A SIMPLE INTRODUCTION TO SOFT RESUMMATION* **

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We provide an elementary pedagogical introduction to some basic concepts and techniques of soft (or Sudakov) resummation, specifically in QCD, paying particular attention to simple but useful tricks of the trade. We briefly review collinear (Altarelli–Parisi) and infrared (eikonal) factorization, cancellation of infrared singularities, and factorization of mass singularities. We recall basic concepts on renormalization group invariance and the solution of renormalization group equations. We then show how threshold resummation can be derived from a renormalization group argument following from the cancellation of infrared singularities. We discuss various equivalent forms of the resummed result, and we briefly present transverse momentum resummation.

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1. Soft resummation from QED to QCD

The all-order resummation of logarithmically enhanced contributions due to the emission of gauge vector bosons, and specifically its use in the context of QCD, is both a standard topic and an active research field. The underpinnings of these techniques in classic QED results are covered both in old [1] and more recent [2] graduate-level textbooks. Some recent developments, such as, for example, webs [3] have been reviewed, and the approach to

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resummation based on soft-collinear effective theories has been the subject of textbook treatments [4]. However, a treatment at the introductory level of the direct QCD approach, rooted in the original papers of Sterman [5], Catani and Trentadue [6], is rather more difficult to find. It is the purpose of these elementary lectures to provide such an introduction.

In order to make our treatment accessible and self-contained, we start by reviewing textbook [7] material on the factorization of soft and collinear emission, discussing the cancellation of infrared singularities, and the factorization of collinear singularities. We then proceed to show the origin of double-logarithmically enhanced contributions, and their all-order exponentiation. We take particular care in explaining some simple computational tricks that are used in order to obtain the desired results, for instance in the treatment of phase space. We then review basic ideas on renormalization group invariance, the Callan–Symanzik renormalization group equation and its solution, again at a basic textbook level. We then show how the standard threshold resummation [5, 6] can be derived using a renormalization group argument [8, 9], explain the underlying physical argument, illustrate the predictive power of the resummed results, and demonstrate the equivalence of various forms that it can take. We conclude by briefly touching upon transverse momentum resummation.

Part I: soft and collinear logarithms

Soft resummation in QCD sums to all perturbative orders contributions that are enhanced by powers of the logarithm of the ratio of a soft scale and the hard scale of the process. They arise because contributions coming from real gluon emission diverge in the threshold limit in which the soft scale vanishes. The divergence is canceled at threshold by virtual corrections, in which the emitted gluon is absent, and the log is the leftover of the cancellation away from threshold. This log gets combined with a second log arising from the region of integration over the transverse momentum of the emitted gluon in which the emitted and emitting parton momenta become collinear (*i.e.* parallel). Resummation is possible due to the factorized nature of the gluon emission process when the gluon is soft (*i.e.* its energy tends to zero) or collinear (*i.e.* its emission angle with respect to the emitting parton tends to zero). Multiple emissions can thus be treated as a branching process with a suitable factorization of phase space. We discuss, in turn, the factorized nature of soft and collinear emission; singularities in the soft limit and the logs they leave behind; and the exponentiation of multiple emission.

2. Factorized emission

The emission of soft gluons and the emission of collinear partons (quark or gluons) from either quarks or gluons factorizes, in the sense that the process with an extra gluon or parton in the final state can be written in terms of that without it, times a universal emission kernel. The factorization happens at the level of amplitudes for soft gluons, and at the level of squared amplitudes for collinear partons — factorization of collinear emission at the amplitude level is rather more subtle and goes beyond the scope of our discussion (see Ref. [10]). We summarize the main results and refer to textbooks [7, 11] and reference works [12, 13] for a more detailed discussion.

2.1. Collinear emission

We consider, for definiteness, the Drell–Yan process, *i.e.* production of a gauge boson of mass (or virtuality) Q^2 in the annihilation of a quark and an antiquark with momenta p_1 and p_2 (see Fig. 1). The leading-order amplitude for this process can be written as

$$M_0(p_1, p_2) = \bar{v}(p_2) A_0(p_1, p_2) u(p_1)\,. \tag{1}$$

We consider a correction to the leading-order amplitude due to the emission of an extra gluon with momentum k from one of the two incoming quark lines. It is convenient to introduce a Sudakov parametrization for the emitted gluon momentum. This consists of writing it in terms of two light-like non-orthogonal vectors p_1 and p_2, and a transverse vector $k_{\rm t}$ orthogonal to both of them, namely as

$$k = (1-z)p_1 + k_{\rm t} + \eta p_2\,, \tag{2}$$

where

$$p_1^2 = p_2^2 = 0\,; \qquad p_1 \cdot k_{\rm t} = p_2 \cdot k_{\rm t} = 0\,; \qquad p_1 \cdot p_2 \neq 0\,. \tag{3}$$

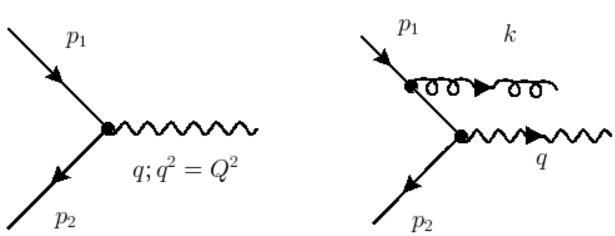


Fig. 1. The leading-order Drell–Yan production process (left) and a next-to-leading order real gluon emission correction to it (right).

The value of η is fixed by the on-shell condition $k^2 = 0$

$$\eta = \frac{-k_t^2/s}{1-z}\,, \tag{4}$$

where s is the center-of-mass energy

$$(p_1 + p_2)^2 = 2p_1 \cdot p_2 = s\,. \tag{5}$$

The collinear limit we are interested in is the limit in which $k_t^2 \to 0$, so also $\eta \to 0$.

As an example, for illustration, take

$$\begin{aligned} p_1 &= (p, 0, 0, p)\,, \\ p_2 &= (p, 0, 0, -p)\,. \end{aligned} \tag{6}$$

The value of p is fixed by the center-of-mass energy, Eq. (5), to be given by

$$p = \frac{\sqrt{s}}{2}\,. \tag{7}$$

Furthermore,

$$k_t = \left(0, \vec{k}_t, 0\right)\,, \tag{8}$$

which shows that $k_t^2 \le 0$.

We assume, for definiteness, that the gluon is radiated by the incoming quark, and we follow the notation and conventions of Ref. [7], to which we also refer for the Feynman rules of QCD. We can write the corresponding contribution to the amplitude M_1 for the process with the extra gluon in terms of the tree-level amplitude M_0 as

$$M_1^q = \bar{v}(p_2) A_0(p_1 - k, p_2) \frac{i(\not{p}_1 - \not{k})}{(p_1 - k)^2} i g_s \gamma^\mu \sum_a t^a \epsilon_\mu^a(k) u(p_1)\,, \tag{9}$$

where t^a are SU(3) gauge group generators, and ϵ_μ^a is the gluon polarization. An analogous calculation yields the contribution of the emission from the antiquark line, $M_1^{\bar{q}}$. We now note that in the collinear limit, the momentum in the intermediate propagator goes on shell. Indeed, we have

$$(p_1 - k)^2 = -2p_1 \cdot k = -s\eta = \frac{k_t^2}{1-z}\,, \tag{10}$$

where in the last step we have used Eq. (4). But for an on-shell (massless) fermion

$$\not{p} = \sum_s u^s(p) \bar{u}^s(p)\,, \tag{11}$$

where $u^s(p)$ is the spin s solution of the Dirac equation. Hence, expanding the intermediate fermion propagator about the collinear limit, we have

$$\frac{i(\not{p}_1 - \not{k})}{(p_1-k)^2} = i\frac{\sum_s u^s(p_1-k)\bar{u}^s(p_1-k)\left[1+O(k_{\rm t}^2/s)\right]}{(p_1-k)^2}\,, \tag{12}$$

where it is understood that the argument $p_1 - k$ of the spinor u_s is an on-shell momentum, $(p_1-k)^2 = 0$, and the denominator is expanded according to Eq. (10) about the collinear limit, in which it vanishes.

Substituting Eq. (12) into Eq. (9) of the amplitude and integrating over the momentum of the radiated gluon, we see that the unpolarized squared matrix element, integrated over the phase space of the radiated gluon,

$$\mathrm{d}\Phi_k = \frac{\mathrm{d}^3k}{(2\pi)^3 2E}\,, \tag{13}$$

factorizes as

$$\sum_s \int |M_1^q|^2\, \mathrm{d}\Phi_k = \frac{\alpha_{\rm s}}{2\pi}\int \frac{\mathrm{d}|k_{\rm t}^2|}{|k_{\rm t}^2|}\mathrm{d}z|M_0(p_1-k,p_2)|^2 P^{\rm r}_{qq}(z)\left[1+O\left(\frac{k_{\rm t}^2}{s}\right)\right]\,, \tag{14}$$

where we have defined $\alpha_{\rm s} = \frac{g_s^2}{4\pi}$, $M_0(p_1-k,p_2)$ is the tree-level Drell–Yan amplitude, but now with the incoming quark carrying momentum

$$p_1 - k = zp_1\left[1+O\left(\frac{k_{\rm t}^2}{s}\right)\right]\,, \tag{15}$$

and $P^{\rm r}_{qq}(z)$ is a dimensionless function, proportional to the squared matrix element for the tree-level emission process $q(p_1) \to q(p_1-k)g(k)$. The function $P^{\rm r}_{qq}(z)$ is called a splitting function; the superscript r denotes the fact that this is only the contribution to the splitting function due to real emission, while there is also a virtual contribution due to gluon loops, which we will discuss in Section 3.3. Note that the fact that the most singular contribution in Eq. (14) as $|k_{\rm t}^2| \to 0$ is proportional to $\frac{1}{|k_{\rm t}^2|}$, and the fact that $P^{\rm r}_{qq}$ is a function of z only, are both entirely fixed by dimensional analysis, as both $|M_0|^2$ and $\int |M_1^q|^2 \mathrm{d}\Phi_k$ have the same dimensions. The integral over $|k_{\rm t}|$ is logarithmic; its upper limit is kinematically bounded by the center-of-mass energy s, while at the lower limit, it is divergent. The way to deal with the divergence will be discussed later.

Now, it should be clear that nowhere in the above argument the explicit form of M_0 was used. Thus, what we have actually proven is that radiation of a gluon from an external quark line produces a logarithmically enhanced contribution to the square amplitude, which is proportional to the original amplitude, but with an incoming quark momentum rescaled by a factor of $z \leq 1$, and a universal splitting function, which only depends on the radiation process. A similar argument also works for gluon radiation from an incoming gluon line, because one may likewise rewrite in the collinear limit the numerator of the intermediate gluon propagator as a sum over on-shell gluon polarization vectors, up to subtleties related to the choice of gauge in the non-Abelian case. Of course, in this case, the splitting function will be different: $P^{\rm r}_{gg}$ now is depending on the $g(p_1) \to g(p_1 - k)g(k)$ matrix element. Furthermore, what is being used in the argument is the on-shell behavior of the intermediate propagator. This means that the argument also works in the off-diagonal case, except that in this case, an incoming quark is replaced by a gluon with factor $P^{\rm r}_{gq}(z)$ and an incoming gluon is replaced by a quark with factor $P^{\rm r}_{qg}(z)$. Finally, it is clear that the argument also works for massive quarks, though in this case, the logarithmic integral at the lower end is regulated by the quark mass.

In short, we have proven that the squared matrix element for radiation from an external line produces a contribution which is enhanced by a logarithm of the hard scale of the process, factorized in terms of universal splitting functions, up to terms with a relative power suppression in the hard scale. The argument of the log is the ratio of the hard scale to the radiator mass, and hence it diverges in the massless limit — a divergence which is usually referred to as collinear or mass singularity.

2.2. Soft radiation

Consider now the case in which a gluon with momentum k is emitted from an external quark or antiquark with momentum p and mass m, and the energy of the gluon, and thus all the components of k tend to zero. In this case, at the amplitude level, assuming for definiteness that the emitting particle is an incoming quark, the external wave function factor $u(p)$ is supplemented by the insertion of a vertex and a propagator factor (see Fig. 2)

$$u(p) \to W^\mu_a(p,k)u(p) = \frac{\not p + \not k + m}{(p-k)^2 - m^2} g_s \gamma^\mu t^a u(p)\,. \tag{16}$$

Expanding about $k = 0$ and using the Dirac equation

$$\begin{aligned} W^\mu_a(p,k)u(p) &= -g_s \frac{2p^\mu - \gamma^\mu(\not p - m)}{2p\cdot k} t^a u(p)\,[1 + O(k)] \\ &= -t^a u(p) g_s \frac{p^\mu}{p\cdot k}\,[1 + O(k)]\,. \end{aligned} \tag{17}$$

The factor $\frac{p^\mu}{p\cdot k}$ is referred to as the eikonal factor, and this approximation to as the eikonal approximation.

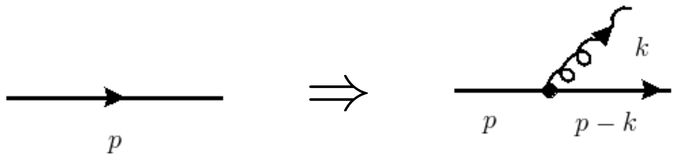


Fig. 2. Radiation of a gluon with momentum k from a quark with momentum p.

Again, a similar argument also works for soft gluon emission from a gluon line. However, unlike for collinear emission, in the case of soft emission, the argument only works for diagonal emission, *i.e.* gluon emission from a quark or a gluon line. On the other hand, factorization now happens at the amplitude level, and the eikonal factor is the same for both emission from a quark and a gluon line. Also in this case, integration over the emitted gluon momentum leads to a logarithm of the hard scale, divergent in the infrared as we now discuss.

3. Singularities and logs

Integration over the momentum of the emitted gluon leads to a double logarithm: a collinear log coming from the lower end of the integration over transverse momentum, and an infrared log coming from the lower end of the integration over energy. Both are divergent; however the way they are treated is substantially different: the infrared divergence cancels against a virtual contribution in the case of infrared logs, while collinear logs can be factored away exploiting their universality properties.

