PARTICLE SEISMOLOGY: MECHANICAL AND GRAVITATIONAL PROPERTIES FROM PARTON–HADRON DUALITY* ** ***

Enrique Ruiz Arriola

Departamento de Física Atómica, Molecular y Nuclear
Universidad de Granada, Av. de la Fuentenueva s/n, 18071 Granada, Spain
and
Instituto Carlos I de Física Teórica y Computacional
Campus Universitario Fuentenueva
Calle Dr. Severo Ochoa, 18071 Granada, Spain
earriola@ugr.es

Wojciech Broniowski

Institute of Nuclear Physics Polish Academy of Sciences
Radzikowskiego 152, 31-342 Kraków, Poland
Wojciech.Broniowski@ifj.edu.pl

Received 29 April 2026, accepted 28 May 2026, published online 16 June 2026

The internal structure of hadrons is characterized by form factors which correspond to matrix elements of currents. Among those, the stress-energy-momentum tensor is a universally conserved quantity providing the gravitational form factors, from which mechanical properties may be derived via the response to the space-time fluctuations. They have received much attention due to their role as moments of the Generalized Parton Distributions, where the stress-energy-momentum tensor couples to two photons, and more recently, due to the explicit lattice QCD determination for the pion and nucleon. In these lectures, we attempt a pedagogical review of the topic from a purely hadronic point of view, based on the notion of dispersion relations, meson dominance, and parton–hadron duality. We show that despite the overwhelming simplicity of the approach, a rather successful description of the lattice QCD data is achieved.

DOI:10.5506/APhysPolB.57.4-A6

* Based on lectures given by E. Ruiz-Arriola at the 65th Jubilee Cracow School of Theoretical Physics, Zakopane, Tatra Mountains, Poland, 14–21 June, 2025.

*** Supported by MICIU (Spain) under grant No. PID2023.147072NB.I00 and Junta de Andalucía FQM225.

1. Introduction

Hadrons are extended quantum objects which feel the strong interaction and have a variety of properties such as mass, spin, charge, radii, magnetic moments, *etc.*, which ultimately characterize them. The finite extension, typically about 0.5–1 fm, suggests that these properties actually correspond to integrated extended distributions which certainly are not homogeneous or isotropic. As a general principle, they become distinctly accessible by noting that hadrons behave differently under different external perturbations. These perturbations must be small enough such that the back reaction on the external perturbation can be neglected, but simultaneously large enough in order to provide a measurable cross section in a scattering process.

Since the discovery of the internal structure of the proton by Hofstadter [1, 2], the main source of experimental information on intrinsic properties of hadrons has been the electron scattering. This allows one to figure out the electric and magnetic distributions under the assumption of the one-photon exchange. Likewise, the neutrino and muon scattering makes it possible to determine the axial and pseudoscalar distributions under the one-Z^0 or $W^\pm$ boson exchange. On the opposite extreme, distributions associated with strong hadronic probes, often described by the pion exchange, are difficult to assess since they distort the probe strongly! Finally, the gravitational interaction, characterized by the one-graviton exchange mechanism, would provide the energy density, pressure, and stress distributions inside the hadron. However, the gravitational interaction, which couples to all objects, is so small that it does not produce any measurable response, hence a one-graviton exchange remains a gedanken process. Thus, a question arises: how can one determine the mechanical and gravitational properties of hadrons, such as the mass, momentum, or pressure densities, without ever explicitly using the gravitons?[1]

In these lectures, we concentrate on the mechanical properties of hadrons and the corresponding gravitational form factors (GFFs), focusing in particular on the pion and the nucleon as prominent examples. While the GFF concept is rather old, with many notable studies done in the past [7–14], it has until recently not been pursued quantitatively and realistically due to the lack of experimental data for the reasons mentioned above.

However, the field experienced a renaissance after the proposals of studying the deeply virtual Compton scattering (DVCS) in terms of the Generalized Parton Distributions (GPDs), from which GFFs arise as moments [15, 16] in the Bjorken-x variable. Then, the D-term was discovered [17] and a semiclassical interpretation was put forward [18, 19].

[1] We disregard here the old problem of quantizing gravity as a fundamental theory in a consistent manner [3, 4], noting that the effective field theory (EFT) approach is sufficient [5] (see [6] for a recent overview).

A second renaissance was triggered by the MIT group's [20, 21] *direct* lattice QCD computation of the relevant matrix elements in the space-like region $0 \leq -t \leq 2$ GeV^2 with an almost physical pion mass (170 MeV), where a benchmarking 5% precision for the nucleon and the pion GFFs has been reached. This greatly improved the seminal studies of the quark parts [22, 23], recently redone with $m_\pi = 250$ MeV [24], the gluonic parts [25], and the gluonic trace anomaly component [26] at larger values of m_π.

From the phenomenological side, a way of extracting GFFs of the pion from the $\gamma\gamma^* \to \pi^0\pi^0$ data [27] was proposed in [28], with further experimental prospects to emerge at Super-KEKB and ILC. For the nucleon case, constraints on GFFs have been obtained via DVCS from CLAS at JLab [29, 30], and from the GlueX [31] data for the J/ψ photoproduction [32]. An extraction of the proton mass radius based on photoproduction of the vector charmoniums was made in [33]. An estimate of the mass radius [34] was computed, and a determination of GFFs from the Compton form factors was made in [35]. Hadronic generalized distribution amplitudes were considered as a gateway to the time-like GFFs in [36–38]. A sophisticated fit to the data was carried out in [39]. In [40], access to GPDs from the Sullivan process was proposed.

Model calculations of the pion GFFs were carried out in numerous approaches, including chiral effective Lagrangians at small [41–43] and finite momenta [44–61].

For the nucleon, the large-N_c scaling was obtained in [62], while the leading chiral corrections were addressed in the heavy baryon [63–66] or covariant [67, 68] frameworks. Further model estimates were made in the Skyrmion [69, 70], the chiral quark soliton model [71], the MIT bag model [72], the holographic QCD model [73, 74], and in AdS/QCD [56, 75–79]. The light-front formulation with valence quarks was considered in [80], and a full QCD light-front modeling was presented in [81]. The QCD sum rules were applied in [82, 83]. A flavor decomposition within the light-cone sum rule approach was carried out in [84]. A decomposition of the nucleon GFFs in terms of quarks and gluons was proposed in [85]. A chiral soliton calculation incorporating dilaton fields was presented in [86]. The parity doubling model was applied in [87]. A classical model of the nucleon was investigated in [88]. A dilaton effective theory was explored in [89, 90]. Finally, a dispersive determination was accomplished in [91].

Importantly, the leading-order perturbative QCD (pQCD) asymptotic behavior of the pion and nucleon GFFs was obtained in [55, 92, 93].

Energy and momentum densities can be directly measured in a classical fluid, be it a gas or a liquid, by placing a thermometer and a barometer or manometer inside the system. This is what is involved in the measurement at any meteorological station used for weather forecasting. Clearly, this procedure is only possible for liquids and gases. In a solid, we cannot place any measuring device inside unless we dig a hole, but one can still study the response to external forces, namely stresses or heating. The situation in a femtoscopic system such as a hadron is even more difficult, since the only way of perturbing the hadron requires a space-time gravitational fluctuation with a shorter wavelength than the hadron size. A recent discussion of these issues at the hadronic level is presented in [94].

In standard Quantum Field Theory textbooks, SEM is routinely described in the introductory chapters as a conserved Noether current corresponding to the symmetry of the arbitrariness of space-time coordinates. Actually, this symmetry may be the only continuous one characterizing the dynamics of a given system, like, for instance, in the case of a neutral spin-0 particle. Due to its connection to gravity, the topic has been regarded as a purely academic subject. The mentioned appearance of the lattice QCD calculations provides a first principles determination of the mechanical properties.

Mathematically, the energy and momentum are identified as group generators of the time and space translations, whereas the angular momentum and relativistic invariance correspond to rotations and boosts. A more physical definition involves the inclusion of test particles, such that the total energy and momentum are conserved if we consider the object and the measuring device as an isolated system.

In these lectures, we review in a pedagogical way some basic facts of SEM in a variety of frameworks, from the classical point mechanics to the quantum field theory, with the purpose of demystifying the concept. The second part is more phenomenological and largely based on [95–98], where a good deal of details and further explanations can be found.

2. Particle seismology

We first focus on and review how hadron masses respond to a space-time deformation of the constant Lorentz metric $\eta^{\mu\nu}$,

$$\eta^{\mu\nu} \to \eta^{\mu\nu} + \Delta g^{\mu\nu}(x)\,. \tag{1}$$

Clearly, the scale of this deformation must be smaller than the hadron size. Then we can visualize it as a "micro-earthquake" inside the hadron, such that the mass changes locally[2]

$$M \to M + \int \mathrm{d}^4 x \Delta g^{\mu\nu}(x) \frac{\delta M}{\delta g^{\mu\nu}(x)} \,. \tag{2}$$

This provides the gravitational densities and stress inside a hadron

$$T_H^{\mu\nu}(x) = -2 \frac{\delta M}{\delta g^{\mu\nu}(x)} \Bigg|_{g^{\mu\nu}=\eta^{\mu\nu}} \equiv \langle H | \Theta^{\mu\nu}(x) | H \rangle \,. \tag{3}$$

The physical normalized hadron state is generally described as a wave packet

$$|H\rangle = \sum_s \int \mathrm{d}^4 p \, \Psi_s(p) \delta_+ \left(p^2 - M^2\right) |p, s\rangle \,, \tag{4}$$

where $\delta_+(p^2 - M^2) = \theta(p \cdot n)\delta(p^2 - M^2)$ is the on-shell spectral condition imposed on a given hypersurface with a normal vector n. Using the translational invariance $\Theta^{\mu\nu}(x) = \mathrm{e}^{iP\cdot x} \Theta^{\mu\nu}(0) \, \mathrm{e}^{-iP\cdot x}$, we get

$$\begin{aligned} T_H^{\mu\nu}(x) \; &= \int \mathrm{d}^4 p \, \mathrm{d}^4 p' \, \mathrm{e}^{ix\cdot(p-p')} \delta^+ \left(p'^2 - M^2\right) \delta^+ \left(p^2 - M^2\right) \\ &\times \sum_{s',s} \phi_s(p')^+ \langle p', s' | \Theta^{\mu\nu}(0) | p, s \rangle \phi_s(p) \,, \end{aligned} \tag{5}$$

where the matrix element can be decomposed into the Lorentz irreducible and symmetric structures

$$\langle p', s' | \Theta^{\mu\nu}(0) | p, s \rangle = \sum_i O_i^{\mu\nu} \left(p', s', p, s\right) G_i \left(q^2\right) \,, \tag{6}$$

with $O_i^{\mu\nu}(p', s', p, s) = O_i^{\nu\mu}(p', s', p, s)$. Besides, if the total system *including* also the metric as a dynamical degree of freedom does not depend on the particular choice of the space-time coordinates (the equivalence principle), then $q_\mu O_i^{\mu\nu}(p', s', p, s) = 0$. The Lorentz-invariant coefficients $G_i(q^2)$ depend only on the momentum transfer due to the on-shell conditions $p^2 = p'^2 = M_H^2$ and are termed the *gravitational form factors* (GFFs).

---

[2] Actually, the proper terminology should probably be "femto-hadron-quake". General relativity literature often uses the notation for weak fields $\Delta g^{\mu\nu}(x) = h^{\mu\nu}(x)$. We make the distintion between the full metric $g^{\mu\nu}(x)$ and the constant flat metric $\eta^{\mu\nu}$ only in this section. For the rest of the lectures, we will take $g^{\mu\nu}$ as the flat metric.

Among these form factors, there is one, D — the Druck term, which turns out to be an intrinsic hadronic property [17] associated with the conserved operator

$$O_D^{\mu\nu}\left(p',p\right) = q^\mu q^\nu - g^{\mu\nu} q^2 \implies D\left(q^2\right) , \tag{7}$$

and which corresponds to a change of the mass against a local variation of the curvature

$$D_H(x) = \frac{\delta M}{\delta R(x)} = \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\, \mathrm{e}^{iq\cdot x} D\left(q^2\right) \implies D(0) = \int \mathrm{d}^4 x \frac{\delta M}{\delta R(x)} , \tag{8}$$

where $R(x) = g^{\mu\nu}(x) R_{\mu\nu}(x)$ is the scalar curvature and $R_{\mu\nu}(x)$ the Ricci curvature tensor (see *e.g.* Ref. [42] for conventions) and $D(q^2)$ is the D-form factor. Note that close to the flat limit, Eq. (1), one has $\Delta R = (\partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^2)\Delta g_{\mu\nu}$. Like any form factor, this function is analytic in the complex q^2-plane except for a branch cut along the positive real axis, $s_0 < s < \infty$, where s_0 is a threshold (for the pion and nucleon $s_0 = 4m_\pi^2$) [3].

In QCD, $D(q^2)$ falls off faster[4] than $1/q^2$ [55, 92, 93], hence satisfies an unsubtracted dispersion relation

$$D(t) = \frac{1}{\pi}\int_{s_0}^{\infty} \mathrm{d}s \frac{\mathrm{Im}\, D(s)}{s-t} \implies D(0) = \frac{1}{\pi}\int_{s_0}^{\infty} \mathrm{d}s \frac{\mathrm{Im}\, D(s)}{s} . \tag{9}$$

The spectral function $\mathrm{Im}\, D(s)$ corresponds to a virtual gravitational hadron–antihadron production $g^* \to H\bar{H}$ in the scalar 0^{++} and tensor 2^{++} quantum number channels. As we will see later, in QCD, it is not positive definite due to the superconvergence sum rules. The value $D(0)$ is a gravitational property of the hadron, which is dynamical and cannot be deduced from a hadronic symmetry, similarly to the anomalous magnetic moments or the axial coupling constant of the nucleon. However, it is finite and unambiguous. To what extent the spectral function $\mathrm{Im}\, D(s)$ can be determined in practice from our knowledge of the meson spectrum in the scalar 0^{++} and tensor 2^{++} channels, will be discussed later on (Section 8).

3. Stress-energy-momentum tensor primer: particles

In order to grasp the meaning of SEM, we start from the classical particles, both non-relativistic and relativistic, and then proceed to field theory

[3] In the nucleon case, there is an additional subthreshold logarithmic singularity at $s_a = 4m_\pi^2 - m_\pi^2/M_N^2$, which stems from the $g \to \pi\pi \to N\bar{N}$ triangle process, and distorts greatly the threshold behavior of the form factor.

[4] For mesons — $D \sim 1/(q^2 \log q^2)$, whereas for baryons — $D \sim 1/(q^2 \log q^2)^2$.

and the interaction of classical particles with fields [99–103]. The bottomline is that *only* point-like interactions satisfy the SEM conservation locally with local densities and currents.

3.1. Classical particles

Classical particles are characterized as being point-like. Thus, the density or concentration of a particle located at point x_0 is just a simple Dirac delta function

$$n(x) = \delta(x - x_0)\,. \tag{10}$$

Correspondingly, for a particle with mass m and charge q, the corresponding mass and charge densities are given by

$$\rho_m(x) \equiv mn(x) = m\delta(x - x_0)\,, \qquad \rho_q(x) \equiv qn(x) = q\delta(x - x_0)\,. \tag{11}$$

For a particle under external (conservative) forces, we have Newton's law

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\nabla V(x)\,, \tag{12}$$

from where we explicitly find the energy conservation

$$E = \frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + V(x) \implies \frac{\mathrm{d}E}{\mathrm{d}t} = 0\,. \tag{13}$$

Thus, a concentration of a collection of moving particles fulfills

$$n(x,t) = \sum_i \delta(x - x_i(t)) \implies \partial_t n = -\sum \frac{\mathrm{d}x_i}{\mathrm{d}t}\nabla\delta(x - x_i(t))\,, \tag{14}$$

which in terms of the current or flux of particles implies the continuity equation

$$\vec{j}(x,t) \equiv \sum_i \frac{\mathrm{d}x_i}{\mathrm{d}t}\delta(x - x_i(t)) \implies \partial_t n + \nabla\cdot\vec{j} = 0\,. \tag{15}$$

From here, one defines the momentum density

$$\vec{\mathcal{P}}(x,t) \equiv m\vec{j}(x,t) = \sum_i m v_i \delta(x - x_i(t))\,, \tag{16}$$

such that

$$\begin{aligned} \partial_t\vec{\mathcal{P}}(x,t) &= \sum_i m\frac{\mathrm{d}^2\vec{x}_i}{\mathrm{d}t^2}\delta(x - x_i(t)) - \sum_i \frac{\mathrm{d}\vec{x}_i}{\mathrm{d}t}\frac{\mathrm{d}\vec{x}_i}{\mathrm{d}t}\cdot\nabla\delta(x - x_i(t)) \\ &= -\vec{\nabla}V(x)n(x,t) - m\vec{\nabla}\overleftrightarrow{T}(x,t)\,, \end{aligned} \tag{17}$$

where we have introduced the stress tensor

$$T_{ab}(x,t) = \sum_i \frac{\mathrm{d}x_i^a}{\mathrm{d}t}\frac{\mathrm{d}x_i^b}{\mathrm{d}t}\delta(x - x_i(t)) \tag{18}$$

and the dyadic product notation $\vec{A}\overleftrightarrow{T}$. Finally, the energy density is naturally defined as

$$\mathcal{H}(x,t) = \sum_i \frac{1}{2}m\left(\frac{\mathrm{d}\vec{x}_i}{\mathrm{d}t}\right)^2 \delta(x - x_i(t)) + V(x)n(x,t)\,, \tag{19}$$

which fulfills

$$\partial_t \mathcal{H}(x,t) + \nabla \cdot J_E(x,t) = -V(x)\nabla j(x,t) \tag{20}$$

in terms of the energy flux

$$\vec{J}_E = \sum_i \sum_i \frac{1}{2}m\left(\frac{\mathrm{d}\vec{x}_i}{\mathrm{d}t}\right)^2 \frac{\mathrm{d}\vec{x}_i}{\mathrm{d}t}\delta(x - x_i(t))\,. \tag{21}$$

3.2. Phase-space point of view

While we are mainly interested in local quantities, formulas get simpler with the phase-space Hamiltonian dynamics, where

$$H(p,x) = E(p) + V(x) \implies \begin{cases} \dot{\vec{x}} = \nabla_p H = \nabla_p E \equiv \vec{v}\,, \\ \dot{\vec{p}} = -\nabla_x H = -\nabla V(x)\,. \end{cases} \tag{22}$$

The phase-space density reads

$$W(x,p,t) = \sum_i \delta(x - x_i(t))\delta(p - p_i(t))\,, \tag{23}$$

which fulfills Liouville's equation

$$\partial_t W + \partial_p H \partial_x W - \nabla_x H \partial_p W = 0 \tag{24}$$

and the Poisson bracket formula

$$\{A,B\} \equiv \partial_x A \partial_p B - \partial_p A \partial_x B \implies \partial_t W + \{H,W\} = 0\,. \tag{25}$$

The local quantities are obtained by

$$A(x,t) = \int \mathrm{d}p A(x,p) W(x,p,t)\,. \tag{26}$$

The correspondence is summarized in Table 1.

Table 1. Phase-space functions and the corresponding local quantities.

$A(x,p)$	1	p	$H(x,p)$	$p_i p_i$	$p_i H$
$O(x,t)$	$n(x,t)$	$\mathcal{P}(x,t)$	$\mathcal{H}(x,t)$	$T_{ij}(x,t)$	$J_E(x,t)$

3.3. Relativistic particles

The previous results make the transition to the relativistic Hamiltonian dynamics straightforward

$$H(p,x) = \sqrt{p^2+m^2} + V(x) \implies \begin{cases} \dot{\vec{x}} = \nabla_p H = \frac{\vec{p}}{\sqrt{p^2+m^2}} \equiv \vec{v}\,, \\ \dot{\vec{p}} = -\nabla_x H = -\nabla V(x)\,. \end{cases} \tag{27}$$

The energy and momentum densities become

$$\mathcal{H}(x,t) = \sum_i \sqrt{p_i^2+m^2}\,\delta(x-x_i) + n(x,t)V(x)\,, \tag{28}$$

$$\mathcal{P}(x,t) = \sum_i p_i \delta(x-x_i)\,, \tag{29}$$

with the continuity equation

$$\partial_t \mathcal{H}(x,t) + \nabla\cdot\vec{\mathcal{P}}(x,t) = V(x)\partial_t n(x,t) = -V\nabla\cdot J\,. \tag{30}$$

Relativistically, the momentum density and the energy flux coincide, since $\vec{v} = \partial_p E$ and thus $\vec{p} = \vec{v}E$ [5]. The stress tensor is now defined as

$$T^{ab}(x,t) = \sum_i \frac{p_i^a p_i^b}{\sqrt{p_i^2+m^2}}\delta(x-x_i(t))$$
$$\implies \partial_t \vec{\mathcal{P}}(x,t) + \nabla T(x,t) = -\nabla V(x) n(x,t)\,. \tag{31}$$

We can combine the previous definitions into a four dimensional SEM

$$T^{\mu\nu} = \begin{pmatrix} \mathcal{H} & \vec{\mathcal{P}} \\ \vec{\mathcal{P}} & T^{ab} \end{pmatrix} \implies \partial_\mu T^{\mu\nu} = f^\nu\,, \tag{32}$$

which is symmetric

$$T^{\mu\nu} = T^{\nu\mu}\,. \tag{33}$$

[5] This is not the case non-relativistically if we ignore the rest mass, *i.e.* we only consider the *kinetic* energy flux $\vec{v}p^2/2m$.

Note that the trace of SEM is given by

$$T^\mu_\mu(x) = \sum_i \frac{m^2}{\sqrt{p_i^2+m^2}}\delta(x-x_i(t)) = \epsilon - 3p \geq 0\,, \tag{34}$$

where ϵ is the energy density and p is the pressure, and is manifestly positive for massive particles and zero for massless particles[6].

3.4. Particle interactions: non-locality and the no-go theorem

The above discussion concerned non-interacting particles in external potentials, hence the next natural step would be to include interactions. This is, however, not so straightforward. In the non-relativistic case for a finite-range two-particle interaction characterized by a potential $v_{12} = V(|\vec{x}_1 - \vec{x}_2|)$, the local conservation law for SEM holds for the point-like particles, but the corresponding currents are non-local *unless* the interaction has zero range [103]. Besides, it turns out that it is impossible to construct a Hamiltonian or Lagrangian description of a system of interacting particles that is both relativistically invariant and contains non-trivial interactions [104]. Thus, the only logical way out is to consider the external fields as dynamical [100].

