AN EXACT UNSTABLE MODE OF SELF-SIMILAR YANG–MILLS SOLUTIONS IN $5+1$ DIMENSIONS

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Self-similar solutions play an important role in the formation of singularities in the Yang–Mills equations. In five space dimensions, these equations admit infinitely many self-similar solutions with an increasing number of unstable modes. It was observed numerically in *Acta Phys. Pol. B* **33**, 1893 (2002) that all excited profiles share the least unstable eigenvalue $\lambda = 5$. We explain this remarkable spectral property by an elementary construction of the corresponding eigenfunction.

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1. Self-similar solutions

The spherically symmetric Yang–Mills equation in five space dimensions is

$$w_{tt} = w_{rr} + \frac{2}{r} w_r + \frac{3}{r^2} w \left(1 - w^2\right) . \tag{1}$$

In terms of the similarity coordinates

$$y = \frac{r}{T - t} , \qquad \tau = -\log(T - t) ,$$

where T is the blowup time, equation (1) becomes

$$w_{\tau\tau} + w_\tau + 2y w_{\tau y} = \left(1 - y^2\right) w_{yy} + \left(\frac{2}{y} - 2y\right) w_y + \frac{3}{y^2} w \left(1 - w^2\right) . \tag{2}$$

By definition, self-similar solutions are independent of τ: $w(\tau, y) = u(y)$. The profile u therefore satisfies

$$\left(1 - y^2\right) u'' + \left(\frac{2}{y} - 2y\right) u' + \frac{3}{y^2} u \left(1 - u^2\right) = 0 , \qquad 0 < y < 1 . \tag{3}$$

It was proved in [1] that there exist infinitely many smooth profiles u_n, $n = 0, 1, 2, \ldots$, where u_n has exactly n zeros on $(0, 1)$. Their endpoint expansions are

$$u_n(y) = 1 - a_n y^2 + O\left(y^4\right), \qquad y \to 0, \tag{4}$$

$$u_n(y) = b_n(1-y) + O\left((1-y)^2\right), \qquad y \to 1, \tag{5}$$

corresponding to the regular center and the similarity horizon, respectively. The fundamental profile, found explicitly in [1], is

$$u_0(y) = \frac{1-y^2}{1+\frac{3}{5}y^2}. \tag{6}$$

Numerical computations in [2] indicated that the fundamental profile u_0 acts as a universal attractor for generic blowup, while the first excited profile u_1 governs the threshold between dispersion and blowup. The nonlinear asymptotic stability of u_0 under small equivariant perturbations was subsequently established in [3, 4].

Further numerical computations revealed a remarkable feature: the least unstable mode has the same eigenvalue $\lambda = 5$ for every excited profile [1], raising the question of its structural origin. In this note, we resolve this little spectral puzzle by constructing the corresponding eigenfunction explicitly in terms of the profile.

2. Linearized eigenvalue problem

Substituting $w(\tau, y) = u_n(y) + \varepsilon\, \mathrm{e}^{\lambda\tau} v(y)$ into (2) and retaining terms linear in ε yields the eigenvalue equation

$$\left(1-y^2\right) v'' + \left(\frac{2}{y} - 2(\lambda+1)y\right) v' + \left[\frac{3\left(1-3u_n^2\right)}{y^2} - \lambda(\lambda+1)\right] v = 0. \tag{7}$$

We seek eigenfunctions $v \in C^\infty([0,1])$.

One such eigenfunction arises from differentiating the family of self-similar solutions with respect to T

$$\partial_T u_n\left(\frac{r}{T-t}\right) = -\mathrm{e}^\tau y u_n'(y),$$

which gives the gauge mode

$$v_g^{(n)}(y) = -y u_n'(y) \qquad \text{with} \quad \lambda = 1. \tag{8}$$

It changes only the blowup time and is therefore not counted as a physical instability.

3. Self-adjoint formulation

For the spectral analysis and the exact-mode construction, it is convenient to introduce the reciprocal coordinate

$$x = \frac{1}{y}\,, \qquad U(x) = u(y)\,, \qquad V(x) = v(y)\,, \qquad 1 \le x < \infty\,.$$

In these variables, equation (3) becomes

$$\left(x^2 - 1\right) U'' + 3U\left(1 - U^2\right) = 0\,. \tag{9}$$

At the similarity horizon $x = 1$ and the center $x = \infty$, the profiles have the endpoint expansions

$$U_n(x) \;=\; b_n(x-1) - \frac{3b_n}{4}(x-1)^2 + \frac{5b_n}{16}(x-1)^3 + O\left((x-1)^4\right)\,, \quad x \to 1\,, \tag{10}$$

$$U_n(x) \;=\; 1 - \frac{a_n}{x^2} + O\left(x^{-4}\right)\,, \quad x \to \infty\,. \tag{11}$$

