)$$

Note a non-trivial appearance of the last term on the right-hand side, originating from $\delta g_{\mu\nu}(x)/\delta g_{\alpha\beta}(x')$.

4. Constituent quarks

As an illustration of the fulfillment of the above formal requirements, in particular the WTIs and the trace identity, we consider a model with free massive (constituent) quarks. Such theories, notably, are interesting in their own right due to numerous applications to the low-energy hadronic phenomenology (for a review see, *e.g.*, [25]), in particular the non-perturbative structure of pions and their interactions. The light-flavor quarks acquire a large (constituent) quark mass M due to the spontaneous chiral symmetry breaking occurring in QCD. At the low-energy quark-model scale, the quarks are assumed to be the only dynamical degrees of freedom, with gluons generated radiatively upon evolution to higher scales [36]. Obviously, the theory is not conformal at non-zero M.

The action of our theory embedded in a curved space-time, written with the help of the Vierbein $V_a^\mu(x)$, is

$$S = \int \mathrm{d}^4x\sqrt{-g}\,\left\{ \tfrac{i}{2}\left[\bar\psi\gamma^a\,V_a^\mu\nabla_\mu\psi - \left(\nabla_\mu\bar\psi\right)\gamma^a\,V_a^\mu\psi\right] - M\bar\psi\psi \right\} . \qquad (20)$$

The Vierbein fields satisfy the relation $V_a^\mu(x)\,V_b^\nu(x)\,g_{\mu\nu}(x) = \eta_{ab}$. The covariant (world) derivative acting on a fermion field is defined as [5] $\nabla_\mu = \partial_\mu + \frac{1}{2}\Sigma^{ab}\,V_a{}^\sigma\,\nabla_\mu V_{b\sigma}$, with $\Sigma^{ab} = \frac{1}{4}[\gamma^a,\gamma^b]$, whereas $\nabla_\nu A^\mu = \partial_\nu A^\mu + \Gamma^\mu_{\nu\rho}A^\rho$ when acting on a vector field. Then, Eqs. (6) and (20) yield the SEM operator

$$\begin{aligned}\theta^{\mu\nu}(x) &= \tfrac{i}{4}\left[\bar\psi(x)\gamma^\nu\nabla^\mu\psi(x) - \left(\nabla^\mu\bar\psi(x)\right)\gamma^\nu\psi(x) + (\mu\leftrightarrow\nu)\right] \\ &-g^{\mu\nu}(x)\tfrac{i}{2}\left\{\left[\bar\psi(x)\gamma_\mu\nabla^\mu\psi(x) - \left(\nabla^\mu\bar\psi(x)\right)\gamma_\mu\psi(x)\right] - M\bar\psi(x)\psi(x)\right\} . \qquad (21)\end{aligned}$$

We shall treat the model at the one-quark-loop level, which corresponds to taking the large-N_c limit. The Feynman diagrams for the one- and two-point correlators are given in Fig. 1. The Feynman rules for the vertices

are as follows: the vertex for the incoming graviton of momentum q and incoming quark of momentum k is

$$V_{gq}^{\mu\nu} = \tfrac{1}{4}\left[\gamma^\mu (2k+q)^\nu - \eta^{\mu\nu}(2\not{k} + \not{q} - 2M)\right] + (\mu \leftrightarrow \nu)\,, \tag{22}$$

whereas for the contact graviton–graviton–quark vertex, we have

$$\begin{aligned} V_{ggq}^{\mu\nu,\alpha\beta} &= (\not{k} - M)\left(\eta^{\alpha\beta}\eta^{\mu\nu} - \eta^{\alpha\nu}\eta^{\beta\mu} - \eta^{\alpha\mu}\eta^{\beta\nu}\right) \\ &+ \left[\left\{\tfrac{1}{8}k^\alpha\left(3\gamma^\nu g^{\beta\mu} + 3\gamma^\mu g^{\beta\nu} - 4\gamma^\beta g^{\mu\nu}\right) + (\alpha \leftrightarrow \beta)\right\} + (\mu\nu \leftrightarrow \alpha\beta)\right]\,. \end{aligned} \tag{23}$$

Here, for the compactness of the notation, the symmetrization in the Lorentz indices has to be carried out as indicated. Also, we have taken the special case occurring in the tadpole diagrams in Fig. 1, where the incoming and outgoing quark momenta are equal to k, which leads to simplifications. The intricate form of Eq. (23) follows from the covariant structure of the action (20) and is crucial for the satisfaction of the WTI (13). For the evaluation of the mixed SEM-scalar correlator, we also need

$$V_{\sigma q} = 1\,, \qquad V_{g\sigma q}^{\mu\nu} = -\eta^{\mu\nu}\,. \tag{24}$$

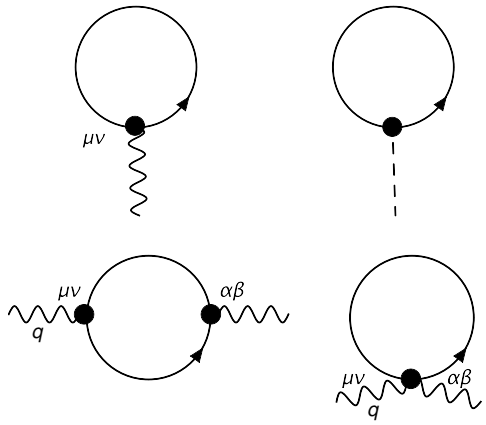


Fig. 1. One-loop Feynman diagrams for the one- and two-point functions in the constituent quark model. The wavy line indicates the graviton, the dashed line the scalar field, and the solid line is the quark propagator with the constituent mass M.