4. Energy momentum tensor primer: fields and test particles

4.1. Electrodynamics and the field energy

The best (and first) known example of dynamical fields coupled to particles is provided by classical electrodynamics and the discovery by Poynting in 1884 [105], where he first realized that fields carry the energy and momentum which can be exchanged with particles. Moreover, he also noticed that both the momentum density and the energy flux coincide (the Poynting vector), in harmony with the relativistic invariance. The main idea is as follows: the particle dynamics is governed by the Lorentz force

$$\dot{\vec{p}_i} = q_i\left[\vec{E} + \vec{v}_i \wedge \vec{B}\right]\,, \qquad \vec{p}_i = \frac{m_i\vec{v}_i}{\sqrt{1-v_i^2}}\,. \tag{35}$$

The electric charge density and currents are now given by

$$\rho_q(x,t) \;=\; \sum_i q_i \delta(x - x_i(t))\,, \tag{36}$$

$$\vec{J}_q(x,t) \;=\; \sum_i q_i v_i \delta(x - x_i(t))\,, \tag{37}$$

[6] Interestingly, this positivity condition does not in general hold in QCD. In particular, the pion violates it at sufficiently small distances (see Ref. [97]).

and satisfy the charge continuity equation

$$\partial_t \rho_q(x,t) + \nabla \cdot \vec{J}_q(x,t) = 0\,. \tag{38}$$

The dynamical electric and magnetic fields obey Maxwell's equations (we use natural units)

$$\nabla \wedge E = -\partial_t B\,, \qquad \nabla \wedge B = J_q + \partial_t E\,, \qquad \nabla \cdot E = \rho_q\,, \qquad \nabla \cdot B = 0\,. \tag{39}$$

From Maxwell's equations we have, after some straightforward manipulations, the local conservation laws

$$\partial_t \mathcal{H} + \nabla \cdot \mathcal{P} \;=\; 0\,, \tag{40}$$

$$\partial_t \mathcal{P} + \nabla \mathcal{T} \;=\; 0\,, \tag{41}$$

where

$$\begin{aligned}
\mathcal{H} &= \frac{1}{2}\left[E^2 + B^2\right] + \sum_i \sqrt{p_i^2 + m_i^2}\,\delta(x - x_i)\,, \\
\vec{\mathcal{P}} &= E \wedge B + \sum_i \vec{p}_i \delta(x - x_i)\,, \\
\mathcal{T}_{ab} &= E_a E_b - \frac{1}{2}\delta_{ab} E^2 + B_a B_b - \frac{1}{2}\delta_{ab} B^2 + \sum_i \frac{p_a p_b}{\sqrt{p^2 + m^2}}\delta(x - x_i)\,.
\end{aligned} \tag{42}$$

Thus, when we place matter, we obtain the total energy and momentum conservation

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3x\, \mathcal{H}_{\text{field}} + \sum_i v_i F_i \;&=\; 0 \implies \mathcal{H} \quad \text{energy density}\,, \\
\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3x\, \mathcal{P}_{\text{field}} + \sum_i p_i \;&=\; 0 \implies \mathcal{P} \quad \text{momentum density}\,.
\end{aligned}$$

This allows one to identify $\mathcal{H}_{\text{field}}$ and $\mathcal{P}_{\text{field}}$ physically *in the vacuum* as the energy and momentum densities *without any ambiguity*. The local character of the SEM conservation law makes it possible to compute scattering of EM waves (such as the Thomson scattering). Besides, the fact that we use test particles, which due to relativity provides a symmetric SEM, enforces the same symmetry feature on the field piece. It is remarkable that even though particles interact through fields, both the total energies and momenta are *additive*. This is, however, a very special feature of electrodynamics. Moreover, we can identify the field energy and momentum by placing *test particles* and checking for the energy and momentum conservation[7].

[7] This does not prevent us from eventually running into contradictions when Maxwell's and Lorentz's equations are solved self-consistently; as a general rule, they can only be used to first-order perturbation theory (see, *e.g.*, Ref. [106]).

4.2. Schrödinger field

The identification of energy and momentum in quantum mechanics usually comes from the correspondence principle, *i.e.*, the fact that for $\hbar \to 0$ quantum mechanics should become classical. Here, we show that classical test particles can be added to a Schrödinger field in such a way that the identification arises from the total energy and momentum conservation[8].

The time-dependent Schrödinger equation in a time-independent potential $V(x)$ has two constants of motion: the probability and the energy,

$$i\partial_t \psi = -\frac{1}{2m}\nabla^2\psi + V\psi \implies \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3x |\psi|^2 = 0\,, \\ \frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3x \left[\frac{1}{2m}|\nabla\psi|^2 + V(x)|\psi|^2\right] = 0\,, \end{cases} \tag{43}$$

as can be explicitly checked. The differential form of these conservation laws can be written in terms of the probability density and the probability flux

$$\begin{cases} n(x,t) = |\psi(x,t)|^2\,, \\ \vec{J}(x,t) = \frac{1}{2mi}\left[\psi^*\nabla\psi - \nabla\psi^*\psi\right]\,, \end{cases} \implies \partial_t n + \nabla\cdot\vec{J} = 0\,, \tag{44}$$

and the energy density and the energy flux

$$\begin{cases} \mathcal{H}(x,t) = \frac{1}{2m}|\nabla\psi|^2 + V(x)|\psi|^2 + \frac{1}{8m}\nabla^2|\psi|^2\,, \\ J_E(x,t) = \frac{1}{2mi}\left[\nabla\psi^*\nabla^2\psi - \nabla^2\psi^*\nabla\psi\right]\,. \end{cases} \tag{45}$$

Considering a collection of classical particles yields also the preservation of the momentum

$$\begin{cases} i\partial_t\psi = -\frac{1}{2m}\nabla^2\psi + \sum_i V(x-x_i)\psi\,, \\ \dot{\vec{p}_i} = -\nabla\int \mathrm{d}^3x\, V(x-x_i(t))|\psi(x,t)]^2\,, \end{cases} \tag{46}$$

$$\implies \begin{cases} P = \sum_i p_i + \int \mathrm{d}^3x\, m\vec{J}(x,t)\,, \\ E = \sum_i \frac{p_i^2}{2M} + \int \mathrm{d}^3x\left[\frac{1}{2m}|\nabla\psi|^2 + \sum_i V(x-x_i)|\psi|^2\right]\,. \end{cases} \tag{47}$$

Note, however, that unlike the EM case, the total conserved energy is non-additive. Finally, we note that a *local* SEM conservation is not possible unless the interaction $V(x-x_i)$ has exactly zero range, in harmony with the classical result (see Section 3.4).

[8] The interaction in quantum and classical systems has been an object of repeated studies in the past, where impediments to a canonical structure have been spelled out [107]. Our setup corresponds to the interaction of a classical field with a classical particle.

4.3. Neutral Klein–Gordon field

The last example presents an interesting case of a scalar neutral field, where the probability is certainly not conserved, but the energy and momentum are. The free Klein–Gordon (KG) equation reads

$$\left(\partial_t^2 - \nabla^2 + m^2\right)\phi = 0\,. \tag{48}$$

Its solutions fulfill the continuity equations

$$\begin{aligned} \partial_t \mathcal{H}_\phi + \nabla \vec{\mathcal{P}}_\phi &= 0\,, \\ \partial_t \mathcal{P}_\phi + \nabla \overleftrightarrow{\mathcal{T}}_\phi &= 0\,, \end{aligned} \tag{49}$$

where

$$\begin{aligned} \mathcal{H}_\phi &= \frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\,, \\ \vec{\mathcal{P}}_\phi &= -\partial_t\phi\nabla\phi\,, \\ \mathcal{T}_{ik} &= \nabla_i\phi\nabla_k\phi + \frac{1}{2}\delta_{ik}\left[(\partial_t\phi)^2 - (\nabla\phi)^2 + m^2\phi^2\right]\,. \end{aligned} \tag{50}$$

In order to properly identify $\mathcal{H}$ and $\mathcal{P}$ as the energy and momentum densities of the scalar field, we include test particles that can exchange energy and momentum with the field ϕ. This can be done by implementing particles as a source term in the KG equation of the form[9]

$$\left(\partial_t^2 - \nabla^2 + m^2\right)\phi = g\sum_i \sqrt{1 - v_i^2}\,\delta(x - x_i)\,, \tag{51}$$

and a single particle Hamiltonian function of the form[10]

$$H(p,x) = \sqrt{p^2 + (M + g\phi(x))^2} \implies \begin{cases} \dot{\vec{x}} = \nabla_p H = \nabla_p E \equiv \vec{v}\,, \\ \dot{\vec{p}} = -\nabla_x H\,. \end{cases} \tag{52}$$

Taking $E_i \equiv H(x_i, p_i)$, we obtain

$$\mathcal{H} = \mathcal{H}_\phi + \sum_i E_i \delta(x - x_i)\,, \tag{53}$$

$$\mathcal{P} = \mathcal{P}_\phi + \sum_i \vec{p}_i \delta(x - x_i)\,, \tag{54}$$

$$\mathcal{T} = \mathcal{T}_\phi + \sum_i \frac{\overleftrightarrow{p_i p_i}}{E_i}\delta(x - x_i)\,, \tag{55}$$

---

[9] For a Lagrangian formulation, see, *e.g.*, [108].

[10] These are analogous to the Lorentz force; here, we have the substitution rule $M_i \to M_i + g\phi(x_i)$.

which fulfill the continuity equation. Note that due to the local mass shift $M_i \to M_i + g\phi(x_i)$ of the test particles, the effect is *not* additive. We get the total energy and momentum conservation

$$E_{\rm tot} = \int \mathrm{d}^3x\, \mathcal{H}_\phi + \sum_i E_i \implies \frac{\mathrm{d}E_{\rm tot}}{\mathrm{d}t} = 0\,, \tag{56}$$

$$\vec{P}_{\rm tot} = \int \mathrm{d}^3x\, \mathcal{P}_\phi + \sum_i \vec{p}_i \implies \frac{\mathrm{d}\vec{P}_{\rm tot}}{\mathrm{d}t} = 0\,, \tag{57}$$

which entitles us to interpret $\mathcal{H}_\phi$ and $\mathcal{P}_\phi$ as the field energy and momentum densities, whereas $\mathcal{T}_\phi$ is the field stress *uniquely*. This mechanical balance offers a possible way of how relativistic point-like classical particles interact via a scalar field.

5. Unitarity and energy versus probability conservation

Unitarity in a scattering process is traditionally and popularly linked to the probability conservation. Among the many interesting properties of SEM, in this section, we show that unitarity also follows from the energy conservation. In order to stress this feature, we consider the simplest case of the elastic scattering of a wave on a static heavy-particle target in the cases of the Schrödinger and neutral scalar fields discussed previously[11].

5.1. Scattering of matter waves

The simplest relevant case appears in almost any textbook on quantum mechanics and is based on the probability conservation. We discuss it here for completeness in a particularly suitable fashion for our purposes. Our starting point is to take a time-dependent wave-packet

$$\psi(x,t) = \int \frac{\mathrm{d}E}{\sqrt{2\pi}} \psi_E(x)\, \mathrm{e}^{-iEt} \implies \left(-\nabla^2 + U\right) \psi_E = E\psi_E\,. \tag{58}$$

The stationary scattering solutions become asymptotically

$$\psi_E(x) \to Z_E \left[\mathrm{e}^{ik\cdot x} + \frac{\mathrm{e}^{ikr}}{r} f\right] \equiv \psi_E^{\rm in}(x) + \psi_E^{\rm out}(x)\,, \qquad r \to \infty\,, \tag{59}$$

where Z_E is a suitable normalization factor and $f_E(\hat{k}', \hat{k})$ is the scattering amplitude for the transition, implied by the change of direction between

[11] The EM proceeds along similar lines but becomes a bit messier due to the vector character of $\vec{E}$ and $\vec{B}$.

the initial velocity and the observation direction $\hat{k} \to \hat{k}' \equiv \hat{x}$. Probability conservation implies

$$\begin{aligned}
\Delta N &\equiv N(\infty) - N(-\infty) = \int_{-\infty}^{\infty} \mathrm{d}t \frac{\mathrm{d}N}{\mathrm{d}t} = \int_{-\infty}^{\infty} \mathrm{d}t \, \mathrm{d}^3x \, \partial_t \rho \\
&= -\int_{-\infty}^{\infty} \mathrm{d}t \, \mathrm{d}^3x \, \vec{\nabla} \cdot \vec{\mathcal{J}} = -\oint \mathrm{d}\vec{S} \cdot \int_{-\infty}^{\infty} \mathrm{d}t \, \vec{\mathcal{J}} = 0 \,, \qquad (60)
\end{aligned}$$

where the divergence integral theorem has been used. Now, using the Plancherel formula for the Fourier transformation and $\mathrm{d}\vec{S} \cdot \hat{r} = r^2 \mathrm{d}\Omega$, we get

$$\begin{aligned}
\Delta N &= \frac{i}{2m} \int \mathrm{d}E \oint \mathrm{d}\vec{S} \cdot (\psi_E^* \nabla \psi_E - \nabla \psi_E^* \psi_E) \\
&= \frac{i}{2m} \int \mathrm{d}E \lim_{r \to \infty} r^2 \int \mathrm{d}\Omega \, (\psi_E^* \partial_r \psi_E - \partial_r \psi_E^* \psi_E) \,, \qquad (61)
\end{aligned}$$

where only the asymptotic wave function enters. With the asymptotic expression (59), the $r \to \infty$ limit selects the forward amplitude, and with averaging over the outgoing directions, we obtain, after some manipulations,

$$0 = \Delta N = \int \mathrm{d}E |Z_E|^2 \left[-\frac{4\pi}{k} \operatorname{Im} f_E \left(\hat{k}, \hat{k}\right) + \int \mathrm{d}\Omega \left| f_E \left(\hat{k}, \hat{x}\right) \right|^2 \right] = 0 \,. \qquad (62)$$

On the other hand,

$$\begin{aligned}
\Delta N_{\mathrm{in}} &= \int \mathrm{d}E |Z_E|^2 \int \mathrm{d}S \frac{k}{m} \,, \\
\Delta N_{\mathrm{out}} &= \frac{1}{m} \int \mathrm{d}E |Z_E|^2 \int \mathrm{d}\Omega \left| f_E \left(\hat{k}, \hat{x}\right) \right|^2 \,. \qquad (63)
\end{aligned}$$

Taking the cross section as a probability transfer, we arrive at

$$\frac{\mathrm{d}\sigma_N}{\mathrm{d}\Omega} = \frac{\Delta N_{\mathrm{out}}/\Delta\Omega}{\Delta N_{\mathrm{in}}/\Delta S} \implies \langle \sigma_N \rangle = \left\langle \frac{4\pi}{k} \operatorname{Im} f \left(\hat{k}, \hat{k}\right) \right\rangle \,, \qquad (64)$$

where the average refers to the wave packet energy decomposition. This is the standard well-known optical theorem result for counting quantum particles hitting a detector.

An analogous result from the energy conservation is

$$\Delta E \equiv E(\infty) - E(-\infty) = \int_{-\infty}^{\infty} \mathrm{d}t \frac{\mathrm{d}E}{\mathrm{d}t} = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{d}^3x\, \partial_t \mathcal{H}$$
$$= - \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{d}^3x\, \vec{\nabla} \cdot \vec{\mathcal{P}} = - \oint \mathrm{d}\vec{S} \cdot \int_{-\infty}^{\infty} \mathrm{d}t\, \vec{\mathcal{P}} = 0\,, \tag{65}$$

therefore, the final outcome looks very similar to Eq. (62), with the modification that we have an additional energy factor E from the energy flux expression

$$0 = \Delta E = \int \mathrm{d}E E |Z|^2 \left[-\frac{4\pi}{k} \mathrm{Im}\, f\left(\hat{k}, \hat{k}\right) + \int \mathrm{d}\Omega \left| f\left(\hat{k}, \hat{x}\right)\right|^2 \right] = 0\,. \tag{66}$$

The relevant cross section appears now via the energy transfer (not the probability transfer)

$$\frac{\mathrm{d}\sigma_E}{\mathrm{d}\Omega} = \frac{\Delta E_{\mathrm{out}}/\Delta\Omega}{\Delta E_{\mathrm{in}}/\Delta S} \implies \langle \sigma_E \rangle = \left\langle \frac{4\pi}{k} \mathrm{Im}\, f_E\left(\hat{k}, \hat{k}\right) E \right\rangle = \langle \sigma_N E \rangle\,. \tag{67}$$

For monochromatic wave packets $|Z_E^2| = A\delta(E - E_0)$, hence $\Delta E_{\mathrm{out}} \sim E_0 \Delta N_{\mathrm{out}}$ and $\Delta E_{\mathrm{in}} \sim E_0 \Delta N_{\mathrm{in}}$, such that

$$\frac{\mathrm{d}\sigma_E}{\mathrm{d}\Omega} = \frac{\Delta E_{\mathrm{out}}/\Delta\Omega}{\Delta E_{\mathrm{in}}/\Delta S} = \frac{\int \mathrm{d}E E |Z_E|^2 \left| f\left(\hat{k}, \hat{x}\right)\right|^2}{\int \mathrm{d}E E |Z_E|^2} = \frac{\Delta N_{\mathrm{out}}/\Delta\Omega}{\Delta N_{\mathrm{in}}/\Delta S} = \frac{\mathrm{d}\sigma_N}{\mathrm{d}\Omega}\,. \tag{68}$$

As we can see, both cross sections coincide *only* for a monochromatic pulse. However, for a given broad energy spectrum, the question arises as to what the physical way of counting an event is. For instance, a calorimeter detector is just a way of absorbing the energy, which in a simplified picture may be viewed as a simple recoiling classical test system[12].

5.2. Neutral and charged Klein–Gordon particle scattering

As already mentioned, the KG equation does not possess the probability conservation. However, it does incorporate the energy conservation and, additionally, the charge conservation (for the charged particle case). From

[12] Actually, most detectors are based on the electric charge transfer.

the former and in the presence of an external field U, we can define an energy norm (for $U + m^2 > 0$)

$$||\phi||_E^2 = \int \mathrm{d}^3x\, \mathcal{H} = \frac{1}{2} \int \mathrm{d}^3x \left[(\partial_t \phi)^2 + (\nabla\phi)^2 + \left(m^2 + U\right) \phi^2 \right] \geq 0\,, \quad (69)$$

and, correspondingly, a conserved energy scalar product

$$\begin{aligned} \langle \phi, \varphi \rangle_E &= \frac{1}{2} \int \mathrm{d}^3x \left[\partial_t\phi \partial_t\varphi + \nabla\phi\nabla\varphi + \left(m^2 + U\right) \phi\varphi \right] \\ &\Longrightarrow \frac{\mathrm{d}}{\mathrm{d}t} \langle \phi, \varphi \rangle_E = 0\,. \end{aligned} \quad (70)$$

The steps to arrive at the energy-weighted theorems are similar to those in the previous section, with

$$\Delta E = -\int_{-\infty}^{\infty} \mathrm{d}t \oint \mathrm{d}\vec{S} \partial_t \phi \nabla \phi\,. \quad (71)$$

For a wave packet with the scattering boundary conditions, we get

$$\begin{aligned} \Delta E &= -\int \mathrm{d}E \int \mathrm{d}\vec{S} iE\phi_E(x) \nabla\phi_E(x)^* |Z_E|^2 \\ &\to r^2 \int \mathrm{d}E\, iE \int \mathrm{d}\Omega \phi_E(x) \partial_r \phi_E(x)^*\,, \end{aligned} \quad (72)$$

such that the (weighted) optical theorem follows:

$$0 = \Delta E = \int \mathrm{d}E\, E |Z|^2 \left[-\frac{4\pi}{k} \mathrm{Im}\, f\left(\hat{k}, \hat{k}\right) + \int \mathrm{d}\Omega \left| f\left(\hat{k}, \hat{x}\right) \right|^2 \right] = 0\,. \quad (73)$$

Thus, the cross section as the energy transfer (not the probability transfer) reads

$$\frac{\mathrm{d}\sigma_E}{\mathrm{d}\Omega} = \frac{\Delta E_{\mathrm{out}}/\Delta\Omega}{\Delta E_{\mathrm{in}}/\Delta S} \Longrightarrow \langle \sigma_E \rangle = \left\langle \frac{4\pi}{k} \mathrm{Im}\, f\left(\hat{k}, \hat{k}\right) E \right\rangle\,. \quad (74)$$

This particular case shows that the optical theorem for a neutral scalar particle has to do with the energy and *not* the probability conservation.

5.3. SEM-based unitarity

The above discussion shows that, in general, the SEM conservation underlies unitarity in a purely quantum-mechanical framework. So, rather than being an exotic object, SEM is an ubiquitous and central quantity. In relativistic field theory, probability is not a conserved quantity since there is no related Noether current. In QCD, for example, one has instead color, quark number, and the SEM conservation. As already mentioned, the only common conserved quantity in *any* field theory is SEM. A formulation embodying these issues seems to be missing, however, the issue only becomes relevant for non-monochromatic beams.

Thus, any conservation law provides a different interpretation of unitarity and hence of cross sections. The distinction of different cross sections in classical transport theory is well known, where the conventional probability cross section plays no role. In the Fokker–Planck approximation of the linear Boltzmann equation, for instance, only the momentum and energy transport coefficients are physically relevant and probability is *not* transported [109].

6. Field theory and local test fields

6.1. Definitions

In field theory, the canonical SEM, $\Theta_{\mu\nu}$, amounts to the conserved Noether current corresponding to the symmetry under the space-time translations [110]. In the simplest case of a scalar field, we have for a general transformation

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x) \implies \phi'(x') = \phi(x) \implies \delta\phi(x) = \epsilon^\mu \partial_\mu \phi \,. \tag{75}$$

The invariance of the Lagrangian yields

$$\begin{aligned} \delta\mathcal{L}(x) &= \epsilon^\mu \partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial \partial^\mu \phi}\delta\partial^\mu \phi \\ &= \partial^\nu \left[\frac{\partial \mathcal{L}}{\partial\partial^\nu \phi}\right]\delta\phi + \frac{\partial \mathcal{L}}{\partial\partial^\mu\phi}\delta\partial^\mu\phi \\ &\implies \epsilon^\nu \partial^\mu \Theta_{\mu\nu} = 0 \,, \end{aligned} \tag{76}$$

therefore, in the scalar theory, the *canonical* SEM reads

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi)^2 - U(\phi) \implies \Theta^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\mathcal{L} \,, \tag{77}$$

which turns out to be symmetric $\Theta^{\nu\mu} = \Theta^{\mu\nu}$.

