PERSONAL THOUGHTS ON MY RESEARCH IN THE STRONG-CORRELATION PHYSICS

Many people have contributed to my understanding of the subject of strongly correlated quantum systems. The suggestion to work in that area came from my Ph.D. supervisor, Professor Janusz Morkowski. As a result of the initial thinking and reading the literature, the idea of the t–J model appeared to me in 1976. The many-hour discussion in 1978 with John Hubbard during my postdoc period at Imperial College convinced me that I was on the right track. Additionally, two discussions with Sir Nevill Mott during that period turned my attention to the physics of Mott localization. Subsequent cooperation with Jurgen (Georg) Honig from Purdue University, starting from 1981, particularly the discussions on metal–insulator transition in V_2O_3 systems, and later on $NiS_{2-x}Se_x$ compounds, resulted in the first thermodynamic theory of the Mott–Hubbard (metal–insulator) phase transition (1983–1987). The period ended with a detailed discussion (1989) on low-temperature properties of what is now known as an almost localized Fermi liquid (1989–1990). It ended with the basic idea of the spin-dependent effective mass in a strongly correlated Fermi liquid (1990). The spin-split masses have been subsequently observed by Ilya Sheikin in Grenoble in 2003 and in the Cambridge quantum matter group (A. McCollam, G. Lonzarich) in 2005, on example in heavy-fermion systems. The spin-split mass concept can be used to test the fundamental topic of the principle of indistinguishability *versus* distinguishability of quantum particles, as I hope to be able to document it soon in a separate work. Additionally, I also profited much from discussions and joint work with experimentalist Andrzej Ślebarski, with whom we proposed a simple scaling law for heavy fermions understood as metals with enormous (renormalized) heavy masses (2005). Also, the discovery of superconductivity in the $CeCu_2Si_2$ heavy-fermion system by Frank Steglich (1979), represented an intriguing puzzle to me and resulted in the Ph.D. thesis of Olga Howczak (2012), with a more or less complete qualitative phase diagram, comprising both metallic and Kondo insulating (paramagnetic and antiferromagnetic states). That work employed the ideas of real-space pairing induced by the Kondo and superexchange interactions.

A decisive development in the physics of strongly correlated fermions comes with the discovery of high-temperature superconductivity by J.G. Bednorz and A.K. Müller (1986), and by P.C.W. Chu and M.-K. Wu, (1987). Right at the beginning of that period, P.W. Anderson (1986) coined the idea that antiferromagnetic kinetic exchange interaction in Mott insulators, which he had derived in 1959, may be operative as the source of superconductive pairing of electrons in those quasi-two-dimensional systems if only they undergo a transition to the metallic phase upon doping the Mott insulator. An almost ideal model for this purpose then appeared to be the *t*–*J* model derived by me in 1976 (published a year later with K.A. Chao and A.M. Oleś). Therefore, in 1987 (published in 1988), I introduced an exact transformation for the kinetic exchange expressed as spin–spin interaction to that with real-space pairing operators and have shown that the coupling constant has a negative sign and hence makes the neighboring pairing interactions attractive. Some confusions arose, in my view, whether this coupling leads to the appearance of a quantum spin-liquid phase. But, instead of dwelling on that issue, I concentrated on the spin-singlet superconductivity resulting from it. I would like to mention here that this really pioneering period ended up with the work of F.C. Zhang and T.M. Rice (1988) on (approximate) justification of the one-band *t*–*J* model starting from the three-band *d*–*p* model. I think that my unpublished extensive notes on the *t*–*J* model as given to Maurice Rice in 1986 may have been partly helpful in that process. Unfortunately, our work has never been quoted there for one reason or another.