5. One-point functions

The one-loop vacuum expectation value of SEM with massive quarks is evaluated according to the top-left diagram in Fig. 1, yielding

$$\langle\theta^{\mu\nu}\rangle \equiv \eta^{\mu\nu}B = -N_c N_f \int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{Tr}\left[V_{gq}^{\mu\nu}S_k\right]\,, \tag{25}$$

where $S_k = i/(\not{k} - M + i\epsilon)$ is the Feynman quark propagator, $N_c = 3$ is the number of colors, N_f is the number of flavors, and the trace is over the Dirac indices. Evaluation of the trace yields

$$\langle\theta^{\mu\nu}\rangle = -4iN_c N_f \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k^\mu k^\nu - \left(k^2 - M^2\right)\eta^{\mu\nu}}{k^2 - M^2 + i\epsilon}\,. \tag{26}$$

Since in our low-energy model we focus on soft contributions, the perturbative contribution, corresponding to the above expression taken at $M = 0$, is subtracted[2]. The result is

$$\tfrac{1}{4}\langle\theta\rangle = B = \frac{N_c N_f}{16\pi^2}M^2 A_0\left(M^2\right)\,, \tag{27}$$

where $A_0(M^2)$ is the one-point Passarino–Veltman function (*cf.* Appendix B).

Analogously, the quark condensate (for a single flavor) is

$$\langle\bar{q}q\rangle = -N_c\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{Tr}\left[V_{\sigma q}S_k\right] = \frac{N_c}{4\pi^2}MA_0\left(M^2\right)\,. \tag{28}$$

Comparing (27) and (28), we notice the relation (holding at the quark-model scale)

$$B = \frac{N_f M}{4}\langle\bar{q}q\rangle\,, \tag{29}$$

or, equivalently,

$$\langle\theta\rangle = MN_f\,\langle\bar{q}q\rangle\,. \tag{30}$$

The typical values of $M \sim 0.3$ GeV are compatible within uncertainties with the QCD sum rule estimates from charmonium decay, $B = -(224^{+35}_{-70}\ \mathrm{MeV})^4$ [37, 38] and $\langle\bar{q}q\rangle = (235 \pm 15\ \mathrm{MeV})^3$. However, Eqs. (27) and (28) are formal in the sense that the quantities are ultraviolet divergent. In particular, $A_0(M^2)$ of Eq. (B.1) is quadratically divergent, hence, as is well known, a suitable regularization is needed to incorporate only the soft physics and make the expressions finite. The procedure disregards the perturbative QCD

[2] This amounts to using the condition $\int \mathrm{d}^4k\,1 = 0$, as in the case of the dimensional regularization.

contributions, which are considered separately. In the Pauli–Villars regularization, as introduced, *e.g.*, in [36], relation (29) holds. In the case of the Spectral Quark Model (SQM) [39] (see Appendix C), we find

$$B = -\frac{N_c N_f M_{\rm V}^4}{192\pi^2}\,, \tag{31}$$

where $M_{\rm V} \simeq 770$ MeV is the vector meson mass. Numerically, we get $B = -(200 - 220\ \text{MeV})^4$ (with $N_f = 3$), in agreement with the QCD sum rules estimate quoted above. Since in SQM [39] the relation

$$f^2 = \frac{N_c M_{\rm V}^2}{24\pi^2} \tag{32}$$

holds[3], where $f \simeq 93$ MeV is the pion weak decay constant, one can also write

$$B = -\frac{3\pi^2 N_f f^4}{N_c}\,. \tag{33}$$

In SQM, the value of the quark condensate is an independent model parameter [39], unrelated to B. While phenomenologically this is a welcome feature, the numerical agreement of B with QCD sum rules should be taken with caution. Since B contains both quarks and gluons, it behaves as $\sim N_c^2$, while numerically the scale is ~ 1 GeV, whereas in the quark models we only have quarks, hence $B \sim N_c N_f$ and the scale is ~ 0.3 GeV. The scale dependence of the quark condensate confirms this feature [25].

6. Two-point functions

The SEM correlations receive contributions from the bottom-row diagrams of Fig. 1, from where we find

$$\left\langle \theta^{\mu\nu}(q)\theta^{\alpha\beta}(-q)\right\rangle = iN_cN_f\int\frac{{\rm d}^4k}{(2\pi)^4}\,{\rm Tr}\left[iV_{gq}^{\mu\nu}S_{k+q}iV_{gq}^{\alpha\beta}S_k + iV_{ggq}^{\mu\nu,\alpha\beta}S_k\right]. \tag{34}$$

The result can be decomposed into transverse spin-0 and spin-2 components, as well as a longitudinal part,

$$\left\langle \theta^{\mu\nu}(q)\theta^{\alpha\beta}(-q)\right\rangle = \langle\theta\theta\rangle_0\left(q^2\right)T_0^{\mu\nu\alpha\beta} + \langle\theta\theta\rangle_2\left(q^2\right)T_2^{\mu\nu\alpha\beta} + \langle\theta\theta\rangle_{\rm L}T_{\rm L}^{\mu\nu\alpha\beta}\,, \tag{35}$$