The canonical or Noether SEM is *not* always symmetric, as for instance when dealing with the Dirac or vector fields [111, 112][13]. Moreover, as

[13] It is possible to redefine SEM in such a way that it becomes symmetric. We refer the reader to old and modern (*cf.* [113] and references therein) literature for a thorough discussion and interpretation of non-symmetric SEM tensors.

usual with the Noether construction, it is not unique since one may add a conserved total-derivative term,

$$\bar{\Theta}^{\mu\nu} = \Theta^{\mu\nu} + \alpha \left[\partial^\mu\partial^\nu - g^{\mu\nu}\partial^2\right]\phi^2 \,, \tag{78}$$

where the parameter α is completely arbitrary.

Given these ambiguities, the question on how one can measure $\Theta^{\mu\nu}$ or, equivalently, how to interpret it, becomes very pertinent, since we would naively expect a physical measurement to be well defined. One simple and natural way is to use the test particle concept at a given space-time location x, similarly to the case of electrodynamics discussed in Section 4.1. Another natural way proposed by Hilbert is via coupling to gravity in a curved space time, in which case the flat and constant metric is distorted, $\eta^{\mu\nu} \to g^{\mu\nu}(x) = \eta^{\mu\nu} + \delta g^{\mu\nu}(x)$, and

$$\begin{aligned}
\Theta^{\mu\nu} &= \frac{-2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}\bigg|_{g^{\mu\nu}=\eta^{\mu\nu}} \,, \\
\eta^{\mu\nu} &= \mathrm{diag}(1,-1,-1,-1) \implies \Theta^{\mu\nu} = \Theta^{\nu\mu} \,.
\end{aligned} \tag{79}$$

In both cases, only the symmetric components of SEM become observable. We stick to this point of view in our presentation. For the Dirac fermions, the Hilbert construction involves tetrads, as discussed in Section 6.7.

Coupling to gravity complies with invariance under general transformations, $x^\mu \to x'^\mu$, therefore, one has to consider an action $S \to \int \mathrm{d}^4x\sqrt{-g}\mathcal{L}$, where the Lorentz-invariant derivatives are replaced by the world derivatives, $\partial^\mu\phi\partial_\mu\phi \to g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, in a minimal way. The ambiguous term in $\theta^{\mu\nu}$ discussed above can be generated in a curved space-time by a non-minimal Lagrangian $\mathcal{L} = \alpha R\phi^2$, which leaves no trace in the flat-space limit[14]. Notably, it induces a change of the Druck term

$$D(0) \to D(0) + \alpha \,. \tag{80}$$

As we will see below, this ambiguity can be fixed by analyzing the production process $g \to \phi\phi$ at high energies and ultimately has to do with taking the field ϕ as fundamental or as a composite field[15].

---

[14] This is similar to the non-minimal gauge-invariant coupling in QED, yielding an anomalous magnetic moment of hadrons.

[15] This corresponds to implementing the SEM improvement of Callan, Coleman, and Jackiw [114], see the discussion in Section 6.6. For hadrons in QCD this is not a fundamental problem (see also the recent claim on the UV-divergent character of $D(0)$ for the Higgs boson [115] or the arbitrariness in soliton models [113]).

6.2. Lorentz properties

Under the Lorentz group, one has the transformation law

$$x^\mu \to \Lambda^\mu_\alpha x^\alpha \implies \Theta^{\mu\nu}(x) \to \Theta'^{\mu\nu}(x') = \Lambda^\mu_\alpha \Lambda^\nu_\beta \Theta^{\alpha\beta}(x)\,, \tag{81}$$

which is a reducible representation under the trace operation; the trace is a scalar

$$\Theta(x) \equiv \Theta^\mu_\mu(x) \to \Theta'(x') = \Theta(x)\,. \tag{82}$$

On the other hand, the (Hilbert) SEM is conserved and symmetric

$$\Theta^{\mu\nu} = \Theta^{\nu\mu}\,, \qquad \partial_\mu \Theta^{\mu\nu} = 0 \implies 6 \text{ independent components}\,. \tag{83}$$

A naive and often considered decomposition into traceless and traceful pieces is not consistent with the conservation law

$$\begin{aligned} \Theta^{\mu\nu} &= \tilde{\Theta}^{\mu\nu}_S + \tilde{\Theta}^{\mu\nu}_T \equiv \frac{1}{4} g^{\mu\nu}\Theta + \left[\Theta^{\mu\nu} - \frac{1}{4} g^{\mu\nu}\Theta\right] \\ &\implies \partial_\mu \tilde{\Theta}^{\mu\nu}_S = \partial^\nu \Theta \neq 0\,. \end{aligned} \tag{84}$$

A consistent decomposition, where the two tensor components are conserved separately and are mutually orthogonal, is given by

$$\Theta^{\mu\nu} = \Theta^{\mu\nu}_S + \Theta^{\mu\nu}_T\,, \tag{85}$$

with

$$\Theta^{\mu\nu}_S = \frac{1}{3}\left[g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\partial^2}\right]\Theta \implies \partial_\mu \Theta^{\mu\nu}_S = 0\,. \tag{86}$$

We will analyze the lattice data using this consistent decomposition. In particular, the meson dominance approach discussed in later sections manifestly displays such a consistent separation explicitly.

The separation into scalar and tensor components of the SEM tensor has also special properties under renormalization.

6.3. Ward–Takahashi identities

At the quantum level, the conservation laws put strong constraints on the time-ordered products, where the appearance of derivatives requires some careful modifications in order to comply with the Lorentz invariance. In this regard, the standard canonical approach is rather cumbersome and plagued with the so-called Schwinger terms [116]. We consider instead the much more transparent path integral approach [117], where the expectation value of a given composite field operator is written as

$$\langle O\rangle_S = \int D\phi O\, \mathrm{e}^{iS[\phi]}\,. \tag{87}$$

In particular, the time-ordered product[16] is

$$\langle 0|T\left[\phi(x_1)\ldots\phi(x_n)\right]|0\rangle = \langle\phi(x_1)\ldots\phi(x_n)\rangle_S = \int D\phi\phi(x_1)\ldots\phi(x_n)\,\mathrm{e}^{iS[\phi]}\,. \tag{88}$$

The invariance under a transformation $\phi\to\phi+\delta\phi$ implies

$$\delta\langle O\rangle_S = \langle\delta O\rangle_S + \langle i\delta S O\rangle_S = 0 \implies \langle\delta O\rangle_S = -i\langle O\delta S\rangle_S\,, \tag{89}$$

which is a functional form of the Feynman–Hellmann theorem. Taking the simplest two-point function as an observable yields

$$\langle 0|T\left[\delta\phi(x_1)\phi(x_2)\right]|0\rangle + \langle 0|T\left[\phi(x_1)\delta\phi(x_2)\right]|0\rangle = -i\langle 0|T\left[\phi(x_1)\phi(x_2)\delta S\right]|0\rangle\,. \tag{90}$$

For a symmetry transformation with a global group generator $\delta\phi(x) = \epsilon A\phi(x)$, with A indicating an operator, the action is invariant, hence $\delta S=0$. The quantum Noether construction with a *local* group generator $\epsilon(x)$ takes $\delta\phi(x)=\epsilon(x)A\phi(x)$ and $\delta S=\int \mathrm{d}^4x\epsilon(x)\partial^\mu J_\mu$, which yields

$$\begin{aligned}&\delta(x-x_1)\langle 0|T\left[A\phi(x_1)\phi(x_2)\right]|0\rangle + \delta(x-x_2)\langle 0|T\left[\phi(x_1)A\phi(x_2)\right]|0\rangle\\ &= -i\langle 0|T\left[\phi(x_1)\phi(x_2)\partial^\mu J_\mu(x)\right]|0\rangle\,.\end{aligned} \tag{91}$$

6.4. Gravitational Ward–Takahashi identity for scalars

The early work on gravitational Ward–Takahashi identities exploited the equivalence principle in the Schwinger formulation [4, 118, 119]. Using the change of the scalar field under general transformation $x\to x' = x+\epsilon(x)$, we get

$$\phi'(x') = \phi(x) \implies \delta\phi(x) = -\epsilon^\mu\partial_\mu\phi(x) \implies \delta S = \int \mathrm{d}^4x\epsilon^\mu\partial^\nu\Theta_{\mu\nu}\,, \tag{92}$$

such that, according to Eq. (91), the gravitational Ward–Takahashi identity reads

$$\begin{aligned}&\delta(x-x_1)\langle 0|T\left[\partial^\mu\phi(x_1)\phi(x_2)\right]|0\rangle + \delta(x-x_2)\langle 0|T\left[\phi(x_1)\partial^\mu\phi(x_2)\right]|0\rangle\\ &= -i\langle 0|T\left[\phi(x_1)\phi(x_2)\partial_\nu\Theta^{\mu\nu}(x)\right]|0\rangle\,.\end{aligned} \tag{93}$$

[16] In general, the so-called T^* product, relevant when derivatives appear to restore the Lorentz invariance, corresponds to taking the derivatives *after* the T-operation, $T^*[\partial_\mu\phi(x)\phi(0)] \equiv \partial_\mu T[\phi(x)\phi(0)]$, a procedure understood here. We keep the T symbol for a cleaner notation.

We introduce the scalar field propagator

$$i\langle 0|T\left[\phi(x_1)\phi(x_2)\right]|0\rangle = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\,\mathrm{e}^{ip\cdot(x_1-x_2)}\Delta(p)\,, \tag{94}$$

and the unamputated 3-point vertex function in the momentum space

$$\Lambda^{\mu\nu}(p',p) = \int \mathrm{d}^4x_1\,\mathrm{d}^2x_2\,\mathrm{e}^{ip'\cdot x_1}\,\mathrm{e}^{-ip\cdot x_2}\langle 0|T\left[\phi(x_1)\phi(x_2)\Theta^{\mu\nu}(0)\right]|0\rangle\,. \tag{95}$$

The amputated vertex function, defined as

$$\Theta^{\mu\nu}(p',p) = D(p')^{-1}\Lambda^{\mu\nu}(p',p)D(p)^{-1}\,, \tag{96}$$

satisfies the following Ward–Takahashi identity in the momentum space:

$$q_\mu\Theta^{\mu\nu}(p+q,p) = p^\nu\Delta^{-1}(p+q) - (p^\nu+q^\nu)\Delta^{-1}(p)\,. \tag{97}$$

6.5. *Gravitational form factor for scalars*

The kinematics is chosen in terms of the variables

$$P^\mu = \frac{1}{2}(p^\mu + p'^\mu)\,, \qquad q^\mu = p'^\mu - p^\mu\,, \tag{98}$$

where the on-shell conditions are

$$p^2 = p'^2 = m^2 \implies \begin{cases} P\cdot q = 0\,, \\ P^2 = 4m^2 - q^2\,. \end{cases} \tag{99}$$

The SEM conservation implies the on-shell condition

$$q_\mu\Theta^{\mu\nu}(p',p) = 0\,. \tag{100}$$

The gravitational form factors for a spin-0 particle are defined via the decomposition

$$\Theta^{\mu\nu}(p',p) \equiv \langle p'|\Theta^{\mu\nu}(0)|p\rangle = 2P^\mu P^\nu A\left(q^2\right) + \frac{1}{2}\left(q^\mu q^\nu - g^{\mu\nu}q^2\right)D\left(q^2\right)\,. \tag{101}$$

The normalization condition for $A(0)$ comes from the Ward–Takahashi identity in the off-shell case (*cf.* [120]), when we first set the on-shell condition $p^2 = m^2$, and then approach the $q\to 0$ limit. In that case

$$\begin{aligned} q_\mu\Theta^{\mu\nu}(p,p)|_{q\to 0} = 2p^\nu p\cdot q \quad &\Longrightarrow \quad \Theta^{\mu\nu}(p,p)|_{p^2=m^2} = 2p^\mu p^\nu \\ &\Longrightarrow \quad A(0) = 1\,. \end{aligned} \tag{102}$$

As mentioned in the Introduction, the value of the Druck form factor at the origin, $D(0)$, is not constrained by symmetries and is a fundamental dynamical quantity of a hadron.

6.6. Scale transformations and trace anomaly

Because of the presence of derivatives, the SEM quantum operator is badly divergent in the ultraviolet limit.

The scale transformations $x \to x' = \mathrm{e}^{\epsilon} x$ form an Abelian group and they generate the corresponding field transformation $\phi(x) \to \phi'(x') = \mathrm{e}^{-\epsilon d_\phi} \phi(\mathrm{e}^{\epsilon} x)$, where $d_\phi = 1$ is the classical dimension. For $U(\phi) = m^2\phi^2/2 + g\phi^4/4!$, the action at the classical level undergoes the change $\delta S = \delta S_m$ with $S_m = -\int \mathrm{d}^4 x m^2 \phi^2/2$, such that for $m = 0$, one has the scale invariance. The corresponding dilaton current is obtained from the Noether construction using an infinitesimal local transformation $\delta\phi(x) = \epsilon\left[d_\phi + x^\mu \partial_\mu\right]\phi(x)$. Then,

$$D_\mu = x_\nu \bar{\Theta}^{\mu\nu} \implies \partial^\mu D_\mu = \bar{\Theta} + m^2\phi^2 \tag{103}$$

in terms of $\bar{\Theta}^{\mu\nu}$, the so-called *improved* SEM tensor, which is also symmetric and conserved

$$\bar{\Theta}^{\mu\nu} = \Theta^{\mu\nu} - \frac{1}{6}\left[\partial^\mu\partial^\nu - g^{\mu\nu}\partial^2\right]\phi^2 \implies \partial_\mu \bar{\Theta}^{\mu\nu} = 0\,. \tag{104}$$

This object was introduced by Callan, Coleman, and Jackiw [114] to improve the high-energy behavior. The corresponding improved action corresponds to adding a curvature term, $\bar{S} = S + \int \mathrm{d}^4 x R\phi^2/6$, and according to Eq. (80), amounts to a change of the Druck form factor at the origin, $D(0) \to D(0) + 1/6$.

Actually, at the quantum level, because of the emergence of a renormalization scale, there is a scale symmetry violation. We can take $\lambda = \mu/\mu_0$, such that the coupling constant g, the mass m, and the dimension d_ϕ change accordingly with the renormalization scale and a (quantum) trace anomaly becomes

$$\partial^\mu D_\mu = \bar{\Theta} = \frac{\beta(g)}{4!}\phi^4 + m^2(1+\gamma_\phi)\phi^2\,, \tag{105}$$

with $\beta(g) \equiv \partial g/\partial \log\mu$ denoting the beta function with the anomalous dimension $\gamma_\phi(g) = \partial \log m/\partial \log\mu$.

6.7. Fermion case

The derivation for spin $1/2$ particles involves the tetrad formalism, which is straightforward but a bit more involved. In the presence of fermions, the symmetric SEM is defined as a functional variation of the action [121]

$$\Theta^{\mu\nu} = \frac{1}{2e(x)}\left[e^\mu_A(x)\frac{\delta S}{\delta\, e^A_\nu(x)} + e^\mu_A(x)\frac{\delta S}{\delta\, e^\nu_A(x)}\right]\,, \tag{106}$$

where $e^\mu_A(x)$ is the vierbein (tetrad) vector, with the metric tensor satisfying $g^{\mu\nu}(x) = e^\mu_A(x) e^\nu_B(x) \eta_{AB}$, while η_{AB} denotes the Minkowski flat metric tensor and $e(x) = \mathrm{Det}[e^\mu_A(x)]$.

The covariant representation of a fermionic matrix element can be written in the form

$$\langle p', s' | \Theta_{\mu\nu}(0) | p, s \rangle = \bar{u}(p', s') \Gamma_{\mu\nu} u(p, s) \,, \tag{107}$$

where $u = u(p, s)$ and $u' = u(p', s')$ are the positive energy Dirac spinors with given momenta and spin projections. The Ward–Takahashi identity has the form [119]

$$q^\mu \Gamma_{\mu\nu}(p', p) = p'_\nu S(p)^{-1} - p_\mu S(p')^{-1} + \frac{i}{2} \left[q^\mu \sigma_{\mu\nu} S(p)^{-1} - q^\nu S(p')^{-1} \sigma_{\mu\nu} \right] \,, \tag{108}$$

where $S(p)$ is the (off-shell) fermion propagator.

Because of the Gordon identity,

$$2m \bar{u}' \gamma^\alpha u = \bar{u}' (2P^\alpha + i\sigma^{\alpha\rho} q_\rho) u \,, \tag{109}$$

one can write three equivalent decompositions

$$\begin{aligned} \Gamma_{\mu\nu} &= A(t)\, \gamma_{\{\mu} P_{\nu\}} + B(t)\, \frac{i\, P_{\{\mu} \sigma_{\nu\}\rho} q^\rho}{2m_N} + D(t)\, \frac{q_\mu q_\nu - g_{\mu\nu} q^2}{4m_N} \\ &= \frac{1}{m_N} \left[A(t) P_\mu P_\nu + J(t)\, i\, P_{\{\mu} \sigma_{\nu\}\rho} q^\rho + D(t)\, \frac{q_\mu q_\nu - g_{\mu\nu} q^2}{4} \right] \\ &= 2J(t)\, \gamma_{\{\mu} P_{\nu\}} - B(t) \frac{P_\mu P_\nu}{m_N} + D(t)\, \frac{q_\mu q_\nu - g_{\mu\nu} q^2}{4m_N} \,, \end{aligned} \tag{110}$$

with the relation

$$B(t) = 2J(t) - A(t) \,. \tag{111}$$

We use the conventions $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ and $a_{\{\mu} b_{\nu\}} = \frac{1}{2}(a_\mu b_\nu + a_\nu b_\mu)$. The $A(t)$ form factor is chirally even, whereas $B(t)$ and $D(t)$ are chirally odd. The relations following from the Ward–Takahashi identity are

$$A(0) = 1 \,, \qquad J(0) = \frac{1}{2} \,. \tag{112}$$

Then relation (111) yields $B(0) = 0$. As in the scalar case, the value of $D(0)$ is not constrained by symmetries.

7. Gravitational form factors in QCD

In this section, we present the definition and the known features of SEM in QCD, whose matrix elements pertain directly to the gravitational form factors of hadrons.

7.1. Stress-energy-momentum tensor

Following previous works on GFFs (note, however, the discussion of a non-symmetric SEM in [122]), we use the Hilbert definition of SEM obtained from the action coupled to gravity, as discussed earlier. In the case of QCD, the Hilbert definition coincides with the Belinfante–Rosenfeld prescription [112], yielding a symmetric SEM in the form

$$\Theta^{\mu\nu} = \frac{i}{4}\bar{\Psi}\left[\gamma^{\mu}\overleftrightarrow{D}^{\mu} + \gamma^{\nu}\overleftrightarrow{D}^{\mu}\right]\Psi - G^{\mu\lambda a}G^{\nu}_{\lambda a} + \frac{1}{4}g^{\mu\nu}G^{\sigma\lambda a}G_{\sigma\lambda a} + \Theta^{\mu\nu}_{\mathrm{GF-EOM}}\,, \tag{113}$$

where Ψ is the Dirac quark field carrying flavor and color, $G^{\mu\nu a}$ represents the gluon field strength tensor with a labeling the color octet representation, and the term $\Theta^{\mu\nu}_{\mathrm{GF-EOM}}$ denotes some extra terms from the gauge fixing and from the use of the equations of motion.

The trace anomaly of QCD is defined as the divergence of the dilatation current $D^{\mu}(x)$

$$\partial^{\mu}D_{\mu} = \Theta^{\mu}_{\mu} \equiv \Theta = \frac{\beta(\alpha)}{2\alpha}G^{\mu\nu a}G^{a}_{\mu\nu} + \sum_{q} m_q\left[1+\gamma_m(\alpha)\right]\bar{q}q\,, \tag{114}$$

where q is the quark field of a given flavor, $\beta(\alpha) = \mu^2\mathrm{d}\alpha/\mathrm{d}\mu^2$ denotes the QCD beta function at the energy scale μ, $\alpha = g^2/(4\pi)$ is the running coupling constant, and $\gamma_m(\alpha) = \mathrm{d}\log m_q/\mathrm{d}\log\mu^2$ is the anomalous dimension of the current quark mass m_q [17]. The form of Eqs. (113) and (114) suggests the decomposition of SEM into the quark and gluon parts,

$$\Theta^{\mu\nu} = \Theta^{\mu\nu}_{q} + \Theta^{\mu\nu}_{g}\,, \tag{115}$$

which has been used to break up the hadron masses into various contributions [123–127]. The decomposition (115) depends on the scale μ and the renormalization scheme. It relates to the well-known feature of the momentum fractions carried by partons in Deep Inelastic Scattering, where the momentum sum rule can be written in the form

$$\langle p|\Theta^{\mu\nu}|p\rangle = 2p^{\mu}p^{\nu}\left[\langle x\rangle_q + \langle x\rangle_g\right] \implies \langle x\rangle_q + \langle x\rangle_g = 1\,. \tag{116}$$

[17] To leading order in perturbation theory, one has $\beta(\alpha) = -\beta_0\alpha^2/(4\pi) + \ldots$ and $\gamma_m(\alpha) = \alpha/(4\pi) + \ldots$ with $\beta_0 = (11N_c - 2N_f)/3$ so that the running strong coupling constant reads $\alpha(\mu^2) = (4\pi/\beta_0)/\log(\mu^2/\Lambda^2)$.

Recent estimates [128, 129] for the quark contributions in the pion and nucleon states read

$$\langle x\rangle^{\pi}_{\rm val} \sim \langle x\rangle^{N}_{\rm val} \sim 0.6 \qquad \text{at} \quad \mu = 2~\text{GeV}\,. \tag{117}$$

In these lectures, we do not analyze the separate contributions of quarks and gluons to GFFs, which are renormalization scale and scheme-dependent, whereas the sum is not, in line with the applied hadronic picture.

7.2. Ward–Takahashi identities for composite particles

Obviously, hadrons are composite objects. Unlike the cases discussed in Section 6, where the field under discussion was the *same* as the one in the Lagrangian, hadrons (mesons and baryons) are associated with Fock-state combinations of the quark and gluon fields. This requires identifying an interpolating field which has good quantum numbers. For instance, the (composite) pion field is usually taken to be

$$\vec{\pi}(x) = Z_\pi \bar{q}(x) i\gamma_5 \vec{\tau} q(x) + \dots\,, \tag{118}$$

where the dots denote *any* combination of fields with *the same* quantum numbers, for instance, $Z'_\pi \bar{q}q\bar{q}i\gamma_5\vec{\tau}q$. Here, we have written only the lowest dimensional cases. While the renormalization constants depend on the renormalization scale μ, it is believed that for typical hadronic renormalization scales $Z_\pi \gg Z'_\pi$, such that the term included in (118) yields the dominant contribution. An interesting and relevant fact is that from the point of view of the Ward–Takahashi identities discussed above, only the transformation properties of the hadron field in question under the corresponding symmetry operation matter, and not the elementary or composite nature of the state. For that reason, the Ward–Takahashi identities assume an identical form as those from a Lagrangian with elementary fields.