In the next stage, in my then-established new research group, first at Warsaw University (Krzysztof Byczuk) and later at Jagiellonian University in the area of strongly correlated systems, we concentrated on solving the *t*–*J* model for the high-T_c cuprates. We started with the Gutzwiller variational approach and have made it statistically consistent, *i.e.*, imposed within the original Gutzwiller variational approach the constraints that the results for physical quantities coming from variational minimization of energy agree with those calculated via a self-consistent procedure. This new factor makes the variational approach similar to the slave-boson theory of Kotliar and Ruckenstein (1986), but avoids introducing the unphysical Bose fields, which lead to a spurious condensation in the mean-field approximation. Here, the help from my Ph.D. students and postdocs: Jan Kaczmarczyk, Jakub Jędrak, Michał Zegrodnik, Maciej Fidrysiak, Marcin Abram, and Marcin Wysokiński has been crucial. The first works (2010–2012), after comparing them to experiment, provided us with an insight to what extent this sort of mean-field theory is insufficient. In the next stage (2013–2023), the statistically consistent Gutzwiller approach (SGA) has been extended systematically into **D**iagrammatic **E**xpansion of the **V**ariational **W**ave **F**unction (DE-GWF), within the scope of which we have made an extensive analysis of theory *versus* experiment, and qualitative *versus* quantitative. In the first stage, a help from J. Bünnemann from Marburg/Cottbus Universities has been very important. In general, one can say that the equilibrium phase diagram as a function of hole concentration agrees quite well, often in a semi-quantitative manner, including the appearance of the pair-density wave in a proper concentration range, encompassing also the paramagnon and acoustic dynamic excitations. All these aspects have been reviewed in our paper: J. Spałek, M. Fidrysiak, M. Zegrodnik, and A. Biborski, «Superconductivity in high- and related strongly correlated systems from a variational perspective: Beyond mean field theory», *Phys. Rep.* **959**, 1 (2022).

Two additional aspects of the work on high-T_c cuprates should be mentioned. First, of a more speculative character, concerns the appearance of the pseudogap in underdoped systems. Namely, I am of the opinion that the systematic DE-GWF approach allows for a definition of two energy scales for particles and distinguishes those near the Fermi surface, which are less correlated, from those located more deeply below the surface. The former are described by the mean-field (SGA) pairing gap and are related to a pseudogap, whereas a true superconducting gap is obtained by DE-GWF and it is smaller than the former.

The second aspect is connected with our work on the t–J–U model in which both the kinetic exchange interaction $\sim J$ and the Hubbard interaction $\sim U$ appear simultaneously (2017–now). This model has been elaborated by us in the last 9 years (with Michał Zegrodnik, Andrzej Biborski, Maciej Fidrysiak, and Tushar Dey). At first sight it seems that including concomitantly both the terms $\sim J$ and $\sim U$ in the starting Hamiltonian, is misleading. Leaving behind the intuitive explanation of this fact in the spirit of a systematic perturbation expansion of the Hubbard model, one can say the following: From a formal point of view, the t–J model limit can be achieved by taking the $U \to \infty$ limit in this more general model. Also, in such a general form of the Hamiltonian, we do not need to utilize the projected fermion-operator formulation, as in the t–J model, which is much more cumbersome to handle. Finally, in the $J = 0$ case, we reach formally the limit of the Hubbard model, in which the paired solution does not appear in the mean-field (SGA) approximation. Therefore, one can discuss then the role of higher-order correlations in stabilizing this superconductivity with real-space pairing. Work along these lines is in progress.

I would also like to mention my idea (2000) of treating the correlations in nano and molecular systems with the help of an exact analytical-numerical approach called EDABI (**E**xact **D**iagonalization **Ab I**nitio). These ideas are connected with the problem of calculating concomitantly single-particle wave function readjustment and correlations induced by the strong interactions on the same footing and in exactly solvable situations. A number of Ph.D. students contributed to this work: Adam Rycerz, Edward M. Görlich, Roman Zahorbeński, Jan Kurzyk, and concerned two groups of problems. The first of them was the work on single-valence electron chains and rings containing up to $N = 18$ atoms (*e.g.*, hydrogen chains). A number of interesting results have been obtained (for review see, *e.g.*, J. Spałek (2020)). The second problem is connected with the work on chemical bonding and introduction of degree of atomicity of electrons composing the covalent bond, even in such a canonical case as H_2 molecule (work with Maciej Hendzel, and Maciej Fidrysiak, 2022–2023). This physical analysis is based on an exact solution of the extended Heitler–London model of the H_2 molecule (J. Spałek, 2001, 2007). Such a combined exact 1^{st} (wave function) — and 2^{nd} (particles interactions) — quantization solution is possible only for relatively small correlated mono-systems. Nonetheless, it is rewarding that it can yield some practical implications about basic principles and the physics involved in those small condensed-matter systems.