[3] It is also obtained when matching chiral quark models to the vector-meson dominance principle, and is reasonably well satisfied numerically with the $M_{\rm V} \simeq m_\rho$.

where the tensors are explicitly given in Appendix A. The evaluation of the one-loop integrals yields the following expressions in terms of the Passarino–Veltman functions (B.1):

$$\begin{aligned}
\langle\theta\theta\rangle_0\left(q^2\right) &= \frac{N_c N_f}{48\pi^2} M^2\left[5A_0\left(M^2\right)+2\left(q^2-4M^2\right)B_0\left(M^2,q^2\right)\right], \\
\langle\theta\theta\rangle_2\left(q^2\right) &= \frac{N_c N_f}{480\pi^2}\left[2\left(46M^2-3q^2\right)A_0\left(M^2\right)\right. \\
&\quad \left.+\left(q^2-4M^2\right)\left(8M^2+3q^2\right)B_0\left(M^2,q^2\right)\right], \\
\langle\theta\theta\rangle_{\mathrm{L}} &= -\frac{N_c N_f}{16\pi^2} M^2 A_0\left(M^2\right) = -B\,. \qquad (36)
\end{aligned}$$

The form of the longitudinal part, with the help of relation (A.4), clearly leads to the fulfillment of the WTI of Eq. (13). This feature is of course expected, as our calculation conforms to all the covariance requirements.

In particular, for the trace–trace correlator, we find

$$\langle\theta\theta\rangle\left(q^2\right) = \frac{N_c N_f}{8\pi^2} M^2\left(q^2-4M^2\right)B_0\left(M^2,q^2\right). \qquad (37)$$

For the SEM–scalar correlator, following the notation of [12], we have the general decomposition into the transverse and longitudinal parts,

$$\langle\theta^{\mu\nu}(q)\bar{q}q(-q)\rangle = t^{\mu\nu}q^2\phi_1\left(q^2\right)+\eta^{\mu\nu}\phi_0\left(q^2\right). \qquad (38)$$

At the one-quark-loop level,

$$\langle\theta^{\mu\nu}(q)\bar{q}q(-q)\rangle = iN_cN_f\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{Tr}\left[iV_{gq}^{\mu\nu}S_{k+q}iV_{\sigma q}S_k+iV_{g\sigma q}^{\mu\nu}S_k\right], \qquad (39)$$

yielding

$$\begin{aligned}
q^2\phi_1 &= \frac{N_cM}{24\pi^2}\left[\left(q^2-4M^2\right)B_0\left(M^2,q^2\right)+4A_0\left(M^2\right)\right], \\
\phi_0 &= -\frac{N_cM}{4\pi^2}A_0\left(M^2\right) = -\langle\bar{q}q\rangle\,. \qquad (40)
\end{aligned}$$

We note that since $\phi_0 = -\langle\bar{q}q\rangle$, the WTI of Eq. (16) is immediately satisfied.

The structure of the one-loop correlators (36) and (40) incorporates absorptive parts, contained in the B_0 function (*cf.* Eq. (B.4)) and appearing in the transverse pieces only. The remaining dispersive pieces are polynomial in q^2. Note that the perturbative renormalization may redefine the polynomial terms but not the absorptive pieces, which are in that sense unique, since they represent a branch cut discontinuity at $q^2 = 4M^2$. In the spectral

regularization, the polynomial pieces q^2 and q^4 with arbitrary coefficients emerge only in $\langle\theta\theta\rangle_2$, as is apparent from Eqs. (C.5) and (C.6), and not in the other considered correlators.

The low-energy model considered here is by construction valid at soft momenta, which also concerns the value of q^2. Yet, we notice that with the relation (C.8) holding in SQM, we readily find that at $q^2 \to \infty$, the transverse part of $\langle\theta^{\mu\nu}\bar{q}q\rangle(q^2)$ tends to

$$q^2\phi_1\left(q^2\right) \to \frac{N_c M}{4\pi^2} A_0\left(M^2\right) = \langle\bar{q}q\rangle\,. \tag{41}$$

Incidentally, this limit agrees with Eqs. (65) and (66) of [12].

Explicit expression in SQM for all the considered correlators are listed in Appendix D.

7. Trace identity for SEM in the quark model

In our quark model valid at the quark model scale, no anomalies are considered, $\mathcal{A} = 0$. Moreover, $\langle\theta^\mu_\mu\rangle = MN_f\langle\bar{q}q\rangle$ in Eq. (18). Then Eq. (19) simplifies to

$$\eta_{\mu\nu}\langle\theta^{\mu\nu}(q)\theta^{\alpha\beta}(-q)\rangle = MN_f\left\langle[\bar{q}q](q)\theta^{\alpha\beta}(-q)\right\rangle + 2\left\langle\theta^{\alpha\beta}\right\rangle. \tag{42}$$

This relation, binding non-trivially the SEM–SEM correlator and the SEM–scalar correlator, is explicitly satisfied by our quark-model expressions (35), (36), (38), and (40), where formulas (A.6) for the contracted tensors are used.