7.3. Pion GFF

The pion, being a spin-0 hadron, is the simplest case, involving two form factors, $A(t)$ and $D(t)$,

$$\langle \pi^a(p')|\Theta^{\mu\nu}(0)|\pi^b(p)\rangle = \delta_{ab}\left[2P^\mu P^\nu A(t) + \frac{1}{2}\left(q^\mu q^\nu - g^{\mu\nu}q^2\right)D(t)\right]\,, \tag{119}$$

with a, b indicating the pion isospin, and the kinematics spelled out in Eq. (98). The corresponding trace form factor is the combination

$$\Theta^\mu_\mu \equiv \Theta\left(q^2\right) = 2\left(m_\pi^2 - \frac{q^2}{4}\right)A\left(q^2\right) - \frac{3}{2}q^2 D\left(q^2\right)\,, \tag{120}$$

from where it follows that

$$\lim_{q^2\to 0} \Theta\left(q^2\right) = 2m_\pi^2 A(0)\,, \tag{121}$$

with $A(0) = 1$, as derived previously.

Raman [10] proposed the decomposition of $\langle \pi^a(p')|\Theta^{\mu\nu}(0)|\pi^b(p)\rangle$ in terms of conserved irreducible tensors corresponding to a well-defined total angular momentum, $J^{PC} = 0^{++}$ (scalar) and 2^{++} (tensor), namely

$$\Theta^{\mu\nu} = \Theta_S^{\mu\nu} + \Theta_T^{\mu\nu}\,, \qquad \begin{cases} \Theta_S^{\mu\nu} = \frac{1}{3}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\Theta\,, \\ \Theta_T^{\mu\nu} = 2\left[P^\mu P^\nu - \frac{P^2}{3}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\right]A\,. \end{cases} \tag{122}$$

Note that both parts are separately conserved, $q_\mu \Theta_S^{\mu\nu} = q_\mu \Theta_T^{\mu\nu} = 0$, and the tensor part is traceless, $\Theta^\mu_{T\mu} = 0$. Such a decomposition was also used in [76].

Since Θ and A carry the information on good J^{PC} channels, from the spin decomposition point of view, they should be regarded as the primary objects, whereas the D form factor, which is the key object in the mechanical considerations, mixes the quantum numbers, with the explicit formula

$$D = -\frac{2}{3t}\left[\Theta - \left(2m_\pi^2 - \frac{1}{2}\,t\right)A\right]. \tag{123}$$

A chiral theorem [14, 130] states that

$$D_\pi(0) = -1 + \mathcal{O}\left(m_\pi^2\right). \tag{124}$$

7.4. Nucleon GFF

We use the covariant normalization $\bar{u}u = 2m_N$. Since the operator $\Gamma_{\mu\nu}$ in question is isosinglet, we omit the isospin index for the nucleon (the corresponding form factors are equal for the proton and the neutron). The condition $J(0) = 1/2$ for the nucleon is referred to as Ji's sum rule [16].

The Raman decomposition for the nucleon case takes the form

$$\begin{aligned} \Gamma_S^{\mu\nu} &= \frac{1}{3} Q^{\mu\nu}\Theta(t)\,, \\ \Gamma_T^{\mu\nu} &= \frac{1}{m_N}\left[P^\mu P^\nu - \frac{1}{m_N}\frac{P^2}{3}Q^{\mu\nu}\right]A(t) + \frac{1}{m_N}\left[i\,P^{\{\mu}\sigma^{\nu\}\rho}q_\rho - \frac{t}{6}Q^{\mu\nu}\right]J(t)\,, \end{aligned} \tag{125}$$

We note that analogously to the pion case, Γ_S and Γ_T are separately conserved, and Γ_T is traceless. The trace anomaly part expressed via the other form factors reads

$$\Theta(t) = \frac{1}{m_N}\left[\left(m_N^2 - \frac{t}{4}\right)A(t) - \frac{3}{4}tD(t) + \frac{1}{2}tJ(t)\right] . \tag{126}$$

With Eq. (112), at $q = 0$, we have

$$\Theta(0) = m_N\,, \qquad D(0) = \frac{4m_N}{3}\left[m_N A'(0) - \Theta'(0)\right] . \tag{127}$$

7.5. MIT lattice data

In recent years, much progress has been accomplished in the lattice simulations of hadronic properties thanks to the application of the Lüscher–Weisz gauge action [131]. In particular, the MIT group has been able to obtain GFFs of the pion [21] and the nucleon [20] (see Fig. 1) to an unprecedented accuracy, and very close to the physical point, namely with $m_\pi = 170$ MeV.

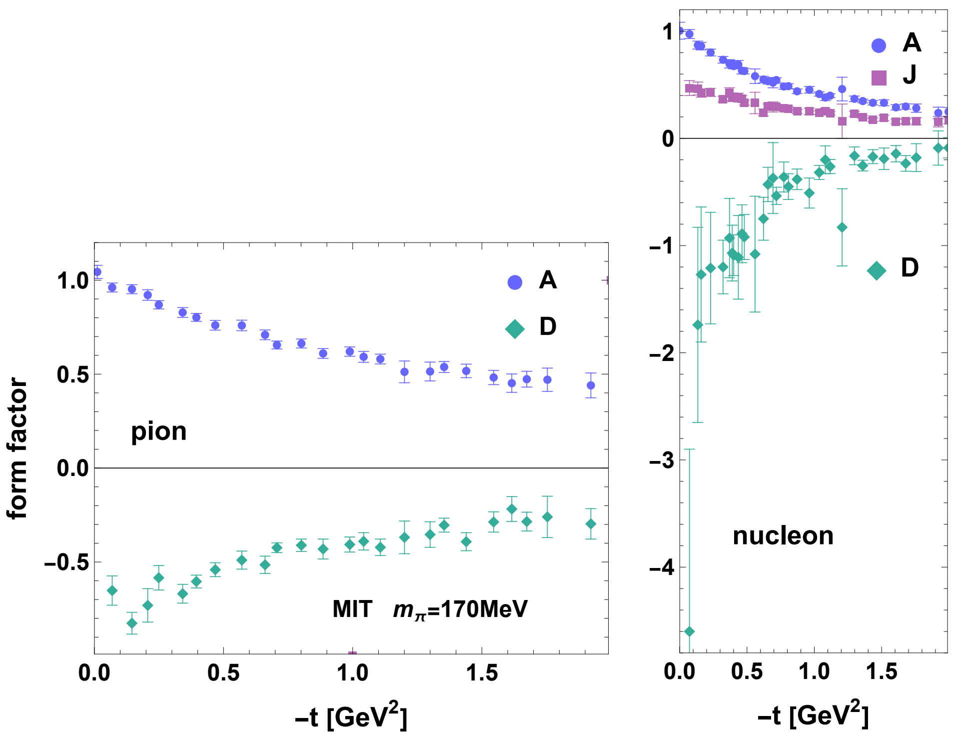

Fig. 1. Lattice results for the gravitational form factors of the pion [21] (left) and nucleon [20] (right), plotted as functions of the space-like momentum transfer $-t$.

The data are for the space-like momenta $0 < Q^2 < 2$ GeV^2, and obtained separately for the quark of three flavors and the gluon components. As already mentioned, in these lectures, however, we use the total (quark+gluon) quantities, since as corresponding to conserved currents, they do not depend on the renormalization scale or scheme. As we will show, these data can be described efficiently with an ansatz whose parameters (meson masses) can be read off *directly* from the Particle Data Group tables.

8. Dispersion relations, sum rules, and meson dominance

The form factors are dynamical quantities which obey important mathematical constraints based on analyticity in the momentum transfer variable extended to the complex plane. For a readable introduction, we refer to [110, 132, 133]. The bottom-line is that form factors are described via corresponding spectral functions whose low- and high-energy behavior is theoretically known.

8.1. Form factor and crossing

Here, we switch for several subsections to a much better known case of the vector (charge) form factor of the pion, since the basic analyticity features extend analogously to the gravitational form factors. From two specific processes, the elastic electron scattering on the pion and the electron–positron annihilation

$$\begin{aligned}
e^-\pi^+ \to e^-\pi^+ \quad &\Longrightarrow \quad \langle \pi^+(p')|J^\mu_Q(0)|\pi^+(p)\rangle = F^Q_\pi\left(q^2\right)(p'^\mu + p^\mu) \\
&\qquad\quad q^2 < 0 \qquad \text{space-like}\,, \\
e^+e^- \to \pi^+\pi^- \quad &\Longrightarrow \quad \langle \pi^+(-p')\pi^-(p)|J^\mu_Q(0)|0\rangle = F^Q_\pi\left(q^2\right)(p'^\mu + p^\mu) \\
&\qquad\quad q^2 > 4m_\pi^2 \qquad \text{time-like}\,,
\end{aligned} \tag{128}$$

one extracts the differential cross section and the total annihilation cross section, respectively, which are related. This relation corresponds to the rotation of the time axis in the corresponding Feynman diagram. Analyticity of the amplitude connects these two processes described with the *same and unique* function of the complex variable s in different domains which are experimentally accessible (*cf.* Fig. 3). This leaves the $0 \leq q^2 \leq 4m_\pi^2$ region as *unphysical*, and it can only be reached via analytical continuation[18]. The analyticity principle states that the form factor $F(s)$ can only have singularities on the real axis.

[18] We are neglecting electromagnetism here; otherwise, $\pi^+\pi^-$ has a bound state.

Since $F(q^2) = F(q^2)^*$ at $q^2 < 0$, we infer that $F(s^*) = F(s)^*$ (the Schwarz reflection principle). From there, it follows that the discontinuity is

$$\mathrm{Disc}\, F(s) = 2i\, \mathrm{Im}\, F(s + i\epsilon)\,, \qquad s > 4m_\pi^2\,. \tag{129}$$

The function exhibits cuts starting at $s > 4m_\pi^2$, which together with the reflection principle guarantees the existence of only two Riemann sheets; in the physical region, they are denoted as $F_{\mathrm{I}}(s)$ and $F_{\mathrm{II}}(s)$, fulfilling $F_{\mathrm{I}}(s + i\epsilon) = F_{\mathrm{II}}(s - i\epsilon)$ and $F_{\mathrm{II}}(s+i\epsilon) = F_{\mathrm{I}}(s-i\epsilon)$ for $s \geq 4m_\pi^2$. Unlike $F(s) \equiv F_{\mathrm{I}}(s)$, the analytical continuation may have singularities such as poles and cuts. Actually, the resonances correspond to poles on the second Riemann sheet, therefore

$$F_{\mathrm{II}}(s) \underbrace{\to}_{s \to m_{\mathrm{R}}^2 - i m_{\mathrm{R}} \Gamma_{\mathrm{R}}} \frac{Z_{\mathrm{R}}}{s - m_{\mathrm{R}}^2 + i m_{\mathrm{R}} \Gamma_{\mathrm{R}}} + \ldots \tag{130}$$

The quantities m_{R}, Γ_{R}, and Z_{R} stand for the resonance mass, width, and the residue at the pole, respectively. Thus, in the analysis of form factors, we select resonances with given quantum numbers (in the case of the pion vector-isovector form factor, it corresponds to the $\rho, \rho', \rho'', \ldots$ states).

8.2. Low energies

In general terms, one has the integral equations of the Bethe–Salpeter form which in an operator form, read (see, *e.g.*, [134])

$$F = \Gamma + \Gamma G_0 T\,, \qquad T = V + V G_0 T\,, \tag{131}$$

and are supposed to hold at sufficiently low energies. A pictorial representation in terms of *hadronic* Feynman diagrams is shown in Fig. 2. In the time-like region e^+e^- becomes *inelastic* for $\sqrt{s} \geq 2m_\pi n$ $(n = 1, 2, \ldots)$ and the $e^+e^- \to \gamma^* \to n(\pi^+\pi^-)$ processes occur.

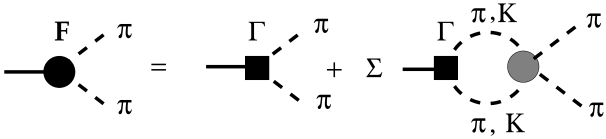

Fig. 2. Diagrammatic representation of the Bethe–Salpeter equation with the $\pi\pi$ and KK production channels.

At energies below the $2\pi^+ + 2\pi^-$ threshold, $s < (4m_\pi)^2$, the discontinuity can be constrained from Watson's theorem (see Appendix A for a simple proof). In practice, all inelastic contributions below the $K\bar{K}$ threshold are neglected, such that

$$\frac{F(s+i\epsilon)}{F(s-i\epsilon)} = \frac{T(s+i\epsilon)}{T(s-i\epsilon)} \implies F(s) = |F(s)|\, \mathrm{e}^{i\delta_{11}(s)}\,, \tag{132}$$

and one gets

$$\mathrm{Im}\, F(s) = |F(s)| \sin\delta_{11}(s) > 0\,, \qquad 4m_\pi^2 < s < 4m_K^2\,, \tag{133}$$

where the last inequality comes from the fact that $\delta_{11}(s) > 0$, *i.e.*, the $\pi\pi$ interaction is *attractive*. From this relation, the known threshold behavior of the scattering phase shift allows one to infer the threshold behavior of the form factor itself, namely

$$\delta_{11}(s) \sim a_{11}\left(s/4 - m_\pi^2\right)^{\frac{3}{2}} \implies \mathrm{Im}\, F(s) \sim \left|F\left(4m_\pi^2\right)\right| a_{11}\left(s/4 - m_\pi^2\right)^{\frac{3}{2}}\,. \tag{134}$$

In the complex plane, Watson's theorem becomes

$$F_{\mathrm{II}}(s) = S_{\mathrm{II}}(s) F_{\mathrm{I}}(s)\,, \tag{135}$$

where $S_{\mathrm{II}}(s)$ is the scattering matrix on the second Riemann sheet of the complex s-plane. While the unitarity condition $S_{\mathrm{II}}(s) = 1/S_{\mathrm{I}}(s)$ ensures that the poles of $S_{\mathrm{II}}(s)$ are zeros of $S_{\mathrm{I}}(s)$, this does not imply anything concerning the zeros of $F_{\mathrm{I}}(s)$.

8.3. Large-momentum behavior from pQCD

Clearly, the hadronic representation is inadequate for large Q^2, where a $q\bar{q}$ state allows for a one-gluon exchange (or more complicated processes suppressed perturbatively). The known leading-order (LO) pQCD asymptotic formula for the space-like vector form factor allows one to obtain the discontinuity along the cut at asymptotically large $s > 0$

$$\begin{aligned} F\left(-Q^2\right) &\simeq \frac{16\pi F_\pi^2 \alpha_{\mathrm{s}}\left(Q^2\right)}{Q^2} \\ &\sim \frac{64\pi^2 F_\pi^2}{\beta_0 Q^2 \log\left(Q^2/\Lambda^2\right)} \underbrace{\to}_{Q^2 \to \mathrm{e}^{-i\pi} s} -\frac{64\pi^2 F_\pi^2}{\beta_0 s\left(\log\left(s/\Lambda^2\right) - i\pi\right)} \\ &\implies \frac{1}{\pi}\mathrm{Im}\, F(s) = -\frac{64\pi^2 F_\pi^2}{\beta_0 s\left(\log^2\left(s/\Lambda^2\right) + \pi^2\right)} < 0\,. \end{aligned} \tag{136}$$

8.4. Dispersion relation

A sketch of the complex-plane structure in the complex variable s is depicted in Fig. 3. Using Cauchy's theorem for the contour indicated in Fig. 3 and the asymptotic behavior, one obtains an unsubtracted dispersion relation of the form

$$F\left(-Q^2\right) = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \mathrm{d}s \frac{\operatorname{Im} F(s)}{s+Q^2}\,, \tag{137}$$

with the value at the origin normalized to the charge of π^+

$$F(0) = 1 = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \mathrm{d}s \frac{\operatorname{Im} F(s)}{s}\,. \tag{138}$$

Besides, a *superconvergent sum rule* [135] follows from a simple observation that $\lim_{Q^2\to\infty} Q^2 F(-Q^2) = 0$ (*cf.* Eq. (136)). Thus,

$$\lim_{Q^2\to\infty} Q^2 F\left(-Q^2\right) = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \mathrm{d}s \operatorname{Im} F(s) = 0\,, \tag{139}$$

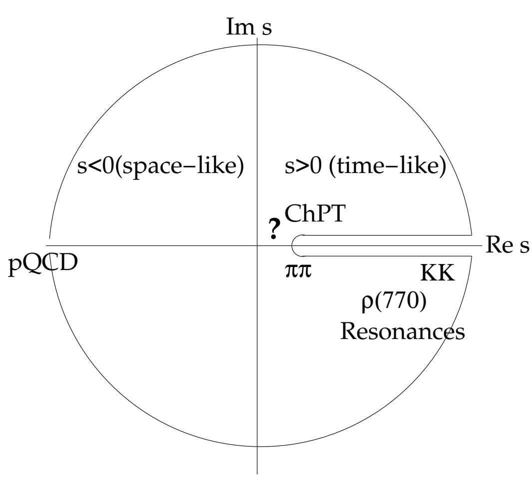

Fig. 3. Analyticity structure of the vector form factor of the pion in the complex $s = q^2$ plane. The question mark corresponds to the unphysical region $0 < s < 4m_\pi^2$. Labels $\pi\pi$ and KK indicate the beginning of the corresponding two-meson production cuts (unitarity cuts).

which implies that $\mathrm{Im}\, F(s)$ must change sign (at least once) along the unitarity cut. Note that from Watson's theorem and the attractive character of the $\pi\pi$ interaction in the 11 channel, the first zero of the spectral function must lie *above* the $K\bar{K}$ threshold.

8.5. Line shapes, finite widths, and space-like momenta

The conditions of analyticity can be solved by using specific parameterizations which are phenomenologically motivated, particularly in the case of resonances, whose position in the complex plane is *process-independent*. However, complex energies cannot be measured, and hence an analytical extrapolation from the experimentally accessible real axis and the complex plane becomes mandatory. The Breit–Wigner or Gounaris–Sakurai [136] functions are popular profiles, often used to model this energy dependence. As such, these models directly comply to unitarity requirements (such as Watson's theorem). Their applicability can only be validated by the data, since the separation between the resonance contribution and the background is *process-dependent*. Fortunately, these subtleties become rather irrelevant in the space-like region, as we argue below.

To show this, we use the Omnès representation of the form factor

$$F(t) = \exp\left[\frac{t}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\mathrm{d}s}{s}\frac{\delta(s)}{s-t}\right], \quad F(0)=1 \implies \frac{F(t+i0)}{F(t-i0)} = \mathrm{e}^{2i\delta(s)}, \tag{140}$$

which complies with Watson's theorem[19]. This, in principle, allows one to *predict* the form factor when the elastic phase-shift is known either experimentally or theoretically. In this context, a resonance is a pole on the second Riemann sheet of the scattering matrix and hence also of the form factor

$$1/S_{\mathrm{II}}(s_{\mathrm{R}}) = S_{\mathrm{I}}(s_{\mathrm{R}}) = 0 \implies 1/F_{\mathrm{II}}(s_{\mathrm{R}}) = F_{\mathrm{I}}(s_{\mathrm{R}})/S_{\mathrm{II}}(s_{\mathrm{R}}) = 0\,. \tag{141}$$

To simplify the discussion, let us consider the energy-dependent Breit–Wigner parametrization in the scattering region $s > 4m^2$

$$\begin{aligned} &N(s) = M^2 - s + iM\Gamma f(s)\,, \qquad S(s) = \mathrm{e}^{2i\delta(s)} = \frac{N(s)}{N(s)^*} \\ &\implies \delta(s) = \tan^{-1}\left[\frac{M\Gamma f(s)}{M^2-s}\right], \qquad \delta\left(M^2\right) = \frac{\pi}{2}, \qquad \Gamma = \frac{1}{M\delta'\left(M^2\right)}, \end{aligned} \tag{142}$$

[19] This solution is not unique, as we can multiply the Omnès function by an arbitrary polynomial $P(s)$. We fix it here to $P(s)=1$ for simplicity and discuss generalizations later.

where $f(M^2) = 1$. In principle, the particular shape of $f(s)$ depends on the background, which is process-dependent. Clearly, in the limit of narrow width, the phase becomes $\delta(s) = \pi\theta(s - M^2)$, and one obtains a simple monopole for the form factor, regardless of the profile function $f(s)$,

$$F(t) \underbrace{\to}_{\Gamma\to 0} \exp\left[t \int_{M^2}^{\infty} \frac{\mathrm{d}s}{s} \frac{1}{s-t}\right] = \frac{M^2}{M^2 - t} \implies \frac{1}{\pi}\mathrm{Im}\, F(s) = \delta\left(s - M^2\right) . \tag{143}$$

The question is to what extent does $F(t)$ resemble a monopole for a *finite* width Γ. We analyze this issue for a much less favorable case: the *widest* known QCD resonance, namely, the 0^{++} isoscalar $f_0(600)$, also called the σ meson. To this end, we propose several profiles for the function in the scattering region $s > 4m^2$

$$\begin{aligned} f_{\mathrm{A}}(s) &= 1\,, \\ f_{\mathrm{B}}(s) &= \sqrt{(s-4m^2)\,/\,(M^2-4m^2)}\,, \\ f_{\mathrm{C}}(s) &= \sqrt{(1-4m^2/s)\,/\,(1-4m^2/M^2)}\,. \end{aligned} \tag{144}$$

The corresponding phase shifts are depicted in Fig. 4 for the numerical values $M = 0.8$ GeV and $\Gamma = 0.7$ GeV, and taking $m = m_\pi$. From these profiles we may obtain, via the Omnès representation of Eq. (140), an analytical function in the complex plane[20]. The results for the FF in the space-like region are presented in Fig. 4 for several choices of the function $f(s)$ and with suitably chosen M and Γ, which resemble qualitatively the realistic benchmark determinations of the $\pi\pi$ scattering phase shift form the solution of Roy equations [137].

We can see that even for a broad S-wave resonance, and for a variety of profiles, the form factor resembles closely a family of monopoles for the space-like momenta. The monopole parameter is roughly the resonance BW mass, with an uncertainty compatible with its width (it is actually much smaller). This is a general feature which does not depend on the chosen partial waves. It becomes particularly helpful when the resonance parameters are known but the phase shifts are not so well known, as is the case away from the resonance region in many processes. A handy possibility is provided by the *halft-width rule* (see Appendix B).