3.1. The Sudakov double log

In order to understand the origin of the double soft-collinear log, consider again the full set of gluon emission corrections to the tree-level Drell–Yan process (see Fig. 3). We are specifically interested in the form of the phase-space integration over the emitted gluon momentum k. To this purpose, it is convenient to use a slightly different version of the Sudakov parametrization, namely the light-cone parametrization, in which instead of p_1 and p_2, we adopt as basis vectors their linear combinations

$$p^{\pm} = \frac{p_1 \pm p_2}{2}\,, \tag{18}$$

namely

$$k = (1-x)p^{+} + yp^{-} + k_{\rm t}\,. \tag{19}$$

The on-shell condition now gives

$$y = \pm\sqrt{(1-x)^2 - \frac{4|k_\mathrm{t}^2|}{s}}\,, \tag{20}$$

where the two solutions correspond to k collinear to either p_1 or p_2 as $k_\mathrm{t}^2 \to 0$. The Lorentz-invariant phase-space integration measure

$$\mathrm{d}\Phi_k = \frac{|k_\mathrm{t}|\,\mathrm{d}|k_\mathrm{t}|\,\mathrm{d}\phi\,\mathrm{d}k_z}{2E(2\pi)^3} \tag{21}$$

can be written in terms of x and k_t^2 by rewriting the longitudinal momentum integral as an energy integral through

$$\frac{\mathrm{d}k_z}{E} = \frac{\mathrm{d}E}{|k_z|}\,, \tag{22}$$

and then noting that with the Sudakov parametrization, Eq. (20),

$$k_z = y\frac{\sqrt{s}}{2}\,; \qquad E = \frac{\sqrt{s}}{2}(1-x)\,. \tag{23}$$

It follows that

$$\mathrm{d}\Phi_k = \frac{1}{4(4\pi^2)}\frac{\mathrm{d}x\,\mathrm{d}|k_\mathrm{t}^2|}{\sqrt{(1-x)^2 - \frac{4|k_\mathrm{t}^2|}{s}}}\,. \tag{24}$$

Because we are interested in the limit in which k is both collinear and soft, we can use the eikonal approximation, Eq. (17). We include all first-order real emission corrections, *i.e.* those corresponding both to radiation from the quark and the antiquark, because the inclusion of the full gauge-invariant set of diagrams avoids some subtleties that would instead arise when computing directly the splitting function corresponding to the emission from a single quark or antiquark line, as in Sections 2.1–2.2. Of course, the full set of diagrams also involves virtual loop corrections that, to this order, interfere with the leading-order process: we will come to these in Section 3.3.

The sum of the two real emission corrections of Fig. 3 to the amplitude is then

$$M_1 = M_1^q + M_1^{\bar{q}} = M_0 t^a g_s\left(\frac{p_2^\mu}{p_2\cdot k} - \frac{p_1^\mu}{p_1\cdot k}\right)\epsilon_\mu^*(k)\left[1 + O(1-x)\right]\,, \tag{25}$$

which is correct (see Eq. (17)) up to terms of the order of the gluon energy $E = \frac{\sqrt{s}}{2}(1-x)$, and thus the square amplitude is

$$|M_1|^2 = |M_0|^2 C_F g_s^2 \frac{2p_1\cdot p_2}{(p_1\cdot k)\,(p_2\cdot k)}\left[1 + O(1-x)\right]\,, \tag{26}$$

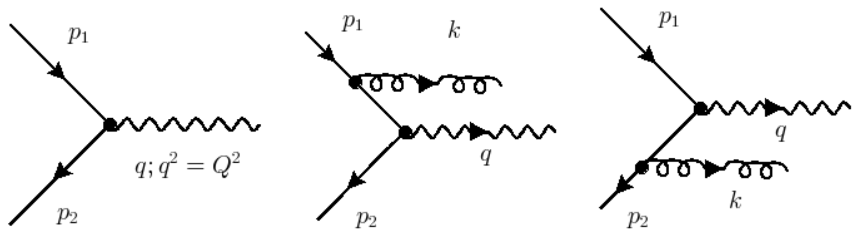


Fig. 3. The leading order Drell–Yan process and the first-order real-emission corrections to it.

where the factor of C_F comes from summing over all gluon colors and using $\sum_a t^a t^b = C_F \mathbb{I}$, and we have used the sum over gluon polarizations $\sum \epsilon^\mu \epsilon^{*\nu} = -g^{\mu\nu}$, as it is appropriate for a gauge-invariant amplitude.

Equation (26) can be written in terms of x and $k_{\rm t}$ using the Sudakov parametrization of k. We have

$$p_1 \cdot p_2 = \frac{s}{2}\,; \qquad p_1 \cdot k = \frac{s}{4}[(1-x) - y]\,; \qquad p_2 \cdot k = \frac{s}{4}[(1-x) + y] \quad (27)$$

so, adding up the two contributions corresponding to the two choices of sign for y, Eq. (20), we get

$$\begin{aligned} |M_1|^2 &= |M_0|^2 C_F g_s^2 \frac{32}{s[(1-x)^2 - y^2]}\,[1 + O(1-x)] \\ &= |M_0|^2 8 C_F g_s^2 \frac{1}{|k_{\rm t}^2|}\,[1 + O(1-x)]\,. \end{aligned} \quad (28)$$

The integration over the phase space of the radiated gluon yields

$$\int {\rm d}\Phi_k |M_1|^2 = |M_0|^2 \frac{2\alpha_{\rm s}}{\pi} \int \frac{{\rm d}x}{\sqrt{(1-x)^2 - \frac{4|k_{\rm t}^2|}{s}}} \frac{{\rm d}|k_{\rm t}^2|}{|k_{\rm t}^2|}\,[1 + O(1-x)]\,. \quad (29)$$

The origin of the double log is now clear: there is a logarithmic integration over $k_{\rm t}^2$, and furthermore, when $k_{\rm t} \to 0$, the energy denominator in the phase space reduces to $\sqrt{(1-x)^2 - \frac{4|k_{\rm t}^2|}{s}} = (1-x)[1 + O(|k_{\rm t}^2|/s)]$, which leads to a further logarithmic integration over x, *i.e.* over energy (recall Eq. (23)).

The double integral can be treated using the distributional identity

$$\frac{1}{\sqrt{(1-x)^2 - \frac{4|k_{\rm t}^2|}{s}}} = \left[\frac{1}{(1-x)_+} - \frac{1}{2}\delta(1-x) \ln \frac{4|k_{\rm t}^2|}{s}\right] \left[1 + O\left(\frac{|k_{\rm t}^2|}{s}\right)\right]. \quad (30)$$

Here, the plus distribution is defined by its action on a generic test function $f(x)$ upon integration over x

$$\int_0^1 \mathrm{d}x \frac{1}{(1-x)_+} f(x) \equiv \int_0^1 \mathrm{d}x \frac{f(x)-f(1)}{1-x} \,. \tag{31}$$

Note that, by construction,

$$\int_0^1 \mathrm{d}x \, \frac{1}{(1-x)_+} = 0 \,. \tag{32}$$

Using Eq. (32), it is apparent that only the second term in Eq. (30) contributes to the integral, which is then straightforwardly giving a squared logarithm (usually called double log). The upper limit of integration is the maximum allowed value of k_t^2, which is proportional to the center-of-mass energy s, while at the lower limit, the integral diverges, so we must introduce some infrared cutoff μ^2 in order to make sense of it. The treatment of this double divergence — one collinear and one soft — will be the subject of the next two sections. We thus get

$$\int \mathrm{d}\Phi_k |M_1|^2 = C_F |M_0|^2 \frac{\alpha}{2\pi} \ln^2 \frac{s}{\mu^2} + \text{less singular} \,, \tag{33}$$

where we have neglected all contributions that are less singular as $\mu^2 \to 0$. This includes both the $O(1-x)$ corrections to the eikonal approximation, which would, in fact, be suppressed by inverse powers of the dimensionful variable s, which contains the large scale of the process, but also lower powers of log, which arise when expressing the maximum value of k_t^2 in terms of s. For example, for the simple $2 \to 2$ process of Fig. 1, one has $k_{\mathrm{t}\,\mathrm{max}}^2 = s/4$ so $\ln^2 k_{\mathrm{t}\,\mathrm{max}}^2 = \ln^2 s - \ln 4 \ln s + \text{non-log}$.

Note that, again, in the whole argument, we never used the explicit form of the zero-emission amplitude M_0, and also that the eikonal approximation also holds for the emission of gluons from gluons. Hence, we conclude that in full generality, the emission of soft gluons from external lines of a generic amplitude leads to a double soft and collinear log when integrating over the phase space of the emitted gluon. In order to make sense of the result, however, we must understand how to treat the soft and collinear singularity, *i.e.* we have to get rid of the cutoff μ^2.

3.2. Mass singularities and their factorization

As mentioned in Section 2.1, collinear emission from a massive parton factorizes with the same universal splitting functions, but the collinear singularity is then regulated by the mass. Indeed, in such case, one finds that the single-emission cross section is proportional to

$$\begin{aligned}\int |M_1^q|^2 \mathrm{d}\Phi_k &= \frac{\alpha_\mathrm{s}}{2\pi} \int\limits_0^{k^2_{\mathrm{t\,max}}} \frac{\mathrm{d}k_\mathrm{t}^2}{k_\mathrm{t}^2 - m^2} \int\limits_0^1 \mathrm{d}x P^\mathrm{r}_{qq}(x) |M_0(xp_1, p_2)|^2 \\ &= \frac{\alpha_\mathrm{s}}{2\pi} \ln \frac{k^2_{\mathrm{t\,max}}}{m^2} \int\limits_0^1 \mathrm{d}x P^\mathrm{r}_{qq}(x) |M_0(xp_1, p_2)|^2 + \text{non-log}\,. \end{aligned} \quad (34)$$

The same conclusion holds if the radiating parton is off-shell, because then the denominator of the intermediate propagator, Eq. (10), contains a contribution equal to the off-shellness p_1^2, and thus it does not vanish when $k_\mathrm{t}^2 \to 0$.

This observation is a key ingredient in solving the problem of the collinear singularity. Indeed, quarks and gluons, of course, cannot exist as free particles, and therefore an incoming parton is always bound in a parent hadron, and consequently, it is not a free massless particle, but rather it has an off-shellness of the order of the characteristic scale of the bound state. This however seems to trade a problem — the collinear singularity — for a worse one — the result depends on a scale which is determined by the non-perturbative physics that describes the binding of quarks and gluons inside hadrons.

The way out emerges when combining the correction due to gluon emission with the leading-order square matrix element $|M_0|^2$. Considering for simplicity the case of the leading-order Drell–Yan process of Fig. 1, the leading order amplitude is fixed by momentum conservation. Defining

$$\tau \equiv \frac{Q^2}{s}\,, \quad (35)$$

it is clear that the leading-order cross section σ_0 is proportional to $\delta(1-\tau)$ by energy-momentum conservation. After gluon emission, the momentum p_1 and thus the center-of-mass energy are rescaled by a factor of x, hence imposing $xs = Q^2$, the mass of the final-state gauge boson, leads to $\tau = x$, so

$$|M_0|^2 \delta(1-\tau) + \int |M_1^q|^2 \mathrm{d}\Phi_k = \sigma_0 \left[\delta(1-\tau) + \frac{\alpha_\mathrm{s}}{2\pi} P^\mathrm{r}_{qq}(\tau) \ln \frac{k^2_{\mathrm{t\,max}}}{m^2}\right] + \text{non-log}\,. \quad (36)$$

But now we note that we can rewrite Eq. (36) as

$$|M_0|^2\delta(1-\tau)+\int|M_1^q|^2\mathrm{d}\Phi_k = \sigma_0\int_\tau^1\frac{\mathrm{d}x}{x}\left[\delta\left(1-\frac{\tau}{x}\right)+\frac{\alpha_\mathrm{s}}{2\pi}P_{qq}^\mathrm{r}\left(\frac{\tau}{x}\right)\ln\frac{k_{\mathrm{t\,max}}^2}{\mu_\mathrm{F}^2}\right] \times\left[\delta(1-x)+\frac{\alpha_\mathrm{s}}{2\pi}P_{qq}^\mathrm{r}(x)\ln\frac{\mu_\mathrm{F}^2}{m^2}\right] +\text{non-log}+O\left(\alpha_\mathrm{s}^2\right)\,, \quad (37)$$

where we introduced a factorization scale μ_F, on which however the result does not depend, up to $O(\alpha_\mathrm{s}^2)$ corrections. We further observe that the hard cross section that we have computed in perturbation theory will always be combined with a parton distribution function (PDF), which provides the appropriate weighting factor for computing the cross section with an incoming parton that carries a momentum p_1 related to the momentum of a parent hadron P_1 as $p_1 = x_1 P_1$.

We then note that the first factor in square brackets in Eq. (37) is identical to our original expression Eq. (36), but now with $\frac{\tau}{x} = 1$, which corresponds to $xs = Q^2$, as appropriate for an incoming quark with momentum xp_1, and also, with m^2 replaced by μ_F^2. Hence, we can include the second factor in square brackets in Eq. (37) in the PDF, effectively including in the PDF all radiation up to scale μ_F^2 that has downgraded the incoming quark momentum to x times what it was before radiation, and endowed the remaining parton with virtuality μ_F^2. We are thus left with a subtracted partonic cross section

$$\hat\sigma_1^q(z) = \sigma_0\left[\delta(1-z)+\frac{\alpha_\mathrm{s}}{2\pi}P_{qq}^\mathrm{r}(z)\ln\frac{k_{\mathrm{t\,max}}^2}{\mu_\mathrm{F}^2}+\text{non-log}\right]\,, \quad (38)$$

where $z = \frac{Q^2}{xs}$ as it should be.

Because the splitting function is universal, this redefinition of the PDF does not depend on the specific process, so the PDFs remain universal — a property of the hadron, and not of the process. On the other hand, the partonic cross section is now both finite and independent of non-perturbative physics, but it has acquired a dependence of the arbitrary factorization scale μ_F. However, this dependence is compensated by an equal and opposite dependence of the PDF, so the physical observables remain independent of it.

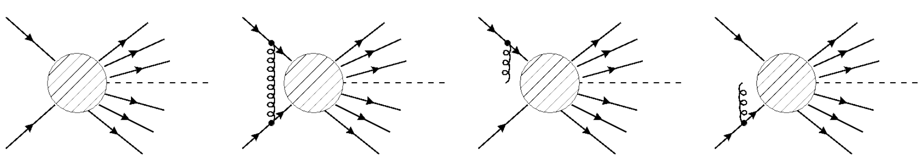
3.3. Cancellation of infrared singularities

After subtraction of the collinear singularity — which has been factorized into the PDF, where it is then regulated by the non-perturbative physics which determines the PDF — we are still left with an infrared singularity. This can be seen clearly by looking at the explicit form of the splitting function $P^{\rm r}_{qq}$

$$P^{\rm r}_{qq}(x) = C_F \frac{1+x^2}{1-x}\,. \tag{39}$$

Recalling the form (34) of the squared matrix element, it is clear that as $x \to 1$, the matrix element $|M_0(xp_1, p_2)|^2$ just reduces to its (regular) value that it had without emission, so the integral over x is logarithmically divergent due to the denominator in Eq. (39). Also recalling that $x \to 1$ is the limit in which the energy, Eq. (23), of the radiated gluon vanishes, this divergence is recognized as the infrared singularity discussed in Sections 2.2 and 3.1.