8. Low-energy theorems *versus* current conservation

In QCD, the trace operator $\theta = \theta^\mu_\mu$ plays an important role. It involves the anomaly and is related to the divergence of the dilatation current $D(x)$ at the quantum level,

$$\partial^\mu D_\mu = \theta = \frac{\beta(\alpha)}{2\alpha} G^{\mu\nu a} G^a_{\mu\nu} + \sum_q m_q\left[1+\gamma_m(\alpha)\right]\bar{q}q\,, \tag{43}$$

where $\beta(\alpha) = \mu^2 \mathrm{d}\alpha/\mathrm{d}\mu^2$ is the QCD beta function at an energy scale μ, $\alpha = g^2/(4\pi)$ is the running coupling constant, G is the gluon field, and $\gamma_m(\alpha) = \mathrm{d}\log m_q/\mathrm{d}\log\mu^2$ denotes the anomalous dimension of the current quark mass m_q, with q summed over the flavors. The trace anomaly θ^μ_μ is renormalization-scheme and scale independent, being free of the UV divergences.

With certain assumptions concerning regularization and renormalization, the following low-energy theorems (LET) or sum rules have been obtained [10]:

$$i\int \mathrm{d}^4x\langle T\theta(x)O(0)\rangle = -d_O\langle O\rangle\,, \tag{44}$$

where O is an arbitrary operator with an (engineering) dimension d_O. The case of $O=\bar{q}q$ from the point of view of LET was discussed in detail in Section VIII of [12].

In particular, for the SEM–scalar correlator, Eq. (44) would mean that (in the momentum space at $q=0$)

$$\langle\theta(0)\bar{q}q(0)\rangle = -3\langle\bar{q}q\rangle\,, \qquad \text{(LET)} \tag{45}$$

with $d_{\bar{q}q}=3$. We note, however, that this formula is incompatible with WTI. Indeed, by contracting Eq. (38) with q_μ and comparing to WTI of Eq. (16), one readily finds that $\phi_0=-\langle\bar{q}q\rangle$. The result follows from the conservation of SEM, hence is based on symmetry preservation and, thus, general. Taking the trace of Eq. (38) yields

$$\langle\theta(q)\bar{q}q(-q)\rangle = 3q^2\phi_1\left(q^2\right)+4\phi_0\left(q^2\right)\,. \tag{46}$$

Since ϕ_1 cannot have a massless pole, this means that

$$\langle\theta(0)\bar{q}q(0)\rangle = -4\langle\bar{q}q\rangle\,, \qquad \text{(WTI)}\,. \tag{47}$$

The LET result of Eq. (45) demands $\phi_0=-\frac{3}{4}\langle\bar{q}q\rangle$ [12], in tension with WTI.

Similarly, according to Eq. (44), LET for the trace–trace correlator reads

$$\langle\theta(0)\theta(0)\rangle = -4\langle\theta\rangle\,, \qquad \text{(LET)} \tag{48}$$

with $d_\theta=4$. On the other hand, WTI of Eq. (14) implies immediately

$$q_\mu\langle\theta(q)\theta^{\mu\nu}(-q)\rangle = -2q^\nu B\,. \tag{49}$$

The generic structure of the correlator is therefore

$$\langle\theta(q)\theta^{\mu\nu}(-q)\rangle = -2g^{\mu\nu}B+\left(q^\mu q^\nu-g^{\mu\nu}q^2\right)C\left(q^2\right)\,. \tag{50}$$

With no massless pole in $C(q^2)$, in the $q^2\to 0$ limit, we find the relation

$$\lim_{q^2\to 0}\langle\theta(q)\theta(-q)\rangle = -8B = -2\langle\theta\rangle\,, \qquad \text{(WTI)}\,. \tag{51}$$

We stress that relations (47) and (51) follow directly from the WTI (current conservation), *i.e.*, from the Lorentz covariance, supplemented with the physical requirement of the absence of zero-mass poles, hence are general, model-independent results.

The explicit results obtained in our quark model naturally comply to Eqs. (47) and (51). We note that they are also compatible with the trace identity (42). Formula (51), with the proper factor of -2 on the right-most side, compared to -4 in Eq. (48), agrees with the sum rule obtained by Brown and Zee [34, 40].

For the $\langle\theta\theta\rangle$ case in QCD, the intriguing conflict between the LET results and the Lorentz symmetry constraints has been explained in Ref. [40].

9. Connection to induced gravity

The Adler–Zee formula [41, 42] for the induced [41–44] Newton's gravitational constant reads (*cf.* Eq. (15) in [40])

$$\frac{1}{16\pi G_{\rm ind}} = \frac{1}{12}\frac{\mathrm{d}}{\mathrm{d}q^2}\langle\theta(q)\theta(-q)\rangle\Big|_{q^2=0}. \tag{52}$$

In SQM (see also [27, 28]), with the help of Eqs. (D.3) and (A.6), we find

$$\frac{1}{16\pi G_{\rm ind}} = -\frac{\langle\theta\rangle}{12M_{\rm V}^2} = \frac{N_c N_f M_{\rm V}^2}{576\pi^2} \simeq 0.001\ \mathrm{GeV}^2. \tag{53}$$

Only the scalar component $\langle\theta\theta\rangle_0(q)$ contributes to Eq. (52). With the relation (32), Eq. (53) can also be written as

$$\frac{1}{16\pi G_{\rm ind}} = \frac{N_f f^2}{24}. \tag{54}$$

The induced cosmological constant[4] is equal to (*cf.* Eq. (5.35) of [43])

$$\Lambda_{\rm ind} = -2\pi G_{\rm ind}\langle\theta\rangle = -8\pi G_{\rm ind} B. \tag{55}$$

In SQM, according to Eq. (31) and (53), we get

$$\Lambda_{\rm ind} = \frac{3M_{\rm V}^2}{2} = \frac{36\pi^2 f^2}{N_c} \simeq 1\ \mathrm{GeV}^2. \tag{56}$$

The huge mismatch compared to the observed value of $\Lambda \simeq 10^{-84}\ \mathrm{GeV}^2$ is a manifestation of the well-known cosmological constant problem [45, 46], experienced in Quantum Field Theory.