[20] Actually, with the exception of case B, the corresponding functions $N(s)$ are not analytical by themselves, which prevents the determination of the resonance pole on the second Riemann sheet. This is not a problem, since the restoration of analyticity of the form factor by the dispersion relations can be done by using its phase, as we do here.

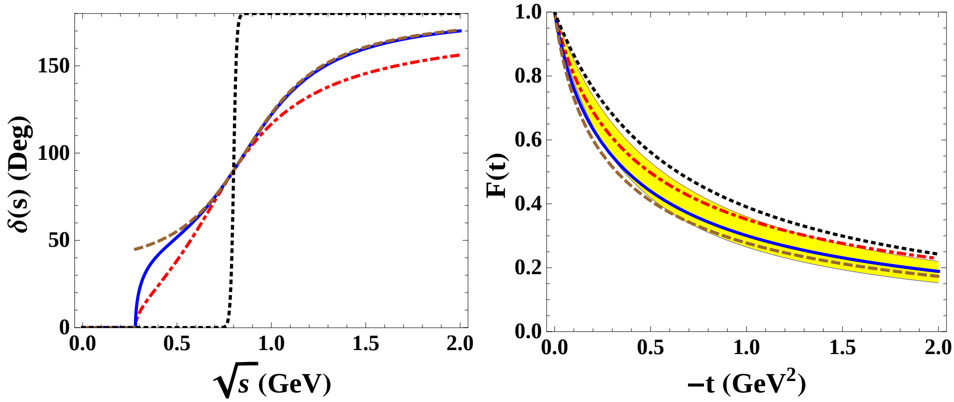

Fig. 4. The case study of the S-wave phase shift $\delta(s)$, and the Omnès form factor in the space-like region, $F(t)$, for several parameterizations (see the main text): A (brown dashed), B (blue solid), C (red dot-dashed), zero-width Monopole with $M = 0.8$ GeV (black dotted). The yellow band represents monopoles with masses in the range 0.60–0.75 GeV.

8.6. The large-N_c limit

It is remarkable that the large-N_c limit of QCD of t'Hooft and Witten [138, 139], *i.e.*, a limit where $N_c \to \infty$ with $\alpha_s N_c$ fixed, provides a rationale for the features discusses above, namely, that resonances are narrow. One can show that

$$\begin{cases} M_B = \mathcal{O}(N_c)\,, & \Gamma_B = \mathcal{O}\left(N_c^0\right) \\ m_M = \mathcal{O}\left(N_c^0\right)\,, & \Gamma_M = \mathcal{O}\left(N_c^{-1}\right) \end{cases} \implies \Gamma/M = \mathcal{O}\left(N_c^{-1}\right)\,. \tag{145}$$

An average value of all PDG resonances [140] yields the so-called Suranyi's ratio [141][21]

$$\left\langle \frac{\Gamma}{M} \right\rangle_{\rm exp} \equiv \sum_i g_i \frac{\Gamma_i}{M_i} = 0.12(8)\,, \tag{146}$$

where g_i is the spin, isospin, and anti-particle multiplicity of a state with resonance mass M_i and width Γ_i. This ratio for mesons and baryons numerically coincides. The uncertainty is mainly from the corresponding variance. The numerical value corroborates the fact that, on average, the QCD resonances are indeed narrow.

[21] The story of this ratio starts back in 1967 and it can be traced from [142, 143].

8.7. The incompleteness problem

Measuring the pion form factor in the space-like region directly from the electron scattering is difficult, since the charged pion decays and cannot be used as a target. However, in the time-like region, the charged pions are created and measured before they decay. The dispersion theory suggests that we may obtain one process from the other by invoking analyticity. Despite the attractive theoretical features, purely dispersive methods encounter unpleasant inconveniences in practice, since we only have experimental access to modules of the form factors on a discrete and finite grid,

$$|f(s_1)|, \ldots, |f(s_N)|\,, \qquad 4m_\pi^2 < s_1 < \cdots < s_N = s_{\max}\,, \tag{147}$$

mostly up to a limited range of momenta, be it space-like or time-like. The phase problem may be solved using a modulus-phase decomposition, after making an assumption on the number of zeros (see, *e.g.*, Ref. [144]). The discrete problem is solved either by interpolation or by using a sufficiently flexible parametrization based on a smoothness assumption. However, the maximum upper boundary energy problem turns out to be more serious: either pQCD applies down to $s_{\max}$ or we have to give up some predictive power.

This is illustrated by a recent dispersive analysis of the BaBar data [145], for the EM pion form factor in the time-like region, where 300 points with 3 MeV separation below $s_{\max} = 9$ GeV^2 allow one to extract the phase [144] and test the fulfillment of the sum rules. From the data, one gets

$$\begin{aligned} \frac{1}{\pi}\int_{4m_\pi^2}^{s_{\max}} \mathrm{d}s \frac{\mathrm{Im}\,F(s)}{s}\Bigg|_{\mathrm{Data}} &= 1.01(1)_{\mathrm{st}} \left({}^{+2}_{-1}\right)_{\mathrm{syst}}\,, \\ \frac{1}{\pi}\int_{4m_\pi^2}^{s_{\max}} \mathrm{d}s\, \mathrm{Im}\,F(s)\Bigg|_{\mathrm{Data}} &= 0.63(2)_{\mathrm{st}} \left({}^{+7}_{-4}\right)_{\mathrm{syst}}\ \mathrm{GeV}^2\,. \end{aligned} \tag{148}$$

As we can see, the mismatch in the charge sum rule is small, at a level of a percent, whereas the superconvergence sum rule is very far from being satisfied. Here, a scale for comparison is $m_\rho^2 = 0.6$ GeV^2, hence we need a large negative value in Eq. (148) from the integration beyond $s_{\max}$.

The pQCD part extrapolated down from infinity to the scale $s_{\max}$ yields

$$\begin{aligned}\frac{1}{\pi}\int_{s_{\max}}^{\infty} \mathrm{d}s\, \frac{\operatorname{Im} F(s)}{s}\Bigg|_{\mathrm{pQCD}} &= -\underbrace{0.0025}_{\mathrm{LO}}-\underbrace{0.0011}_{\mathrm{NLO}}-\underbrace{0.0006}_{\mathrm{NNLO}}, \\ \frac{1}{\pi}\int_{s_{\max}}^{\infty} \mathrm{d}s\, \operatorname{Im} F(s)\Bigg|_{\mathrm{pQCD}} &= -\underbrace{0.114}_{\mathrm{LO}}-\underbrace{0.030}_{\mathrm{NLO}}-\underbrace{0.013}_{\mathrm{NNLO}}\ \mathrm{GeV}^2 . \end{aligned} \tag{149}$$

We thus note that it has a tiny negative contribution to the charge sum rule, and only about a fifth of the needed magnitude in the superconvergence sum rule. Therefore, while superconvergence is a theorem, pQCD is far away, even if we use it all the way down to $s_{\max} = 9\ \mathrm{GeV}^2$.

A possible remedy is to use subtracted dispersion relations, but this requires the use of unknown subtraction constants, hence the predictive power is diminished. For example, with two subtractions, we get explicitly

$$F\left(-Q^2\right) = 1 - Q^2 F'(0) + \frac{1}{\pi}\left[\int_{4m_\pi^2}^{s_{\max}} + \int_{s_{\max}}^{\infty}\right] \mathrm{d}s \frac{Q^4}{s^2}\frac{\operatorname{Im} F(s)}{s+Q^2}\,, \tag{150}$$

where $F(0) = 1$ due to the charge conservation. The last term is $\mathcal{O}(Q^4/s_{\max}^2)$, hence it is suppressed for $Q^2 \ll s_{\max}$, and we can *ignore* the high-energy tail, but then $F'(0)$ cannot be predicted with the data. The situation worsens for form factors for which $F(0)$ is not constrained by a conservation law.

8.8. Extended meson dominance

At the field theoretical level, the effective narrowness of the resonance in the space-like region can be efficiently implemented in terms of a current-field identity proposed by Sakurai [146]

$$J_3^\mu = f_\rho m_\rho^2 \rho_3^\mu\,, \tag{151}$$

which yields a monopole form factor for the pion

$$F\left(-Q^2\right) = \frac{m_\rho^2}{m_\rho^2 + Q^2}\,. \tag{152}$$

The space-like vector form factor of the pion is very well approximated with this single meson dominance ansatz after the half-width rule is implemented as a conservative uncertainty estimate. This ansatz, however, does not comply strictly to the superconvergence sum rule, an issue related to the incompleteness problem discussed above.

At this place, after a somewhat lengthy but simpler and more pedagogical discussion of the pion EM form factor, we return to the properties of SEM and GFFs, the principal topic of these lectures.

The generalized meson-dominance construction for GFFs follows the derivation of the previous section, but now for different quantum number for the intermediate meson states. Saturation with the 0^{++} and 2^{++} isoscalar states (note that the Raman decomposition is manifest) yields the structure for the currents of the form [147–149]

$$\Theta^{\mu\nu} = \sum_S \frac{1}{3} f_S \left(\partial^\mu \partial^\nu - g^{\mu\nu}\partial^2\right) S + \sum_T f_T m_T^2 T^{\mu\nu} ,$$

where S and $T^{\mu\nu}$ are scalar 0^{++} and tensor 2^{++} fields, respectively. Certainly, $T^{\mu\nu} = T^{\nu\mu}$ and $T^\mu_\mu = 0$. On-shell, they have masses m_S and m_T, respectively, and $\partial^\mu T_{\mu\nu} = 0$ (for the complete Lagrangian see, *e.g.*, [150, 151]). Denoting the corresponding sources as J_S and $J_T^{\mu\nu}$, and using the equations of motion

$$\left(\partial^2 + m_S^2\right) S = J_S , \qquad \left(\partial^2 + m_T^2\right) T^{\mu\nu} = \tilde{J}_T^{\mu\nu} , \tag{153}$$

we get formally (up to polynomials in q) the following expressions for the matrix elements:

$$\begin{aligned} \langle A|\Theta^{\mu\nu}|B\rangle \;=\; & \sum_S \frac{f_S}{3} \frac{g^{\mu\nu}q^2 - q^\mu q^\nu}{m_S^2 - q^2 - i\epsilon} \langle A|J_S|B\rangle \\ & + \sum_T f_T \frac{m_T^2}{m_T^2 - q^2 - i\epsilon} \langle A| \sum_\lambda \epsilon_\lambda^{\mu\nu} \epsilon_{\alpha\beta}^\lambda J_T^{\alpha\beta} |B\rangle . \end{aligned} \tag{154}$$

From a field theory point of view, the above meson-dominance formula should not be taken literally, as it does not incorporate the notion of subtractions or the pQCD high momentum behavior. Besides, it is well known that higher spin fields have problems, particularly due to the role played by the off-shell behavior of propagators. The simplest way to avoid these issues is to use, in the spirit of dispersion relations, the meson dominance for the absorptive parts, where, by construction, the mesons are on the mass shell. Then, the information from pQCD is used to apply a minimal needed number of subtractions as the short distance constraints.

The absorptive part of the form factor in the time-like region, $q^2 \to s + i\epsilon$, where we have the process $g \to R \to A\bar{B}$, reads

$$\frac{1}{\pi} \mathrm{Im}\, \langle A\bar{B}|\Theta^{\mu\nu}|0\rangle = \sum_R \langle A\bar{B}|R\rangle \langle R|\Theta^{\mu\nu}|0\rangle \delta\left(m_{\mathrm{R}}^2 - s\right) . \tag{155}$$

With this form, one can reconstruct the dispersive part from the dispersion relation with suitable subtraction constants.

The vacuum-to-hadron transition amplitudes are parametrized as

$$\langle S|\Theta^{\mu\nu}|0\rangle = \tfrac{1}{3} f_S q^2 Q^{\mu\nu}\,, \qquad \langle T|\Theta^{\mu\nu}|0\rangle = f_T m_T^2 \epsilon_\lambda^{\mu\nu}\,, \tag{156}$$

where $\epsilon_\lambda^{\mu\nu}$ is the spin-2 polarization tensor, which is symmetric, $\epsilon_\lambda^{\mu\nu} = \epsilon_\lambda^{\nu\mu}$, traceless, $g_{\mu\nu}\epsilon_\lambda^{\mu\nu} = 0$, and transverse, $q_\mu \epsilon_\lambda^{\mu\nu} = 0$. The extra factor of 1/3 in the scalar case is conventional, chosen such that $\langle S|\Theta|0\rangle = f_S q^2$. The tensor

$$Q^{\mu\nu} \equiv g^{\mu\nu} - q^\mu q^\nu / q^2 \tag{157}$$

fulfills $Q^\mu_\mu = 3$ and the conservation law $q^\mu Q_{\mu\nu} = 0$.

The sum over the tensor polarizations is given by [152, 153]

$$\sum_\lambda \epsilon_\lambda^{\alpha\beta} \epsilon_\lambda^{\mu\nu} = \frac{1}{2}\left(Q^{\mu\alpha}Q^{\nu\beta} + Q^{\nu\alpha}Q^{\mu\beta}\right) - \frac{1}{3} Q^{\mu\nu}Q^{\alpha\beta}\,. \tag{158}$$

The on-shell condition $P \cdot q = 0$ implies $P_\alpha Q^{\alpha,\beta} = P^\beta$, whereas $\bar{u}' \not{q} u = 0$ implies $\gamma_\alpha Q^{\alpha,\beta} = \gamma^\beta$, hence we obtain

$$\begin{aligned}
\sum_\lambda \epsilon_\lambda^{\alpha\beta} P_\alpha P_\beta \epsilon_\lambda^{\mu\nu} &= P^\mu P^\nu - \frac{1}{3} P^2 Q^{\mu\nu}\,, \\
\sum_\lambda \epsilon_\lambda^{\alpha\beta} P^{\{\alpha}\gamma^{\beta\}} \epsilon_\lambda^{\mu\nu} &= P^{\{\mu}\gamma^{\nu\}} - \frac{1}{3} Q^{\mu\nu} \not{P}
\end{aligned} \tag{159}$$

(*cf.* the analogous tensor structure in Eq. (125)).

8.9. Spectral properties and GFFs of the pion

In pQCD, one derives the asymptotic formulas [55, 92, 93]

$$A(t) = -3D(t)\,(1 + \mathcal{O}(\alpha)) = -\frac{48\pi\alpha(t) f_\pi^2}{t}\,(1 + \mathcal{O}(\alpha))\;, \tag{160}$$

hence the dispersion relations and their ramifications hold similarly to the case of the vector form factor, *cf.* Eq. (136). In the present case, Watson's theorem implies that at $4m_\pi^2 < s < 4m_K^2$, one has

$$\mathrm{Im}\,\Theta(s) = |\Theta(s)| \sin\delta_{00}(s)\,, \qquad \mathrm{Im}\,A(s) = |A(s)| \sin\delta_{02}(s)\,. \tag{161}$$

The *on-shell* couplings of the resonances to the $\pi\pi$ continuum are denoted as

$$\begin{aligned}
\langle S|\pi\pi\rangle &= g_{S\pi\pi}\,, \\
\langle T|\pi\pi\rangle &= g_{T\pi\pi}\epsilon_\lambda^{\alpha\beta} P^\alpha P^\beta = g_{T\pi\pi}\epsilon_\lambda^{\alpha\beta} p'^\alpha p^\beta\,.
\end{aligned} \tag{162}$$

Thus, we get

$$\begin{aligned}\frac{1}{\pi}\mathrm{Im}\,\langle\pi\pi|\Theta^{\mu\nu}|0\rangle &= \sum_S \frac{g_{S\pi\pi}f_S}{3}\delta\left(m_S^2-q^2\right)\left(g^{\mu\nu}q^2-q^\mu q^\nu\right) \\ &\quad+\sum_{T,\lambda}\epsilon_\lambda^{\alpha\beta}P^\alpha P^\beta \epsilon_\lambda^{\mu\nu} g_{T\pi\pi}f_T\delta\left(m_T^2-q^2\right)\,, \end{aligned} \tag{163}$$

which naturally complies with a separate conservation for each spin channel contribution when contracting with q^μ. Therefore, in the narrow resonance large-N_c-motivated limit, the result is

$$\begin{aligned}\frac{1}{\pi}\mathrm{Im}\,A(s) &= \frac{1}{2}\sum_T g_{T\pi\pi}f_T\delta\left(m_T^2-q^2\right)\,, \\ \frac{1}{\pi}\mathrm{Im}\,\Theta(s) &= \sum_S g_{S\pi\pi}f_S m_S^2\delta\left(m_S^2-q^2\right)\,. \end{aligned} \tag{164}$$

As expected, A and Θ get contributions exclusively from the 2^{++} and 0^{++} states, respectively.

The minimal hadronic ansatz in a channel with good quantum numbers corresponds to the simplest meson dominance of zero-width resonances compatible with normalization conditions and pQCD (modulo α_S corrections). In our case, we have [95]

$$A(t)=\frac{m_{f_2}^2}{m_{f_2}^2-t}\,, \qquad \Theta(t)=2m_\pi^2+\frac{m_\sigma^2 t}{m_\sigma^2-t}\,. \tag{165}$$

From here, the D-term becomes

$$D_\pi(0)=-1+\frac{4m_\pi^2}{3m_{f_2}^2}\,. \tag{166}$$

The fit to the MIT lattice data [21] yields $m_{f_2}^*=1.24(3)$ GeV and $m_\sigma^*=0.65(3)$ GeV [95]. The asterisks indicate that in the comparison/fit, we have taken the lattice value of $m_\pi=170$ MeV. The Druck form factor is obtained from Eq. (123) with the earlier fitted $A(t)$ and $\Theta(t)$. We can see from Fig. 5 that the fit with Eq. (165) is well within the uncertainties of the lattice determination.

To relate the form factors at *different* pion masses, a mass-independent renormalization scheme is needed, such as $\overline{\mathrm{MS}}$ in Chiral Perturbation Theory, where the so-called gravitational low-energy constants L_{11}, L_{12}, L_{13} [14] are

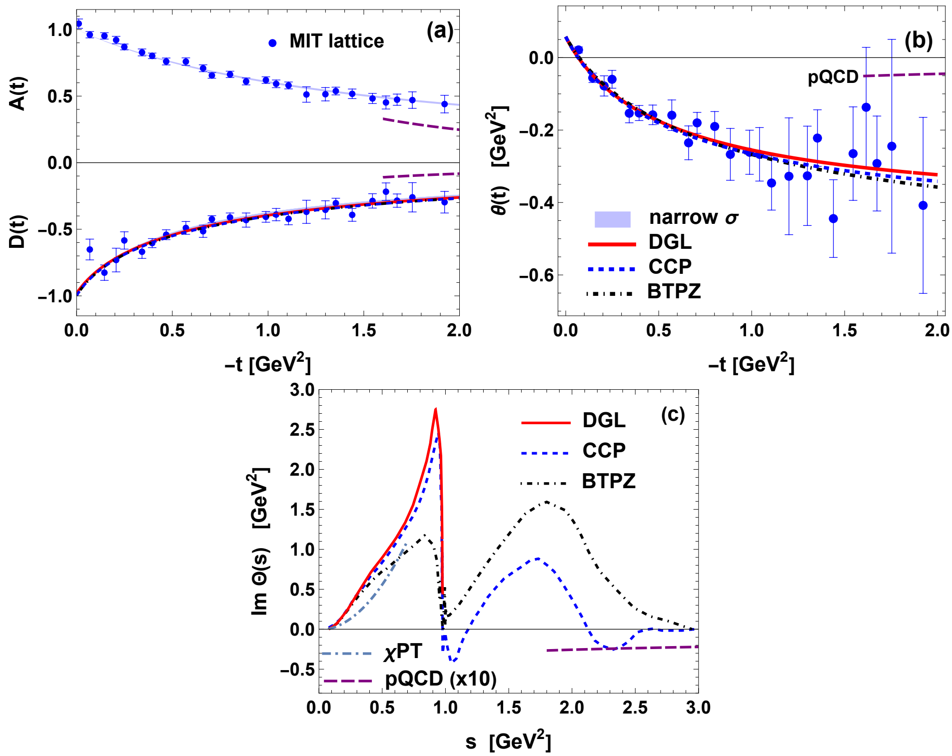

Fig. 5. The GFFs of the pion: (a) $A(t)$ and $D(t)$, (b) $\Theta(t)$, and (c) the spectral 0^{++} functions. The legend indicates various spectral models used in the scalar sector. The long-dashed lines indicate the LO pQCD formulas, while the blue dot-dashed line in (c) corresponds to the χPT result.

needed. The analysis with the MIT lattice data yields the following change when going from $m_\pi = 170$ MeV to the physical value of $m_\pi = 140$ MeV [95]:

$$m_\sigma^* = 0.65(3) \to m_\sigma = 0.63(6)\,, \qquad m_{f_2}^* = 1.24(3) \to m_{f_2} = 1.27(4)\,. \tag{167}$$

The Druck term at $m_\pi = 140$ MeV becomes, accordingly,

$$D(0) = -0.95(3)\,. \tag{168}$$

8.10. Spectral properties and GFFs of the nucleon

From pQCD [92, 93] at large $Q^2 = -t$, one has

$$A(t) \sim +\frac{\alpha(t)^2}{(-t)^2}\,, \quad J(t) \sim +\frac{\alpha(t)^2}{(-t)^2}\,, \quad B(t) \sim -\frac{\alpha(t)^2}{(-t)^3}\,, \quad D(t) \sim -\frac{\alpha(t)^2}{(-t)^3}\,. \tag{169}$$

Correspondingly, the discontinuities at large s have the behavior

$$\begin{aligned} \mathrm{Im}\, A(s) &\sim +\frac{1}{s^2L^3}\,, \qquad \mathrm{Im}\, J(s) \sim +\frac{1}{s^2L^3}\,, \\ \mathrm{Im}\, B(s) &\sim +\frac{1}{s^3L^3}\,, \qquad \mathrm{Im}\, D(s) \sim +\frac{1}{s^3L^3}\,, \end{aligned} \tag{170}$$

where $L = \log s/\Lambda^2_{\mathrm{QCD}}$. This asymptotics shows that one can use the unsubtracted dispersion relations in the GFF analysis. Moreover, the superconvergence sum rules hold.