A classic result of quantum electrodynamics, the Kinoshita–Lee–Nauenberg (KLN) theorem, guarantees that, once one consistently includes loop corrections along with real emission contributions, this singularity cancels. The argument generalizes in QCD for colorless final states [13]. The cancellation is illustrated graphically in Fig. 4. The argument of Section 3.1 shows that the infrared divergent contribution to the cross section can be obtained from the interference of the contributions to the amplitude coming from gluons emitted from different quark lines. Indeed, the infrared singularity comes from the $x \to 1$ limit of the denominator in the amplitude, Eq. (28), which in turn arises from the interference of the two eikonal factors proportional to p_1^μ and p_2^μ in Eq. (25). The KLN theorem ensures that this singularity cancels against an equal and opposite singularity, which is present in the virtual contribution that corresponds to the gluon loop obtained by connecting the endpoints of the two interfering emitted real gluons. This loop correction has, of course, the same final-state particles as the amplitude without either



(a) (b) (c) (d)

Fig. 4. Cancellation of infrared singularities: the interference between the leading-order process (a) and the loop correction (b) cancels the infrared singularity arising due to interference between the two real-emission contributions (c) and (d).

of the two interfering final-state real emitted gluons. Thus it interferes with it when calculating the square modulus of the amplitude, thereby producing a contribution of the same order as the square of either of the real emission terms (see Fig. 4).

A proof of the KLN theorem and its extension to QCD goes beyond the scope of our discussion. Suffice it to say that it can be explicitly verified, and indeed inclusion of both the real emission and the loop correction leads to a splitting function which is free of infrared singularities and takes the form

$$P_{qq}(x) = C_F \left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x) \right] . \tag{40}$$

Note that in terms of the single gluon emission process that is used in order to compute the splitting function, the loop correction has the kinematics of the process without emission. Therefore, the contribution coming from it corresponds to the momentum of the parton before and after emission being the same, *i.e.* it is localized at $x=1$: it only contributes to distributions at $x=1$ such as the delta and the plus distribution in Eq. (40).

The plus distribution in Eq. (40) integrates to zero according to Eq. (32), but it leads to logarithmically enhanced contribution when the partonic cross section is combined with a parton distribution, as we shall see explicitly in Section 6.1. Indeed, the virtual correction cancels the singularity at the endpoint $x=1$ but it leaves behind a $\frac{1}{1-x}$ behavior away from the endpoint that leads to logarithmic behavior upon integration over the momentum fraction x.

After factorization of the collinear singularity and cancellation of the infrared singularity, we are thus left with a finite double-logarithmic contribution. Indeed, as we shall show explicitly in Section 6.2, the upper limit of the transverse momentum integral is

$$k^2_{\mathrm{t\,max}} = Q^2 \frac{(1-z)^2}{4z} . \tag{41}$$

Substituting this into the expression (38) of the partonic cross section, and adding the virtual contribution, one ends up with

$$\hat{\sigma}(z) = \sigma_0 \left[\delta(1-z) + \frac{\alpha_{\mathrm{s}}}{2\pi} \left(P_{qq}(z) \ln \frac{Q^2}{\mu_{\mathrm{F}}^2} + \left[P^{\mathrm{r}}_{qq}(z) \ln \frac{(1-z)^2}{4z} \right]_+ \right) \right] . \tag{42}$$

Hence, the partonic cross section contains a double-logarithmic contribution, coming from interplay between the leftover of the infrared singularity, that produces a $\frac{1}{1-z}$ behavior, and the leftover of the collinear singularity, that produces a $\ln k^2_{\mathrm{t\,max}}$ behavior, which, in turn, is sensitive to the infrared scale $Q^2(1-z)^2$.

4. Multiple emission and exponentiation

So far, we have considered a single emission. Resummation sums multiple emission to all orders, and it is thus useful to understand what happens when several consecutive emissions take place. In such case, it is necessary to take into account the overall momentum conservation of the radiation process. Once this is done, it turns out that multiple emission leads to higher order powers of logs from a suitable region of the final phase-space integration.

4.1. Momentum conservation and phase-space factorization

In a multiple emission process, an incoming parton with the momentum p_1 emits a first gluon with the momentum k_1 and the outgoing parton after the first emission has momentum given by $p_1 - k$. This is then the incoming parton for the second emission, and so forth (see Fig. 5). Using the Sudakov parametrization, Eq. (2), it is clear that after the first emission of a gluon with momentum

$$k_1 = (1 - x_1)p_1 + k_{\mathrm{t}1} + \eta_1 p_2 \,, \tag{43}$$

the quark longitudinal momentum has become $x_1 p_1$ and its transverse momentum $-k_{\mathrm{t}1}$. After the second emission, the quark longitudinal momentum is $x_1 x_2 p_1$ and the transverse momentum $-k_{\mathrm{t}1} - k_{\mathrm{t}2}$, and so on.

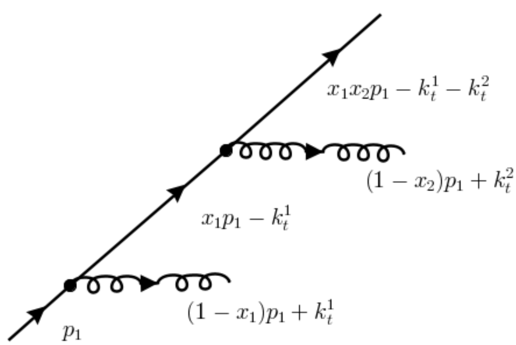


Fig. 5. Kinematics of multiple emission.

In a total cross section, one integrates over the transverse momentum of the final state, namely the transverse component of q, the momentum of the final-state gauge boson, in the case of Drell–Yan production shown in Fig. 1. Therefore, all the integrals over transverse momentum of the emitted gluons can be taken as independent, by simply shifting the integration variable. On the other hand, the longitudinal momentum is fixed by the condition that the final momentum p_{n+1} after n emissions must take the value that is necessary in order to produce the desired final state. So, for instance, in the Drell–Yan production after one collinear emission from parton with momentum p_1, the center-of-mass energy of the collision that produces the gauge boson is (recall Eq. (35)) $2x_1 p_1 \cdot p_2 = Q^2 = s\tau$, after two emissions, it is $2x_1 x_2 p_1 \cdot p_2 = Q^2 = s\tau$, and so on.

Therefore, when combining two subsequent emissions, the two splitting functions that appear in the partonic cross section according to Eq. (38), in order to guarantee longitudinal momentum conservation, must combine according to

$$\begin{aligned} &\int \mathrm{d}x_1\, \mathrm{d}x_2 \delta(x_2 x_1 - \tau) P_{qq}(x_1) P_{qq}(x_2) \\ &= \int_0^1 \frac{\mathrm{d}x_1}{x_1} \mathrm{d}x_2 \delta\left(x_2 - \frac{\tau}{x_1}\right) P_{qq}(x_1) P_{qq}(x_2) \\ &= \int_\tau^1 \frac{\mathrm{d}x_1}{x_1} P_{qq}\left(\frac{\tau}{x_1}\right) P_{qq}(x_1) \equiv \left[P_{qq} \otimes P_{qq}\right](\tau)\,, \end{aligned} \tag{44}$$

where in the last step, we have defined the convolution symbol $\otimes$.

It is clear that, consequently, the longitudinal momentum integrations for the separate gluon emissions are not independent, due to the momentum conservation constraint: phase space does not factorize. However, it does factorize if one takes a suitable integral transform: the Mellin transformation, that turns convolutions into ordinary products. This means that if we define the anomalous dimension

$$\gamma_{qq}(N) \equiv \int_0^1 \mathrm{d}x\, x^{N-1} P_{qq}(x)\,, \tag{45}$$

then the convolution in Eq. (44) transforms into an ordinary product

$$\int_0^1 \mathrm{d}\tau\, \tau^{N-1} \left[P_{qq} \otimes P_{qq}\right](\tau) = \gamma_{qq}(N) \gamma_{qq}(N)\,. \tag{46}$$

This implies that, by taking a Mellin transform, we can treat emissions as independent, which will then enable us to actually sum them to all orders.

4.2. Ordered regions

We have seen that the transverse momentum integrations can be taken as independent. However, we have also seen in Section 3.2 that if the collinear radiation happens from an off-shell parton, then the off-shellness cuts off the momentum integration in the infrared, thereby providing the lower scale of the collinear log. However, this off-shellness is, of course in turn, given by transverse momentum of the previous emission according to Eq. (10). Hence, the lower limit of the transverse momentum integral for the second emission is set by the value of the transverse momentum from the previous emission and so on

$$\int_{\mu^2}^{k^2_{\mathrm{t\,max}}} \frac{\mathrm{d}k^2_{\mathrm{t}1}}{k^2_{\mathrm{t}1}} \int_{k^2_{\mathrm{t}1}}^{k^2_{\mathrm{t\,max}}} \frac{\mathrm{d}k^2_{\mathrm{t}2}}{k^2_{\mathrm{t}2}} = \int_{\mu^2}^{k^2_{\mathrm{t\,max}}} \frac{\mathrm{d}k^2_{\mathrm{t}1}}{k^2_{\mathrm{t}1}} \ln \frac{k^2_{\mathrm{t\,max}}}{k^2_{\mathrm{t}1}} = \frac{1}{2} \ln^2 \frac{k^2_{\mathrm{t\,max}}}{\mu^2} \,. \tag{47}$$

Note that, of course, the integral over $k^2_{\mathrm{t}2}$, the transverse momentum of the second emission, also includes the region in which $k^2_{\mathrm{t}2} < k^2_{\mathrm{t}1}$, but in this region, there is no collinear singularity because the singularity of the propagator after the second emission is screened by the larger virtuality of the first emission, so the contribution from this integration region is not logarithmic.

The argument generalizes to the case of n emissions

$$\int_{\mu^2}^{k^2_{\mathrm{t\,max}}} \frac{\mathrm{d}k^2_{\mathrm{t}1}}{k^2_{\mathrm{t}1}} \cdots \int_{k^2_{\mathrm{t}n-1}}^{k^2_{\mathrm{t\,max}}} \frac{\mathrm{d}k^2_{\mathrm{t}n}}{k^2_{\mathrm{t}n}} = \frac{1}{n!} \ln^n \frac{k^2_{\mathrm{t\,max}}}{\mu^2} \,. \tag{48}$$

In Mellin space, each subsequent emission is accompanied by a factor of the anomalous dimension, and due to the factorization, Eq. (46), the n-emission term simply contains the n^{th} power of the anomalous dimension. Hence, the sequence of collinear emissions in the ordered transverse momentum region in which $k^2_{\mathrm{t}1} < k^2_{\mathrm{t}2} < \cdots < k^2_{\mathrm{t}n}$ exponentiates in Mellin space. Resummation to all orders of collinear emission then leads to an exponential series of logarithmic contributions.

The factorization of the logarithms and subtraction of the collinear singularity according to Eq. (37) then simply amounts to splitting this exponentiated result into the product of two exponentials, one including all transverse momentum integrations above μ^2_{F}, included in the partonic cross section, and the other with the region below μ^2_{F} included in the PDF. The dependence of the PDF on the scale μ^2_{F} is the well-known Altarelli–Parisi evolution.

Soft resummation, that we now turn to, amounts to showing that the double logs which are present in the soft limit, in which emitted gluons are not only collinear but also soft, also exponentiate. The exponentiation of collinear logs coming from multiple collinear emission, that we have derived here by studying the kinematics of the emission process, can be derived using a renormalization group argument. A similar renormalization group argument can then be used in order to also obtain the exponentiation and resummation of infrared logs.

Part II: resummation from renormalization group improvement

The exponentiation of collinear logs that we discussed at the end of the previous section can be understood as a consequence of the independence of physical predictions on the factorization scale that we introduced in Section 3.2. The requirement of independence can be cast in the form of a differential equation which is, in fact, an instance of the general renormalization group equation that expresses the scale dependence of predictions in quantum field theory. Soft resummation, *i.e.* the exponentiation of soft logs, can be also derived from a renormalization group argument, which generalizes the renormalization group argument leading to collinear exponentiation.

5. Renormalization group basics

The basic idea of renormalization is that in a quantum field theory, all couplings and fields must be defined in terms of a reference scale, in such a way that all quantum fluctuations at distances shorter than that of the reference scale are absorbed in the definition of the couplings and fields. The fields and couplings must, consequently, be defined by relating them to a reference scale. The quantum field theory then provides relations between observable quantities, with the couplings and fields used as a way to relate observables to each other. Infinities only appear if one tries to express observable quantities in terms of unobservable parameters defined at infinitely short distance scales.

For example, one may define the electron charge in quantum electrodynamics in terms of the Coulomb force at a large distance, *i.e.* at the scale of the (rest) electron mass. The (say) elastic electron–electron scattering cross section is finite once re-expressed in terms of the electron charge thus defined. Renormalization group invariance is then the consequence of the fact that physical predictions do not depend on the renormalization scale that has been chosen to define the couplings. Since the dependence of the couplings on this scale is a universal property of the theory, this independence expresses the scale dependence of physical observables in terms of this universal scale dependence of the coupling.

The basic idea of deriving collinear exponentiation from renormalization group invariance, that we will present shortly, takes this one step further, based on the observation that in QCD, physically observable cross sections are expressed in factorized form in terms of parton distributions, convoluted with partonic cross sections. This factorization requires introducing a scale, as discussed in Section 3.2 for the subtraction of collinear singularities. As discussed in Section 4.2, this can be viewed as a scale that splits in two the parton radiation process, with radiation below the scale included in the parton distribution, and radiation above it included in the hard matrix element. Physical predictions are independent of this scale, hence this scale dependence must cancel between the PDF and the hard cross section: indeed, we have seen that it is expressed in terms of a universal, process-independent splitting function, or rather, its Mellin transform, the anomalous dimension, Eq. (45). But the PDF and the hard cross section depend on different variables: the hard cross section depends on the specific hard scale of the process, while the PDF does not. As we shall see, dimensional analysis then implies that the cancellation of the dependence on the factorization scale can only go through the scale dependence of the coupling constant: it is calculable in terms of it, and the universal anomalous dimension. It can then be integrated away, leading to all-order resummation and exponentiation of logs of the hard physical scale.

5.1. Renormalization group invariance and the renormalization group equation

The simplest application of renormalization group invariance is to a physical observable that depends on a single scale. The prototypical example is the R ratio, defined as the cross section for an electron–positron pair to produce any hadronic final state, normalized to the QED cross section for production of a muon–antimuon pair

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu-)}\,. \tag{49}$$

The numerator receives contributions from any Feynman diagram with partons in the final state, see Fig. 6.