[4] We use the convention $\mathcal{L}_{\rm grav} = \frac{1}{16\pi G}\int \mathrm{d}^4x(R-2\Lambda)$.

Similarly, the numerical value of $G_{\rm ind}$ from Eq. (53) is

$$G_{\rm ind} \simeq 20\ {\rm GeV}^{-2}\,, \tag{57}$$

compared to the experimental value of the Newton gravitational constant, $G \simeq 6.7 \times 10^{-39}\ {\rm GeV}^{-2}$.

10. Dispersion relations and temporal correlations

Temporal correlations of two operators, O_1 and O_2 (such as used in lattice QCD studies [13, 14]), are defined as

$$\langle O_1 O_2\rangle(t) \equiv i\int {\rm d}^3x \langle 0|TO_1(\vec{x},t)O_2(0)|0\rangle = \int \frac{{\rm d}q_0}{2\pi}\,{\rm e}^{iq_0t}\langle O_1O_2\rangle\left(q_0^2\right)\,. \tag{58}$$

In our case O_i are the SEM tensors or $\bar{q}q$. These correlators, being analytic functions, satisfy suitably subtracted dispersion relations. However, it is clear from Eq. (58) that the polynomial pieces in q_0 generate the Dirac $\delta(t)$ function and its derivatives, hence they do not show up at non-zero t. Therefore, in the analysis of the temporal SEM correlations, we only need the discontinuities of Eqs. (36) and (40) residing in the Passarino–Veltman B_0 functions, which have the explicit form

$$\begin{aligned}
{\rm disc}\,\Pi_0(s) &= MN_f s\,{\rm disc}\,\phi_1(s)\\
&= \frac{2\pi i N_c N_f}{24\pi^2}\theta\left(s-4M^2\right)\sqrt{1-\frac{4M^2}{s}}M^2\left(s-4M^2\right)\,,\\
{\rm disc}\,\Pi_2(s) &= \frac{2\pi i N_c N_f}{480\pi^2}\theta\left(s-4M^2\right)\sqrt{1-\frac{4M^2}{s}}\left(s-4M^2\right)\left(3s+8M^2\right)\,,
\end{aligned} \tag{59}$$

where $s=q^2=q_0^2$. Essentially, the model provides the origin of the $q\bar{q}$ production cut at $s=4M^2$ and a specific dependence on M in the multiplying polynomials.

Taking the Fourier transform of (59) yields, at large Euclidean times $\tau = it$, the behavior

$$\begin{aligned}
\Pi_{0,2}(\tau) &\sim \frac{N_c N_f M^3\,{\rm e}^{-2M\tau}}{12\pi^2\tau^2}\,,\\
\phi_1(\tau) &\sim \frac{N_c\,{\rm e}^{-2M\tau}}{48\pi^2\tau^2}\,.
\end{aligned} \tag{60}$$

A similar asymptotic form for $\Pi_0(\tau)$, $\Pi_2(\tau)$, and $\phi_1(\tau)$ reflects the same limit of the corresponding discontinuities in (59) near the $\bar{q}q$ production threshold $s = 4M^2$. The slowly-varying function, which above has the form $\sim 1/\tau^2$, is altered with regularization. In particular, in SQM, we obtain

$$\begin{aligned} \Pi_{0,2}(\tau) &\sim \frac{N_c N_f M_{\rm V}^4 \, {\rm e}^{-M_{\rm V}\tau}}{192\pi^3\tau}\,, \\ \phi_1(\tau) &\sim -\frac{\langle\bar{q}q\rangle \, {\rm e}^{-M_{\rm S}\tau}}{4\pi M_{\rm S}^2\tau}\,. \end{aligned} \tag{61}$$

It would be a bit naive to expect that the nuances of the slowly varying functions in the temporal correlators could be sorted out on the lattice, where one expects meson dominance in channels with the corresponding quantum numbers. Nevertheless, the derived expressions can be used to estimate the magnitude of the non-resonant (constituent) quark continuum to these correlators.

11. Conclusions

In this paper, we have considered, using standard procedures of the path integral formulation of field theory in curved space-time, the Ward–Takashi and trace identities for the two-point correlators (Green's functions) of the stress-energy-momentum tensor for a general non-conformal case. The identities were then illustrated on a relevant example of the free massive Dirac fermion theory, treated at the one-loop level. Physically, the model corresponds to chiral (constituent) quarks, used in numerous phenomenological applications in the soft non-perturbative regime of QCD. We have discussed the constraints following from WTIs in reference to the low-energy sum rules involving the trace of the stress-energy-momentum tensor at zero momentum, underlying the key role of these symmetry constraints. We have also obtained the temporal representation of the two-point correlators in the quark model, which provides a possibly useful benchmark for the non-perturbative quark continuum contribution in future lattice QCD studies.