From Watson's theorem, in the range of $4m_\pi^2 < s < 4m_K^2$, one obtains

$$\begin{aligned} \mathrm{Im}\, \Theta(t) &= \frac{3\sigma_\pi |f_{0,+}(t)||\Theta_\pi(t)|}{2\left(4m_N^2 - t\right)} > 0\,, \\ \mathrm{Im}\, J(t) &= \frac{3t^2\sigma_\pi^5}{64\sqrt{6}} |f_{2,-}(t)||A_\pi(t)| > 0\,, \\ \mathrm{Im}\, A(t) + \frac{2t\mathrm{Im}\, J(t)}{4m_N^2 - t} &= \frac{3t^2 m_N \sigma_\pi^5}{32\sqrt{6}} |f_{2,+}(t)||A_\pi(t)| > 0\,, \end{aligned} \tag{171}$$

where $\sigma_\pi = \sqrt{1-4m_\pi^2/t}$, and $f_{l\pm}$ indicate the helicity non-flip and helicity flip amplitudes for the partial wave l in the $\pi\pi \to N\bar{N}$ process. These formulas allow one to obtain the threshold behavior of GFFs from the known threshold behavior of the amplitudes $f_{l\pm}$ [98].

The *on-shell* couplings of the resonances to the $N\bar{N}$ continuum are taken as [154], re-written in a suitable form[22]

$$\langle S|N\bar{N}\rangle = g_{SNN}\,, \tag{172}$$

$$\langle T|N\bar{N}\rangle = \epsilon_\lambda^{\alpha\beta}\bar{v}\left[g_{TNN}P^{\{\alpha}\gamma^{\beta\}} + \frac{i\,P_{\{\mu}\sigma_{\nu\}\rho}q^\rho}{2m_N} f_{TNN}\right] u\,. \tag{173}$$

Thus, we get

$$\begin{aligned} \frac{1}{\pi}\mathrm{Im}\,\langle N\bar{N}|\,\Theta^{\mu\nu}\,|0\rangle &= \sum_S \frac{g_{SNN}f_S}{3}\delta\left(m_S^2 - q^2\right) m_S^2 Q^{\mu\nu} \\ &+ \sum_{T,\lambda} \epsilon_\lambda^{\alpha\beta}\left[g_{TNN}P^{\{\alpha}\gamma^{\beta\}} + \frac{i\,P_{\{\mu}\sigma_{\nu\}\rho}q^\rho}{2m_N} f_{TNN}\right] \epsilon_\lambda^{\mu\nu} f_T \delta\left(m_T^2 - q^2\right)\,, \end{aligned} \tag{174}$$

[22] These Dirac–Pauli couplings correspond to the replacements generated by the Gordon identity $g_{TNN} = g'_{TNN} + m_N f'_{TNN}$ and $f_{TNN} = -m_N f'_{TNN}$ with the primed couplings those from Ref. [98].

which naturally complies with separate conservation for each term, yielding zero when contracted with q^μ. Therefore, in the narrow resonance large-N_c-motivated approach we get

$$\frac{1}{\pi}\mathrm{Im}\,A(s) = \sum_T g_{TNN} f_T \delta\left(m_T^2 - q^2\right) , \tag{175}$$

$$\frac{1}{\pi}\mathrm{Im}\,B(s) = \sum_T f_{TNN} f_T \delta\left(m_T^2 - q^2\right) , \tag{176}$$

$$\frac{1}{\pi}\mathrm{Im}\,\Theta(s) = \sum_S g_{SNN} f_S m_S^2 \delta\left(m_S^2 - q^2\right) , \tag{177}$$

where, as expected, A and Θ get contributions exclusively from the 2^{++} and 0^{++} states, respectively. The normalization is

$$A(0) = 1\,, \qquad B(0) = 0\,, \qquad \Theta(0) = m_N\,, \tag{178}$$

while the high-energy behavior follows from Eq. (169)

$$A(t) \sim \frac{\alpha^2}{t^2}\,, \qquad B(t) \sim \frac{\alpha^2}{t^3}\,, \qquad \Theta(t) \sim \frac{\alpha^2}{t^2}\,. \tag{179}$$

The minimal hadronic ansatz complying to the above requirements, and taking $B(t) = 0$,[23] yields

$$\begin{aligned} A(t) &= 2J(t) = \frac{1}{\left(1 - t/m_{f_2}^2\right)\left(1 - t/m_{f_2'}^2\right)}\,, \\ B(t) &= 0\,, \\ \Theta(t) &= \frac{m_N}{\left(1 - t/m_\sigma^2\right)\left(1 - t/m_{f_0}^2\right)}\,. \end{aligned} \tag{180}$$

Correspondingly, the D-term becomes

$$D_N(0) = \frac{4m_N^2}{3}\left(-\frac{1}{m_\sigma^2} - \frac{1}{m_{f_0}^2} + \frac{1}{m_{f_2}^2} + \frac{1}{m_{f_2'}^2}\right)\,. \tag{181}$$

We use the PDG [140] for the masses of $f_0(980)$, $f_2(1270)$, and $f_2'(1525)$ (no fit is carried out here) and $m_\sigma = 650(50)$ MeV (consistent with the value

[23] The somewhat surprising smallness of $B(t)$ (compatible with zero within the uncertainties of the MIT lattice data at the unphysical pion mass value $m_\pi = 170$ MeV), has been disclosed in [98]. At present, it is unclear if $B(t)$ becomes larger or smaller for the physical pion mass. For a recent attempt to explain the smallness of $B(t)$, see [155].

obtained in the pion case). The result is shown in Fig. 6. As we can see, the agreement is good, with the model curves falling within the error bands of the MIT data. Ansatz (180) can be improved by relaxing the condition $B(t) = 0$, which yields an even better agreement [98].

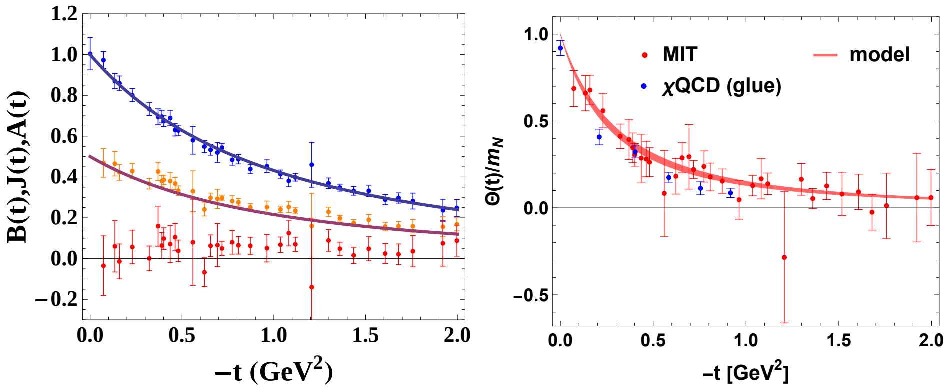

Fig. 6. The minimal hadronic ansatz for GFFs of the nucleon, compared to the MIT lattice data of [20] (points) at $m_\pi = 170$ MeV. For additional comparison, we also display the glue contribution to the trace anomaly form factor at $m_\pi = 253$ MeV from the χQCD Collaboration [26] (blue points in the right panel).

8.11. The incompleteness problem for GFFs revisited

Involved calculations [91] of the pion and nucleon GGFs use the Roy equations and the Roy–Steiner equations, respectively. These are rigorous approaches, resting on crossing, unitarity, and analyticity below a given maximal CM energy, which typically corresponds to the $N\bar{N}$ production threshold. Although the approach is theoretically quite appealing, since the low-intermediate energy region is accurately described, the superconvergence sum rules are broken to a large extent. Certain phenomenological contributions mimicking infinitely many narrow resonance via the radial Regge trajectories may mend the violations [156].

9. Transverse distributions

The physical interpretation of form factors is a subject discussed recurrently since the early works. According to our discussion in Section 2, a pictorial manner to grasp the meaning of form factors is by looking directly at the hadron mass variation due to an external gravitational field[24].

[24] The same applies of course to other currents; the electroweak form factor corresponds to a variation with respect to the electroweak fields.

Moreover, the light-front 2D transverse viewpoint is by definition Lorentz-invariant and can be regarded as a genuinely intrinsic hadron property. For a detailed discussion and derivations, we refer to [97, 157–163], where the transverse momentum density, energy density, and pressure are defined and discussed. Here, we focus on the transverse distributions which are naturally formulated in the light-front formulation and have a simple partonic probabilistic interpretation.

We take the conventions $p^{\pm} = (p^0 \pm p^3)/\sqrt{2} = p_{\mp}$, such that $x \cdot p = p^+x^- + p^-x^+ - p_\perp \cdot x_\perp$ and $\mathrm{d}^4p = \mathrm{d}p^+\mathrm{d}p^-\mathrm{d}^2p_\perp$. Also, for the metric tensor $\eta^{++} = \eta^{--} = 0$ and $\eta^{+-} = \eta^{-+} = 1$. Besides, we use the light-cone spinors defined via projections

$$\begin{aligned} &\Psi_\pm = \mathcal{P}_\pm \Psi\,, \qquad \mathcal{P}_\pm = \gamma^0\gamma^\pm = \left(1 \pm \gamma^0\gamma^3\right)/\sqrt{2} \\ &\Longrightarrow \mathcal{P}_+ + \mathcal{P}_- = 1\,, \qquad \mathcal{P}_\pm^2 = \mathcal{P}_\pm = \mathcal{P}_\pm^\dagger\,, \qquad \mathcal{P}_\pm\mathcal{P}_\mp = 0\,. \end{aligned} \tag{182}$$

In QCD, the EM current and SEM written in LC coordinates and in the gauge $A^+ = 0$ (which is ghost free) have the form

$$\begin{aligned} J^+ &= \Psi_+^\dagger Q \Psi_+\,, \\ \Theta_q^{++} &= \frac{i}{2}\left(\Psi_+^\dagger \partial^+ \Psi_+ - \partial^+\Psi_+^\dagger \Psi_+\right)\,, \qquad \Theta_g^{++} = \left(\partial^+ A_\perp^a\right)^2\,, \\ \Theta^{++} &= \Theta_q^{++} + \Theta_g^{++}\,. \end{aligned} \tag{183}$$

The field expansion for the quark field in the transverse coordinate space [159] at $x^+ = 0$ is

$$\begin{aligned} q_+(b, x^-) = &\int_0^\infty \frac{\mathrm{d}p^+}{4\pi p^+} \sum_\lambda \\ &\times \left[b_\lambda(b, p^+)\, u_{\lambda,+}(p^+)\mathrm{e}^{-ip^+x^-} + d_\lambda^\dagger(b, p^+)\, v_{\lambda,+}(p^+)\mathrm{e}^{ip^+x^-}\right], \end{aligned} \tag{184}$$

with $b_\lambda^\dagger(b, p^+)$ and $d_\lambda^\dagger(b, p^+)$ denoting the quark and antiquark creation operators with LC helicity λ. Then

$$\begin{aligned} \int \mathrm{d}x^-\, q_+^\dagger q_+ &= \sum_\lambda \int \frac{\mathrm{d}p^+}{4\pi p^+}\left[n\left(b, p^+\right) - \bar{n}_\lambda\left(b, p^+\right)\right]\,, \\ \int \mathrm{d}x^-\, q_+^\dagger i\partial^+ q_+ &= \sum_\lambda \int \frac{\mathrm{d}p^+}{4\pi p^+}\left[p^+ n_\lambda\left(b, p^+\right) - p^+\bar{n}_\lambda\left(b, p^+\right)\right]\,, \end{aligned} \tag{185}$$

where $n_\lambda(b, p^+) = b_\lambda^\dagger(b, p^+)b_\lambda(b, p^+)$ and $\bar{n}_\lambda(b, p^+) = d_\lambda^\dagger(b, p^+)d_\lambda(b, p^+)$ denote the particle and antiparticle number operators, respectively.

Thus, for $\pi^+ = u\bar{d}$ taken for definiteness,

$$\int \mathrm{d}x^- \, J^+(b,x^-) \underbrace{\to}_{\pi^+} \sum_\lambda \int \frac{\mathrm{d}p^+}{4\pi p^+} \left[\frac{2}{3} n_{u,\lambda}\left(b,p^+\right) + \frac{1}{3} n_{\bar{d},\lambda}\left(b,p^+\right) \right] . \quad (186)$$

Since $q_+^\dagger q_+$ is positive for quarks and negative for antiquarks, Eq. (186), and consequently $F(b)$ (the Fourier transform of the charge form factor in the space-like momentum space, Eq. (193)), are positive definite. For Θ_q^{++}, one also finds positivity in an analogous way

$$\int \mathrm{d}x^- \frac{i}{2}\left(\Psi_+^\dagger \partial^+ \Psi_+ - \partial^+ \Psi_+^\dagger \Psi_+\right) = i \int \mathrm{d}x^- \Psi_+^\dagger \partial^+ \Psi_+$$
$$\underbrace{\to}_{\pi^+} \sum_\lambda \int \frac{\mathrm{d}p^+}{4\pi p^+} \left[p^+ n_{u,\lambda}\left(b,p^+\right) + p^+ n_{\bar{d},\lambda}\left(b,p^+\right) \right] . \quad (187)$$

9.1. Wave packets on the light-front

Consider a normalized state as a wave packet

$$|\phi\rangle = \int \frac{\mathrm{d}^2 p_\perp \, \mathrm{d}p^+}{(2\pi)^3 2p^+} \tilde{\phi}\left(p_\perp, p^+\right) \left|p_\perp, p^+\right\rangle , \quad (188)$$

from where the scalar product is

$$\begin{aligned} \langle \phi | \psi \rangle &= \int \frac{\mathrm{d}^2 p_\perp \, \mathrm{d}p^+}{(2\pi)^3 2p^+} \tilde{\phi}\left(p_\perp, p^+\right)^* \tilde{\psi}\left(p_\perp, p^+\right) \\ &= \int \mathrm{d}^2 x_\perp \, \mathrm{d}x^- \phi\left(x_\perp, x^-\right)^* \psi\left(x_\perp, x^-\right) . \end{aligned} \quad (189)$$

The coordinate and momentum representations are related via the Fourier transform

$$\psi\left(x_\perp, x^-\right) = \int \frac{\mathrm{d}^2 p_\perp \, \mathrm{d}p^+}{\sqrt{(2\pi)^3 2p^+}} \tilde{\psi}\left(p_\perp, p^+\right) \mathrm{e}^{i\left(x_\perp \cdot p_\perp - p^+ x^-\right)} . \quad (190)$$

The integration over the x^- coordinate in the local operator allows one to define the transverse wave packet distribution in the transverse coordinate, $b = x_\perp$, as follows:

$$n_\psi(b) = \int \mathrm{d}x^- \left|\psi\left(b, x^-\right)\right|^2 = \int_0^\infty \frac{\mathrm{d}p^+}{4\pi p^+} \left| \int \frac{\mathrm{d}^2 p_\perp}{(2\pi)^2} \mathrm{e}^{ib\cdot p_\perp} \tilde{\psi}\left(p_\perp, p^+\right) \right|^2 . \quad (191)$$

We take the $x^+ = 0$ quantization surface. Using translational invariance, after some straightforward manipulations, one obtains an intuitive formula for the expectation value of the electromagnetic current J^+

$$\langle\psi| \int \mathrm{d}x^- \, J^+(b, x^-)|\psi\rangle = \int \mathrm{d}^2 b' n_\psi \left(b - b'\right) F\left(b'\right) , \tag{192}$$

where $F(b)$ is the Fourier transform of the charge form factor in the space-like momentum space

$$F(b) = \int \frac{\mathrm{d}^2 q_\perp}{(2\pi)^2} F\left(-q_\perp^2\right) \, \mathrm{e}^{-iq_\perp \cdot b} . \tag{193}$$

For a localized wave packet $n_\psi(b) \to \delta^{(2)}(b)$ and $n_\psi^+(b) \to p^+\delta^{(2)}(b)$, hence one has

$$\langle\psi| \int \mathrm{d}x^- \, J^+\left(b, x^-\right)|\psi\rangle \to F(b) . \tag{194}$$

Transverse charge density is invariant under longitudinal boosts.

9.2. Transverse distributions of GFFs

Similarly, it is straightforward to show that $A(b)$ is the relative distribution of P^+ in the transverse coordinate space

$$\begin{aligned} \Theta^{++}(b) &= \int \frac{\mathrm{d}^2 q_\perp}{2P^+(2\pi)^2} \, \mathrm{e}^{-iq_\perp \cdot b} \, 2{P^+}^2 A\left(q_\perp^2\right) = P^+ A(b) , \\ \int \mathrm{d}^2 b \, \Theta^{++}(b) &= P^+ . \end{aligned} \tag{195}$$

The transverse energy density is

$$\Theta^{+-}(b) = \int \frac{\mathrm{d}^2 q_\perp}{2P^+(2\pi)^2} \, \mathrm{e}^{-iq_\perp \cdot b} \left[2P^+P^- A\left(q_\perp^2\right) + \frac{1}{2} q_\perp^2 D\left(q_\perp^2\right)\right] , \tag{196}$$

and does not possess definite positivity.

9.3. Transverse densities and mechanical properties

The form factor D determines the transverse pressure $p(b)$ and the shear forces $s(b)$ as follows [18, 19]:

$$\begin{aligned} \Theta^{ij}(b) &= \frac{1}{2P^+} \int \frac{\mathrm{d}^2 q_\perp}{(2\pi)^2} \, \mathrm{e}^{-iq_\perp \cdot b} \frac{1}{2} \left[q_\perp^i q_\perp^j - \delta^{ij} q_\perp^2\right] D\left(q_\perp^2\right) \\ &= \delta^{ij} p(b) + \left[\frac{b^i b^j}{b^2} - \frac{1}{2}\delta^{ij}\right] s(b) . \end{aligned}$$

The trace of GFF is given by

$$\begin{aligned}\Theta^{\mu}_{\mu}(b) &= 2\Theta^{+-}(b) - \Theta^{11}(b) - \Theta^{22}(b) = \epsilon(b) - 2p(b) \\ &\times \frac{1}{2P^+}\int \frac{\mathrm{d}^2 q_\perp}{(2\pi)^2}\,\mathrm{e}^{-iq_\perp\cdot b}\left[2\left(m_\pi^2+\frac{1}{4}q_\perp^2\right)A\left(q_\perp^2\right)+\frac{3}{2}q_\perp^2 D\left(q_\perp^2\right)\right] \\ &= \frac{1}{2P^+}\int \frac{\mathrm{d}^2 q_\perp}{(2\pi)^2}\,\mathrm{e}^{-iq_\perp\cdot b}\Theta\left(q_\perp^2\right) = \frac{1}{2P^+}\Theta(b)\,. \end{aligned} \tag{197}$$

We note that $\int_0^\infty 2\pi b\,\mathrm{d}b\,p(b) = 0$, as expected from classical mechanical stability. Also [18, 19],

$$D(0) = 2m_N \int_0^\infty 2\pi b\,\mathrm{d}b\,b^2 p(b)\,. \tag{198}$$

Interestingly, one can decompose the pressure as follows [97, 98]:

$$p(b) = \frac{m_N}{6}A(b) + \frac{1}{24m_N}\nabla_b^2 B(b) - \frac{1}{6}\Theta(b)\,, \tag{199}$$

which is displayed in Fig. 7. The contribution of A is positive according to the general argument of Eq. (195), thus repulsive and short-range, reflecting the large mass of f_2. The term from B is small and with no definite sign (here we use the meson dominance model parametrization from [98], where B is small but non-zero). Importantly, the contribution of Θ is attractive and

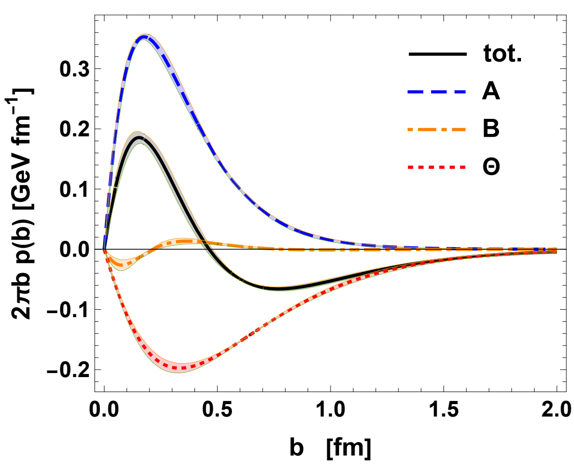

Fig. 7. Anatomy of the transverse pressure (multiplied by $2\pi b$) from the decomposition of Eq. (199). The attractive part comes from Θ, the repulsive part from A, while the contribution of B is small and changes the sign.

long-range, reflecting the smallness of the σ mass. Therefore, the meson dominance offers naturally a simple picture with a 2^{++} repulsion in the core and a 0^{++} attraction in the tail, reflecting the hierarchy of masses. Qualitatively similar results were obtained in [85, 86, 164], where the 0^{++} component office SEM is associated with the gluon contribution to the trace anomaly.

9.4. Radii

The transverse radii are defined as

$$\left\langle b^2 \right\rangle_F = \frac{\int_0^\infty 2\pi b\, b^2 F(b)}{\int_0^\infty 2\pi b\, F(b)} = \frac{4}{F(0)} \left. \frac{\mathrm{d}F(t)}{\mathrm{d}t} \right|_{t=0} . \tag{200}$$

In the model parametrization of [98],

$$\left\langle b^2 \right\rangle_A = 4 \left(-c_A + \frac{1}{m_{f_2}^2} + \frac{1}{m_{f_2'}^2} + \frac{1}{m_{f_2''}^2} + \frac{1}{m_{f_2'''}^2} \right) = [0.34(1)\ \mathrm{fm}]^2 ,$$

c_A approximately cancels the contribution $1/m_{f_2''}^2 + 1/m_{f_2'''}^2$

$$\left\langle b^2 \right\rangle_\Theta = 4 \left(\frac{1}{m_\sigma^2} + \frac{1}{m_{f_0}^2} \right) = [0.60(3)\ \mathrm{fm}]^2 ,$$

$$\left\langle b^2 \right\rangle_{\mathrm{mech}} = \frac{\int_0^\infty 2\pi b\, b^2 [p(b) + \frac{1}{2} s(b)]}{\int_0^\infty 2\pi b [p(b) + \frac{1}{2} s(b)]} = \frac{4D(0)}{\int_0^\infty \mathrm{d}(-t) D(t)} = [0.48(3)\ \mathrm{fm}]^2 .$$

Hierarchy of the radii reflects the meson mass pattern:

$$\begin{aligned} \left\langle b^2 \right\rangle_A^{1/2} &< \left\langle b^2 \right\rangle_{\mathrm{mech}}^{1/2} < \left\langle b^2 \right\rangle_\Theta^{1/2} , \\ 0.34(1) &< 0.48(3) < 0.60(3)\ [\mathrm{fm}] . \end{aligned} \tag{201}$$

For the case of the nucleon, one can use the Abel transform [165, 166] to obtain the relation of the transverse (2D) and the radial (3D) distributions. Note that for the pion, the relation does not hold, as the 3D distributions do not reflect the intrinsic structure. The corresponding 3D radii obey the hierarchy [98]

$$\begin{aligned} \left\langle r^2 \right\rangle_A^{1/2} &< \left\langle r^2 \right\rangle_J^{1/2} < \left\langle r^2 \right\rangle_E^{1/2} < \left\langle r^2 \right\rangle_{\mathrm{mech}}^{1/2} < \left\langle r^2 \right\rangle_\Theta^{1/2} , \\ 0.51(1) &< 0.57(3) < 0.67(2) < 0.72(5) < 0.90(4)\ [\mathrm{fm}] . \end{aligned} \tag{202}$$

As a benchmark, the charge radius of the proton is 0.84 fm, while its magnetic radius is 0.85 fm.