It is a pleasure to note at the end that two "side" topics were undertaken during my career. First was on diluted magnetic (semimagnetic) semiconductors with Tomasz Dietl, Robert R. Gałązka, Jacek Furdyma, Andrzej Golnik, and Andrzej Lewicki, quite rewarding (1982–1987). It discussed the influence of classical magnetic fluctuations on quantum electronic states of large (Rydberg) donors (the so-called problem of bound magnetic polaron), as well as that on the role of superexchange in those Mott semiconductors. The second of them is the ongoing work on nonstandard statistics for strongly correlated particles. It started a long time ago (1988–1994) together with Włodek Wójcik and Krzysztof Byczuk and is now continuing with Piotr Kuterba and undergraduate student Mateusz Wójcik.

Thanks to all of them! Without their help and inventiveness, much of this works would never be finalized. I thank also my family: Ewa, Basia, Leszek, but that goes without any further saying.

Finally, I would like to cordially thank my coworkers who edited these proceedings: Danka Goc-Jagło, Maciek Fidrysiak, and Piotrek Kuterba. They even designed the cover page, as well as struggled with the authors (including J.S.), missing the successive deadlines.

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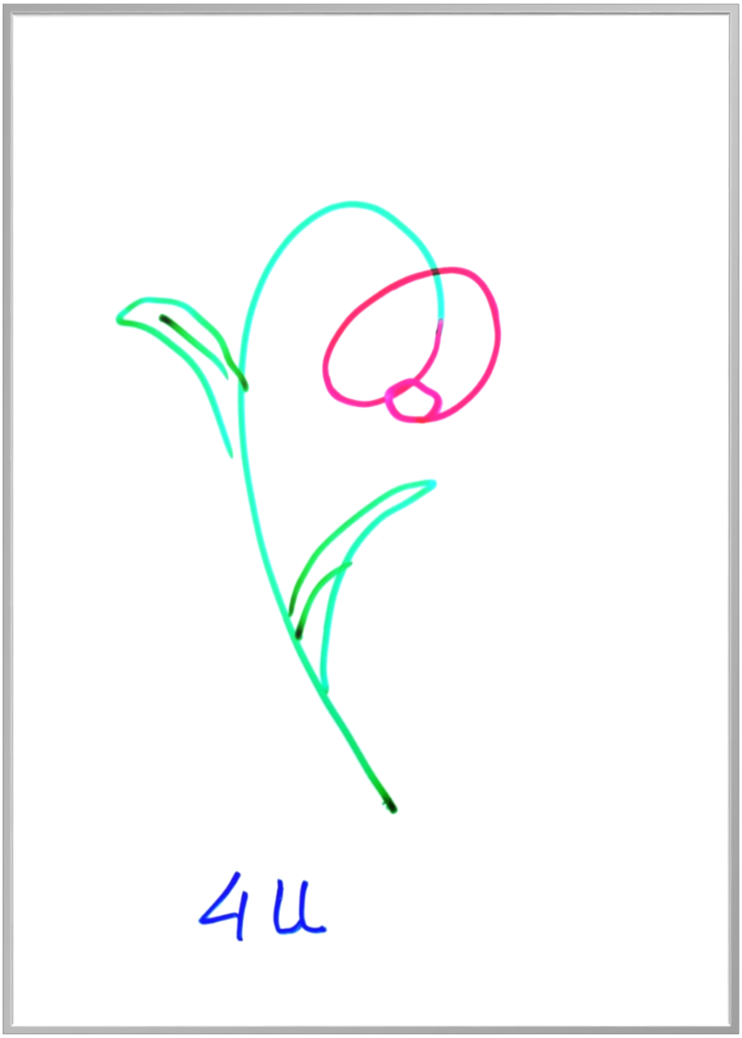
Józef Spałek

Kraków, May 7, 2026

P.S. Here is the motto of my work described above:

“*One of my strongest stylistic prejudices*
in science is that many of the facts Nature
confronts us with are so implausible, given
the simplicities of non-relativistic quantum
mechanics and statistical mechanics, that the
mere demonstration of a reasonable mechanism
leaves no doubt of the correct explanation. This
is so especially if it also correctly predicts
unexpected facts (...) Very often such, a simplified
model throws more light on the real workings of
nature than any number of “ab initio” calculations of
individual situations, which even where correct
often contain so much detail as to conceal
rather than reveal reality.”

P.W. Anderson, The Nobel Lecture (1977)



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Józef Spałek