Since R is a ratio of total cross sections, it is dimensionless, and it depends only on the center-of-mass energy squared s of the e^+e^- collision. Hence, at energies high enough that quark masses can be neglected, by dimensional analysis, it can be only a function of the coupling with dimensionless coefficients. In a classical theory, one would conclude that these coefficients must be numerical constants, but in the quantized theory, all predictions depend on a renormalization scale $\mu_{\rm R}$, hence, one can form

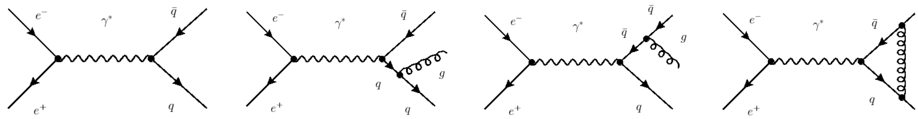


Fig. 6. Feynman diagrams contributing to the the production of partonic final states up to NLO in QCD.

a dimensionless ratio $s/\mu_{\rm R}^2$ on which the coefficients may also depend, so $R = R\left(\frac{s}{\mu_{\rm R}^2}, \alpha_{\rm s}(\mu_{\rm R})\right)$. However, physical predictions cannot depend on the choice of $\mu_{\rm R}$ and thus R must satisfy the renormalization group equation

$$\mu_{\rm R}^2 \frac{{\rm d}}{{\rm d}\mu_{\rm R}^2} R\left(\frac{s}{\mu_{\rm R}^2}, \alpha_{\rm s}\left(\mu_{\rm R}^2\right)\right) = \left[\mu_{\rm R}^2 \frac{\partial}{\partial \mu_{\rm R}^2} + \beta(\alpha_{\rm s}) \frac{\partial}{\partial \alpha_{\rm s}}\right] R\left(\frac{s}{\mu_{\rm R}^2}, \alpha_{\rm s}\left(\mu_{\rm R}^2\right)\right) = 0\,, \tag{50}$$

where we have introduced the beta function

$$\mu_{\rm R}^2 \frac{{\rm d}}{{\rm d}\mu_{\rm R}^2} \alpha_{\rm s}\left(\mu_{\rm R}^2\right) = \beta\left(\alpha_{\rm s}\left(\mu_{\rm R}^2\right)\right) = -\beta_0 \alpha_{\rm s}^2 + O\left(\alpha^3\right)\,, \tag{51}$$

which is a universal property of the theory, and perturbatively computable.

Equation (50) is easy to solve directly, however, the general solution can be found based on a simple observation. Namely that, because R does not depend on $\mu_{\rm R}^2$, one can choose $\mu_{\rm R}^2 = s$. With this choice, we find that

$$R = R\left(1, \alpha_{\rm s}(s)\right)\,, \tag{52}$$

namely that R is a function of $\alpha_{\rm s}(s)$ with numerical coefficients. Integrating Eq. (51), one finds that $\alpha_{\rm s}(s)$ in terms of $\alpha_{\rm s}(\mu_{\rm R}^2)$ is given by

$$\alpha_{\rm s}(s) = \frac{\alpha_{\rm s}\left(\mu_{\rm R}^2\right)}{1 + \beta_0 \alpha_{\rm s}\left(\mu_{\rm R}^2\right) \ln \frac{s}{\mu_{\rm R}^2}}\,, \tag{53}$$

so with the choice of $\mu_{\rm R}^2 = s$, we are exploiting renormalization group invariance to sum to all orders in $\alpha_{\rm s}(\mu_{\rm R}^2)$ logs of the scale ratio $\frac{s}{\mu_{\rm R}^2}$, which is large when s is large.

Having worked out this simple case, we can now turn to the exponentiation of collinear singularities. The starting point is the factorized expression of the physical cross section in terms of parton distributions and a partonic cross section. We consider again for definiteness as an explicit example the Drell–Yan process of Section 3.2, but now with incoming protons with momenta P_1 and P_2 and center-of-mass energy $s = (P_1 + P_2)^2$. The physically measurable cross section depends only on the invariant mass Q^2 of the final-state gauge boson and on the dimensionless variable τ, Eq. (35). Assuming that the two incoming partons carry, respectively, fractions x_1, x_2 of the momenta of the two incoming protons, the center-of-mass energy of the partonic collision is $x_1 x_2 s$, so by momentum conservation $x_1 x_2 s = Q^2$. Integrating over the incoming parton momentum fractions, the measurable cross section σ can thus be written as

$$\sigma(\tau) = \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2\, \delta(\tau - x_1 x_2 z) q_1(x_1) q_2(x_2) \hat{\sigma}(z)\,, \tag{54}$$

where $q_i(x_i)$ denote the two PDFs, $\hat{\sigma}(z)$ is the cross section computed with incoming partons with momenta p_i and with $z = \frac{Q^2}{2p_1 \cdot p_2}$ (partonic cross section, henceforth), and we have for the time being not indicated explicitly the dependence of the various quantities on Q^2 and the renormalization and factorization scales. Also, for simplicity, we have only indicated a single partonic initial state, while in general, the right-hand side of Eq. (54) will contain a sum over all possible initial-state parton pairs that can lead to the desired final state.

The integral over the momentum fractions in Eq. (54) is actually a double convolution

$$\begin{aligned} \sigma(\tau) &= \int_\tau^1 \frac{\mathrm{d}x_1}{x_1} \int_{\tau/x_1}^1 \frac{\mathrm{d}x_2}{x_2} q_1(x_1) q_2(x_2) \hat{\sigma}\left(\frac{\tau}{x_1 x_2}\right) \\ &= \int_\tau^1 \frac{\mathrm{d}y}{y} \mathcal{L}(y) \hat{\sigma}\left(\frac{\tau}{y}\right) = [\mathcal{L} \otimes \hat{\sigma}](\tau)\,, \end{aligned} \tag{55}$$

with $\mathcal{L}$, the parton luminosity, defined as

$$\mathcal{L}(y) \equiv \int_y^1 \frac{\mathrm{d}x_1}{x_1} q_1(y) q_2\left(\frac{y}{x_1}\right) = [q_1 \otimes q_2](y)\,. \tag{56}$$

It follows that, upon the Mellin transformation, the cross section can be written as the ordinary product of the partonic cross section and the parton distributions. Writing now explicitly the dependence on all kinematic variables we have

$$\sigma\left(N, Q^2\right) = \hat{\sigma}_0\left(Q^2\right) C\left(\frac{Q^2}{\mu_{\mathrm{F}}^2}, N, \alpha_{\mathrm{s}}\left(\mu_{\mathrm{R}}^2\right), \frac{\mu_{\mathrm{R}}^2}{\mu_{\mathrm{F}}^2}\right) q_1\left(N, \mu_{\mathrm{F}}^2\right) q_2\left(N, \mu_{\mathrm{F}}^2\right) \tag{57}$$

$$= \hat{\sigma}_0\left(Q^2\right) C\left(\frac{Q^2}{\mu_{\mathrm{F}}^2}, N, \alpha_{\mathrm{s}}\left(\mu_{\mathrm{R}}^2\right), \frac{\mu_{\mathrm{R}}^2}{\mu_{\mathrm{F}}^2}\right) \mathcal{L}\left(N, \mu_{\mathrm{F}}^2\right)\,, \tag{58}$$

where

$$\mathcal{L}\left(N,\mu_{\rm F}^2\right) = q_1\left(N,\mu_{\rm F}^2\right) q_2\left(N,\mu_{\rm F}^2\right) \,, \tag{59}$$

and we have written the partonic cross section $\hat{\sigma}$ in terms of a coefficient function C by factoring out its leading-order expression $\hat{\sigma}_0(Q^2)$, which does not depend on the strong coupling as the leading-order process is electroweak. The coefficient function C is consequently dimensionless and thus can only depend on dimensionless ratios of dimensionful variables, that include Q^2, and the renormalization and factorization scales that were, respectively, introduced in order to define the renormalized theory and to treat the collinear singularities.

In Eq. (57), by slight abuse of notation, we have used the same symbol to denote a function and its Mellin transform

$$\sigma(N) = \int_0^1 {\rm d}x\, x^{N-1}\sigma(x)\,, \tag{60}$$

$$\hat{\sigma}_0 C(N) = \int_0^1 {\rm d}x\, x^{N-1}\hat{\sigma}(x)\,, \tag{61}$$

$$q_{1,2}(N) = \int_0^1 {\rm d}x\, x^{N-1} q_{1,2}(x)\,. \tag{62}$$

The physical cross section can only depend on the physical kinematic variables, *i.e.* Q^2, and the Mellin variable N, conjugate to τ Eq. (35). We have introduced a factorization scale according to the argument of Section 3.2, on which both the partonic cross section and the parton distributions depend, though the physical cross section does not. We have finally introduced a renormalization scale $\mu_{\rm R}$ according to the argument of the previous section, on which the perturbatively computable coefficient function must depend in order to compensate the dependence on it of the renormalized strong coupling $\alpha_{\rm s}(\mu_{\rm R}^2)$ in such a way that the physical cross section does not depend on it.

We now impose the condition that physical observables must be independent of the factorization scale

$$\mu_{\rm F}^2 \frac{\partial}{\partial \mu_{\rm F}^2} \ln\left[C\left(\frac{Q^2}{\mu_{\rm F}^2}, N, \alpha_{\rm s}\left(\mu_{\rm R}^2\right), \frac{\mu_{\rm R}^2}{\mu_{\rm F}^2}\right)\mathcal{L}\left(N,\mu_{\rm F}^2\right)\right] = 0\,. \tag{63}$$

The condition is imposed on the logarithmic derivative because we know from Section 3.2 that the dependence on the factorization scale is logarithmic, and we have imposed it on the logarithm of the physical cross section because this immediately leads to a condition relating the coefficient function and the luminosity (*u.e.* the PDFs)

$$\begin{aligned}\mu_{\rm F}^2 \frac{\partial}{\partial \mu_{\rm F}^2} \ln C\left(\frac{Q^2}{\mu_{\rm F}^2}, N, \alpha_{\rm s}\left(\mu_{\rm R}^2\right), \frac{\mu_{\rm R}^2}{\mu_{\rm F}^2}\right) &= -\mu_{\rm F}^2 \frac{\partial}{\partial \mu_{\rm F}^2} \ln \mathcal{L}\left(N, \mu_{\rm F}^2\right) \\ &\equiv -2\gamma_{qq}\left(N, \alpha_{\rm s}\left(\mu_{\rm R}^2\right)\right) . \end{aligned} \tag{64}$$

Furthermore, exploiting the independence of result on $\mu_{\rm R}$, we perform the partial derivative with respect to $\mu_{\rm F}^2$ in Eq. (64) at fixed $\frac{\mu_{\rm R}}{\mu_{\rm F}}$, because, with this choice, solving the renormalization group equation will simultaneously sum all large scale ratios.

In the last step, we have defined γ_{qq} as the logarithmic derivative of the quark PDF, and the factor of 2 is a consequence of the fact that the Mellin transform of the luminosity, Eq. (56), is the product of two-quark PDFs. We have further made use of the fact that γ_{qq} cannot depend on Q^2: hence, on dimensional grounds, it can only be a function of the strong coupling, which, in turn, is evaluated at scale $\mu_{\rm R}$ as a consequence of the renormalization process. Comparing Eq. (64) to the collinear subtracted partonic cross section given by Eq. (38), we identify γ_{qq} with the Mellin transformed expression (45) of the splitting function.

We can thus view Eq. (63) as an equation satisfied by the coefficient function: the Callan–Symanzik equation

$$\begin{aligned}&\mu_{\rm F}^2 \frac{\partial}{\partial \mu_{\rm F}^2} C\left(\frac{Q^2}{\mu_{\rm F}^2}, N, \alpha_{\rm s}\left(\mu_{\rm R}^2\right), \frac{\mu_{\rm R}^2}{\mu_{\rm F}^2}\right) \\ &= -2\gamma_{qq}\left(N, \alpha_{\rm s}\left(\mu_{\rm R}^2\right)\right) C\left(\frac{Q^2}{\mu_{\rm F}^2}, N, \alpha_{\rm s}\left(\mu_{\rm R}^2\right), \frac{\mu_{\rm R}^2}{\mu_{\rm F}^2}\right) ,\end{aligned} \tag{65}$$

with the partial derivative performed at fixed $\frac{\mu_{\rm R}}{\mu_{\rm F}}$.

5.2. Collinear resummation

The exponentiation of collinear logarithms follows from solving the renormalization group equation, Eq. (65). We can solve it in two steps. First, we trade the derivative with respect to the factorization scale with a derivative with respect to the physical scale, noting that the coefficient function for fixed $\frac{\mu_{\rm R}}{\mu_{\rm F}}$ only depends on $\mu_{\rm F}$ through the ratio $\frac{Q^2}{\mu_{\rm F}^2}$

$$\begin{aligned}&Q^2 \frac{\partial}{\partial Q^2} C\left(\frac{Q^2}{\mu_{\rm F}^2}, N, \alpha_{\rm s}\left(\mu_{\rm R}^2\right), \frac{\mu_{\rm R}^2}{\mu_{\rm F}^2}\right) \\ &= 2\gamma_{qq}\left(N, \alpha_{\rm s}\left(\mu_{\rm R}^2\right)\right) C\left(\frac{Q^2}{\mu_{\rm F}^2}, N, \alpha_{\rm s}\left(\mu_{\rm R}^2\right), \frac{\mu_{\rm R}^2}{\mu_{\rm F}^2}\right) .\end{aligned} \tag{66}$$

Next, as we did when solving the renormalization group equation satisfied by the R ratio, we exploit the independence of results on the renormalization scale and choose $\mu_{\mathrm{R}}^2 = Q^2$. With this choice, we get

$$Q^2 \frac{\partial}{\partial Q^2} C\left(\frac{Q^2}{\mu_{\mathrm{F}}^2}, N, \alpha_{\mathrm{s}}\left(Q^2\right), \frac{Q^2}{\mu_{\mathrm{F}}^2}\right) = 2\gamma_{qq}\left(N, \alpha_{\mathrm{s}}\left(Q^2\right)\right) C\left(\frac{Q^2}{\mu_{\mathrm{F}}^2}, N, \alpha_{\mathrm{s}}\left(Q^2\right), \frac{Q^2}{\mu_{\mathrm{F}}^2}\right) . \tag{67}$$

This is now straightforward to solve, as it is a first-order ordinary differential equation, with solution

$$C\left(\frac{Q^2}{\mu_{\mathrm{F}}^2}, N, \alpha_{\mathrm{s}}\left(Q^2\right), \frac{Q^2}{\mu_{\mathrm{F}}^2}\right) = C\left(1, N, \alpha_{\mathrm{s}}\left(Q^2\right), 1\right) \exp \int_{\mu_{\mathrm{F}}^2}^{Q^2} \frac{\mathrm{d}\mu^2}{\mu^2} 2\gamma_{qq}\left(N, \alpha_{\mathrm{s}}\left(\mu^2\right)\right) . \tag{68}$$

This solution manifestly displays the independence of physical observables on the factorization scale: indeed, the dependence of $C\left(\frac{Q^2}{\mu_{\mathrm{F}}^2}, N, \alpha_{\mathrm{s}}(Q^2), \frac{Q^2}{\mu_{\mathrm{F}}^2}\right)$, Eq. (68), on the scale μ_{F} is through the lower extreme of integration in the exponential, and this exactly matches the scale dependence of the luminosity, as given by Eq. (64). Note also that the further dependence on the scale through α_{s}, which is present in Eq. (64) because the derivative is taken at fixed $\frac{\mu_{\mathrm{R}}}{\mu_{\mathrm{F}}}$, has been reabsorbed when choosing the renormalization scale $\mu_{\mathrm{R}}^2 = Q^2$, so that the derivative in Eq. (67) is a partial derivative, *i.e.* only acts on the direct dependence of $C\left(\frac{Q^2}{\mu_{\mathrm{F}}^2}, N, \alpha_{\mathrm{s}}(Q^2), \frac{Q^2}{\mu_{\mathrm{F}}^2}\right)$ on $\frac{Q^2}{\mu_{\mathrm{F}}^2}$, for fixed α_{s}. This generalizes the procedure that we followed when solving the renormalization group equation (50) which we can now view as the special case of the Callan–Symanzik equation (65) in which the anomalous dimension vanishes.