10. Summary

In these lectures, we have attempted to broadly discuss some general features of the stress-energy-momentum tensor and its matrix elements, starting from classical mechanics and classical field theory concepts, and ending up with QCD and the meson-dominance explanation of the recent lattice data for the hadronic gravitational form factors. We hope to have convinced the reader of the very basic nature and phenomenological importance of these issues.

The principal points concerning the results for the gravitational form factors of the pion and nucleon are following:

1. The gravitational form factors provide insight into the matter distribution inside hadrons, in particular the mass and forces. They are related to the generalized parton distributions as low-momentum transfers, which makes them accessible experimentally.

2. The MIT lattice benchmark data provide high-accuracy gravitational form factors for the pion and nucleon directly in the "intermediate" space-like region up to $Q^2 = 2$ GeV^2. These data are fully compatible with the meson dominance approach.

3. It is important to carry out the form factor analysis in good spin channels.

4. The matter extension (radius) is large due to the small value of the σ meson mass, $m_\sigma = 0.64(4)$ GeV. Precise modeling involves a broad σ described with an appropriate spectral function, but the description of the space-like data is largely insensitive to the spectral details.

5. The form factor $D(t)$ (the Druck term) is a combination of the good spin form factors, 0^{++} and 2^{++}, with the meson dominance applied to the MIT lattice data yielding
$$D_\pi(0) = -0.95(3)\,, \qquad D_N(0) = -3.0(4)\,. \tag{203}$$

6. The gravitational transverse distributions are intrinsic properties of hadrons. The meson dominance provides an efficient description of the transverse distributions at not too small transverse radii, $b \gtrsim 0.1$ fm.

The main general features of GFFs of the pion and nucleon are collected in Tables 2 and 3, containing the properties of the spectral densities, form factors for the space-like momenta, and the transverse densities. The signs of the low and high values of the arguments are indicated with $\pm$. For the case of spectral densities, "low" means the behavior near the 2π production threshold, while "high" denotes the asymptotic limit. For the remaining cases, "low" means the zero argument. Labels pQCD, 2π, and sym. indicate the reason for the listed behavior: perturbative QCD, the two-pion threshold, and the symmetry (the Ward–Takahashi identity), respectively.

Table 2. Summary of basic properties of GFFs of the pion.

Quantity	Low limit		Intermediate range	High limit	
$\mathrm{Im}\, A(s)$	$+$	2π	changes sign	$-$	pQCD
$\mathrm{Im}\, D(s)$	$-$		changes sign	$+$	
$\mathrm{Im}\, \Theta(s)$	$+$		changes sign	$-$	
$A\left(-Q^2\right)$	1	sym.		$+$	pQCD
$D\left(-Q^2\right)$	$-1+\mathcal{O}\left(m_\pi^2\right)$			$-$	
$\Theta\left(-Q^2\right)$	$2m_\pi^2$		changes sign	$-$	
$A(b)$	$+\infty$	pQCD	positive definite	$+$	2π
$\Theta(b)$	$-\infty$		changes sign	$+$	
$p(b)$	$+\infty$		changes sign	$-$	

Table 3. Summary of basic properties of GFFs of the nucleon.

Quantity	Low limit		Intermediate range	High limit	
$\mathrm{Im}\, A(s)$	$+$	2π	changes sign	$+$	pQCD
$\mathrm{Im}\, J(s)$	$+$		changes sign	$+$	
$\mathrm{Im}\, B(s)$	$+$		changes sign	$+$	
$\mathrm{Im}\, D(s)$	$-$		changes sign	$+$	
$\mathrm{Im}\, \Theta(s)$	$+$		changes sign	$-$	
$A\left(-Q^2\right)$	1	sym.		$+$	pQCD
$J\left(-Q^2\right)$	$\frac{1}{2}$			$+$	
$B\left(-Q^2\right)$	0			$-$	
$D\left(-Q^2\right)$				$-$	
$\Theta\left(-Q^2\right)$	m_N		changes sign	$-$	
$A(b)$	$+$		positive definite	$+$	2π
$\Theta(b)$				$+$	
$p(b)$			changes sign	$-$	

We are grateful to the authors of Refs. [20] and [26] for providing us with the data used in the figures. We also thank Pablo Sanchez-Puertas for many discussions. E.R.A. was supported by the Spanish MINECO and European FEDER funds grant and by project No. PID2023-147072NB-I00 funded by MCIN/AEI/10.13039/501100011033, and by the Junta de Andalucía grant FQM-225.

Appendix A

Watson's theorem

We discuss the unitarity conditions in coupled channels when one channel is closed. The goal is to address the $\pi\pi$ effect in the processes $g^*, \gamma^* \to N\bar{N}$ above the $\pi\pi$ threshold but below the $N\bar{N}$ threshold. The unitarity of the S-matrix as a sum over components reads

$$SS^\dagger = 1 \implies \sum_n S_{in} S^*_{fn} = \delta_{if} \, .$$

In general, we have

$$S_{if} = \delta_{if} + 2\pi i \delta \left(E_f - E_i \right) T_{if} \, ,$$

which implies the generalized optical theorem

$$T_{if} - T^*_{fi} = 2\pi i \sum_n T_{in} T^*_{fn} \delta \left(E_n - E_i \right) \, ,$$

where n are the open-channels. Due to the time-reversal symmetry, the S-matrix is symmetric, *i.e.*, $S^T = S$. The channels may be open or closed, such that the S-matrix acquires a block diagonal form where the closed channels submatrix have a purely real S-matrix. Therefore,

$$\sum_n S_{in} S^*_n = \delta_{if} \, .$$

In our case, $n = e^+e^-$, $\pi^+\pi^-$, $K\bar{K}$, $N\bar{N}$, *etc.* In the case of only one channel open, say $\pi\pi \to \pi\pi$, we have $S_{11} = \mathrm{e}^{2i\delta_1}$, with δ_1 denoting the phase-shift.

As a warm up, let us consider first the case of two channels, $1 = \pi\pi$ and $2 = K\bar{K}$, and deduce the unitarity condition on the transition $\pi\pi \to K\bar{K}$ below the $K\bar{K}$ threshold. Then

$$SS^\dagger = SS^* = \begin{pmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{pmatrix} \begin{pmatrix} S^*_{11} & S^*_{12} \\ S^*_{12} & S^*_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\implies \begin{cases} |S_{11}|^2 + |S_{12}|^2 = 1 \\ S_{11} S^*_{12} + S_{12} S^*_{22} = 0 \\ |S_{12}|^2 + |S_{22}|^2 = 1 \end{cases} \implies \frac{S_{12}}{S^*_{12}} = -\frac{S_{11}}{S^*_{22}} \, . \qquad \text{(A.1)}$$

Using the fact that channel 2 is closed, $S_{22} = S_{22}^*$, and $S_{11} = |S_{11}|\, \mathrm{e}^{2i\delta_1} \equiv \eta_1\, \mathrm{e}^{2i\delta_1}$, we find that

$$\frac{S_{12}}{S_{12}^*} = -\frac{\eta_1\, \mathrm{e}^{2i\delta_1}}{S_{22}} \implies S_{12} = \pm i |S_{12}|\, \mathrm{e}^{i\delta_1} \,.$$

Therefore, in this case, we have

$$S = \begin{pmatrix} \eta_1\, \mathrm{e}^{2i\delta_1} & +i\sqrt{1-\eta_1^2}\, \mathrm{e}^{i\delta_1} \\ +i\sqrt{1-\eta_1^2}\, \mathrm{e}^{i\delta_1} & \eta_1 \end{pmatrix} ,$$

which corresponds to the case of both channels opened

$$S = \begin{pmatrix} \eta_1\, \mathrm{e}^{2i\delta_1} & +i\sqrt{1-\eta_1^2}\, \mathrm{e}^{i(\delta_1+\delta_2)} \\ +i\sqrt{1-\eta_1^2}\, \mathrm{e}^{i(\delta_1+\delta_2)} & \eta_1\, \mathrm{e}^{2i\delta_2} \end{pmatrix} ,$$

when $\delta_2 \to 0$.

Next, we consider the case where both channels are open, but one is weakly coupled (for instance, $1 = \pi\pi$ and $2 = e^+e^-$), such that $S_{22} = 1 + \ldots$ and $S_{12} = +iF + \ldots$. Then we find to the first order in F that

$$SS^\dagger = SS^* = \begin{pmatrix} S_{11} & iF \\ iF & 1 \end{pmatrix} \begin{pmatrix} S_{11}^* & -iF^* \\ -iF^* & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A.2}$$

$$\implies \begin{cases} |S_{11}|^2 = 1 \\ -F^* S_{11} + F = 0 \end{cases}$$

$$\implies \frac{F}{F^*} = S_{11} = e^{2i\delta_1} \implies F = |F|\, \mathrm{e}^{i\delta_1} \implies \mathrm{Im}\, F = |F| \sin\delta_1 ,$$

which is Watson's theorem for one channel. Note that for attractive interactions $\delta_1 > 0$, hence $\mathrm{Im}\, F > 0$.

Appendix B

The half-width rule

As we have mentioned, PDG [140] provides a summary of estimates of resonance masses and widths for mesons with given J^{PC}, but the more detailed information about phase-shifts is not always available in the literature. So, what is a reasonable numerical mass value we have to take when mapping a resonance into a monopole form factor? A rather conservative estimate of the uncertainty is given by the *half-width rule*. Quite generally, we have

$$\text{Amplitude} = \text{Background} + \text{Resonance}\,.$$

Realistically, one would thus have a sum of the two contributions

$$\rho(s) = \rho_{\rm B}(s) + Z_{\rm R}\rho_{\rm R}(s)\,. \tag{B.1}$$

For a BW resonance profile,

$$\rho_{\rm R}(s) = \frac{1}{\pi}\frac{M\Gamma}{\left(M^2-s\right)^2+\Gamma^2M^2}\,, \qquad \int \rho(s) = 1\,, \tag{B.2}$$

where the normalization assumes that we integrate for simplicity over the whole axis. In a probabilistic interpretation, we have

$$\frac{\rho\left(M_{\rm R}^2 \pm \Gamma_{\rm R}M_{\rm R}\right)}{\rho\left(M_{\rm R}^2\right)} = \frac{1}{2}\,. \tag{B.3}$$

In Fig. 8, we compare a Gaussian and BW shapes. As we can see, the line shapes are very similar within the half-width rule interval.

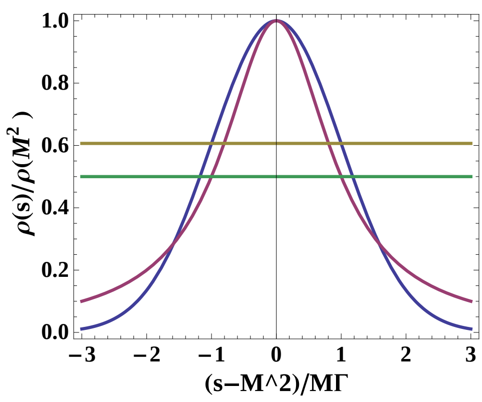

Fig. 8. Illustration of the half-width rule comparing a Gaussian and a BW distribution.

REFERENCES

[1] E.E. Chambers, R. Hofstadter, «Structure of the Proton», *Phys. Rev.* **103**, 1454 (1956).

[2] E.B. Hughes *et al.*, «Neutron Form Factors from Inelastic Electron–Deuteron Scattering», *Phys. Rev.* **139**, B458 (1965).

[3] R.P. Feynman, «Quantum Theory of Gravitation», *Acta Phys. Pol.* **24**, 697 (1963); *Erratum ibid.* **25**, 855 (1964).

[4] B.S. DeWitt, «Quantum Theory of Gravity. III. Applications of the Covariant Theory», *Phys. Rev.* **162**, 1239 (1967).

[5] J.F. Donoghue, «General relativity as an effective field theory: The leading quantum corrections», *Phys. Rev. D* **50**, 3874 (1994), arXiv:gr-qc/9405057.

[6] L. Buoninfante *et al.*, «Visions in quantum gravity», *SciPost Phys. Comm. Rep.* **11**, 1 (2025), arXiv:2412.08696 [hep-th].

[7] I.Y. Kobzarev, L.B. Okun, «Gravitational Interaction of Fermions», *Zh. Eksp. Teor. Fiz.* **43**, 1904 (1962).

[8] D.H. Sharp, W.G. Wagner, «Asymptotic Behavior of Nucleon–Nucleon Scattering», *Phys. Rev.* **131**, 2226 (1963).

[9] H. Pagels, «Energy-Momentum Structure Form Factors of Particles», *Phys. Rev.* **144**, 1250 (1966).

[10] K. Raman, «Gravitational Form Factors of Pseudoscalar Mesons, Stress-Tensor-Current Commutation Relations, and Deviations from Tensor- and Scalar-Meson Dominance», *Phys. Rev. D* **4**, 476 (1971).

[11] M.G. Hare, G. Papini, «Mass Radius of the Nucleon», *Can. J. Phys.* **50**, 1163 (1972).

[12] T.N. Truong, R.S. Willey, «Branching ratios for decays of light Higgs bosons», *Phys. Rev. D* **40**, 3635 (1989).

[13] J. Gasser, U.G. Meißner, «Chiral expansion of pion form factors beyond one loop», *Nucl. Phys. B* **357**, 90 (1991).

[14] J.F. Donoghue, H. Leutwyler, «Energy and momentum in chiral theories», *Z. Phys. C* **52**, 343 (1991).

[15] X.D. Ji, «QCD Analysis of the Mass Structure of the Nucleon», *Phys. Rev. Lett.* **74**, 1071 (1995), arXiv:hep-ph/9410274.

[16] X.D. Ji, «Deeply virtual Compton scattering», *Phys. Rev. D* **55**, 7114 (1997), arXiv:hep-ph/9609381.

[17] M.V. Polyakov, C. Weiss, «Skewed and double distributions in the pion and the nucleon», *Phys. Rev. D* **60**, 114017 (1999), arXiv:hep-ph/9902451.

[18] M.V. Polyakov, «Generalized parton distributions and strong forces inside nucleons and nuclei», *Phys. Lett. B* **555**, 57 (2003), arXiv:hep-ph/0210165.

[19] M.V. Polyakov, P. Schweitzer, «Forces inside hadrons: Pressure, surface tension, mechanical radius, and all that», *Int. J. Mod. Phys. A* **33**, 1830025 (2018), arXiv:1805.06596 [hep-ph].

[20] D.C. Hackett, D.A. Pefkou, P.E. Shanahan, «Gravitational Form Factors of the Proton from Lattice QCD», *Phys. Rev. Lett.* **132**, 251904 (2024), arXiv:2310.08484 [hep-lat].

[21] D.C. Hackett, P.R. Oare, D.A. Pefkou, P.E. Shanahan, «Gravitational form factors of the pion from lattice QCD», *Phys. Rev. D* **108**, 114504 (2023), arXiv:2307.11707 [hep-lat].

[22] D. Brömmel, «Pion Structure from the Lattice», Ph.D. Thesis, Regensburg University, 2007.

[23] QCDSF and UKQCD collaborations (D. Brömmel *et al.*), «Spin Structure of the Pion», *Phys. Rev. Lett.* **101**, 122001 (2008), arXiv:0708.2249 [hep-lat].

[24] J. Delmar *et al.*, «Generalized form factors of the pion and kaon using twisted mass fermions», *PoS* **(LATTICE2023)**, 308 (2024), arXiv:2401.04080 [hep-lat].

[25] P.E. Shanahan, W. Detmold, «Gluon gravitational form factors of the nucleon and the pion from lattice QCD», *Phys. Rev. D* **99**, 014511 (2019), arXiv:1810.04626 [hep-lat].

[26] χQCD Collaboration (B. Wang *et al.*), «Trace anomaly form factors from lattice QCD», *Phys. Rev. D* **109**, 094504 (2024), arXiv:2401.05496 [hep-lat].

[27] Belle Collaboration (M. Masuda *et al.*), «Study of π^0 pair production in single-tag two-photon collisions», *Phys. Rev. D* **93**, 032003 (2016), arXiv:1508.06757 [hep-ex].

[28] S. Kumano, Q.T. Song, O.V. Teryaev, «Hadron tomography by generalized distribution amplitudes in the pion-pair production process $\gamma^*\gamma \to \pi^0\pi^0$ and gravitational form factors for pion», *Phys. Rev. D* **97**, 014020 (2018), arXiv:1711.08088 [hep-ph].

[29] CLAS Collaboration (H.S. Jo *et al.*), «Cross Sections for the Exclusive Photon Electroproduction on the Proton and Generalized Parton Distributions», *Phys. Rev. Lett.* **115**, 212003 (2015), arXiv:1504.02009 [hep-ex].

[30] V.D. Burkert, L. Elouadrhiri, F.X. Girod, «The pressure distribution inside the proton», *Nature* **557**, 396 (2018).

[31] GlueX Collaboration (A. Ali *et al.*), «First Measurement of Near-Threshold J/Ψ Exclusive Photoproduction off the Proton», *Phys. Rev. Lett.* **123**, 072001 (2019), arXiv:1905.10811 [nucl-ex].

[32] X.Y. Wang, F. Zeng, Q. Wang, «Systematic analysis of the proton mass radius based on photoproduction of vector charmoniums», *Phys. Rev. D* **105**, 096033 (2022), arXiv:2204.07294 [hep-ph].

[33] Y. Guo, F. Yuan, W. Zhao, «Bayesian Inferring Nucleon Gravitational Form Factors via Near-Threshold J/Ψ Photoproduction», *Phys. Rev. Lett.* **135**, 111902 (2025), arXiv:2501.10532 [hep-ph].

[34] D.E. Kharzeev, «Mass radius of the proton», *Phys. Rev. D* **104**, 054015 (2021), arXiv:2102.00110 [hep-ph].

[35] MMGPDs Collaboration (M. Goharipour *et al.*), «Mechanical properties of the nucleon from the generalized parton distributions», *Phys. Rev. D* **112**, 014016 (2025), arXiv:2501.16257 [hep-ph].

[36] Q.T. Song, O.V. Teryaev, S. Yoshida, «Gravitational form factors in the perturbative limit», *Phys. Lett. B* **868**, 139797 (2025), arXiv:2503.11316 [hep-ph].

[37] J. Han, B. Pire, Q.T. Song, «Baryon–antibaryon generalized distribution amplitudes and $e^+e^- \to B\bar{B}\gamma$», *Phys. Rev. D* **112**, 014048 (2025), arXiv:2506.09854 [hep-ph].

[38] J. Han, B. Pire, Q.T. Song, «Accessing baryon–antibaryon generalized distribution amplitudes in $e^\pm\gamma \to e^\pm B\bar{B}$», *Phys. Rev. D* **113**, 014027 (2026), arXiv:2511.05970 [hep-ph].

[39] H. Alharazin, J.Y. Panteleeva, «Reconstruction of gravitational form factors using generative machine learning», *Phys. Rev. D* **113**, 116007 (2026), arXiv:2602.19267 [hep-ph].

[40] Y. Hatta, J. Schoenleber, «Sullivan Process Near Threshold and the Pion Gravitational Form Factors», *Phys. Rev. Lett.* **134**, 251901 (2025), arXiv:2502.12061 [hep-ph].

[41] A.A. Andrianov, V.A. Andrianov, V.L. Yudichev, «Chiral bosonization of $U_A(1)$-currents and the energy-momentum tensor in quantum chromodynamics», *J. Math. Sci.* **88**, 142 (1998).

[42] E. Megias, E. Ruiz Arriola, L.L. Salcedo, W. Broniowski, «Low-energy chiral Lagrangian from the spectral quark model», *Phys. Rev. D* **70**, 034031 (2004).

[43] E. Megias, E. Ruiz Arriola, L.L. Salcedo, «Energy momentum tensor of chiral quark models at low energies», *Phys. Rev. D* **72**, 014001 (2005).

[44] W. Broniowski, E. Ruiz Arriola, K. Golec-Biernat, «Generalized parton distributions of the pion in chiral quark models and their QCD evolution», *Phys. Rev. D* **77**, 034023 (2008), arXiv:0712.1012 [hep-ph].

[45] W. Broniowski, E. Ruiz Arriola, «Gravitational and higher-order form factors of the pion in chiral quark models», *Phys. Rev. D* **78**, 094011 (2008), arXiv:0809.1744 [hep-ph].

[46] T. Frederico, E. Pace, B. Pasquini, G. Salmé, «Pion generalized parton distributions with covariant and light-front constituent quark models», *Phys. Rev. D* **80**, 054021 (2009), arXiv:0907.5566 [hep-ph].

[47] P. Masjuan, E. Ruiz Arriola, W. Broniowski, «Meson dominance of hadron form factors and large-N_c phenomenology», *Phys. Rev. D* **87**, 014005 (2013), arXiv:1210.0760 [hep-ph].

[48] C. Fanelli *et al.*, «Pion generalized parton distributions within a fully covariant constituent quark model», *Eur. Phys. J. C* **76**, 253 (2016), arXiv:1603.04598 [hep-ph].

[49] A. Freese, I.C. Cloët, «Gravitational form factors of light mesons», *Phys. Rev. C* **100**, 015201 (2019); *Erratum ibid.* **105**, 059901 (2022).

[50] A.F. Krutov, V.E. Troitsky, «Pion gravitational form factors in a relativistic theory of composite particles», *Phys. Rev. D* **103**, 014029 (2021), arXiv:2010.11640 [hep-ph].

[51] Z. Xing, M. Ding, L. Chang, «Glimpse into the pion gravitational form factor», *Phys. Rev. D* **107**, L031502 (2023), arXiv:2211.06635 [hep-ph].

[52] Y.Z. Xu *et al.*, «Pion and kaon electromagnetic and gravitational form factors», *Eur. Phys. J. C* **84**, 191 (2024), arXiv:2311.14832 [hep-ph].