The fundamental profile (6) becomes

$$U_0(x) = \frac{x^2 - 1}{x^2 + \frac{3}{5}}\,. \tag{12}$$

Equation (7) is transformed into

$$pV'' + \frac{2\lambda}{x}V' + \left[3\left(1 - 3U_n^2\right) - \frac{\lambda(\lambda+1)}{x^2}\right] V = 0\,, \qquad p = x^2 - 1\,. \tag{13}$$

Substituting

$$V = x^{\lambda} p^{\frac{1-\lambda}{2}} \phi \tag{14}$$

into equation (13), we obtain the self-adjoint Sturm–Liouville equation for $\phi \in L^2((1,\infty), p^{-1}\,\mathrm{d}x)$

$$-\left(p\phi'\right)' - 3\left(1 - 3U_n^2\right)\phi = E p^{-1}\phi\,, \qquad E = -(\lambda - 1)^2\,. \tag{15}$$

Hereafter, we refer to E as the energy and to λ as the similarity eigenvalue. Although the transformation (14) relates the two differential equations for every λ, it identifies smooth similarity modes with square-integrable Sturm–Liouville eigenfunctions only when $\mathrm{Re}(\lambda) > 1$. To see this, consider a smooth mode with $V(1) \neq 0$. Near $x = 1$, its transform satisfies

$$\phi \sim C(x-1)^{(\lambda-1)/2}\,,$$

which is square integrable with weight p^{-1} precisely for $\mathrm{Re}(\lambda) > 1$. In that half-plane, the second branch is not square integrable. At infinity, the regular branch $V = O(x^{-2})$ corresponds to $\phi = O(x^{-3})$, whereas the singular branch $V \sim Cx^3$ corresponds to $\phi \sim Cx^2$. Only the regular branch is square integrable with weight p^{-1} at infinity.

Thus, for $\mathrm{Re}(\lambda) > 1$, the similarity eigenmodes of (13) are in one-to-one correspondence with the eigenfunctions of the problem (15), allowing us to apply the Sturm–Liouville theory. In particular, the self-adjointness of the Sturm–Liouville operator implies that E is real and hence that λ is real. The analogous correspondence for wave maps is discussed in [5]. This equivalence does not apply to similarity modes with $\mathrm{Re}(\lambda) < 1$.

The gauge mode (8) becomes $V_g^{(n)}(x) = xU_n'(x)$, which under (14) gives the zero-energy resonance (a bounded but not square-integrable solution)

$$\phi_g^{(n)} = U_n' \,.$$

From the proof of existence in [1], it follows that the profile U_n and its derivative each have exactly n interior zeros, which interlace. Therefore, the Sturm oscillation theorem implies that the spectral problem (15) has exactly n negative energy levels. We order them as

$$0 > E_1^{(n)} > \cdots > E_n^{(n)} \,, \tag{16}$$

where $E_k^{(n)}$, $k = 1, \ldots, n$, denote the negative energy levels for the profile U_n. The corresponding eigenfunctions are denoted by $\phi_k^{(n)}(x)$. The associated unstable similarity eigenvalues are ordered as

$$1 < \lambda_1^{(n)} < \cdots < \lambda_n^{(n)} \,, \tag{17}$$

with the corresponding eigenfunctions denoted by $V_k^{(n)}(x) = v_k^{(n)}(y)$.

Table 1 presents the numerically computed eigenvalues for the first five excited profiles, together with their limiting values as $n \to \infty$. The striking feature of these numerical results, first observed in [1], is that the least unstable eigenvalue $\lambda_1^{(n)} = 5$ is common to all excited profiles, whereas the remaining eigenvalues depend on n. Equivalently, every excited profile shares the same highest negative energy level $E_1^{(n)} = -16$.