This proves the exponentiation of the collinear logarithms: the coefficient function is given by a function that depends on scale only through the coupling (like the R ratio) times the exponential of an integral over scale that sums to all order collinear logarithms, along with the running with the coupling.

It is useful to rewrite the solution (68) in an equivalent form by changing variable of integration from $\mu_{\rm F}^2$ to $\alpha_{\rm s}(\mu_{\rm F}^2)$ using Eq. (51)

$$\ln C\left(\frac{Q^2}{\mu_{\rm F}^2}, N, \alpha_{\rm s}\left(Q^2\right), \frac{Q^2}{\mu_{\rm F}^2}\right)$$
$$= \ln C\left(1, N, \alpha_{\rm s}\left(Q^2\right), 1\right) + 2\int\limits_{\alpha_{\rm s}(\mu_{\rm F}^2)}^{\alpha_{\rm s}(Q^2)} \frac{{\rm d}\alpha}{\beta(\alpha)}\gamma_{qq}(N,\alpha)\,, \tag{69}$$

where we have also written the logarithm of the coefficient function, in order to ease the comparison with resummed results that we will derive in the next section. Using the leading-order beta function, Eq. (51), the exponentiated collinear factor becomes

$$\exp\left[-\frac{2\gamma_N^{(0)}}{\beta_0}\int\limits_{\alpha_{\rm s}(\mu_{\rm F}^2)}^{\alpha_{\rm s}(Q^2)}\frac{{\rm d}\alpha}{\alpha}\right] = \left(\frac{\alpha_{\rm s}\left(Q^2\right)}{\alpha_{\rm s}\left(\mu_{\rm F}^2\right)}\right)^{-\frac{2\gamma_N^{(0)}}{\beta_0}}\,. \tag{70}$$

Further, using Eq. (53) to express $\alpha_{\rm s}(Q^2)$ in terms of $\alpha_{\rm s}(\mu_{\rm F}^2)$, we get

$$\exp\left[-\frac{2\gamma_N^{(0)}}{\beta_0}\int\limits_{\alpha_{\rm s}(\mu_{\rm F}^2)}^{\alpha_{\rm s}(Q^2)}\frac{{\rm d}\alpha}{\alpha}\right] = \left(1+\beta_0\alpha_{\rm s}\left(\mu_{\rm F}^2\right)\ln\frac{Q^2}{\mu_{\rm F}^2}\right)^{\frac{2\gamma_N^{(0)}}{\beta_0}} + O\left[\alpha^2\left(\mu_{\rm F}^2\right)\ln\frac{Q^2}{\mu_{\rm F}^2}\right] \tag{71}$$

$$= 1 + 2\alpha_{\rm s}\left(\mu_{\rm F}^2\right)\gamma_N^{(0)}\ln\frac{Q^2}{\mu_{\rm F}^2} + O\left(\alpha^2\left(\mu_{\rm F}^2\right)\right)\,. \tag{72}$$

Equation (71) explicitly shows that expression (70) sums to all orders in $\alpha_{\rm s}(\mu_{\rm F}^2)$ the collinear logs of $\frac{Q^2}{\mu_{\rm F}^2}$: we have indeed exploited renormalization group invaraince to sum all large scale ratios. On the other hand, Eq. (72) shows that expanding this out to first order in $\alpha_{\rm s}(\mu_{\rm F}^2)$ leads back to the single-emission result, Eq. (38). Note that, because one can always express $\alpha_{\rm s}(Q^2)$ in terms of $\alpha_{\rm s}(\mu_{\rm F}^2)$ or conversely, we can equivalently view the coefficient function as a function of $\frac{Q^2}{\mu_{\rm F}^2}$ and either $\alpha_{\rm s}(\mu_{\rm F}^2)$ or $\alpha_{\rm s}(Q^2)$, as Eqs. (69)–(53) explicitly demonstrate.

6. Soft resummation from RG invariance

Soft resummation, *i.e.* the all-order resummation and exponentiation of the double soft-collinear logs discussed in Section 3.1, follows from a similar argument to the collinear exponentiation derived in the previous section. Namely, the real emission and virtual contributions to the hard cross section discussed in Section 3.3 can be argued to analogously factorize. The infrared singularities then cancel between these two contributions, which however depend on different kinematic variables: the real emission depends on the maximum transverse momentum $k^2_{\mathrm{t\,max}}$, while the virtual contribution does not. Again, dimensional analysis then implies that the cancellation of the singularity must happen through the scale dependence of the coupling, and it is calculable in terms of it and of a universal function. Once integrated away, it leads to all-order resummation and exponentiation of logs of $k^2_{\mathrm{t\,max}}$, which close to threshold is a soft scale. We will discuss here the case of processes, such as deep-inelastic scattering, or the invariant mass distribution for the Drell–Yan process that only depend on a single dimensionful variable and a dimensionless ratio. For the Drell–Yan process that, following the discussion in Part 1, we can take as a reference example, these variables are the invariant mass Q^2 of the final-state gauge boson, and its ratio $x = Q^2/s$ to the center-of-mass energy s of the collision

6.1. The soft limit in Mellin space

Since we wish to perform exponentiation in Mellin space, where the phase space factorizes, we have first to understand how soft logs look like in Mellin space. To this purpose, we note that

$$\int\limits_0^1 \mathrm{d}x\, x^{N-1} \left[\frac{\ln^p(1-x)}{1-x}\right]_+$$

$$= \frac{1}{(p+1)} \sum_{k=0}^{p+1} \binom{p+1}{k} \Gamma^{(k)}(1) \left(\ln\frac{1}{N}\right)^{p+1-k} + O\left(\frac{1}{N}\right) \quad (73)$$

$$= \frac{1}{p+1} \ln^{p+1}\frac{1}{N} + O\left(\ln^p\frac{1}{N}\right) + O\left(\frac{1}{N}\right), \quad (74)$$

where $\Gamma^{(k)}(1)$ denotes the k^{th} derivative of the Euler Gamma function $\Gamma(x)$ evaluated at $x = 1$. Equation (73) is easy to prove by noting that

$$\int\limits_0^1 \mathrm{d}x\, x^{N-1}(1-x)^{-1+\epsilon} = \frac{\Gamma[\epsilon]\Gamma[N]}{\Gamma[N+\epsilon]} = \frac{1}{\epsilon}\Gamma[1+\epsilon]N^{-\epsilon}\left[1 + O\left(\frac{1}{N}\right)\right], \quad (75)$$

and expanding both sides about $\epsilon = 0$. Equation (73) explicitly shows that, as stated in Section 3.3, a contribution of the form $\frac{1}{(1-x)_+}$ is logarithmic upon integration with a test function: its Mellin transform, up to power-suppressed corrections, is just $-\ln N$.

We further note that the $x \to 1$ threshold limit corresponds to the $\mathrm{Re}\, N \to \infty$ limit: indeed $\mathrm{d}x\, x^{N-1} = \mathrm{d}\left(\ln \frac{1}{x}\right) \exp\left(-N \ln \frac{1}{x}\right)$ so in the $N \to \infty$ limit, only the point at $x = 1$ survives. It is, in fact, easy to prove that any real function has a Mellin transform that vanishes as $N \to \infty$. The partonic cross section $\hat{\sigma}(x)$, however, contains both contributions that are real functions of x, but also contributions proportional to distributions localized at $x = 1$, such as those discussed in Section 3.3, whose Mellin transform survives the $N \to \infty$ limit, as Eq. (73) demonstrates. In fact

$$\int_0^1 \mathrm{d}x\, x^{N-1} \delta(1-x) = 1\,. \tag{76}$$

We conclude that the only contributions to the Mellin space partonic cross section that do not vanish as $N \to \infty$ are the Mellin transform of distributions localized at $x = 1$. The Dirac delta distribution transforms into a constant, the $\frac{1}{(1-x)_+}$ distribution leads to a single power of $\ln N$, and each extra power of $\ln(1-x)$ leads to an extra power of $\ln N$. Hence, the Mellin transform of the highest power of $\ln(1-x)$ is the same power of $\ln N$, with $\frac{1}{(1-x)_+}$ counting as the first log. However, the Mellin transform also includes lower powers of $\ln N$, all the way down to the constant.

The Mellin transform γ_{qq} of the splitting function P_{qq}, as given by Eq. (40), is proportional to $\frac{1}{(1-x)_+}$. It follows that for large N, it behaves as $\ln N$ as $N \to \infty$, plus a constant, up to terms that vanish as $N \to \infty$. Each collinear log of Q^2 which is summed by the collinear exponentiation of Eq. (72) is consequently accompanied by $\ln N$. This is an infrared log, corresponding to the $x \to 1$ limit of the splitting function. However, as discussed in Section 3.3, the collinear log itself is

$$\ln \frac{k^2_{\mathrm{t\,max}}}{\mu^2_{\mathrm{F}}} = \ln \frac{Q^2}{\mu^2_{\mathrm{F}}} \frac{(1-x)^2}{4x} = \ln \frac{Q^2}{\mu^2_{\mathrm{F}}} + 2\ln(1-x) + \text{non-log}\,, \tag{77}$$

thereby producing an extra $\ln(1-x)$, and thus leading, upon the Mellin transformation, to an extra power of $\ln N$. Hence, each collinear emission leads to a power of $\ln^2 N$ — the Sudakov double log, now in Mellin space.

The goal of Sudakov resummation is thus to determine all contributions to the partonic cross section that survive the $N \to \infty$ limit, specifically showing that they exponentiate.

6.2. Real–virtual factorization and the soft scale

Exponentiation of Sudakov double logs can be derived from a resummation argument based on two ingredients. The first is the factorization of the coefficient function into real emission and virtual contribution, neither of which is separately finite in the infrared, and the identification of the scales on which they depend. The second is the renormalization group improvement of this scale dependence.

The factorization is based on the (partly conjectural) fact that real emission from the internal lines of a diagram is regular in the infrared: *i.e.*, in comparison to the eikonal contribution (17), it does not diverge as $\frac{1}{p\cdot k}$ as $k \to 0$. In fact, we have already tacitly assumed this to be the case when stating, towards the end of Section 3.1, that the argument presented there does not depend on the form of the amplitude M_0. Indeed, for a generic amplitude M_0, on top of the contribution from radiation from external lines shown in Fig. 3, there are, in general, further real-emission contributions coming from the internal lines of M_0 that we neglected in the argument of Section 3.1. The suppression of radiation from internal lines then leads to factorization, because soft radiation from external lines factorizes due to the eikonal argument of Section 2.2. This suppression is relatively easy to prove in QED: the argument, originally given in Ref. [14], is summarized in Appendix A. A general proof in QCD is rather less straightforward, though the result is likely to hold at least with processes with a colorless final state. The result is represented pictorially for the Drell–Yan process in Fig. 7.

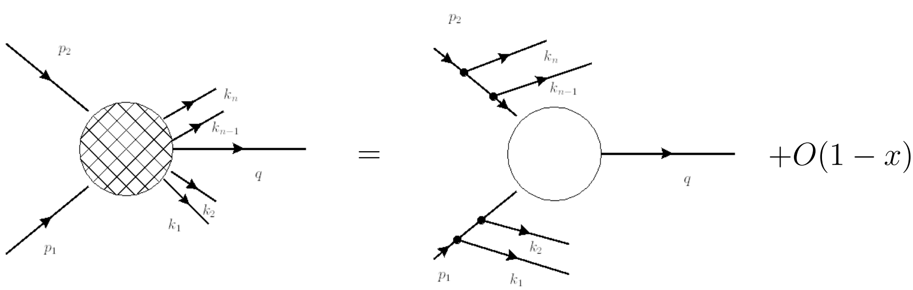


Fig. 7. Graphical representation of the suppression of radiation from internal lines in the soft limit. It is assumed that the line with momentum q is colorless and cannot radiate like in the Drell–Yan process.

Upon this assumption, the partonic cross section factorizes as

$$\hat{\sigma}\left(Q^2, x; \epsilon\right) = H\left(Q^2; \epsilon\right) J\left(Q^2, x; \epsilon\right) \left[1 + O(1-x)\right], \tag{78}$$

where $H(Q^2;\epsilon)$ includes all the loop corrections and has therefore the kinematics of the leading-order process, and thus depends only on Q^2, while

$J(Q^2, x; \epsilon)$ is a dimensionless function that includes the radiation from external lines and thus the non-trivial kinematic dependence on $x = \frac{Q^2}{s}$, with s the center-of-mass energy of the partonic collision. We have explicitly indicated a dependence on a (dimensional) regulator ϵ because H and J are separately divergent, hence Eq. (78) only makes sense at a regularized level.