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Appendix A

Tensors

We introduce the basic tensors appearing in our analysis, where q is the graviton momentum. Let

$$t^{\mu\nu} = \eta^{\mu\nu} - q^{\mu\nu}\,, \qquad q^{\mu\nu} = \frac{q^\mu q^\nu}{q^2}\,, \tag{A.1}$$

with $t^{\mu\nu}q_\nu = 0$. The spin-0 and spin-2 combinations form the basis for the transverse parts of SEM,

$$T_0^{\mu\nu\alpha\beta} = \tfrac{1}{3}t^{\mu\nu}t^{\alpha\beta}\,, \qquad T_2^{\mu\nu\alpha\beta} = \tfrac{1}{2}t^{\mu\alpha}t^{\nu\beta} + \tfrac{1}{2}t^{\mu\beta}t^{\nu\alpha} - \tfrac{1}{3}t^{\mu\nu}t^{\alpha\beta}\,. \tag{A.2}$$

In addition, we need the tensor appearing in the longitudinal part of SEM,

$$T_{\rm L}^{\mu\nu\alpha\beta} = q^{\alpha\beta}q^{\mu\nu} + \eta^{\mu\nu}q^{\alpha\beta} + \eta^{\alpha\beta}q^{\mu\nu} - \eta^{\beta\nu}q^{\alpha\mu} - \eta^{\alpha\nu}q^{\beta\mu} - \eta^{\beta\mu}q^{\alpha\nu} - \eta^{\alpha\mu}q^{\beta\nu}\,, \tag{A.3}$$

with the property

$$q_\nu T_{\rm L}^{\mu\nu\alpha\beta} = q^\mu\eta^{\alpha\beta} - q^\alpha\eta^{\mu\beta} - q^\beta\eta^{\mu\alpha}\,. \tag{A.4}$$

The transverse tensors exhibit the projective property

$$\begin{aligned} T_0^{\mu\nu\alpha\beta}T_{0\,\alpha\beta}{}^{\sigma\tau} &= T_0^{\mu\nu\sigma\tau}\,,\\ T_2^{\mu\nu\alpha\beta}T_{2\,\alpha\beta}{}^{\sigma\tau} &= T_2^{\mu\nu\sigma\tau}\,,\\ T_0^{\mu\nu\alpha\beta}T_{2\,\alpha\beta}{}^{\sigma\tau} &= 0\,. \end{aligned} \tag{A.5}$$

The tensors contracted with $\eta_{\mu\nu}$ become

$$\begin{aligned} \eta_{\mu\nu}T_0^{\mu\nu\alpha\beta} &= t^{\alpha\beta}\,,\\ \eta_{\mu\nu}T_2^{\mu\nu\alpha\beta} &= 0\,,\\ \eta_{\mu\nu}T_{\rm L}^{\mu\nu\alpha\beta} &= \eta^{\alpha\beta} + q^{\alpha\beta}\,,\\ \eta_{\mu\nu}\eta_{\alpha\beta}T_0^{\mu\nu\alpha\beta} &= 3\,,\\ \eta_{\mu\nu}\eta_{\alpha\beta}T_{\rm L}^{\mu\nu\alpha\beta} &= 5\,. \end{aligned} \tag{A.6}$$

Appendix B

Passarino–Veltman functions

The adopted convention for the Passarino–Veltman one-loop functions is

$$\begin{aligned} i\pi^2 A_0\left(M^2\right) &= \int \mathrm{d}^4k \frac{1}{k^2-M^2+i\epsilon}\,, \\ i\pi^2 B_0\left(M^2,q^2\right) &= \int \mathrm{d}^4k \frac{1}{[(k+q)^2-M^2+i\epsilon]\,[k^2-M^2+i\epsilon]}\,, \end{aligned} \tag{B.1}$$

or, with the Euclidean momenta upon the Wick rotation $K_0=-ik_0$,

$$\begin{aligned} \pi^2 A_0\left(M^2\right) &= -\int \mathrm{d}^4K \frac{1}{K^2+M^2}\,, \\ \pi^2 B_0\left(M^2,q^2=-Q^2\right) &= \int \mathrm{d}^4K \frac{1}{[(K+Q)^2+M^2]\,[K^2+M^2]}\,. \end{aligned} \tag{B.2}$$

These functions are, correspondingly, quadratically and logarithmically divergent. With the Feynman parametrization of the denominators, one formally finds

$$B_0\left(M^2,q^2\right) = -\int\limits_0^1 \mathrm{d}x \log\left(\frac{M^2-x(1-x)q^2}{\lambda^2}\right)\,, \tag{B.3}$$

where λ is a constant. The discontinuity of B_0 is well defined (finite) and equals to

$$\operatorname{disc} B_0\left(M^2,q^2\right) = 2i\,\mathrm{Im}\,B_0\left(M^2,q^2\right) = 2\pi i\,\theta\left(q^2-4M^2\right)\sqrt{1-\frac{4M^2}{q^2}}\,. \tag{B.4}$$

Appendix C

Spectral regularization

The details and a physical justification of the method defining SQM are provided in [39]. The spectral function for quarks of a (complex) mass ω (it corresponds to the mass M in the body of the text) is

$$\begin{aligned} \rho(\omega) &= \rho_{\mathrm{V}}(\omega)+\rho_{\mathrm{S}}(\omega)\,, \\ \rho_{\mathrm{V}}(\omega) &= \frac{1}{2\pi i}\frac{1}{\omega}\frac{1}{\left(1-4\omega^2/M_{\mathrm{V}}^2\right)^{5/2}}\,, \\ \rho_{\mathrm{S}}(\omega) &= \frac{1}{2\pi i}\frac{-48\pi^2 M_{\mathrm{S}}\langle\bar{q}q\rangle}{N_c\left(1-4\omega^2/M_{\mathrm{S}}^2\right)^{5/2}}\,, \end{aligned} \tag{C.1}$$

where $M_{\rm V}$ and $M_{\rm S}$ are model parameters (vector and scalar masses, fixed with certain physical observables [39]), accompanied with the complex-plane integration contour shown in Fig. 2. The V and S parts give, correspondingly, the even and odd moments of the Passarino–Veltman functions in ω, as listed below.