Table 1. Unstable similarity eigenvalues $\lambda_k^{(n)}$ for $n = 1, \dots, 5$ and their fixed-k limits as $n \to \infty$. The limiting row is obtained from (18).

n	$\lambda_1^{(n)}$	$\lambda_2^{(n)}$	$\lambda_3^{(n)}$	$\lambda_4^{(n)}$	$\lambda_5^{(n)}$
1	5	—	—	—	—
2	5	28.407078	—	—	—
3	5	28.378841	183.492964	—	—
4	5	28.374054	183.179999	1214.561254	—
5	5	28.373323	183.129311	1212.339587	8070.020343
∞	5	28.373193	183.120247	1211.917424	8052.273337

The limiting values $\lambda_k^{(\infty)} = \lim_{n\to\infty} \lambda_k^{(n)}$ were determined in [1] by selecting an appropriate self-adjoint extension of the limiting Schrödinger operator. Writing $\lambda_k^{(\infty)} = 1 + \alpha_k$, they satisfy

$$\operatorname{Im}\left[\log\Gamma\left(\frac{1}{2} + \alpha_k + \frac{i\sqrt{11}}{2}\right) - \log\Gamma\left(\frac{1}{2} + \frac{i\sqrt{11}}{2}\right)\right] = k\pi\,, \quad k = 1, 2, \dots\,, \tag{18}$$

where $\log\Gamma$ denotes the analytic branch that is real on the positive real axis. In particular, $\alpha_1 = 4$ is an exact root, giving $\lambda_1^{(\infty)} = 5$. However, this limiting result leaves open why $\lambda_1^{(n)} = 5$ holds exactly for every finite $n \geq 1$. We now resolve this question by constructing the associated eigenfunction explicitly and showing that it corresponds to the highest negative energy level.

4. Construction of the exact mode

The same energy level for every excited profile suggests seeking a corresponding eigenfunction given by a universal first-order expression in the profile (here we suppress the index n on U_n)

$$\phi = A(x)U' + B(x)U\,, \tag{19}$$

where A and B are independent of the particular profile U. The gauge mode $\phi_g = U'$ is the simplest example of such an expression. We now seek a nonzero energy for which an ansatz of this form solves the linearized equation for every profile.

Substituting the ansatz (19) into equation (15) and using the profile equation (9) and its derivative to eliminate U'' and U''', we obtain

$$\left(pA'' + p'A' + 2pB' + p'B + \frac{E}{p}A\right)U' + \left(pB'' + p'B' + \frac{E}{p}B - 6A'\right)U + 6\left(A' - B\right)U^3 = 0\,. \tag{20}$$

A sufficient condition for this expression to vanish for every profile is that the three coefficients vanish separately. Setting the coefficient of the cubic term to zero gives $B = A'$, so the ansatz (19) takes the total-derivative form $\phi = (AU)'$, while the remaining coefficient equations reduce to

$$3pA'' + 2p'A' + \frac{E}{p}A = 0\,, \qquad pA''' + p'A'' + \left(\frac{E}{p} - 6\right)A' = 0\,. \tag{21}$$

Taking A to be constant recovers the gauge mode with $E = 0$. To find a nonconstant solution, we seek A bounded at infinity and allow a simple pole at $x = 1$, where every profile vanishes. The simplest rational ansatz in $p = x^2 - 1$ with these properties is

$$A = a + \frac{b}{p}\,, \qquad b \neq 0\,.$$

Substituting this into equation (21) and clearing denominators gives

$$(Ea + 10b)p + (E + 16)b = 0\,, \qquad 2bx(E + 16) = 0\,.$$

Since $b \neq 0$, the second equation yields $E = -16$, and then the first one gives $a = 5b/8$. Normalizing $a = 1$, we obtain

$$A = 1 + \frac{8}{5p} = \frac{5x^2 + 3}{5(x^2 - 1)} = \frac{1}{U_0}\,.$$

Consequently, for every excited profile U_n, $n \geq 1$, the first-order ansatz yields the mode, provisionally denoted by ϕ_n,

$$\boxed{\phi_n = \left(\frac{U_n}{U_0}\right)'}\,, \tag{22}$$

which satisfies the eigenvalue equation (15) with $E = -16$, corresponding to $\lambda = 5$.

It remains to verify the behavior of ϕ_n at the endpoints. The endpoint expansions (10) and (11) give

$$\phi_n = O\left((x-1)^2\right) \quad \text{as } x \to 1\,, \qquad \phi_n = O\left(x^{-3}\right) \quad \text{as } x \to \infty\,.$$

Hence, $\phi_n \in L^2((1,\infty), p^{-1}\,\mathrm{d}x)$, completing the proof that ϕ_n is an eigenfunction of the spectral problem (15) with energy $E = -16$ for every $n \geq 1$.