Furthermore, recalling the argument from Sections 3.2–3.3, we note that the radiation contribution $J(Q^2, x; \epsilon)$, up to non-logarithmic terms, can only depend on the kinematic variables Q^2 and x through the upper limit $k^2_{\mathrm{t\,max}}$ of the integration over the transverse momentum of the emitted partons. This means that J actually depends on a fixed combination of the kinematic variables Q^2 and x, namely by $k^2_{\mathrm{t\,max}}(Q^2, x)$. The value of $k^2_{\mathrm{t\,max}}$ for the Drell–Yan process, already given in Eq. (41), can be determined as follows. In the general case in which in the final state we have a gauge boson with momentum q and a set of partons with total momentum k', momentum conservation implies

$$p_1 + p_2 = k' + q\,, \tag{79}$$

where p_i are the momenta of the incoming partons and $q^2 = Q^2$. In the center-of-mass reference frame, the energy of the collision is given by

$$s = (p_1 + p_2)^2 = \left(q_0 + k'_0\right)^2 = \left(\sqrt{Q^2 + q_{\mathrm{t}}^2 + q_z^2} + \sqrt{k'^2 + q_{\mathrm{t}}^2 + q_z^2}\right)^2\,, \tag{80}$$

where in the last step $q_{\mathrm{t}} = |\vec{q}_{\mathrm{t}}|$ and q_z are, respectively, the transverse and longitudinal components of the momentum of the gauge boson. It follows that for fixed s, q_{t} is maximum when $k'^2 = q_z^2 = 0$. In this case,

$$s = \left(\sqrt{Q^2 + q^2_{\mathrm{t\,max}}} + q^2_{\mathrm{t\,max}}\right)^2\,. \tag{81}$$

Solving for $q^2_{\mathrm{t\,max}}$, and noting that, of course, $k^2_{\mathrm{t\,max}} = q^2_{\mathrm{t\,max}}$, we get

$$k^2_{\mathrm{t\,max}} = q^2_{\mathrm{t\,max}} = Q^2 \frac{(1-x)^2}{4x}\,, \tag{82}$$

where $x = \frac{Q^2}{s}$, as anticipated in Eq. (41).

The resummation is performed in Mellin space, where, of course, the coefficient function, given by Eq. (61), in turn factorizes into the product of a function of Q^2 and a function that depends on a fixed combination of Q^2 and N. Equation (75) immediately implies that with $k^2_{\mathrm{t\,max}} = Q^2 \frac{(1-x)^2}{4x}$, the Mellin space coefficient function only depends on N through the soft scale

$$\Lambda^2_{\mathrm{soft}} = \frac{Q^2}{N^2}\,, \tag{83}$$

up to corrections that vanish when $N \to \infty$. This, in turn, implies that the Mellin transform of $J(Q^2, x; \epsilon)$, which contains all the N dependence of the coefficient function, only depends on Q^2 and N through $\Lambda^2_{\rm soft}$. Hence, upon the Mellin transformation $H(Q^2; \epsilon)$, which is N-independent, remains unchanged; $J(Q^2, x; \epsilon)$ is transformed into a function of $\frac{Q^2}{N^2}$; and the Mellin space coefficient function is the product of these two factors.

6.3. Renormalization group improving the soft scale

The renormalization group argument that leads to resummation is performed in terms of a physical anomalous dimension, defined as

$$\gamma^{\rm phys}\left(N, \alpha_{\rm s}\left(Q^2\right)\right) \equiv Q^2 \frac{\rm d}{{\rm d}Q^2} \ln C\left(\frac{Q^2}{\mu_{\rm F}^2}, N, \alpha_{\rm s}\left(Q^2\right), \frac{Q^2}{\mu_{\rm F}^2}\right). \tag{84}$$

Note that it follows from Eq. (68) that $\gamma^{\rm phys}$ does not depend on the factorization scale $\mu_{\rm F}$. This, in particular, means that even though the coefficient function itself diverges when $\mu_{\rm F} \to 0$, and indeed $\mu_{\rm F}$ was introduced in Section 3.2 in order to regulate the collinear singularity, the scale dependence of the log of the coefficient function remains finite in the limit. This result is clear from the expression of the coefficient function, Eq. (68), and it is thus a direct consequence of the renormalization group equation (63) that it satisfies. Note also that $\gamma^{\rm phys}$ differs from the anomalous dimension $2\gamma_{qq}$: it provides the dependence of the physically measurable cross section on the physical scale Q^2 (hence its name), and it also receives a contribution from the first factor on the right-hand side of Eq. (68).

Exploiting the factorization, Eq. (78), the physical anomalous dimension can be written as a sum of two contributions, which correspond, respectively, to the logarithmic derivative with respect to the physical scale of the virtual and real factors, H and J, respectively, with the latter computed in Mellin space (and the former N-independent). Even though the physical anomalous dimension is finite, the real and virtual contributions to it are not separately finite, and thus this decomposition can only be performed in terms of regularized quantities, which requires introducing a scale. It is the independence of the physical anomalous dimension of this scale that leads to soft resummation, just like the independence of the physical cross section of the factorization scale led to collinear resummation in Section 5.2.

Indeed, recalling the definition (61) of the coefficient function, and substituting the Mellin space version of the factorized expression of the partonic cross section of Eq. (78) in the definition (84) of the physical anomalous dimension, we get

$$\gamma^{\rm phys}\left(N,\alpha_{\rm s}\left(Q^2\right)\right)$$
$$= \lim_{\epsilon\to 0}\left[\gamma^{(c)}\left(\frac{Q^2}{\mu^2},\alpha_{\rm s}\left(\mu^2\right);\epsilon\right)+\gamma^{(l)}\left(\frac{Q^2/N^2}{\mu^2},\alpha_{\rm s}\left(\mu^2\right);\epsilon\right)\right], \quad (85)$$

where $\gamma^{(c)} = Q^2\frac{\rm d}{{\rm d}Q^2}\ln H$ and $\gamma^{(l)} = Q^2\frac{\rm d}{{\rm d}Q^2}\ln J$, and all expressions on the right-hand side are computed in d dimensions, so necessarily $\alpha_{\rm s}$ must be μ-dependent. Also, we have made use of the fact that H and correspondingly $\gamma^{(c)}$ only depends on Q^2, while J, and correspondingly $\gamma^{(l)}$, only depends on $\frac{Q^2}{N^2}$. Since $\gamma^{(c)}$ and $\gamma^{(l)}$ are both dimensionless, they can depend on Q^2 or, respectively, Q^2/N only through their ratio to the regularization scale μ^2. The N dependence, and thus the logs, is contained in $\gamma^{(l)}$, while $\gamma^{(c)}$ is constant, *i.e.* N-independent.

Following the argument of Section 5.2, we could now introduce a suitable scale, analogous to the factorization scale, in order to renormalize $\gamma^{(l)}$ and $\gamma^{(c)}$ in such a way that they become simultaneously finite in the infrared. However, this is not necessary, as we are really only interested in their sum, namely the physical anomalous dimension, and we can instead exploit the condition of μ-independence of $\gamma^{\rm phys}$

$$\lim_{\epsilon\to 0}\left[\mu^2\frac{\rm d}{{\rm d}\mu^2}\gamma^{(l)}\left(\frac{Q^2/N^2}{\mu^2},\alpha_{\rm s}\left(\mu^2\right);\epsilon\right)+\mu^2\frac{\rm d}{{\rm d}\mu^2}\gamma^{(c)}\left(\frac{Q^2}{\mu^2},\alpha_{\rm s}\left(\mu^2\right);\epsilon\right)\right]=0\,. \quad (86)$$

But now we note that $\gamma^{(c)}$ and $\gamma^{(l)}$ depend on different kinematic variables, hence the only way Eq. (86) can hold is if in the $\epsilon\to 0$ limit, the scale derivatives of both $\gamma^{(c)}$ and $\gamma^{(l)}$, that must be equal up to the sign, are functions of $\alpha_{\rm s}(\mu^2)$ with coefficients that do not depend on either Q^2 or $\frac{Q^2}{N^2}$

$$\begin{aligned}\lim_{\epsilon\to 0}\mu^2\frac{\rm d}{{\rm d}\mu^2}\gamma^{(l)}\left(\frac{Q^2/N^2}{\mu^2},\alpha_{\rm s}\left(\mu^2\right);\epsilon\right) &= -\lim_{\epsilon\to 0}\mu^2\frac{\rm d}{{\rm d}\mu^2}\gamma^{(c)}\left(\frac{Q^2}{\mu^2},\alpha_{\rm s}\left(\mu^2\right);\epsilon\right)\\ &= -g\left(\alpha_{\rm s}\left(\mu^2\right)\right)\,,\end{aligned} \quad (87)$$

with

$$g(\alpha_{\rm s}) = g_1\alpha_{\rm s}+g_2\alpha_{\rm s}^2+\ldots \quad (88)$$

We also note that the coefficients g_i are necessarily finite, as we shall see explicitly shortly.

We now observe that Eq. (87) looks exactly like the renormalization group equation (64) satisfied by the logarithm of the coefficient function: it can be consequently solved in the same way, leading to a solution of the form of Eq. (69), both for $\gamma^{(l)}$ and $\gamma^{(c)}$, which only differ due to the scale on which they depend and the sign of their scale dependence. We get

$$\gamma^{(c)}\left(\frac{Q^2}{\mu^2},\alpha_s\left(\mu^2\right);\epsilon\right) = g_0^{(c)}\left(\alpha_s\left(Q^2\right);\epsilon\right) - \int_{\mu^2}^{Q^2}\frac{\mathrm{d}\lambda^2}{\lambda^2}g\left(\alpha_s\left(\lambda^2\right)\right) , \tag{89}$$

$$\gamma^{(l)}\left(\frac{Q^2/N^2}{\mu^2},\alpha_s\left(\mu^2\right);\epsilon\right) = g_0^{(l)}\left(\alpha_s\left(Q^2/N^2\right);\epsilon\right) + \int_{\mu^2}^{Q^2/N^2}\frac{\mathrm{d}\lambda^2}{\lambda^2}g\left(\alpha_s\left(\lambda^2\right)\right) . \tag{90}$$

Substituting these solutions into the expression of Eq. (85) of the physical anomalous dimension, we find

$$\begin{aligned}\gamma^{\mathrm{phys}}\left(N,\alpha_s\left(Q^2\right)\right) &= \lim_{\epsilon\to 0}\left[g_0^{(c)}\left(\alpha_s\left(Q^2\right);\epsilon\right) + g_0^{(l)}\left(\alpha_s\left(\frac{Q^2}{N^2}\right);\epsilon\right)\right] \\ &\quad + \int_{Q^2}^{Q^2/N^2}\frac{\mathrm{d}\lambda^2}{\lambda^2}g\left(\alpha_s\left(\lambda^2\right)\right) .\end{aligned} \tag{91}$$

This is manifestly independent of the scale μ, and furthermore, it shows that because γ^{phys} is finite, both the function $g(\alpha_s)$, and the sum of the initial conditions $g_0^{(c)}(\alpha_s(Q^2);\epsilon)$ and $g_0^{(l)}\left(\alpha_s(Q^2/N^2);\epsilon\right)$ must also be finite.

We can rewrite the solution, Eq. (91), in a more compact form by noting that we can relate the strong coupling at different scales by integrating up the beta function, Eq. (51)

$$\begin{aligned}\gamma^{\mathrm{phys}}\left(N,\alpha_s\left(Q^2\right)\right) &= \lim_{\epsilon\to 0}\left[g_0^{(c)}\left(\alpha_s\left(Q^2\right);\epsilon\right) + g_0^{(l)}\left(\alpha_s\left(Q^2\right);\epsilon\right)\right] \\ &\quad + \int_{Q^2}^{Q^2/N^2}\frac{\mathrm{d}\lambda^2}{\lambda^2}\left[g\left(\alpha_s\left(\lambda^2\right)\right) + \beta\left(\alpha_s\left(\lambda^2\right)\right)\frac{\mathrm{d}g_0^{(l)}}{\mathrm{d}\alpha_s}\left(\alpha_s\left(\lambda^2\right)\right)\right] \\ &= \bar{g}_0\left(\alpha_s\left(Q^2\right)\right) + \int_{Q^2}^{Q^2/N^2}\frac{\mathrm{d}\lambda^2}{\lambda^2}\bar{g}\left(\alpha_s\left(\lambda^2\right)\right) ,\end{aligned} \tag{92}$$

where $\bar{g}_0$ and $\bar{g}$ are perturbative series in α_s. This expression of the physical anomalous dimension is our final result. Renormalization group invariance and the cancellation of infrared singularities between factorized real emission contribution, that only depends on the soft scale $\Lambda^2_{\mathrm{soft}} = \frac{Q^2}{N^2}$, and virtual contribution, that only depends on the hard scale Q^2, leads to a prediction of the full logarithmic dependence on the Mellin variable N to all orders in $\alpha_s(Q^2)$. As we shall see shortly, knowledge of the first-order coefficient in the expansion of $\bar{g}$ fully determines leading log resummation, knowledge of the next coefficient together with the first correction to $\bar{g}_0$ determines the next-to-leading log result, and so on. These coefficients can be determined by comparing to fixed-order results, thereby allowing for prediction of logarithmic terms to all orders based on fixed-order knowledge.

7. The resummed coefficient function

The resummed result given in Eq. (92) provides an expression for the physical anomalous dimension, *i.e.* essentially for the scale dependence of the coefficient function. Hence, a little more work is required in order to arrive at a resummed expression for the coefficient function itself. This is done by separating off the part of the Q^2 dependence of the coefficient function that goes through the ratio to the factorization scale Q^2/μ_F^2. The final resummed result can then be cast in various equivalent forms, and shown to sum to all orders in $\alpha_s(Q^2)$ contributions corresponding to a given fixed logarithmic accuracy by including a suitable finite number of terms in the expansion of the coefficients that enter its expression.

7.1. The soft scale and the factorization scale

In order to go from the physical anomalous dimension to the coefficient function, we start recalling that contributions to the coefficient function that survive in the large-N limit are either constant, *i.e.* N-independent functions of α_s only, or logarithmic, *i.e.* powers of $\ln N$. The two contributions to the physical anomalous dimension on the right-hand side of Eq. (92) then, respectively, provide us with an expression for the scale dependence of constant and logarithmic contributions to the coefficient function. Namely, we can write the coefficient function as

$$C\left(N, \frac{Q^2}{\mu_F^2}, \alpha_s\left(Q^2\right)\right) = C^{(c)}\left(\alpha_s\left(Q^2\right)\right) C^{(l)}\left(\frac{Q^2/N^2}{\mu_F^2}, \alpha_s\left(Q^2\right)\right) + O\left(\frac{1}{N}\right), \tag{93}$$

where

$$Q^2 \frac{\mathrm{d}}{\mathrm{d}Q^2} C^{(c)}\left(\alpha_s\left(Q^2\right)\right) = \bar{g}_0\left(\alpha_s\left(Q^2\right)\right) C^{(c)}\left(\alpha_s\left(Q^2\right)\right), \tag{94}$$

$$Q^2 \frac{\partial}{\partial Q^2} C^{(l)}\left(\frac{Q^2/N^2}{\mu_F^2}, \alpha_s\left(Q^2\right)\right) = \left[\int_{Q^2}^{Q^2/N^2} \frac{\mathrm{d}\lambda^2}{\lambda^2} \bar{g}\left(\alpha_s\left(\lambda^2\right)\right)\right] C^{(l)}\left(\frac{Q^2/N^2}{\mu_F^2}, \alpha_s\left(Q^2\right)\right). \tag{95}$$

The two factors $C^{(c)}$ and $C^{(l)}$ in the right-hand side of Eq. (93) can be thought of as providing a factorization of the coefficient function analogous to that of Eq. (78) that we started from, with the function of the soft scale $C^{(l)}$ playing the role of J and the function of the hard scale $C^{(c)}$ playing the role of H. However, the two factors are now individually finite.