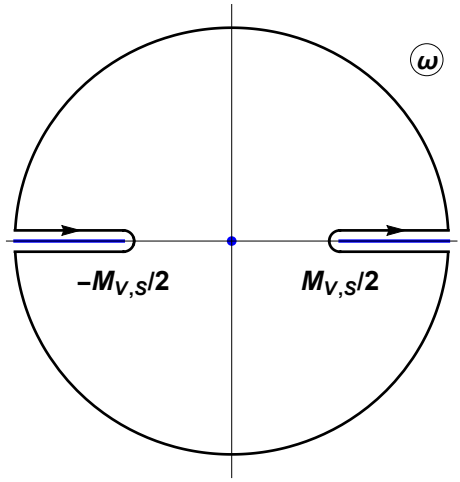


Fig. 2. Contour C of the integration in the complex quark-mass plane for the spectral regularization method. The cuts, starting at $\pm\frac{1}{2}M_{\rm V,S}$, and the pole at $\omega = 0$, present in the case of $\rho_{\rm V}(\omega)$, are indicated. The circular part of the contour C tends to infinity.

To regularize the quark-model formulas for the one- and two-body functions appearing in the body of this paper, several lowest order spectral moments of the Passarino–Veltman functions (B.2) are needed, denoted as

$$\begin{aligned} \left\langle w^n A_0\left(w^2\right)\right\rangle &= \int\limits_C \mathrm{d}\omega\rho(\omega)\omega^n A_0\left(\omega^2\right)\,, \\ \left\langle w^n B_0\left(w^2,q^2\right)\right\rangle &= \int\limits_C \mathrm{d}\omega\rho(\omega)\omega^n B_0\left(\omega^2,q^2\right)\,. \end{aligned} \tag{C.2}$$

For A_0, we obtain them from the Cauchy theorem as a sum of residues at $\omega = \pm i\frac{1}{2}M_{\rm V,S}$ and also, for the case of the $n = 0$ moment, at $\omega = 0$, *cf.* the $1/\omega$ pole in $\rho_{\rm V}$ in Eq. (C.1). Explicitly,

$$\left\langle wA_0\left(w^2\right)\right\rangle = \frac{4\pi^2\langle\bar{q}q\rangle}{N_c}\,, \qquad \left\langle w^2A_0\left(w^2\right)\right\rangle = -\frac{M_{\rm V}^4}{12}\,. \tag{C.3}$$

For the $n=0$ moment of A_0, the residue at $\omega=0$ yields an extra (perturbative) contribution which is quadratically divergent,

$$-\frac{1}{\pi^2}\int \mathrm{d}^4K\,\frac{1}{K^2}\,. \tag{C.4}$$

Upon renormalization, it brings an arbitrary constant μ_A of the dimension of the energy squared, hence we can write

$$\left\langle A_0\left(w^2\right)\right\rangle = \frac{M_{\rm V}^2}{6}+\mu_A^2\,, \tag{C.5}$$

where the first term is the non-perturbative contribution.

Similarly, for the even moments of B_0, we find

$$\begin{aligned}
\left\langle B_0\left(w^2,q^2\right)\right\rangle &= \frac{2}{3}\frac{M_{\rm V}^2}{M_{\rm V}^2-q^2}-\log\left(\frac{M_{\rm V}^2-q^2}{\mu_B^2}\right)\,,\\
\left\langle w^2B_0\left(w^2,q^2\right)\right\rangle &= \frac{1}{6}\frac{M_{\rm V}^4}{M_{\rm V}^2-q^2}\ \ ,\\
\left\langle w^4B_0\left(w^2,q^2\right)\right\rangle &= \frac{1}{48}M_{\rm V}^4\left[\frac{2M_{\rm V}^2}{M_{\rm V}^2-q^2}-\frac{3M_{\rm V}}{q}\log\left(\frac{M_{\rm V}+q}{M_{\rm V}-q}\right)\right]\,,
\end{aligned} \tag{C.6}$$

where μ_B is an arbitrary energy scale originating from the renormalization of the perturbative contribution in $\langle B_0(w^2,q^2)\rangle$. For the odd moments, we have

$$\begin{aligned}
\left\langle wB_0\left(w^2,q^2\right)\right\rangle &= -\frac{8\pi^2\langle\bar qq\rangle}{N_c\left(M_{\rm S}^2-q^2\right)}\,,\\
\left\langle w^3B_0\left(w^2,q^2\right)\right\rangle &= -\frac{\pi^2\langle\bar qq\rangle}{N_c}\left[\frac{2M_{\rm S}^2}{M_{\rm S}^2-q^2}-\frac{3M_{\rm S}}{q}\log\left(\frac{M_{\rm S}+q}{M_{\rm S}-q}\right)\right]\,.
\end{aligned} \tag{C.7}$$