[53] Y. Li, J.P. Vary, «Stress inside the pion in holographic light-front QCD», *Phys. Rev. D* **109**, L051501 (2024), arXiv:2312.02543 [hep-th].

[54] W.Y. Liu, E. Shuryak, C. Weiss, I. Zahed, «Pion gravitational form factors in the QCD instanton vacuum. I», *Phys. Rev. D* **110**, 054021 (2024), arXiv:2405.14026 [hep-ph].

[55] W.Y. Liu, E. Shuryak, I. Zahed, «Pion gravitational form factors in the QCD instanton vacuum. II», *Phys. Rev. D* **110**, 054022 (2024), arXiv:2405.16269 [hep-ph].

[56] X. Wang *et al.*, «Bridging Electromagnetic and Gravitational Form Factors: Insights from LFHQCD», arXiv:2406.09644 [hep-ph].

[57] M.A. Sultan *et al.*, «Gravitational form factors of pseudoscalar mesons in a contact interaction», *Phys. Rev. D* **110**, 054034 (2024), arXiv:2407.10437 [hep-ph].

[58] D. Fujii, A. Iwanaka, M. Tanaka, «Gravitational form factors of pion from top-down holographic QCD», *Phys. Rev. D* **110**, L091501 (2024), arXiv:2407.21113 [hep-ph].

[59] A.F. Krutov, V.E. Troitsky, «Step toward estimation of the neutral-hadron size: The gravitational mass radius of π^0 meson in a relativistic theory of composite particles», *Phys. Rev. D* **111**, 034034 (2025), arXiv:2410.17570 [hep-ph].

[60] Y. Choi, H.D. Son, H.M. Choi, «Gravitational form factors of the pion in the self-consistent light-front quark model», *Phys. Rev. D* **112**, 014043 (2025), arXiv:2504.14997 [hep-ph].

[61] S. Puhan, S. Sharma, N. Kumar, H. Dahiya, «Understanding the Valence Quark Structure of the Pion through Generalized Transverse Momentum-Dependent Parton Distributions», *Prog. Theor. Exp. Phys.* **2025**, 083B02 (2025), arXiv:2504.14982 [hep-ph].

[62] K. Goeke, M.V. Polyakov, M. Vanderhaeghen, «Hard exclusive reactions and the structure of hadrons», *Prog. Part. Nucl. Phys.* **47**, 401 (2001), arXiv:hep-ph/0106012.

[63] A.V. Belitsky, X. Ji, «Chiral structure of nucleon gravitational form factors», *Phys. Lett. B* **538**, 289 (2002), arXiv:hep-ph/0203276.

[64] S.i. Ando, J.W. Chen, C.W. Kao, «Leading chiral corrections to the nucleon generalized parton distributions», *Phys. Rev. D* **74**, 094013 (2006), arXiv:hep-ph/0602200.

[65] M. Diehl, A. Manashov, A. Schafer, «Chiral perturbation theory for nucleon generalized parton distributions», *Eur. Phys. J. A* **29**, 315 (2006); *Erratum ibid.* **56**, 220 (2020).

[66] A.M. Moiseeva, A.A. Vladimirov, «On chiral corrections to nucleon GPD», *Eur. Phys. J. A* **49**, 23 (2013), arXiv:1208.1714 [hep-ph].

[67] M. Dorati, T.A. Gail, T.R. Hemmert, «Chiral perturbation theory and the first moments of the generalized parton distributions in a nucleon», *Nucl. Phys. A* **798**, 96 (2008), arXiv:nucl-th/0703073.

[68] H. Alharazin, D. Djukanovic, J. Gegelia, M.V. Polyakov, «Chiral theory of nucleons and pions in the presence of an external gravitational field», *Phys. Rev. D* **102**, 076023 (2020), arXiv:2006.05890 [hep-ph].

[69] C. Cebulla, K. Goeke, J. Ossmann, P. Schweitzer, «The nucleon form-factors of the energy–momentum tensor in the Skyrme model», *Nucl. Phys. A* **794**, 87 (2007), arXiv:hep-ph/0703025.

[70] M. Tanaka, D. Fujii, M. Kawaguchi, «Gravitational form factors of the nucleon in the Skyrme model based on scale-invariant chiral perturbation theory», *Phys. Rev. D* **112**, 054048 (2025), arXiv:2507.21220 [hep-ph].

[71] K. Goeke *et al.*, «Nucleon form factors of the energy-momentum tensor in the chiral quark-soliton model», *Phys. Rev. D* **75**, 094021 (2007), arXiv:hep-ph/0702030.

[72] M.J. Neubelt *et al.*, «Energy momentum tensor and the D term in the bag model», *Phys. Rev. D* **101**, 034013 (2020), arXiv:1911.08906 [hep-ph].

[73] Z. Abidin, C.E. Carlson, «Nucleon electromagnetic and gravitational form factors from holography», *Phys. Rev. D* **79**, 115003 (2009), arXiv:0903.4818 [hep-ph].

[74] K.A. Mamo, I. Zahed, «J/ψ near threshold in holographic QCD: A and D gravitational form factors», *Phys. Rev. D* **106**, 086004 (2022), arXiv:2204.08857 [hep-ph].

[75] C. Mondal, «Longitudinal momentum densities in transverse plane for nucleons», *Eur. Phys. J. C* **76**, 74 (2016), arXiv:1511.01736 [hep-ph].

[76] M. Fujita, Y. Hatta, S. Sugimoto, T. Ueda, «Nucleon D-term in holographic quantum chromodynamics», *Prog. Theor. Exp. Phys.* **2022**, 093B06 (2022), arXiv:2206.06578 [hep-th].

[77] J. Deng, D. Hou, «Nucleon structure from an AdS/QCD model in the Veneziano limit», *Phys. Rev. D* **112**, 036011 (2025), arXiv:2502.00771 [nucl-th].

[78] K.A. Mamo, «Entanglement, trace anomaly, and confinement in QCD», *Phys. Rev. D* **112**, L111506 (2025), arXiv:2507.00176 [hep-ph].

[79] K.A. Mamo, «Radius-Flow Entanglement in Hadron States and Gravitational Form Factors», arXiv:2603.03064 [hep-ph].

[80] BLFQ Collaboration (S. Nair *et al.*), «Gravitational form factors and mechanical properties of quarks in protons: A basis light-front quantization approach», *Phys. Rev. D* **110**, 056027 (2024), arXiv:2403.11702 [hep-ph].

[81] BLFQ Collaboration (S. Xu *et al.*), «Towards a first principles light-front Hamiltonian for the nucleon», *Phys. Lett. B* **867**, 139599 (2025), arXiv:2408.11298 [hep-ph].

[82] K. Azizi, U. Özdem, «Nucleon's energy-momentum tensor form factors in light-cone QCD», *Eur. Phys. J. C* **80**, 104 (2020), arXiv:1908.06143 [hep-ph].

[83] I.V. Anikin, «Gravitational form factors within light-cone sum rules at leading order», *Phys. Rev. D* **99**, 094026 (2019), arXiv:1902.00094 [hep-ph].

[84] Z. Dehghan, F. Almaksusi, K. Azizi, «Mechanical properties of proton using flavor-decomposed gravitational form factors», *J. High Energy Phys.* **2025**, 025 (2025), arXiv:2502.16689 [hep-ph].

[85] X. Ji, C. Yang, «Momentum Flow Mechanisms and Color-Lorentz Forces on Quarks in the Nucleon», *Research* **9**, 1155 (2026), arXiv:2503.01991 [hep-ph].

[86] D. Fujii, M. Kawaguchi, M. Tanaka, «Dominance of gluonic scale anomaly in confining pressure inside nucleon and D-term», *Phys. Lett. B* **866**, 139559 (2025), arXiv:2503.09686 [hep-ph].

[87] M. Kawaguchi, M. Harada, Y.L. Ma, «Origin of hadron mass from gravitational D-form factor and neutron star measurements», *Phys. Lett. B* **876**, 140400 (2026), arXiv:2512.23937 [hep-ph].

[88] A. Mejia, P. Schweitzer, «Energy-momentum tensor form factor $D(t)$ of the proton and neutron», *Phys. Rev. D* **113**, 054016 (2026), arXiv:2511.21916 [hep-ph].

[89] R. Stegeman, R. Zwicky, «Gravitational D-form factor: the σ-meson as a dilaton confronted with lattice QCD data I», *J. High Energy Phys.* **2026**, 184 (2026), arXiv:2508.18537 [hep-ph].

[90] R. Stegeman, R. Zwicky, «Gluon gravitational D-form factor: the σ-meson as a dilaton confronted with lattice data II», *J. High Energy Phys.* **2026**, 159 (2026), arXiv:2512.12315 [hep-ph].

[91] X.-H. Cao, F.-K. Guo, Q.-Z. Li, D.-L. Yao, «Dispersive determination of nucleon gravitational form factors», *Nat. Commun.* **16**, 6979 (2025), arXiv:2411.13398 [hep-ph].

[92] X.B. Tong, J.P. Ma, F. Yuan, «Gluon gravitational form factors at large momentum transfer», *Phys. Lett. B* **823**, 136751 (2021), arXiv:2101.02395 [hep-ph].

[93] X.B. Tong, J.P. Ma, F. Yuan, «Perturbative calculations of gravitational form factors at large momentum transfer», *J. High Energy Phys.* **2022**, 46 (2022), arXiv:2203.13493 [hep-ph].

[94] X. Ji, C. Yang, «A journey of seeking pressure and forces in the nucleon», *Nucl. Phys. B* **1024**, 117342 (2026), arXiv:2508.16727 [hep-ph].

[95] W. Broniowski, E. Ruiz Arriola, «Gravitational form factors of the pion and meson dominance», *Phys. Lett. B* **859**, 139138 (2024), arXiv:2405.07815 [hep-ph].

[96] E. Ruiz Arriola, W. Broniowski, «Scalar and tensor meson dominance and gravitational form factors of the pion», *PoS* **(QNP2024)**, 068 (2025), arXiv:2411.10354 [hep-ph].

[97] W. Broniowski, E. Ruiz Arriola, «Transverse Densities of the Energy-momentum Tensor and the Gravitational Form Factors of the Pion», *Acta Phys. Pol. B* **56**, 3-A18 (2025), arXiv:2412.00848 [hep-ph].

[98] W. Broniowski, E. Ruiz Arriola, «Gravitational form factors and mechanical properties of the nucleon in a meson dominance approach», *Phys. Rev. D* **112**, 054028 (2025), arXiv:2503.09297 [hep-ph].

[99] S. Weinberg, «Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity», *John Wiley and Sons*, New York 1972.

[100] E.C.G. Sudarshan, N. Mukunda, «Classical Dynamics: A Modern Perspective», *World Scientific*, 1974.

[101] A.O. Barut, «Electrodynamics and Classical Theory of Fields & Particles», *Courier Corporation*, 1980.

[102] K. Huang, «Introduction to Statistical Physics», *Chapman and Hall/CRC*, 2009.

[103] D.E. Soper, «Classical Field Theory», *Courier Dover Publications*, 2008.

[104] H. Leutwyler, «A no-interaction theorem in classical relativistic Hamiltonian particle mechanics», *Nuovo Cim.* **37**, 556 (1965).

[105] J.H. Poynting, «XV. On the transfer of energy in the electromagnetic field», *Phil. Trans. R. Soc.* **175**, 343 (1884).

[106] J.D. Jackson, «Classical Electrodynamics», *John Wiley & Sons*, 2012.

[107] J. Caro, L.L. Salcedo, «Impediments to mixing classical and quantum dynamics», *Phys. Rev. A* **60**, 842 (1999).

[108] G. Kalman, «Lagrangian Formalism in Relativistic Dynamics», *Phys. Rev.* **123**, 384 (1961).

[109] L.P. Pitaevskii, E.M. Lifshitz, «Physical Kinetics», Vol. 10, *Butterworth–Heinemann*, 2012.

[110] J.D. Bjorken, S.D. Drell, «Relativistic Quantum Fields», International Series In Pure and Applied Physics, *McGraw-Hill*, New York 1965.

[111] D.Z. Freedman, I.J. Muzinich, E.J. Weinberg, «On the energy-momentum tensor in gauge field theories», *Ann. Phys.* **87**, 95 (1974).

[112] S. Pokorski, «Gauge Field Theories», *Cambridge University Press*, 2005.

[113] K. Fukushima, T. Uji, «Energy-momentum tensor form factors and spin density distribution in the nucleon calculated in a quantized Skyrme model with vector mesons», arXiv:2603.11704 [hep-ph].

[114] C.G. Callan, Jr., S.R. Coleman, R. Jackiw, «A new improved energy-momentum tensor», *Ann. Phys.* **59**, 42 (1970).

[115] P. Beißner, B.-D. Sun, E. Epelbaum, J. Gegelia, «Gravitational form factors of the Higgs boson», *Eur. Phys. J. C* **85**, 1471 (2025), arXiv:2508.19821 [hep-ph].

[116] D.G. Boulware, S. Deser, «Stress-Tensor Commutators and Schwinger Terms», *J. Math. Phys.* **8**, 1468 (1967).

[117] H. Suura, B.L. Young, «Derivation of General Conservation Laws and Ward–Takahashi Identities in the Functional Integration Method», *Phys. Rev. D* **8**, 4353 (1973).

[118] R. Brout, F. Englert, «Gravitational Ward Identity and the Principle of Equivalence», *Phys. Rev.* **141**, 1231 (1966).

[119] L. Bessler, T. Muta, H. Umezawa, «Nucleon Mass and the Gravitational Ward–Takahashi Identity», *Phys. Rev.* **180**, 1604 (1969).

[120] W. Broniowski, V. Shastry, E. Ruiz Arriola, «Off-shell generalized parton distributions and form factors of the pion», *Phys. Lett. B* **840**, 137872 (2023), arXiv:2211.11067 [hep-ph].

[121] N.D. Birrell, P.C.W. Davies, «Quantum Fields in Curved Space», Cambridge Monographs on Mathematical Physics, *Cambridge Univ. Press*, Cambridge, UK 1984.

[122] H.Y. Won, C. Lorcé, «Relativistic energy-momentum tensor distributions in a polarized nucleon», *Phys. Rev. D* **111**, 094021 (2025), arXiv:2503.07382 [hep-ph].

[123] X.D. Ji, «Breakup of hadron masses and the energy-momentum tensor of QCD», *Phys. Rev. D* **52**, 271 (1995), arXiv:hep-ph/9502213.

[124] X.D. Ji, «Gauge-Invariant Decomposition of Nucleon Spin», *Phys. Rev. Lett.* **78**, 610 (1997), arXiv:hep-ph/9603249.

[125] E. Leader, C. Lorcé, «The angular momentum controversy: What's it all about and does it matter?», *Phys. Rep.* **541**, 163 (2014), arXiv:1309.4235 [hep-ph].

[126] C. Lorcé, «On the hadron mass decomposition», *Eur. Phys. J. C* **78**, 120 (2018), arXiv:1706.05853 [hep-ph].

[127] Y. Hatta, A. Rajan, K. Tanaka, «Quark and gluon contributions to the QCD trace anomaly», *J. High Energy Phys.* **2018**, 008 (2018), arXiv:1810.05116 [hep-ph].

[128] Extended Twisted Mass Collaboration (C. Alexandrou *et al.*), «Quark and Gluon Momentum Fractions in the Pion and in the Kaon», *Phys. Rev. Lett.* **134**, 131902 (2025), arXiv:2405.08529 [hep-lat].

[129] Extended Twisted Mass Collaboration (C. Alexandrou *et al.*), «Complete flavor decomposition of the spin and momentum fraction of the proton using lattice QCD simulations at physical pion mass», *Phys. Rev. D* **101**, 094513 (2020), arXiv:2003.08486 [hep-lat].

[130] V.A. Novikov, M.A. Shifman, «Comment on the $\psi' \to J/\psi\pi\pi$ decay», *Z. Phys. C* **8**, 43 (1981).

[131] M. Luscher, P. Weisz, «On-shell improved lattice gauge theories», *Commun. Math. Phys.* **97**, 59 (1985), *Erratum ibid.* **98**, 433 (1985).

[132] G. Barton, «Introduction to Dispersion Techniques in Field Theory», *W.A. Benjamin*, New York 1965.

[133] K. Nishijima, «Fields and Particles: Field Theory and Dispersion Relations», *W.A. Benjamin*, New York 1969.

[134] J. Nieves, E. Ruiz Arriola, «Bethe–Salpeter approach for unitarized chiral perturbation theory», *Nucl. Phys. A* **679**, 57 (2000), arXiv:hep-ph/9907469.

[135] J.F. Donoghue, E.S. Na, «Asymptotic limits and the structure of the pion form factor», *Phys. Rev. D* **56**, 7073 (1997), arXiv:hep-ph/9611418.

[136] G.J. Gounaris, J.J. Sakurai, «Finite-Width Corrections to the Vector-Meson-Dominance Prediction for $\rho \to e^+e^-$», *Phys. Rev. Lett.* **21**, 244 (1968).

[137] R. García-Martín *et al.*, «Pion–pion scattering amplitude. IV. Improved analysis with once subtracted Roy-like equations up to 1100 MeV», *Phys. Rev. D* **83**, 074004 (2011), arXiv:1102.2183 [hep-ph].

[138] G. 't Hooft, «A planar diagram theory for strong interactions», *Nucl. Phys. B* **72**, 461 (1974).

[139] E. Witten, «Baryons in the $1N$ expansion», *Nucl. Phys. B* **160**, 57 (1979).

[140] Particle Data Group (S. Navas *et al.*), «Review of Particle Physics», *Phys. Rev. D* **110**, 030001 (2024).

[141] E. Ruiz Arriola, W. Broniowski, «0^{++} states in a large-N_c Regge approach», in: «Proceedings of Mini-Workshop on Understanding hadronic spectra», Bled, Slovenia, July 3–10, 2011, arXiv:1110.2863 [hep-ph].

[142] P. Masjuan, E. Ruiz Arriola, «Regge trajectories of excited baryons, quark–diquark models, and quark–hadron duality», *Phys. Rev. D* **96**, 054006 (2017).

[143] E. Ruiz Arriola, P. Masjuan, W. Broniowski, «Excited Hadrons and Quark–Hadron Duality», *Acta Phys. Pol. B Proc. Suppl.* **10**, 1079 (2017).

[144] E. Ruiz Arriola, P. Sanchez-Puertas, «Phase of the electromagnetic form factor of the pion», *Phys. Rev. D* **110**, 054003 (2024), arXiv:2403.07121 [hep-ph].

[145] BaBar Collaboration (J.P. Lees *et al.*), «Precise measurement of the $e^+e^- \to \pi^+\pi^-(\gamma)$ cross section with the initial-state radiation method at BaBar», *Phys. Rev. D* **86**, 032013 (2012), arXiv:1205.2228 [hep-ex].

[146] J.J. Sakurai, «Currents and Mesons», *University of Chicago Press*, 1969.

[147] W. Królikowski, «Partial conservation and the 2^+ mesons», *Phys. Lett. B* **24**, 305 (1967).

[148] K. Raman, «Spin-Two Mesons, the Stress Tensor, and a Field-Source Identity. I», *Phys. Rev. D* **2**, 1577 (1970).

[149] K. Raman, «Some Consequences of Tensor-Meson Dominance», *Phys. Rev. D* **3**, 2900 (1971).

[150] D. Toublan, «Lowest tensor-meson resonances contributions to the chiral perturbation theory low-energy coupling constants», *Phys. Rev. D* **53**, 6602 (1996); *Erratum ibid.* **57**, 4495 (1998).

[151] G. Ecker, C. Zauner, «Tensor meson exchange at low energies», *Eur. Phys. J. C* **52**, 315 (2007), arXiv:0705.0624 [hep-ph].

[152] M.D. Scadron, «Covariant Propagators and Vertex Functions for Any Spin», *Phys. Rev.* **165**, 1640 (1968).

[153] Y.V. Novozhilov, «Introduction to Elementary Particle Theory», International Series of Monographs in Natural Philosophy, *Pergamon Press*, Oxford, UK 1975.

[154] M.M. Nagels *et al.*, «Compilation of coupling constants and low-energy parameters», *Nucl. Phys. B* **109**, 1 (1976).

[155] X. Cao *et al.*, «Origin of the nucleon gravitational form factor $B_N(t)$: Exposition in light-front holographic QCD», *Phys. Rev. D* **113**, L071503 (2026), arXiv:2601.19141 [hep-ph].

[156] E. Ruiz Arriola, P. Sanchez-Puertas, W. Broniowski, «Hadronic form factors in QCD and the incompleteness problem in the time-like region», arXiv:2604.09185 [hep-ph].

[157] D.E. Soper, «Parton model and the Bethe–Salpeter wave function», *Phys. Rev. D* **15**, 1141 (1977).

[158] M. Burkardt, «Impact parameter dependent parton distributions and off-forward parton distributions for $\vec{\zeta}0$», *Phys. Rev. D* **62**, 071503(R) (2000); *Erratum ibid.* **66**, 119903 (2002).

[159] M. Diehl, «Generalized parton distributions in impact parameter space», *Eur. Phys. J. C* **25**, 223 (2002); *Erratum ibid.* **31**, 277 (2003).

[160] M. Burkardt, «Impact Parameter Space Interpretation for Generalized Parton Distributions», *Int. J. Mod. Phys. A* **18**, 173 (2003), arXiv:hep-ph/0207047.

[161] G.A. Miller, «Transverse Charge Densities», *Annu. Rev. Nucl. Part. Sci.* **60**, 1 (2010), arXiv:1002.0355 [nucl-th].

[162] A. Freese, G.A. Miller, «Convolution formalism for defining densities of hadrons», *Phys. Rev. D* **108**, 034008 (2023), arXiv:2210.03807 [hep-ph].

[163] A. Freese, «Mechanical form factors and densities of nonrelativistic fermions», *Phys. Rev. D* **112**, 034037 (2025), arXiv:2505.06135 [hep-ph].

[164] D. Fujii, M. Tanaka, «Scale-anomaly-induced binding pressure in hadrons», *Phys. Lett. B* **870**, 139872 (2025), arXiv:2507.23786 [hep-ph].

[165] J.Y. Panteleeva, M.V. Polyakov, «Forces inside the nucleon on the light front from 3D Breit frame force distributions: Abel tomography case», *Phys. Rev. D* **104**, 014008 (2021), arXiv:2102.10902 [hep-ph].

[166] A. Freese, G.A. Miller, «Unified formalism for electromagnetic and gravitational probes: Densities», *Phys. Rev. D* **105**, 014003 (2022), arXiv:2108.03301 [hep-ph].