Specifically, Eq. (95) tells us that logarithmically enhanced terms, contained in $C^{(l)}$, satisfy

$$\ln C^{(l)}\left(\frac{Q^2/N^2}{\mu^2},\alpha_{\rm s}\left(Q^2\right)\right)-\ln C^{(l)}\left(\frac{Q_0^2/N^2}{\mu^2},\alpha_{\rm s}\left(Q_0^2\right)\right)$$
$$=\int_{Q_0^2}^{Q^2}\frac{{\rm d}k^2}{k^2}\int_{k^2}^{k^2/N^2}\frac{{\rm d}\lambda^2}{\lambda^2}\bar g\left(\alpha_{\rm s}\left(\lambda^2\right)\right)=-\int_1^{N^2}\frac{{\rm d}n}{n}\int_{Q_0^2/n}^{Q^2/n}\frac{{\rm d}k^2}{k^2}\,\bar g\left(\alpha_{\rm s}\left(k^2\right)\right)\,, \quad (96)$$

where Q_0^2 is some reference value of the hard scale Q^2. We may be tempted to interpret Q_0 as the factorization scale $\mu_{\rm F}$, but then we realize that, in actual fact, the coefficient function has the form given by Eq. (68): hence the logarithmic part of the coefficient function is, in turn, given by the product of two factors, one of which depends on the factorization scale, and the other which does not.

In order to cast the result, Eq. (96), in the desired form, we let

$$\bar g(\alpha_{\rm s})=A(\alpha_{\rm s})-\frac{\partial B\left(\alpha_{\rm s}\left(k^2\right)\right)}{\partial\ln k^2} \quad (97)$$

so that, substituting in Eq. (96), $C^{(l)}$ takes the form

$$C^{(l)}\left(\frac{Q^2/N^2}{\mu_{\rm F}^2},\alpha_{\rm s}\left(Q^2\right)\right)$$
$$=\exp\left\{\int_1^{N^2}\frac{{\rm d}n}{n}\left[\left(-\int_{\mu^2(\mu_{\rm F})}^{Q^2/N^2}\frac{{\rm d}k^2}{k^2}A\left(\alpha_{\rm s}\left(k^2\right)\right)\right)+B\left(\alpha_{\rm s}\left(\frac{Q^2}{n}\right)\right)\right]\right\}\,, \quad (98)$$

where $\mu^2(\mu_{\rm F})$ is some reference scale that depends on $\mu_{\rm F}$.

We can determine both the value of the scale μ and of the function $A(\alpha_{\rm s}(k^2))$ in Eq. (98) by demanding that the $\mu_{\rm F}$ dependence of $C^{(l)}$ is given by the renormalization group equation (65), as we now show. First, we note that Eq. (40), together with the Mellin transformation formula (73), implies that γ_{qq} is proportional to $\ln N$, up to constants and terms that vanish as $N\to\infty$. This turns out to be the case to all perturbative orders (in the $\overline{\rm MS}$

factorization scheme)

$$\gamma_{qq}\left(N,\alpha_{\mathrm{s}}\left(\mu^2\right)\right)=-\ln N\sum_k\left(\frac{\alpha_{\mathrm{s}}\left(\mu^2\right)}{\pi}\right)^k A_k+O\left(N^0\right)+O\left(\frac{1}{N}\right). \quad (99)$$

The coefficient of the logarithmically enhanced contribution to the anomalous dimension is known as the cusp anomalous dimension. This logarithmically enhanced contribution is present in the γ_{qq} and γ_{gg} anomalous dimensions, that, respectively, characterize collinear emission of a gluon from a quark or a gluon line. Collinear emission of a quark from a gluon or a quark is not logarithmically enhanced, and in fact the corresponding anomalous dimensions vanish as $N\to\infty$. For the Drell–Yan process, on which we are focusing, only the quark cusp anomalous dimension is thus relevant.

The desired result now follows noting that if we let $\mu^2(\mu_{\mathrm{F}})=\mu_{\mathrm{F}}^2$ in Eq. (98), we immediately get

$$\mu_{\mathrm{F}}^2\frac{\mathrm{d}}{\mathrm{d}\mu_{\mathrm{F}}^2}\ln C^{(l)}\left(\frac{Q^2/N^2}{\mu_{\mathrm{F}}^2},\alpha_{\mathrm{s}}\left(Q^2\right)\right)=\ln N^2 A\left(\alpha_{\mathrm{s}}\left(\mu_{\mathrm{F}}^2\right)\right). \quad (100)$$

Note taht the factor of 2 from $\ln N^2=2\ln N$ exactly provides the factor of 2 needed to obtain twice $-\gamma_{qq}$ as in Eq. (64). Comparing to Eq. (65) with the anomalous dimension given by Eq. (99) (since $C^{(l)}$ only includes logarithmically enhanced contributions), we find that the resummation function $A(\alpha_{\mathrm{s}}(\mu^2))$ coincides with the cusp anomalous dimension

$$A(\alpha_{\mathrm{s}})=\sum_k\left(\frac{\alpha_{\mathrm{s}}}{\pi}\right)^k A_k. \quad (101)$$

7.2. The resummed result and its accuracy

We are now ready to present the resummed coefficient function and study the accuracy that the resummation has achieved. The final expression for the resummed coefficient function, collecting all results, is

$$C\left(N,\frac{Q^2}{\mu_{\mathrm{F}}^2},\alpha_{\mathrm{s}}\left(Q^2\right)\right)$$
$$=C^{(c)}\left(\alpha_{\mathrm{s}}\left(Q^2\right)\right)\exp\int_1^{N^2}\frac{\mathrm{d}n}{n}\left[\left(-\int_{\mu_{\mathrm{F}}^2}^{Q^2/n}\frac{\mathrm{d}k^2}{k^2}A\left(\alpha_{\mathrm{s}}\left(k^2\right)\right)\right)+B\left(\alpha_{\mathrm{s}}\left(\frac{Q^2}{n}\right)\right)\right], \quad (102)$$

with A given by the cusp anomalous dimension of Eq. (101), and the functions $C^{(c)}$ and B given as power series in α_s

$$C^{(c)}(\alpha_s) = \left(\frac{\alpha_s}{\pi}\right)^k C_k^{(c)}, \quad (103)$$

$$B(\alpha_s) = \sum_k \left(\frac{\alpha_s}{\pi}\right)^k B_k. \quad (104)$$

This resummed result shows that the double logs associated to the multiple emissions of Section 4.2 do indeed exponentiate: the cusp anomalous dimension is the most singular part of the splitting function in the infrared limit, and the splitting function is, in turn, the coefficient of the most singular contribution in the collinear limit, with the infrared and collinear singularities, respectively, regulated by the plus prescription and the factorization scale μ_F. Indeed,

$$A_1 = C_F, \quad (105)$$

the coefficient of the double log, Eq. (33), that now is exponentiated. Higher-order contributions to the cusp anomalous dimension correspond to double logs associated to higher-order corrections to the splitting function and the anomalous dimension in this soft-collinear limit. Finally, the function $B(\alpha_s)$ resums to all orders contributions that are associated to soft radiation, but not in the ordered collinear region of Section 4.2.

The leading logs, which correspond to an extra power of log squared at each order in α_s come from the A_1 contribution to Eq. (102), while B starts contributing at the next-to-leading logarithmic order. Also, beyond leading log, one must include contributions to the function $C^{(l)}$, because even though not logarithmically enhanced, by interference with logarithmically enhanced contributions coming from A and B, they produce subleading logarithmic contributions.

An efficient way of keeping track of the logarithmic orders is to write the coefficient function as

$$C(N, \alpha_s) = g_0(\alpha_s) \exp\left[\frac{1}{\alpha_s} g_1(\alpha_s \ln N) + g_2(\alpha_s \ln N) + \alpha_s g_3(\alpha_s \ln N) + \ldots\right], \quad (106)$$

where all functions g_i are power series in their respective arguments

$$g_i(x) = \sum_{k=k_i^{\min}}^{\infty} g_i^k x^k. \quad (107)$$

Here, $k_0^{\min} = 0$, $k_1^{\min} = 2$, and $k_i^{\min} = 1$ for all $i \geq 2$: the coefficient function starts with 1 at order α_s^0 (as it must be, given that the leading order σ_0 has been factored out, Eq. (61)), and the g_1 term starts at $O(\alpha_s)$ with a double log.

Subsequent contributions g_i in the exponent correspond to the leading, next-to-leading, ..., logarithmic approximation, where at the leading order, the power of α_s is always by unit lower than the power of $\ln N$, at next-to-leading, it is the same, at next-to-next-to-leading, one more and so on. In the coefficient function at the leading logarithmic level, each order in α_s is accompanied by an extra power of $\ln^2 N$, and it is fully predicted by the knowledge of g_1, which, in turn, is entirely predicted by the knowledge of the A_1 coefficient, *i.e.* the leading-order cusp anomalous dimension. The function g_2 is predicted by the knowledge of A_2 and B_1. The knowledge of g_2 then predicts at each order in α_s the coefficient of the next-lower order power of $\ln N$. However, if this is supplemented also by the first order coefficient g_0^1 in the expansion of the prefactor function g_0, at each order in α_s the coefficients of two next lower powers of $\ln N$ are actually predicted. This is usually referred to as next-to-leading logarithmic (NLL) approximation in the QCD literature. Confusingly, in the SCET literature, it is usually called NLL', while NLL is the approximation without the g_0^1 coefficient, in which the coefficient of one less power of $\ln N$ is predicted. The pattern continues at next orders, and it is summarized in Table 1.

Table 1. Summary of the coefficients in the resummation formulae required to achieve a given logarithmic accuracy.

Log accuracy	Accuracy of C: $\alpha_s^n L^k$	Accuracy of $\ln C$: $\alpha_s^n L^k$	g_0: α_s^k	g_j order	A: α_s^i	B: α_s^i
LL	$k = 2n$	$k = n + 1$	0	1	1	0
NLL	$2n - 2 \leq k \leq 2n$	$k = n$	1	2	2	1
NNLL	$2n - 4 \leq k \leq 2n$	$k = n - 1$	2	3	3	2

The coefficients A_k, B_k, and $C_k^{(c)}$ can be determined by matching the resummed result to a fixed-order computation. Specifically, the leading cusp anomalous dimension A_1 is determined by the coefficient of the highest power of log of a next-to-leading (NLO) order computation. Comparison to the NLO also determines the coefficients B_1 and $C_1^{(c)}$ that enter the next-to-leading logarithmic (NLL) resummation. However, the NLO cusp anomalous dimension A_2 only appears in an NNLO fixed order computation. It is easy to check that this pattern persists to all orders: the coefficients B_k and $C_k^{(c)}$

that enter the N^{k}LL resummation are fully determined by a fixed N^{k}LO fixed-order result, but the N^{k}LO cusp anomalous dimension only appears in a N^{k+1}LO fixed-order computation. On the other hand, the cusp anomalous dimension is process-independent, so once the cusp anomalous dimension is known to N^{k}LO, an N^{k}LO fixed-order computation of a process fully determines the coefficients needed for the N^{k}LL resummation.

The resummed result of Eq. (102) takes an especially simple form when choosing a factorization scale that coincides with the hard scale, $\mu_{\mathrm{F}}^2 = Q^2$. With this choice, the exponential evolution factor is absent in the expression (68) of the coefficient function, because it is entirely reabsorbed in the parton luminosity, so the coefficient function becomes a function of N and $\alpha_{\mathrm{s}}(Q^2)$ only. In this case, its resummed expression becomes

$$C\left(N,1,\alpha_{\mathrm{s}}\left(Q^2\right)\right) = C^{(c)}\left(\alpha_{\mathrm{s}}\left(Q^2\right)\right)\exp\int\limits_1^{N^2}\frac{\mathrm{d}n}{n}\left[\left(-\int\limits_{Q^2}^{Q^2/n}\frac{\mathrm{d}k^2}{k^2}A\left(\alpha_{\mathrm{s}}\left(k^2\right)\right)\right)+B\left(\alpha_{\mathrm{s}}\left(\frac{Q^2}{n}\right)\right)\right]. \tag{108}$$

This can be rewritten in an interesting way by switching the order of the two integrations and performing the n integral

$$C\left(N,1,\alpha_{\mathrm{s}}\left(Q^2\right)\right) = C^{(c)}\left(\alpha_{\mathrm{s}}\left(Q^2\right)\right)\exp\int\limits_{Q^2}^{Q^2/N^2}\frac{\mathrm{d}k^2}{k^2}\left[A\left(\alpha_{\mathrm{s}}\left(k^2\right)\right)\int\limits_{N^2}^{Q^2/k^2}\frac{\mathrm{d}n}{n}+B\left(\alpha_{\mathrm{s}}\left(k^2\right)\right)\right] \tag{109}$$

$$= C^{(c)}\left(\alpha_{\mathrm{s}}\left(Q^2\right)\right)\exp\int\limits_{Q^2}^{Q^2/N^2}\frac{\mathrm{d}k^2}{k^2}\left[A\left(\alpha_{\mathrm{s}}\left(k^2\right)\right)\ln\frac{Q^2/N^2}{k^2}+B\left(\alpha_{\mathrm{s}}\left(k^2\right)\right)\right]. \tag{110}$$

This form of the resummed result shows that the A term effectively resums logs by performing a further perturbative evolution with the large-N anomalous dimension given by Eq. (99) from the hard scale Q^2 to the soft scale Q^2/N^2.

This can be made even more explicit by rewriting the resummation exponent using the identity

$$\int_0^1 \mathrm{d}x \, \frac{x^{N-1}-1}{1-x} \ln^k(1-x) = -\sum_{n=0}^{\infty} \frac{\Gamma^{(n)}(1)}{n!} \frac{\mathrm{d}^n}{\mathrm{d}L^n} \int_0^{1-1/N} \frac{\mathrm{d}x}{1-x} \ln^k(1-x) + O\left(\frac{1}{N}\right), \tag{111}$$

where

$$L = \ln \frac{1}{N} . \tag{112}$$

Using this in Eq. (109), we get

$$C\left(N, 1, \alpha_{\mathrm{s}}\left(Q^2\right)\right) = C^{(c)}\left(\alpha_{\mathrm{s}}\left(Q^2\right)\right) \times \exp\left[\int_0^1 \mathrm{d}x \frac{x^{N-1}-1}{1-x} \left(\int_{Q^2}^{(1-x)^2 Q^2} \frac{\mathrm{d}k^2}{k^2} 2A\left(\alpha_{\mathrm{s}}\left(k^2\right)\right) + \bar{B}\left(\alpha_{\mathrm{s}}\left((1-x)^2 Q^2\right)\right)\right)\right], \tag{113}$$

where the function $\bar{B}$ is a series in α_{s} whose coefficients are determined order-by-order using Eq. (111) in terms of the coefficients A_k and B_k: for instance, $\bar{B}_1 = B_1 + 2\gamma_E A_1$ and so on. This is the form of the resummation that was given in the original paper [6]. The term proportional to A in the exponent is just the Mellin transform of $P_{qq}(\alpha_{\mathrm{s}}(k^2))$, integrated from scale Q^2 to scale Q^2/N, thereby exposing the physical meaning of the resummation. On the other hand, it should be noted that the integral over k^2 in Eq. (113) is ill-defined, because the strong coupling $\alpha_{\mathrm{s}}(k^2)$ diverges at the so-called Landau pole, *i.e.* when k^2 is small enough that the denominator in Eq. (53) vanishes. Hence, Eq. (113) is only meaningful insofar as it is equal to Eq. (109) up to terms that vanish in the $N \to \infty$ limit.