Various low- and high-q^2 limits of the above formulas can be readily obtained. In particular, at $q^2\to\infty$, one finds

$$q^2\left\langle\omega B_0\left(\omega^2,q^2\right)\right\rangle = 2\left\langle\omega A_0\left(\omega^2\right)\right\rangle+\mathcal{O}\left(\frac{1}{q^2}\right)\,. \tag{C.8}$$

Appendix D

Correlators in the Spectral Quark Model

The explicit formulas for the SEM–SEM correlators of Eq. (36) in SQM, with expressions for the spectral moments from Appendix C inserted, acquire the simple forms

$$\begin{aligned}
\langle\theta\theta\rangle_0\left(q^2\right) &= \frac{\langle\theta\rangle}{4}\left[-\frac{2M_{\rm V}}{q}\log\left(\frac{M_{\rm V}+q}{M_{\rm V}-q}\right)+3\right],\\
\langle\theta\theta\rangle_2\left(q^2\right) &= \frac{\langle\theta\rangle}{10}\left[3\frac{q^4}{M_{\rm V}^4}\log\left(\frac{M_{\rm V}^2-q^2}{\mu_B^2}\right)\right.\\
&\quad\left.-\frac{2M_{\rm V}}{q}\log\left(\frac{M_{\rm V}+q}{M_{\rm V}-q}\right)+9+q^2\frac{\nu_A^2}{M_{\rm V}^4}\right],\\
\langle\theta\theta\rangle_{\rm L}\left(q^2\right) &= \frac{\langle\theta\rangle}{4}=-B\,,
\end{aligned}\tag{D.1}$$

where Eq. (31) has been used. Here, $\nu_A^2=6\mu_A^2+3M_{\rm V}^2$ (see Appendix C) and μ_B are renormalization constants for the perturbative contributions. Note that the terms with the pole at $q^2=M_{\rm V}^2$, present in Eq. (C.6), cancel out, while the branch cuts appear. The perturbative contributions show up only in $\langle\theta\theta\rangle_0$, with the renormalization constants giving rise to the arbitrary polynomial pieces q^2 and q^4.

Analogously to the above expressions, for the SEM–scalar correlator, we get

$$\begin{aligned}
q^2\phi_1\left(q^2\right) &= \langle\bar{q}q\rangle\left[1-\frac{M_{\rm S}}{2q}\log\left(\frac{M_{\rm S}+q}{M_{\rm S}-q}\right)\right],\\
\phi_0\left(q^2\right) &= -\langle\bar{q}q\rangle\,.
\end{aligned}\tag{D.2}$$

The low-momentum expansion of the above functions yields

$$\begin{aligned}
\langle\theta\theta\rangle_0\left(q^2\right) &= \langle\theta\rangle\left[-\frac{1}{4}-\frac{q^2}{3M_{\rm V}^2}+\mathcal{O}\left(\frac{q^4}{M_{\rm V}^4}\right)\right],\\
\langle\theta\theta\rangle_2\left(q^2\right) &= \langle\theta\rangle\left[\frac{1}{2}+\frac{q^2\left(18\mu_A^2+5M_{\rm V}^2\right)}{30M_{\rm V}^2}+\mathcal{O}\left(\frac{q^4}{M_{\rm V}^4}\right)\right],\\
\phi_1\left(q^2\right) &= -\langle\bar{q}q\rangle\left[\frac{1}{3M_{\rm S}^2}+\frac{q^2}{5M_{\rm S}^4}+\mathcal{O}\left(\frac{q^4}{M_{\rm S}^4}\right)\right],
\end{aligned}\tag{D.3}$$

while at asymptotically high $Q^2 = -q^2$, we find

$$\begin{aligned}
\langle\theta\theta\rangle_0\left(q^2\right) &= \frac{3}{4}\langle\theta\rangle + \mathcal{O}\left(\frac{M_{\rm V}}{|q|}\right), \\
\langle\theta\theta\rangle_2\left(q^2\right) &= \langle\theta\rangle\left[\frac{3q^4}{10M_{\rm V}^4}\log\left(\frac{-q^2}{\mu_B^2}\right) + \frac{3\mu_A^2}{5M_{\rm V}^4}q^2 + \frac{3}{4} + \mathcal{O}\left(\frac{M_{\rm V}}{|q|}\right)\right], \\
q^2\phi_1\left(Q^2\right) &= \langle\bar{q}q\rangle(1 + \mathcal{O}\left(\frac{M_{\rm S}}{|q|}\right). \qquad \text{(D.4)}
\end{aligned}$$

The leading behavior of the spin-2 (traceless) term can be rewritten as

$$\langle\theta\theta\rangle_2\left(q^2\right) = -\frac{N_c N_f}{160\pi^2}\, q^4 \log\left(\frac{-q^2}{\mu_B^2}\right) + \mathcal{O}\left(\frac{q^2}{M_{\rm V}^2}\right), \qquad \text{(D.5)}$$

which agrees with the leading quark term in OPE (see, *e.g.*, Eq. (4.1) in [47])[5]. This is not incidental, as we incorporate the perturbative contribution via the massless pole in the quark spectral function (C.1). In the spin-0 (traceful) or the longitudinal correlators, such terms do not appear in a non-interacting theory due to the conformal symmetry in the massless quark limit ($M_{\rm V}, M_{\rm S} \to 0$).

We also note that the large dynamical scales $M_{\rm V}$ and $M_{\rm S}$ allow for terms in the large-momentum expansion which are absent in the standard OPE.

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