To locate this energy level in the spectrum, let us count the nodes of ϕ_n. Since $U_0 > 0$ on $(1,\infty)$, the quotient U_n/U_0 has exactly n zeros. Therefore, Rolle's theorem gives at least $n-1$ distinct interior zeros of ϕ_n. On the other hand, the spectral problem (15) has exactly n negative energy levels, so the Sturm oscillation theorem allows for at most $n-1$ interior zeros for an eigenfunction of negative energy. Thus, ϕ_n has exactly $n-1$ nodes, and $E = -16$ is the highest negative energy level. In the notation (16), this reads

$$E_1^{(n)} = -16\,, \qquad \phi_1^{(n)}(x) = \left(\frac{U_n(x)}{U_0(x)}\right)'\,. \tag{23}$$

The corresponding similarity eigenfunction with $\lambda_1^{(n)} = 5$, expressed in the original coordinate y, is

$$v_1^{(n)}(y) = -\frac{y}{5\,(1-y^2)^4}\left[\left(5+3y^2\right)\left(1-y^2\right)u_n'(y) + 16y\,u_n(y)\right]\,. \tag{24}$$

Remark. We emphasize that, although the linearized operators share the eigenvalue $E_1^{(n)} = -16$, they do not form an isospectral hierarchy. As Table 1 illustrates, the remaining discrete eigenvalues depend on n: increasing n introduces an additional bound state below $E = -16$, but also shifts the other levels. This differs from a Darboux transformation that creates a bound state while preserving the existing bound-state energies. Our construction provides an exact eigenfunction at one distinguished energy level directly in terms of each nonlinear profile, rather than an intertwining map between the linearized operators associated with different profiles.

5. An underlying differential identity

It is instructive to recast the preceding calculation without imposing the profile equation. Writing

$$\mathcal{E}[U] = pU'' + 3U\left(1-U^2\right)\,, \qquad \mathcal{L}_{U,E} = p\partial_x^2 + p'\partial_x + 3 - 9U^2 + \frac{E}{p}\,,$$

we obtain, for arbitrary sufficiently smooth functions A and U,

$$\mathcal{L}_{U,E}\left[(AU)'\right] = \left(A\partial_x + 3A'\right)\mathcal{E}[U] + \left(3pA'' + 2p'A' + \frac{E}{p}A\right)U' + \left(pA''' + p'A'' + \left(\frac{E}{p} - 6\right)A'\right)U\,.$$

With $A = U_0^{-1}$ and $E = -16$, the last two terms vanish by (21), yielding the identity

$$\boxed{\mathcal{L}_{U,-16}\left[\left(\frac{U}{U_0}\right)'\right] = U_0^{-1}\left(\partial_x - \frac{3U_0'}{U_0}\right)\mathcal{E}[U]\,,} \tag{25}$$

which is valid for any sufficiently smooth function $U(x)$. In particular, for every excited profile U_n, the identity shows that $(U_n/U_0)'$ satisfies the eigenvalue equation with $E = -16$, recovering the exact mode constructed above. Whether this identity reflects a deeper structure is unclear to us.

6. Special modes in $d = 7$ and $d = 9$

A natural question is whether similar special eigenvalues occurs in higher dimensions. Below we repeat the argument given in [5] for wave maps in higher odd dimensions. In dimension d, the change of variables

$$v(y) = y^{-\frac{d-3}{2}}\left(1-y^2\right)^{\frac{d-3-2\lambda}{4}}\phi(x)\,, \qquad x = \frac{1}{y}\,, \tag{26}$$

transforms the similarity-mode equation to a Sturm–Liouville equation analogous to (15) with

$$E = -\left(\lambda - \frac{d-3}{2}\right)^2\,.$$

The gauge eigenvalue $\lambda = 1$ and its reflection $\tilde\lambda = d-4$ give the same energy $E = -(d-5)^2/4$. Thus, the same solution ϕ that gives the gauge mode $v_g = -yu'$ for $\lambda = 1$ through (26) gives the formal reflected mode

$$\tilde v_g = -\frac{yu'}{(1-y^2)^{\frac{d-5}{2}}} \qquad \text{with } \tilde\lambda = d-4\,.$$

For $d = 5$, the reflection fixes $\tilde\lambda = \lambda = 1$, so it yields no new mode. However, for $d = 7$ and $d = 9$ there exist infinitely many smooth profiles u_n that vanish at $y = 1$ and satisfy

$$u_n(y) \sim b_n(1-y)^{\frac{d-3}{2}}\,,$$

so their reflected modes $\tilde{v}_g^{(n)}$ are smooth at $y = 1$, and hence admissible. Therefore all such profiles share the eigenvalue $\tilde{\lambda} = 3$ in $d = 7$ and $\tilde{\lambda} = 5$ in $d = 9$, corresponding to the energies $E = -1$ and $E = -4$, respectively.

The above reflected modes in $d = 7$ and $d = 9$ can also be obtained directly from the first-order ansatz (19), which in these cases reduces to the derivative term alone ($B = 0$). In contrast, for the case $d = 5$ considered here, the ansatz produces a mode that reflection does not.

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