8. Transverse momentum resummation

The resummation formalism that we have discussed so far is conceptually interesting and transparent, but of limited phenomenological interest, because measured cross sections vanish at threshold, and become very small close to threshold. Hence, in the region where resummation effects are significant, the measured cross section is quite small. A related resummation formalism which is instead very relevant phenomenologically is that for transverse-momentum distributions, such as the Drell–Yan cross section, but now differential not only with respect to the gauge boson mass, but also with respect to its transverse momentum q_{t}. This can be thought of as the differential counterpart of the cross section that we discussed so far. The transverse momentum distribution if computed at fixed perturbative order diverges in the $q_\mathrm{t} \to 0$ limit. This divergence is akin to the divergence of the real emission cross section of Section 3.1: it is related to a soft-collinear Sudakov double-log, that can be similarly resummed to all orders.

A treatment of transverse momentum resummation goes beyond the scope of these lectures. It will suffice here to make some general remarks and quote its final form. The same sort of argument that led us to perform threshold resummation in Mellin space leads to transverse momentum resummation in Fourier space, conjugate to transverse momentum itself. Indeed, the real emission contribution to the transverse momentum distribution of the gauge boson respects the conservation of transverse momentum and, consequently, the phase space $\mathrm{d}\Phi_n$ for n gluon emission contains a momentum conservation delta

$$\mathrm{d}\Phi_n \propto \mathrm{d}^2 k_{\mathrm{t}1}\, \mathrm{d}^2 k_{\mathrm{t}2} \dots \mathrm{d}^2 k_{\mathrm{t}n} \delta^{(2)}\left(\vec{k}_{\mathrm{t}1} + \dots + \vec{k}_{\mathrm{t}n} - \vec{q}_\mathrm{t}\right) , \tag{114}$$

where k_t^i are the transverse momenta of the emitted gluons and q_t is the transverse momentum of the gauge boson.

Resummation requires as the first step the factorization of the real emission contributions which are then exponentiated, but factorization of the phase space is broken by the transverse momentum conserving delta. Factorization of the phase space is achieved by performing a Fourier transformation, just like (recall Section 4.1) the factorization of longitudinal phase space (*i.e.* the integrals over the momentum fractions x_i, Eq. (44)) is achieved through the Mellin transformation. Indeed, we have

$$\begin{aligned}
&\mathrm{d}^2 q_\mathrm{t}\, \mathrm{d}^2 k_{\mathrm{t}1} \dots \mathrm{d}^2 k_{\mathrm{t}n} \delta\left(\vec{k}_{\mathrm{t}1} + \dots + \vec{k}_{\mathrm{t}n} - \vec{q}_\mathrm{t}\right) \\
&= \mathrm{d}^2 q_\mathrm{t} \int \frac{\mathrm{d}^2 b}{(2\pi)^2}\, \mathrm{e}^{i\vec{b}\cdot\vec{q}_\mathrm{t}} \mathrm{d}^2 k_{\mathrm{t}1}\, \mathrm{e}^{-i\vec{b}\cdot\vec{k}_{\mathrm{t}1}} \dots \mathrm{d}^2 k_{\mathrm{t}n}\, \mathrm{e}^{-i\vec{b}\cdot\vec{k}_\mathrm{t}^n} & (115)\\
&= \mathrm{d}^2 q_\mathrm{t} \int \frac{\mathrm{d}|\vec{b}|^2}{4\pi} J_0\left(|\vec{b}||\vec{q}_\mathrm{t}|\right) \mathrm{d}^2 k_{\mathrm{t}1}\, \mathrm{e}^{-i\vec{b}\cdot\vec{k}_{\mathrm{t}1}} \dots \mathrm{d}^2 k_{\mathrm{t}n}\, \mathrm{e}^{-i\vec{b}\cdot\vec{k}_\mathrm{t}^n} , & (116)
\end{aligned}$$

where in the last step, we have performed the integration over the azimuthal angle of $\vec{q}_\mathrm{t}$ and J_0 is a Bessel function. Since transverse momentum is two-dimensional and longitudinal momentum one-dimensional, transverse momentum resummation is accordingly somewhat more cumbersome than threshold resummation.

A second complication is related to the fact that in transverse momentum resummation, the soft and collinear logs arise as logarithms of q_t^2 which, as we saw in Section 2.1, are present for all kinds of parton emission, hence all partonic subchannels contribute, and not only the diagonal channel as in the case of threshold resummation. With these preliminary considerations, we are now ready to present the expression for the resummed transverse momentum distribution in the ij partonic subchannel, with factorization scale set equal to Q^2

$$\frac{\mathrm{d}\hat{\sigma}_{ij}}{\mathrm{d}q_{\mathrm{t}}^2}\left(N, q_{\mathrm{t}}, \alpha_{\mathrm{s}}\left(Q^2\right), Q^2\right) = \sigma_0 \int\limits_0^\infty \mathrm{d}b\, \frac{b}{2}\, J_0\left(bq_{\mathrm{t}}\right) H_{ij}\left(N, \alpha_{\mathrm{s}}\left(Q^2\right)\right)$$
$$\times \exp\left[-\int\limits_{\frac{1}{b^2}}^{Q^2} \frac{\mathrm{d}q^2}{q^2}\left[A^{q_{\mathrm{t}}}\left(\alpha_{\mathrm{s}}\left(q^2\right)\right)\ln\frac{Q^2}{q^2} + B^{q_{\mathrm{t}}}\left(\alpha_{\mathrm{s}}\left(q^2\right), N\right)\right]\right]. \quad (117)$$

Note that the cross section still depends on Q^2 and the dimensionless ratio $x = Q^2/s$, and now also on q_{t}, and the resummation is performed in Fourier–Mellin space in order to factorize both the longitudinal and the transverse phase space, so the resummed result depends on N conjugate to x and b conjugate to q_{t}.

The similarity of Eq. (117) to the threshold resummation of Eq. (109) should be clear. The soft resummation scale is now $1/b^2$, and the exponent takes the form of an integral from the hard scale Q^2 to this soft scale, with a double-logarithmic contribution $A^{q_{\mathrm{t}}}$ and a single-logarithmic contribution $B^{q_{\mathrm{t}}}$. Note that $A^{q_{\mathrm{t}}}$ coincides with the cusp anomalous dimension only up to order α^3, but beyond this order (corresponding to NNLL resummation), it differs from it. The functions $H_{ij}\left(N, \alpha_{\mathrm{s}}(M^2)\right)$ do not depend on the soft resummation scale $1/b^2$, and are thus analogous to the prefactor $C^{(c)}\left(\alpha_{\mathrm{s}}(Q^2)\right)$ in the threshold resummation formula. They are universal functions which do not depend on the specific process and only depend on the parton flavors ij.

9. Conclusion

In these lectures, we have provided a simple introduction to basic, elementary concepts of soft resummation. The lectures barely scratch the surface of what has become, since the seminal papers of more than thirty years ago, a vast and active research subject. A subject to which, among other things, several workshops are devoted, and that has close connection to the Monte Carlo parton showers, that are fundamental for collider phenomenology, and that implement resummation numerically.

On top of the transverse momentum resummation that we have briefly touched upon in Section 8, other applications of the resummation formalism, actively pursued in recent years, are related to the resummation of observables with hadronic final states and thus depend on color, specifically those that characterize jets and their substructure [15, 16]. Also, as mentioned in the introduction, in the past two decades many resummation results have been re-derived, and some derived for the first time, using the formalism of soft-collinear effective theory, in which the fundamental QCD Lagrangian is replaced by an effective Lagrangian in which the fundamental fields are decomposed as the sum of fields with different scaling properties that describe hard, soft, and collinear excitations [4].

We hope that this brief introduction will stimulate the readers to deepen their knowledge of this fascinating, old but still very active research topic.

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Appendix A

Emission from internal lines

We provide a brief summary of the classic argument from Ref. [14] that proves that in QED, only radiation from external lines leads to infrared divergences.

We consider a generic amplitude in QED with an incoming fermion line carrying momentum p, an outgoing fermion line carrying momentum p', and a number of external photons with momenta q_i

$$M_0 = \bar{u}\left(p'\right) \Gamma(p,q) u(p)\,. \tag{A.1}$$

Some of the photons with momenta $q_1, \ldots, q_n$ may be virtual; in such case, an integral over the loop momentum is understood. We now consider the same amplitude, with one additional emitted photon with momentum k in the soft $k \to 0$ limit. The additional photon may be emitted either by the incoming fermion line, or by the outgoing fermion line, or by some internal line. Thus,

$$M_1 = M_1^\mu \epsilon_\mu^*(k)\,, \tag{A.2}$$

with

$$\begin{aligned} M_1^\mu &= \bar{u}\left(p'\right)\left[\gamma^\mu \frac{\not{p}' + \not{k} + m}{\left(p'+k\right)^2 - m^2}\Gamma(p,q)\right. \\ &\left. + \Gamma(p-k,q)\frac{\not{p} - \not{k} + m}{(p-k)^2 - m^2}\gamma^\mu + \Gamma^\mu(p,q,k)\right] u(p)\,. \end{aligned} \tag{A.3}$$

Some slight redefinition of the momenta q_i is needed in order to keep p and p' at the same values, while emitting one more photon, but this is irrelevant in the small-k limit. We now observe that the QED Ward identity

$$k_\mu M_1^\mu = 0 \tag{A.4}$$

gives

$$\Gamma(p-k,q) = \Gamma(p,q) + k_\mu \Gamma^\mu(p,q,k)\,. \tag{A.5}$$

Equation (A.5) can be used to eliminate $\Gamma(p-k,q)$ from Eq. (A.3). We obtain

$$\begin{aligned} M_1^\mu &= \bar{u}\left(p'\right)\left[\gamma^\mu \frac{\not{p}' + \not{k} + m}{\left(p'+k\right)^2 - m^2}\Gamma(p,q) + \Gamma(p,q)\frac{\not{p} - \not{k} + m}{(p-k)^2 - m^2}\gamma^\mu\right. \\ &\left. + \Gamma_\nu(p,q,k)\left(g^{\nu\mu} + k^\nu \frac{\not{p} - \not{k} + m}{(p-k)^2 - m^2}\gamma^\mu\right)\right] u(p)\,. \end{aligned} \tag{A.6}$$

We may now adopt the procedure that led to Eq. (17), to get

$$\begin{aligned} M_1^\mu &= \bar{u}\left(p'\right)\left[\Gamma(p,q)\left(\frac{p'^\mu}{p'\cdot k} - \frac{p^\mu}{p\cdot k}\right)\right. \\ &\left. + \Gamma_\nu(p,q,k)\left(g^{\nu\mu} - \frac{k^\nu p^\mu}{p\cdot k}\right)\right] u(p)\left[1 + O(k)\right]\,. \end{aligned} \tag{A.7}$$

The first term in Eq. (A.7) is the usual eikonal approximation for the emission of one additional photon in the soft limit. We now want to show that the second term is suppressed in comparison to the eikonal term when $k \to 0$. It is not *a priori* obvious that this is the case. Indeed, assume that the photon with momentum k is emitted by an internal fermion line carrying momentum $p + Q$, where Q is some partial sum of the q_i. We may write

$$\Gamma(p,q) = \Gamma_2(p,q)\frac{1}{\not{p} + \not{Q} - m}\Gamma_1(p,q) \tag{A.8}$$

and

$$\Gamma_\nu(p,q,k) = \Gamma_2(p-k,q)\frac{1}{\not{p}+\not{Q}-\not{k}-m}\gamma_\nu\frac{1}{\not{p}+\not{Q}-m}\Gamma_1(p,q)\,. \tag{A.9}$$

If both Q and k are small, Γ_ν contains two small denominators, compared to the single one in Eq. (A.8); as a consequence, the second term in Eq. (A.7) may be as singular as the first one.

We now show that this is not the case, thanks to a cancellation between the two terms in round brackets in the second term of Eq. (A.7). We have

$$\begin{aligned}
&\frac{1}{\not{p}+\not{Q}-\not{k}-m}\gamma_\nu\left(g^{\mu\nu}-\frac{p^\mu k^\nu}{p\cdot k}\right) = \frac{\not{p}+\not{Q}-\not{k}+m}{(p+Q-k)^2-m^2}\gamma_\nu\left(g^{\mu\nu}-\frac{p^\mu k^\nu}{p\cdot k}\right)\\
&= \frac{1}{(p+Q-k)^2-m^2}\left[-\gamma_\nu(\not{p}+\not{Q}-\not{k}-m)+2(p+Q-k)_\nu\right]\left(g^{\mu\nu}-\frac{p^\mu k^\nu}{p\cdot k}\right)\\
&= \frac{1}{(p+Q-k)^2-m^2}\\
&\times\left[-\gamma_\nu\left(g^{\mu\nu}-\frac{p^\mu k^\nu}{p\cdot k}\right)(\not{p}+\not{Q}-m)-\not{k}\gamma^\mu+2Q^\mu-2p^\mu\frac{Q\cdot k}{p\cdot k}\right].
\end{aligned} \tag{A.10}$$

The factor of $\not{p}+\not{Q}-m$ in the first term cancels one of the potentially small denominators in Eq. (A.9); the remaining terms are proportional to either k or Q, and therefore soften the overall singularity. This argument holds irrespective of the fermion mass m, and therefore in particular when $m=0$, which is the case of interest in QCD.

We conclude that the last term in Eq. (A.7) is at most as singular as M_0, and therefore indeed less singular than the eikonal term. A crucial role is played by the Ward identity: in the last step in Eq. (A.10), a cancellation between two factors of p^μ has taken place. Quite obviously, a direct extension of the same argument to QCD is not possible. Gluons are emitted both from fermion and gluon lines; furthermore, the structure of Slavnov–Taylor identities is far more complicated in the non-Abelian case.

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